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ORIGINAL RESEARCH

Quasi-contraction operators with common fixed point results and applications to stability, well-posedness Ostrowski property

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Abstract The purpose of this paper is to present some common fixed point results for two quasi-contraction operators in complete metric space which provide a positive answer to the question of Mihiţ, Moţ and Petruşel [16], concerning common fixed point results. Here we have established the existence and uniqueness of the common fixed point results to apply in stability analysis. More precisely, we will also see under which conditions the common fixed point problem satisfies well-posedness, Ulam-Hyers stability, and Ostrowski's property. Illustrative examples and an application are given to support the obtained results, which extend some of the theorems in the recent literature.

Keywords Fixed point · $ComFix$ · Quasi-contraction · Well-posed · Ulam-Hyers stability · Ostrowski property

Mathematics Subject Classification 47H10 · 54H25

1 Introduction

In 1922, Banach [2] established the existence and uniqueness of fixed points of certain types of maps in metric space. This result was a remarkable contribution due to its wide theoretical and numerical applications. Let $S : X \to X$ be a α-contraction i.e., $\exists\, \alpha \in (0, 1)$ such that

$$\rho(S(\vartheta), S(\varrho)) \leq \alpha\rho(\vartheta, \varrho) \ \ \forall \vartheta, \varrho \in X.$$

Then any α-contraction mapping $S : X \to X$ has a unique fixed point on a complete metric space (X, ρ) and the Picard sequence $\{S^n(\vartheta)\}$ iterates generated from any element $\vartheta \in X$ converges to a fixed point which is unique.

Definition 1.1 Let (X, ρ) be a metric space. Then, $S : X \to X$ is called a Picard operator on X if:

(i) $Fix(S) = \{\vartheta : S(\vartheta) = \vartheta\} = \{\vartheta^*\}$;
(ii) $\{S^n(\vartheta)\} \to \vartheta^*$ as $n \to \infty, \forall\, \vartheta \in X$.

Definition 1.2 Let (X, ρ) be a metric space. Then, $S : X \to X$ is called a weakly Picard operator if, $\forall\, \vartheta \in X$, the sequence $\{S^n(\vartheta)\}$ converges in S and the limit is a fixed point of S.

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Definition 1.3 $S : X \to X$ is said to be a ψ-weakly Picard operator on the metric space (X, ρ) if:

(i) S is a weakly Picard operator;
(ii) $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing and continuous at 0 with $\psi(0) = 0$;
(iii) $\rho(\vartheta, S^\infty(\vartheta)) \leq \psi(\rho(\vartheta, S(\vartheta)))$, $\forall \vartheta \in X$, where $S^\infty(\vartheta) = \lim_{n\to\infty} S^n(\vartheta)$.

In particular, if S is a Picard operator and $\vartheta^* \in X$ denotes its unique fixed point, then S is said to be ψ-Picard operator if

$$\rho(\vartheta, \vartheta^*) \leq \psi(\rho(\vartheta, S(\vartheta))),\ \forall\ \vartheta \in X.$$

If $\psi(t) = ct$, then S is called a c-weakly Picard operator and c-Picard operator respectively. For example, any α-contraction S on a complete metric space (X, ρ) is $\frac{1}{1-\alpha}$-Picard operator. For complementary and related results see, [3], [20], [21], [23].

Many authors, such as Kannan [12], Reich [23], Chatterjea [5], Hardy and Rogers [9], and Ćirić [6], [7] have extended the Banach contraction principles in various spaces. In this context, we have mainly focused on Ćirić type quasi-contractions [10] and generalized some of the related results. In 1971, Ćirić [6] has introduced a class of operators on a metric space (X, ρ) which satisfy the following condition:

$$\rho(S\vartheta, S\varrho) \leq q \max\{\rho(\vartheta, \varrho), \rho(\vartheta, S\vartheta), \rho(\varrho, S\varrho), \rho(\vartheta, S\varrho), \rho(\varrho, S\vartheta)\}\ \ \forall \vartheta, \varrho \in X \text{ and } 0 \leq q < 1. \tag{1.1}$$

The map satisfying (1.1) is said to be quasi-contraction condition.

Throughout this paper, we use some notations given by:

(i) $Fix(S) = \{\vartheta : S(\vartheta) = \vartheta\}$ as fixed point set of S and
(ii) $ComFix(S, T) = Fix(S) \cap Fix(T)$ as the common fixed point set of S and T.

We also denote the graph of operator S by $Graph(S) = \{(\vartheta, S(\vartheta)) : \vartheta \in X\}$.

1.1 Motivation and work done

Recently in the article [16], Mihiţ, Moţ and Petruşel have presented a new Ćirić type of operators to get common fixed point results for two operators in a complete metric space. They have also proposed a problem for a similar type of $ComFix$ point theory using a pair of operators in complete metric space. In this paper we address the open problem mentioned in [16] using a pair of quasi-contraction operators [25]. We have also given the sufficient conditions for the $ComFix$ point problem that satisfy well-posedness [24], Ulam-Hyers stability [18,26] and Ostrowski property [17]. An application and illustrative examples are added to support the obtained results. Our results extend some of the theorems in the recent literature [6,8,16,19–22].

