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ORIGINAL RESEARCH

Action of three generalized derivations on Lie ideals in prime rings with n-Engel condition

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Abstract Let R be a prime ring with extended centroid C and characteristic of R be different from 2. Suppose that U is the Utumi quotient ring of R and $\mathcal{L}$ a noncentral Lie ideal of R. Assuming $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ three generalized derivations of R, we investigate the identity

$$[\mathcal{T}_2(u)\mathcal{T}_1(u) - \mathcal{T}_1(u)u + u\mathcal{T}_3(u), u]_n = 0$$

for all $u \in \mathcal{L}$, and then we describe the forms of the maps, where $n \geq 1$ is a fixed integer.

Keywords Derivation · Generalized derivation · Prime ring

Mathematics Subject Classification 16W25 · 16N60

1 Introduction

Throughout this paper R will be a prime ring with center $Z(R)$, extended centroid C and U be its Utumi quotient ring. The standard polynomial of degree 4 is defined as

$$s_4(\xi_1, \xi_2, \xi_3, \xi_4) = \sum_{\sigma \in S_4} (-1)^\sigma \xi_{\sigma(1)}\xi_{\sigma(2)}\xi_{\sigma(3)}\xi_{\sigma(4)},$$

where $(-1)^\sigma$ takes value either $(+1)$ or (-1) according as σ is an even or odd permutation in the symmetric group S_4. By R satisfies s_4, we mean that $s_4(\xi_1, \xi_2, \xi_3, \xi_4) = 0$ for all $\xi_1, \xi_2, \xi_3, \xi_4 \in R$. A subgroup $\mathcal{L}$ of R is called Lie ideal of R, if $\mathcal{L}$ is additive subgroup of R such that $[\mathcal{L}, R] \subseteq \mathcal{L}$. We recall that an additive mapping $F : R \to R$ associated to a derivation d is a generalized derivation of R, if $\mathcal{F}(xy) = \mathcal{F}(x)y + xd(y)$ for all $x, y \in R$. For some $a, b \in R$, $\mathcal{F}(x) = ax + xb$ for all $x \in R$ is called inner generalized derivation. The symbol $[a, b]$ stands for commutator of the elements $a, b \in R$ which is defined by $[a, b] = ab - ba$. By $[a, b]_n$, we mean the the n-th commutator of $a, b \in R$ which is $[[a, b]_{n-1}, b]$ for $n \geq 1$.

Let S be a nonempty subset of R. A mapping $\mathcal{F} : R \to R$ is said to be commuting on S if $[\mathcal{F}(\xi), \xi] = 0$ for all $\xi \in S$ and centralizing on S if $[\mathcal{F}(\xi), \xi] \in Z(R)$ for all $\xi \in S$. Posner [22] initiated the study of commuting and centralizing maps on prime rings. After then, this result has been extended by a number of authors in many directions.

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The mappings $\mathcal{F} : R \to R$ and $\mathcal{G} : R \to R$ are said to be co-commuting (co-centralizing) on S if $\mathcal{F}(\xi)\xi - \xi G(\xi) = 0$ for all $\xi \in S$ (resp. $\mathcal{F}(\xi)\xi - \xi\mathcal{G}(\xi) \in Z(R)$ for all $\xi \in S$).

Brešar proved in [4] that if R is a prime ring and d and δ are two nonzero co-centralizing derivations (i.e., $d(\xi)\xi - \xi\delta(\xi) \in Z(R)$ for all $\xi \in R$), then R must be commutative. Later Lee and Wong [20] studied above condition on noncentral Lie ideal $\mathcal{L}$ of a prime ring R. Lee and Wong [20] proved that if $d(\xi)\xi - \xi\delta(\xi) \in Z(R)$ for all $\xi \in \mathcal{L}$, then either $d = \delta = 0$ or R satisfies s_4.

Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is noncentral valued on R. Let $\mathcal{F}$ and $\mathcal{G}$ be two nonzero generalized derivations of R. Let S be a nonempty subset of R. Denote the set $f(S) = \{f(\xi_1, \ldots, \xi_n)|\xi_1, \ldots, \xi_n \in S\}$. In [1], Argaç and De Filippis studied the structure of the maps $\mathcal{F}$ and $\mathcal{G}$, when $\mathcal{F}(\xi)\xi - \xi\mathcal{G}(\xi) = 0$ for all $\xi \in f(I)$, where I is an ideal of R. In [8], De Filippis and Dhara described the forms of the maps $\mathcal{F}$ and $\mathcal{G}$, when $\mathcal{F}(\xi)\xi - \xi\mathcal{G}(\xi) \in C$ for all $\xi \in f(I)$.

Argac et al. [2] studied the Engel condition acting on noncentral Lie ideal $\mathcal{L}$ in prime ring. They described the forms of the generalized derivation $\mathcal{F}$, when $[\mathcal{F}(u), u]_n = 0$ for all $u \in \mathcal{L}$, $n \geq 1$ a fixed integer.