2 Preliminary results

Let $S : X \to X$ be an operator on the metric space (X, ρ). Then, S is called graphic k-contraction [21] if $\exists$ $k \in (0, 1)$ such that

$$\rho(S(\vartheta), S^2(\vartheta)) \leq k\rho(\vartheta, S(\vartheta)),\ \ \forall\ \vartheta \in X.$$

Theorem 2.1 *[21] Let (X, ρ) be a complete metric space and the closed graph of $S : X \to X$ be a graphic k-contraction. Then the following holds:*

(i) the Picard sequence $\{S^n(\vartheta)\}$ converges to a fixed point ϑ^* of S;
(ii) S is $\frac{1}{1-k}$-weakly Picard operator;
(iii) $Fix(S) = \{\vartheta : S(\vartheta) = \vartheta\}$ satisfy the well-posed condition of Reich and Zaslavski, i.e., $Fix(S) = \{\vartheta^*\}$ and for any sequence $\{v_n\} \subseteq X$ with $\rho(v_n, S(v_n)) \to 0$ as $n \to \infty$, implies that $v_n \to \vartheta^*$ as $n \to \infty$;
(iv) $Fix(S) = \{\vartheta : S(\vartheta) = \vartheta\}$ has Ulam-Hyers stablity, i.e., $\exists\ c > 0$ such that $\forall\ \epsilon > 0$ and $\forall\ \tilde{\vartheta} \in X$ satisfying $\rho(\tilde{\vartheta}, S(\tilde{\vartheta})) \leq \epsilon$, implies that $\rho(\tilde{\vartheta}, \vartheta^*) \leq c\epsilon$.

Let $S : X \to X$ be an operator on the metric space (X, ρ) such that $Fix(S) = \{\vartheta^*\}$. Then, S is called β-quasi-contraction if $\exists\, \beta \in (0, 1)$ such that

$$\rho(S(\vartheta), \vartheta^*) \leq \beta\rho(\vartheta, \vartheta^*)\ \ \forall\, \vartheta \in X.$$

Theorem 2.2 *[21] Let $S : X \to X$ be a β-quasi-contraction on the metric space (X, ρ) such that $Fix(S) = \{\vartheta^*\}$. Then S has the Ostrowski stability property, i.e., $Fix(S) = \{\vartheta^*\}$ and for any $\{w_n\} \subset X$ with $\rho(w_{n+1}, S(w_n)) \to 0$ implies that $w_n \to \vartheta^*$ as $n \to \infty$.*

3 Main results

Theorem 3.1 *Let $S, T : X \to X$ be two operators on a complete metric space (X, ρ) and suppose that $\exists$ $\alpha \in (0, \frac{1}{3})$ and $\forall \vartheta, \varrho \in X$, S and T satisfy the condition:*

$$\rho(S(\vartheta), T(\varrho)) \leq \alpha \max\Big\{\rho(\vartheta, \varrho), \rho(\vartheta, S(\vartheta)), \rho(\vartheta, T(\varrho)), \rho(\varrho, S(\vartheta)), \rho(\varrho, T(\varrho))\Big\}. \tag{3.1}$$

Then the following conclusions hold:

(i) $ComFix(S, T) = Fix(S) = Fix(T) = \{\vartheta^*\}$;

(ii) $\forall\, \vartheta_0 \in X$, the sequence $\{\vartheta_n\}$ given by $\vartheta_{2n+1} = S(\vartheta_{2n})$, $\vartheta_{2n+2} = T(\vartheta_{2n+1})\ \forall\, n \in \mathbb{N}$, converges to ϑ^* as $n \to \infty$;

(iii) $\forall\, \varrho_0 \in X$, the sequence $\{\varrho_n\}$ given by $\varrho_{2n+1} = T(\varrho_{2n})$, $\varrho_{2n+2} = S(\varrho_{2n+1})\ \forall\, n \in \mathbb{N}$, converges to ϑ^* as $n \to \infty$.

Proof (i) Let $\vartheta^* \in Fix(S)$ therefore $S(\vartheta^*) = \vartheta^*$. Therefore from the condition (3.1),

$$\begin{aligned}\rho(\vartheta^*, T(\vartheta^*)) &= \rho(S(\vartheta^*), T(\vartheta^*))\\ &\leq \alpha \max\{\rho(\vartheta^*, \vartheta^*), \rho(\vartheta^*, S(\vartheta^*)), \rho(\vartheta^*, T(\vartheta^*)), \rho(\vartheta^*, S(\vartheta^*)), \rho(\vartheta^*, T(\vartheta^*))\}\\ &= \alpha \max\{0, \rho(\vartheta^*, T(\vartheta^*))\}. \qquad [\text{ since } S(\vartheta^*) = \vartheta^*\,]\end{aligned}$$

As such $\rho(\vartheta^*, T(\vartheta^*)) \leq \alpha\rho(\vartheta^*, T(\vartheta^*))$. If $\rho(\vartheta^*, T(\vartheta^*)) \neq 0$, then $\alpha \geq 1$, which is a contradiction.

Therefore $\rho(\vartheta^*, T(\vartheta^*)) = 0$, which implies that $\vartheta^* = T(\vartheta^*)$, and so $\vartheta^* \in Fix(T)$, that is $Fix(S) \subseteq Fix(T)$.

Similarly, it can be shown that $Fix(T) \subseteq Fix(S)$. Therefore $Fix(S) = Fix(T)$.

Now we prove that S and T have at most one $ComFix$ point. If possible let, $\vartheta^*, \varrho^* \in Fix(S) \cap Fix(T)$ then by the relation (3.1), $\rho(\vartheta^*, \varrho^*) \leq \alpha\rho(\vartheta^*, \varrho^*)$.

If $\rho(\vartheta^*, \varrho^*) \neq 0$ then $\alpha \geq 1$, which is a contradiction. Therefore $\rho(\vartheta^*, \varrho^*) = 0$, which implies that $\vartheta^* = \varrho^*$.

Hence we conclude that $ComFix(S, T) = Fix(S) = Fix(T) = \{\vartheta^*\}$.

(ii) For arbitrary $\vartheta_0 \in X$, the sequence $\{\vartheta_n\}$ given by $\vartheta_{2n+1} = S(\vartheta_{2n})$, $\vartheta_{2n+2} = T(\vartheta_{2n+1})\ \forall\, n \in \mathbb{N}$.

Now,

$$\begin{aligned}\rho(\vartheta_1, \vartheta_2) = \rho(S(\vartheta_0), T(\vartheta_1)) &\leq \alpha \max\{\rho(\vartheta_0, \vartheta_1), \rho(\vartheta_0, S(\vartheta_0)), \rho(\vartheta_0, T(\vartheta_1)), \rho(\vartheta_1, S(\vartheta_0)), \rho(\vartheta_1, T(\vartheta_1))\}\\ &= \alpha \max\{\rho(\vartheta_0, \vartheta_1), \rho(\vartheta_0, \vartheta_2), \rho(\vartheta_1, \vartheta_2)\}\\ &\qquad [\text{Here if } \max\{\rho(\vartheta_0, \vartheta_1), \rho(\vartheta_0, \vartheta_2), \rho(\vartheta_1, \vartheta_2)\} = \rho(\vartheta_1, \vartheta_2)\\ &\qquad \text{then } \alpha \geq 1, \text{ which is a contradiction}]\\ &\leq \alpha \max\{\rho(\vartheta_0, \vartheta_1), \rho(\vartheta_0, \vartheta_2)\}\\ &\leq \alpha\{\rho(\vartheta_0, \vartheta_1) + \rho(\vartheta_0, \vartheta_2)\}\\ &\leq \alpha\{\rho(\vartheta_0, \vartheta_1) + \rho(\vartheta_0, \vartheta_1) + \rho(\vartheta_1, \vartheta_2)\}. \qquad [\text{By using triangle inequality }]\end{aligned}$$

Therefore, $\rho(\vartheta_1, \vartheta_2) \leq \frac{2\alpha}{1-\alpha}\rho(\vartheta_0, \vartheta_1)$. Now using the principle of mathematical induction we get,

$$\rho(\vartheta_n, \vartheta_{n+1}) \leq (\frac{2\alpha}{1-\alpha})^n \rho(\vartheta_0, \vartheta_1). \tag{3.2}$$

Since $\alpha \in (0, \frac{1}{3})$, therefore $\frac{2\alpha}{1-\alpha} \leq 1$. Now taking limit $n \to \infty$ on both sides of (3.2) we get,

$$\lim_{n\to\infty} \rho(\vartheta_{n+1}, \vartheta_n) = 0.$$

Let us consider $m > n$.

$$\begin{aligned} \text{Now,} \quad \rho(\vartheta_n, \vartheta_m) &\leq \rho(\vartheta_n, \vartheta_{n+1}) + \rho(\vartheta_{n+1}, \vartheta_{n+2}) + \cdots + \rho(\vartheta_{m-1}, \vartheta_m) \\ &\leq \{(\frac{2\alpha}{1-\alpha})^n + (\frac{2\alpha}{1-\alpha})^{n+1} + \cdots + (\frac{2\alpha}{1-\alpha})^{m-1}\}\rho(\vartheta_0, \vartheta_1) \\ &= (\frac{2\alpha}{1-\alpha})^n\{1 + \frac{2\alpha}{1-\alpha} + \cdots + (\frac{2\alpha}{1-\alpha})^{m-n-1}\}\rho(\vartheta_0, \vartheta_1) \\ &= (\frac{1-\alpha}{1-3\alpha})\{(\frac{2\alpha}{1-\alpha})^n - (\frac{2\alpha}{1-\alpha})^m\}\rho(\vartheta_0, \vartheta_1). \end{aligned}$$

The right hand side goes to 0 as $m, n \to \infty$, therefore, $\lim_{n,m\to\infty} \rho(\vartheta_n, \vartheta_m) = 0$, which implies $\{\vartheta_n\}$ is a Cauchy sequence in X. Let it converge to ϑ^*.