De Filippis and Rania [10] studied also the Engel condition $[\mathcal{F}(u)u - u\mathcal{G}(u), u]_n = 0$ for all $u \in \mathcal{L}$, $n \geq 1$ a fixed integer and then described the forms of the maps.

In [7], Carini et al. initiated to study the product of two generalized derivations. They studied $\mathcal{F}(u)\mathcal{G}(u) = 0$ for all $u \in f(R)$ and then described the structure of the maps.

Recently, in [6], De Filippis et al. studied product of two generalized derivation with 1-Engel condition in prime ring. More precisely, they obtained the following result:

Let R be a prime ring with extended centroid C, U be its Utumi ring of quotients. Let $\mathcal{L}$ be a non-central Lie ideal of R, $\mathcal{F}$ and $\mathcal{G}$ be two non-zero generalized derivations of R. If $char(R) \neq 2$ such that

$$[\mathcal{F}(u)\mathcal{G}(u), u] = 0$$

for all $u \in \mathcal{L}$, then one below mentioned conclusion holds:

(1) there exist $p, q \in U$ such that $\mathcal{F}(x) = xp$ and $\mathcal{G}(x) = qx$ with $pq \in C$;
(2) R satisfies s_4.

Recently, Dhara et al. [11] studied the situation $[\mathcal{F}(u)\mathcal{G}(u) - u\mathcal{H}(u), u] = 0$ for all $u \in \mathcal{L}$, where $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are three generalized derivations of R and $\mathcal{L}$ a non-central Lie ideal of R and then described the possible forms of the maps.

Moreover, recently Tiwari [23] studied the case $\mathcal{T}_2(u)\mathcal{T}_1(u) - \mathcal{T}_1(u)u + u\mathcal{T}_3(u) = 0$ for all $u \in f(R)$, where $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are three generalized derivations of R and then obtained possible forms of the maps. In the present article, our interest is to study the situation with n-Engel condition. More precisely, we will prove the following Theorem.

Theorem 1.1 *Let R be a prime ring with extended centroid C, U its Utumi quotient ring and L a noncentral Lie ideal of R. Suppose that $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ are three generalized derivations of R and $n \geq 1$ a fixed integer. If char $R \neq 2$ such that*

$$[\mathcal{T}_2(u)\mathcal{T}_1(u) - \mathcal{T}_1(u)u + u\mathcal{T}_3(u), u]_n = 0$$

for all $u \in \mathcal{L}$, then one below mentioned conclusion holds:

(1) there exists $\mu \in C$ such that $\mathcal{T}_1 = 0$, $\mathcal{T}_2$ is any map, $\mathcal{T}_3(x) = \mu x$ for all $x \in R$;
(2) there exist $\gamma \in C$ and $b, c \in U$ such that $\mathcal{T}_1(x) = \gamma x$, $\mathcal{T}_2(x) = xb$ and $\mathcal{T}_3(x) = cx$ for all $x \in R$ with $\gamma b + c \in C$;
(3) there exist $\gamma \in C$ and $a, c, m \in U$ such that $\mathcal{T}_1(x) = xa$, $\mathcal{T}_2(x) = \gamma x$ and $\mathcal{T}_3(x) = cx + xm$ for all $x \in R$ with $c - a, \gamma a + m \in C$;
(4) R satisfies s_4, the standard identity of degree 4.

2 The case: inner generalized derivations

Let $a, b, c, p, q, m \in U$ and $\mathcal{T}_1(x) = ax + xb$, $\mathcal{T}_2(x) = px + xq$ and $\mathcal{T}_3(x) = cx + xm$ for all $x \in R$. Since $\mathcal{L}$ is a noncentral Lie ideal of R, we may assume that $\mathcal{L} = [R, R]$. Then $[\mathcal{T}_2(u)\mathcal{T}_1(u) - \mathcal{T}_1(u)u + u\mathcal{T}_3(u), u]_n = 0$

for all $u \in [R, R]$ yields

$$\Big[p[\xi_1, \xi_2]a[\xi_1, \xi_2] + p[\xi_1, \xi_2]^2 b + [\xi_1, \xi_2]qa[\xi_1, \xi_2] + [\xi_1, \xi_2]q[\xi_1, \xi_2]b - a[\xi_1, \xi_2]^2 - [\xi_1, \xi_2]b[\xi_1, \xi_2] + [\xi_1, \xi_2]c[\xi_1, \xi_2] + [\xi_1, \xi_2]^2 m, [\xi_1, \xi_2]\Big]_n = 0 \tag{1}$$

for all $\xi_1, \xi_2 \in R$.

Now we begin to investigate this generalized polynomial identity (GPI).