Now,

$$\begin{aligned} \rho(\vartheta^*, S(\vartheta^*)) &\leq \rho(\vartheta^*, \vartheta_{2n+2}) + \rho(\vartheta_{2n+2}, S(\vartheta^*)) \\ &\leq \rho(\vartheta^*, \vartheta_{2n+2}) + \rho(T(\vartheta_{2n+1}), S(\vartheta^*)) \\ &\leq \rho(\vartheta^*, \vartheta_{2n+2}) + \alpha \max\Big\{\rho(\vartheta_{2n+1}, \vartheta^*), \rho(\vartheta_{2n+1}, T(\vartheta_{2n+1})), \rho(\vartheta_{2n+1}, S(\vartheta^*)), \\ &\qquad \rho(\vartheta^*, T(\vartheta_{2n+1})), \rho(\vartheta^*, S(\vartheta^*))\Big\} \\ &\leq \rho(\vartheta^*, \vartheta_{2n+2}) + \alpha \max\Big\{\rho(\vartheta_{2n+1}, \vartheta^*), \rho(\vartheta_{2n+1}, \vartheta_{2n+2}), \rho(\vartheta_{2n+1}, S(\vartheta^*)), \\ &\qquad \rho(\vartheta^*, \vartheta_{2n+2}), \rho(\vartheta^*, S(\vartheta^*))\Big\}. \end{aligned}$$

Now taking limit $n \to \infty$ on both sides we get, $\rho(\vartheta^*, S(\vartheta^*)) \leq \alpha\rho(\vartheta^*, S(\vartheta^*))$.

If $\rho(\vartheta^*, S(\vartheta^*)) \neq 0$, then $\alpha \geq 1$, which is a contradiction. Therefore we have $\rho(\vartheta^*, S(\vartheta^*)) = 0$, which implies $\vartheta^* = S(\vartheta^*)$.

Again,

$$\begin{aligned} \rho(\vartheta^*, T(\vartheta^*)) &\leq \rho(\vartheta^*, \varrho_{2n+1}) + \rho(\varrho_{2n+1}, T(\vartheta^*)) \\ &= \rho(\vartheta^*, \varrho_{2n+1}) + \rho(S(\varrho_{2n}), T(\vartheta^*)) \\ &\leq \rho(\vartheta^*, \varrho_{2n+1}) + \alpha \max\Big\{\rho(\varrho_{2n}, \vartheta^*), \rho(\varrho_{2n}, S(\varrho_{2n})), \rho(\varrho_{2n}, T(\vartheta^*)), \\ &\qquad \rho(\vartheta^*, S(\varrho_{2n})), \rho(\vartheta^*, T(\vartheta^*))\Big\} \\ &\leq \rho(\vartheta^*, \varrho_{2n+1}) + \alpha \max\Big\{\rho(\varrho_{2n}, \vartheta^*), \rho(\varrho_{2n}, \varrho_{2n+1}), \rho(\varrho_{2n}, T(\vartheta^*)), \rho(\vartheta^*, \varrho_{2n+1}), \rho(\vartheta^*, T(\vartheta^*))\Big\}. \end{aligned}$$

Now taking limit $n \to \infty$ on both sides we get, $\rho(\vartheta^*, T(\vartheta^*)) \leq \alpha\rho(\vartheta^*, T(\vartheta^*))$.

If $\rho(\vartheta^*, T(\vartheta^*)) \neq 0$, then $\alpha \geq 1$, which is a contradiction. Therefore we have $\rho(\vartheta^*, T(\vartheta^*)) = 0$, which implies $\vartheta^* = T(\vartheta^*)$.

Hence, $\vartheta_{2n+1} = S(\vartheta_{2n})$, $\vartheta_{2n+2} = T(\vartheta_{2n+1})$ $\forall\, n \in \mathbb{N}$ as such the sequence $\{\vartheta_n\}$ converge to ϑ^* as $n \to \infty$.

(iii) Similarly for arbitrary $\varrho_0 \in X$, the sequence $\{\varrho_n\}$ given by $\varrho_{2n+1} = T(\varrho_{2n})$, $\varrho_{2n+2} = S(\varrho_{2n+1})$ $\forall\, n \in \mathbb{N}$, it can be shown that $\{\varrho_n\}$ is a Cauchy sequence in X and it converges to ϱ^*.

Now,

$$\begin{aligned} \rho(\varrho^*, T(\varrho^*)) &\leq \rho(\varrho^*, \varrho_{2n+2}) + \rho(\varrho_{2n+2}, T(\varrho^*)) \\ &= \rho(\varrho^*, \varrho_{2n+2}) + \rho(T(\varrho_{2n+1}), T(\varrho^*)) \\ &\leq \rho(\varrho^*, \varrho_{2n+2}) + \alpha \max\Big\{\rho(\varrho_{2n+1}, \varrho^*), \rho(\varrho_{2n+1}, T(\varrho_{2n+1})), \rho(\varrho_{2n+1}, T(\varrho^*)), \\ &\qquad \rho(\varrho^*, T(\varrho_{2n+1})), \rho(\varrho^*, T(\varrho^*))\Big\} \\ &\leq \rho(\varrho^*, \varrho_{2n+2}) + \alpha \max\Big\{\rho(\varrho_{2n+1}, \varrho^*), \rho(\varrho_{2n+1}, \varrho_{2n+2}), \rho(\varrho_{2n+1}, T(\varrho^*)), \\ &\qquad \rho(\varrho^*, \varrho_{2n+2}), \rho(\varrho^*, T(\varrho^*))\Big\}. \end{aligned}$$

Now taking limit $n \to \infty$ on both sides we get, $\rho(\varrho^*, T(\varrho^*)) \leq \alpha\rho(\varrho^*, T(\varrho^*))$.

If $\rho(\varrho^*, T(\varrho^*)) \neq 0$, then $\alpha \geq 1$, which is a contradiction. Therefore $\rho(\varrho^*, T(\varrho^*)) = 0$, which implies $\varrho^* = T(\varrho^*)$.

Since, $ComFix(S, T) = Fix(S) = Fix(T)$, therefore $\vartheta^* = \varrho^*$.

Hence $\varrho_{2n+1} = T(\varrho_{2n})$, $\varrho_{2n+2} = S(\varrho_{2n+1})$ $\forall\, n \in \mathbb{N}$ and that the sequence $\{\varrho_n\}$ converge to ϑ^* as $n \to \infty$.

Example 3.1 Suppose $S, T : [0, 2] \to [0, 2]$ be two operators such that

$$S(\vartheta) = \begin{cases} \frac{\vartheta}{10}, & \text{if } 0 \leq \vartheta \leq 1 \\ \frac{\vartheta}{5}, & \text{if } \vartheta > 1 \end{cases}$$

and

$$T(\varrho) = \frac{\varrho^2 + \varrho}{10}, \ \ \varrho \in [0, 2].$$

Choose $\vartheta = \frac{999}{1000}$ and $\varrho = \frac{1001}{1000}$, then $\rho(S(\vartheta), T(\varrho)) = 0.1004001$ and $\rho(\vartheta, \varrho) = 0.002$.

Therefore $\rho(S(\vartheta), T(\varrho)) \leq \alpha\, \rho(\vartheta, \varrho)$ imply that $\alpha \geq 50.20005$, which is a contradiction. Therefore we can not assume the condition that $\exists\, \alpha < 1$ such that $\rho(S(\vartheta), T(\varrho)) \leq \alpha\, \rho(\vartheta, \varrho)$ $\forall \vartheta, \varrho \in [0, 2]$.

On the other hand, the maximum value of $\rho(S(\vartheta), T(\varrho))$ for all values $\vartheta, \varrho \in [0, 2]$ is $= \frac{3}{5}$ and

$$\max\Big\{\rho(\vartheta, \varrho), \rho(\vartheta, S(\vartheta)), \rho(\vartheta, T(\varrho)), \rho(\varrho, S(\vartheta)), \rho(\varrho, T(\varrho))\Big\} = 2\ \forall\, \vartheta, \varrho \in [0, 2].$$

Now, if we choose $\alpha \in (\frac{3}{10}, \frac{1}{3})$ then the operators S and T satisfy the relation (3.1) and [0, 2] is a complete metric space.

Therefore from the Theorem 3.1, we can conclude that the operators S and T have a $ComFix$ point. Moreover S and T have $ComFix(S, T) = 0$.

Theorem 3.2 *Let $S, T : X \to X$ be two operators on a complete metric space (X, ρ) such that $\exists\, \alpha \in (0, \frac{1}{3})$ and $\forall \vartheta, \varrho \in X$, satisfying the condition:*

$$\rho(S(\vartheta), T(\varrho)) \leq \alpha \max\Big\{\rho(\vartheta, \varrho), \rho(\vartheta, S(\vartheta)), \rho(\vartheta, T(\varrho)), \rho(\varrho, S(\vartheta)), \rho(\varrho, T(\varrho))\Big\}.$$

Then we have the following conclusions:

(i) if $\alpha < \frac{3-\sqrt{5}}{4}$, then S and T are graph contractions;
(ii) if $\alpha < \frac{1}{4}$, then S and T are β-quasi-contractions;
(iii) if $\alpha < \frac{3-\sqrt{5}}{4}$, then S and T are c-Picard operators, with $c = \frac{(1-\alpha)^2}{1-4\alpha-\alpha^2}$.

Proof Let $\vartheta \in X$.

$$\begin{aligned} \rho(S^2(\vartheta), S(\vartheta)) &\leq \rho(S^2(\vartheta), T(\vartheta^*)) + \rho(S(\vartheta), T(\vartheta^*)) \\ &\leq \alpha \max\Big\{\rho(S(\vartheta), \vartheta^*), \rho(S(\vartheta), S^2(\vartheta)), \rho(S(\vartheta), T(\vartheta^*)), \rho(\vartheta^*, S^2(\vartheta)), \rho(\vartheta^*, T(\vartheta^*))\Big\} + \\ &\qquad \alpha \max\Big\{\rho(\vartheta, \vartheta^*), \rho(\vartheta, S(\vartheta)), \rho(\vartheta, T(\vartheta^*)), \rho(\vartheta^*, S(\vartheta)), \rho(\vartheta^*, T(\vartheta^*))\Big\} \\ &\leq \alpha \max\Big\{\rho(S(\vartheta), \vartheta^*), \rho(S(\vartheta), S^2(\vartheta)), \rho(\vartheta^*, S^2(\vartheta))\Big\} + \\ &\qquad \alpha \max\Big\{\rho(\vartheta, \vartheta^*), \rho(\vartheta, S(\vartheta)), \rho(\vartheta^*, S(\vartheta))\Big\} \\ &\leq \alpha[\rho(S(\vartheta), \vartheta^*) + \rho(S(\vartheta), S^2(\vartheta)) + \rho(\vartheta^*, S^2(\vartheta))] + \alpha[\rho(\vartheta, \vartheta^*) + \rho(\vartheta, S(\vartheta)) + \rho(\vartheta^*, S(\vartheta))]. \end{aligned}$$

Therefore,

$$\rho(S^2(\vartheta), S(\vartheta)) \leq \alpha[\rho(\vartheta, \vartheta^*) + 2\rho(S(\vartheta), \vartheta^*) + \rho(S^2(\vartheta), \vartheta^*) + \rho(S(\vartheta), S^2(\vartheta)) + \rho(\vartheta, S(\vartheta))]. \tag{3.3}$$

$$\begin{aligned} \text{Now,}\quad \rho(S^2(\vartheta), \vartheta^*) &= \rho(S^2(\vartheta), S(\vartheta^*)) \leq \alpha \max\{\rho(S(\vartheta), \vartheta^*), \rho(S^2(\vartheta), S(\vartheta)), \rho(S^2(\vartheta), \vartheta^*)\} \\ &\leq \alpha[\rho(S(\vartheta), \vartheta^*) + \rho(S^2(\vartheta), S(\vartheta)) + \rho(S^2(\vartheta), \vartheta^*)]. \end{aligned}$$

Therefore,

$$\rho(S^2(\vartheta), \vartheta^*) \leq \frac{\alpha}{1-\alpha}[\rho(S(\vartheta), \vartheta^*) + \rho(S^2(\vartheta), S(\vartheta))]. \tag{3.4}$$

Now using relations (3.3) and (3.4) we get,

$$\rho(S^2(\vartheta), S(\vartheta)) \leq \alpha\Big[\rho(\vartheta, \vartheta^*) + 2\rho(S(\vartheta), \vartheta^*) + \frac{\alpha}{1-\alpha}[\rho(S(\vartheta), \vartheta^*) + \rho(S^2(\vartheta), S(\vartheta))] + \rho(S(\vartheta), S^2(\vartheta)) + \rho(S(\vartheta), \vartheta)\Big].$$

Therefore,

$$\rho(S^2(\vartheta), S(\vartheta)) \leq \frac{\alpha(1-\alpha)}{1-2\alpha}\Big[\rho(\vartheta, \vartheta^*) + \frac{2-\alpha}{1-\alpha}\rho(S(\vartheta), \vartheta^*) + \rho(S(\vartheta), \vartheta)\Big]. \tag{3.5}$$

Again,

$$\begin{aligned}\rho(S(\vartheta), \vartheta^*) = \rho(S(\vartheta), S(\vartheta^*)) &\leq \alpha \max\{\rho(\vartheta, \vartheta^*), \rho(S(\vartheta), \vartheta), \rho(S(\vartheta), \vartheta^*)\} \\ &\leq \alpha[\rho(\vartheta, \vartheta^*) + \rho(S(\vartheta), \vartheta) + \rho(S(\vartheta), \vartheta^*)].\end{aligned}$$

Therefore,

$$\rho(S(\vartheta), \vartheta^*) \leq \frac{\alpha}{1-\alpha}[\rho(\vartheta, \vartheta^*) + \rho(S(\vartheta), \vartheta)]. \tag{3.6}$$

Again,

$$\rho(\vartheta, \vartheta^*) \leq \rho(S(\vartheta), \vartheta) + \rho(S(\vartheta), \vartheta^*). \tag{3.7}$$

Therefore from above relations (3.6) and (3.7) we have,

$$\rho(S(\vartheta), \vartheta^*) \leq \frac{2\alpha}{1-2\alpha}\rho(S(\vartheta), \vartheta). \tag{3.8}$$

$$\rho(\vartheta, \vartheta^*) \leq \frac{1}{1-2\alpha}\rho(S(\vartheta), \vartheta). \tag{3.9}$$

Therefore using the relations (3.5), (3.8) and (3.9) we obtain,

$$\rho(S^2(\vartheta), S(\vartheta)) \leq \frac{2\alpha}{(1-2\alpha)^2}\rho(S(\vartheta), \vartheta). \tag{3.10}$$

If $\alpha < \frac{3-\sqrt{5}}{4}$, then $\frac{2\alpha}{(1-2\alpha)^2} < 1$, i.e., is a graphic contraction.

(ii) Again, from the relation (3.6) we have,

$$\begin{aligned}\rho(S(\vartheta), \vartheta^*) &\leq \frac{\alpha}{1-\alpha}[\rho(\vartheta, \vartheta^*) + \rho(S(\vartheta), \vartheta)] \\ &\leq \frac{\alpha}{1-\alpha}[\rho(\vartheta, \vartheta^*) + \rho(S(\vartheta), \vartheta^*) + \rho(\vartheta, \vartheta^*)].\end{aligned}$$

Therefore,

$$\rho(S(\vartheta), \vartheta^*) \leq \frac{2\alpha}{1-2\alpha}\rho(\vartheta, \vartheta^*). \tag{3.11}$$

If $\alpha < \frac{1}{4}$ then $\frac{2\alpha}{1-2\alpha} < 1$, and so is a β-quasi-contraction.

(iii) Again from the relation (3.10), we can say that S is graphic $\tau = \frac{2\alpha}{(1-2\alpha)^2}$ contraction.

Therefore from the Theorem 2.1, we obtain that S is a $\frac{1}{1-\tau} = \frac{4\alpha^2-4\alpha+1}{4\alpha^2-6\alpha+1}$ Picard operator.

Hence the theorem.

Theorem 3.3 *Let $S, T : X \to X$ be two operators on the complete metric space (X, ρ) such that $\exists\, \alpha \in (0, \frac{1}{3})$ and $\forall \vartheta, \varrho \in X$, satisfying the condition:*

$$\rho(S(\vartheta), T(\varrho)) \leq \alpha \max\Big\{\rho(\vartheta, \varrho), \rho(\vartheta, S(\vartheta)), \rho(\vartheta, T(\varrho)), \rho(\varrho, S(\vartheta)), \rho(\varrho, T(\varrho))\Big\}.$$

Then the following conclusions hold:

(i) if $\alpha < \frac{3-\sqrt{5}}{4}$, then $Fix(S) = \{\vartheta : S(\vartheta) = \vartheta\}$ and $Fix(T) = \{\vartheta : T(\vartheta) = \vartheta\}$ will satisfy well-posed conditions of Reich and Zaslavski;

(ii) if $\alpha < \frac{3-\sqrt{5}}{4}$, then $Fix(S) = \{\vartheta : S(\vartheta) = \vartheta\}$ and $Fix(T) = \{\vartheta : T(\vartheta) = \vartheta\}$ have Ulam-Hyers stability;

(iii) if $\alpha < \frac{1}{4}$, then S and T have the Ostrowski property.

Proof (i) The operators S and T are both graphic contractions for ($\alpha < \frac{3-\sqrt{5}}{4}$) [see Theorem 3.2]. Therefore by the Theorem 2.1, we can conclude that $Fix(S)$ and $Fix(T)$ satisfy Reich and Zaslavski conditions for well-posed.

(ii) Also from the Theorem 2.1 and Theorem 3.2, we can conclude that $Fix(S)$ and $Fix(T)$ have Ulam-Hyers stability when $\alpha < \frac{3-\sqrt{5}}{4}$.

(iii) The operators S and T are both β-quasi-contractions for ($\alpha < \frac{1}{4}$) [see Theorem 3.2]. Therefore by the Theorem 2.2, we can conclude that $\alpha < \frac{1}{4}$, as such S and T have the Ostrowski property.

4 An application

Let us construct the following operational problem: find $(\vartheta, \varrho) \in X \times Y$ which satisfies the successive relations

$$\begin{aligned} \vartheta &= S(\varrho) \\ \varrho &= T(\vartheta) \\ (\vartheta, \varrho) &= h(\vartheta, \varrho), \end{aligned}$$

where $S : Y \to X, T : X \to Y$ and $h = (h_1, h_2) : X \times Y \to X \times Y$ are given operators and X, Y are non-empty and closed subset of a complete metric space (M, ρ).

We consider the following hypotheses:

$$(i)\quad \exists\, \beta \in (0, \frac{1}{3}) \quad \text{such that } \rho(h_1(\vartheta_1, \varrho_1), S(\varrho_2)) \leq \beta \max\{\rho(\varrho_1, \varrho_2), \rho(\vartheta_1, h_1(\vartheta_1, \varrho_1)), \rho(\vartheta_1, S(\varrho_2))\};$$

$$(ii)\quad \exists\, \lambda \in (0, \frac{1}{3}) \quad \text{such that } \rho(h_2(\vartheta_1, \varrho_1), T(\vartheta_2)) \leq \lambda \max\{\rho(\vartheta_1, \vartheta_2), \rho(\varrho_1, h_2(\vartheta_1, \varrho_1)), \rho(\varrho_1, T(\vartheta_2))\}.$$

Let us define the metric $\tilde{\rho}$ on $X \times Y$ for $z_1 = (\vartheta_1, \varrho_1), z_2 = (\vartheta_2, \varrho_2)$ by

$$\tilde{\rho}(z_1, z_2) = \max\{\rho(\vartheta_1, \vartheta_2), \rho(\varrho_1, \varrho_2)\}.$$

We denote $t : X \times Y \to X \times Y$, $t(\vartheta, \varrho) = (S(\varrho), T(\vartheta))$ and $\alpha = \max\{\beta, \lambda\}$.

From the above constructions, our problem becomes a $ComFix$ point problem of the following form

$$(\vartheta, \varrho) = t(\vartheta, \varrho) = h(\vartheta, \varrho).$$

Therefore we have,

$$\begin{aligned}
\tilde{\rho}(h(z_1), t(z_2)) &= \max\Big\{\rho(h_1(\vartheta_1, \varrho_1), S(\varrho_2)), \rho(h_2(\vartheta_1, \varrho_1), T(\vartheta_2))\Big\} \\
&\leq \max\Big\{\beta \max\{\rho(\varrho_1, \varrho_2), \rho(\vartheta_1, h_1(\vartheta_1, \varrho_1)), \rho(\vartheta_1, S(\varrho_2))\}, \\
&\qquad\qquad \lambda \max\{\rho(\vartheta_1, \vartheta_2), \rho(\varrho_1, h_2(\vartheta_1, \varrho_1)), \rho(\varrho_1, T(\vartheta_2))\}\Big\} \\
&\leq \alpha \max\Big\{ \max\{\rho(\vartheta_1, \vartheta_2), \rho(\varrho_1, \varrho_2)\}, \max\{\rho(\vartheta_1, h_1(\vartheta_1, \varrho_1)), \rho(\varrho_1, h_2(\vartheta_1, \varrho_1))\}, \\
&\qquad\qquad \max\{\rho(\vartheta_1, S(\varrho_2)), \rho(\varrho_1, T(\vartheta_2))\}\Big\} \\
&\leq \alpha \max\Big\{\tilde{\rho}(z_1, z_2), \tilde{\rho}(z_1, h(z_1)), \tilde{\rho}(z_1, t(z_2))\Big\} \\
&\leq \alpha \max\Big\{\tilde{\rho}(z_1, z_2), \tilde{\rho}(z_1, h(z_1)), \tilde{\rho}(z_1, t(z_2)), \tilde{\rho}(z_2, h(z_1)), \tilde{\rho}(z_2, t(z_2))\Big\}.
\end{aligned}$$

Thus h and t satisfy the condition (3.1). Therefore, using the Theorem 3.1, we can get $ComFix$ point of h and t i.e., the existence and uniqueness of the solutions, and also can apply all the stability properties using Theorem 3.3.

This constructive problem is also called altering point problems [27]. The above constructive model can also be used in the hierarchical system of nonlinear variational inequality problems [27], [28].

5 Conclusion

In this work, we have generalized the $ComFix$ point problem for a pair of operators satisfying quasi-contraction-type metric conditions. We have proved the existence and uniqueness of the $ComFix$ point problem using this contraction. After that, with the help of graphical contraction and β-quasi contraction, we have proved the sufficient condition for the $ComFix$ point problem to be well-posed and to have Ulam-Hyers stability and Ostrowski property.

We can construct a similar type of $ComFix$ point theory in a complete metric space (X, ρ) for a pair of operators using convex contraction instead of graphic k-contraction. We recall that, an operator $S : X \to X$ on (X, ρ), is said to be (l, k)-convex contraction if $\exists\, l, k \in \mathbb{R}_+$ with $l + k < 1$ such that

$$\rho(S^2(\vartheta), S^2(\varrho)) \leq l\rho(S(\vartheta), S(\varrho)) + k\rho(\vartheta, \varrho) \quad \forall\ \vartheta, \varrho \in X.$$

Since any continuous (l, k)-convex contraction on a complete metric space (X, ρ) is a Picard operator. See, [1], [4], [11], [15] for related results and extensions.

The following open problem may arise here: Deduce a similar type of $ComFix$ point theory in a complete metric space (X, ρ) for a pair of self-operators satisfying the generalized quasi-contraction. We recall that a mapping S is said to be a generalized quasi-contraction if $\exists\, q \in [0, 1)$ such that $\forall\ \vartheta, \varrho \in X$,

$$\begin{aligned}\rho(S\vartheta, S\varrho) \leq q \max\{&\rho(\vartheta, \varrho), \rho(\vartheta, S\vartheta), \rho(\varrho, S\varrho), \rho(\vartheta, S\varrho), \rho(\varrho, S\vartheta),\\ &\rho(S^2\vartheta, \vartheta), \rho(S^2\vartheta, S\vartheta), \rho(S^2\vartheta, \varrho), \rho(S^2\vartheta, S\varrho)\}.\end{aligned}$$

For related results and extensions, see, [13], [14].

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Declarations

Conflicts of interest The authors have no conflict of interest in this paper.

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