Lemma 2.1 *Let $a, b, c, p, q, m \in U$ and R be a prime ring of char $R \neq 2$. If*

$$\Big[p[\xi_1, \xi_2]a[\xi_1, \xi_2] + p[\xi_1, \xi_2]^2 b + [\xi_1, \xi_2]qa[\xi_1, \xi_2] + [\xi_1, \xi_2]q[\xi_1, \xi_2]b - a[\xi_1, \xi_2]^2 - [\xi_1, \xi_2]b[\xi_1, \xi_2] + [\xi_1, \xi_2]c[\xi_1, \xi_2] + [\xi_1, \xi_2]^2 m, [\xi_1, \xi_2]\Big]_n = 0$$

is a trivial GPI for R, then one of the following holds:

(1) $a, b \in C$;
(2) $p, q \in C$;
(3) $p, b \in C$.

Proof By [5], we know the fact that R and U satisfy the same GPIs. Moreover, by hypothesis the given GPI is a trivial GPI for R, and hence

$$\Big[p[\xi_1, \xi_2]a[\xi_1, \xi_2] + p[\xi_1, \xi_2]^2 b + [\xi_1, \xi_2]qa[\xi_1, \xi_2] + [\xi_1, \xi_2]q[\xi_1, \xi_2]b - a[\xi_1, \xi_2]^2 - [\xi_1, \xi_2]b[\xi_1, \xi_2] + [\xi_1, \xi_2]c[\xi_1, \xi_2] + [\xi_1, \xi_2]^2 m, [\xi_1, \xi_2]\Big]_n$$

is the zero element in the free product $T = U *_C C\{\xi_1, \xi_2\}$, that is,

$$\Big[p[\xi_1, \xi_2]a[\xi_1, \xi_2] + p[\xi_1, \xi_2]^2 b + [\xi_1, \xi_2]qa[\xi_1, \xi_2] + [\xi_1, \xi_2]q[\xi_1, \xi_2]b - a[\xi_1, \xi_2]^2 - [\xi_1, \xi_2]b[\xi_1, \xi_2] + [\xi_1, \xi_2]c[\xi_1, \xi_2] + [\xi_1, \xi_2]^2 m, [\xi_1, \xi_2]\Big]_n = 0 \in T. \tag{2}$$

This implies $\{p, a, 1\}$ linearly C-dependent, because if they are linearly C-independent, then

$$[a[\xi_1, \xi_2]^2, [\xi_1, \xi_2]]_n = 0 \in T$$

implying $a \in C$, a contradiction. Thus there exist $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 p + \alpha_2 a + \alpha_3 = 0$.

Now we consider the following two cases:

Case-1: *Let* $\alpha_1 = 0$.

Then $\alpha_2 \neq 0$ and hence $a \in C$. Then (2) reduces to

$$\Big[p[\xi_1, \xi_2]^2 a + p[\xi_1, \xi_2]^2 b + [\xi_1, \xi_2](aq - b + c)[\xi_1, \xi_2] + [\xi_1, \xi_2]q[\xi_1, \xi_2]b + [\xi_1, \xi_2]^2(m - a), [\xi_1, \xi_2]\Big]_n = 0 \in T. \tag{3}$$

If $p \notin C$, then $p[\xi_1, \xi_2]^2(a + b)[\xi_1, \xi_2]^n = 0 \in T$ implying $a + b = 0$. Thus we have obtained $a, b \in C$ as desired in conclusion (1).

If $p \in C$, then by (3)

$$[\xi_1, \xi_2]\Big[(aq - b + c + pa - a)[\xi_1, \xi_2] + (p+q)[\xi_1, \xi_2]b + [\xi_1, \xi_2]m, [\xi_1, \xi_2]\Big]_n = 0 \in T. \tag{4}$$

This implies $\{m, b, 1\}$ is linearly C-dependent, because if they are linearly C-independent, then

$$[\xi_1, \xi_2]^{n+1}(p+q)[\xi_1, \xi_2]b = 0 \in T \tag{5}$$

implying $p + q = 0$, i.e., $q = -p \in C$ as desired in conclusion (2). Thus there exists $\beta_1, \beta_2, \beta_3 \in C$ such that $\beta_1 m + \beta_2 b + \beta_3 = 0$. If $\beta_1 = 0$, then $b \in C$ and hence conclusion (3) is obtained. Let $\beta_1 \neq 0$. Then $m = \alpha' b + \alpha''$ for some $\alpha', \alpha'' \in C$. By (4)

$$[\xi_1, \xi_2]\Big[(aq - b + c + pa - a)[\xi_1, \xi_2] + (p+q)[\xi_1, \xi_2]b + [\xi_1, \xi_2](\alpha' b + \alpha''), [\xi_1, \xi_2]\Big]_n = 0 \in T.$$

This implies either $b \in C$ or

$$[\xi_1, \xi_2]^{n+1}(p + q + \alpha')[\xi_1, \xi_2]b = 0 \in T$$

implying $q = -(p + \alpha') \in C$. In any cases, we have conclusions.

Case-2: *Let* $\alpha_1 \neq 0$.

Then $p = \beta_1 a + \beta_2$. By (2)

$$\Big[(\beta_1 a + \beta_2)[\xi_1, \xi_2]a[\xi_1, \xi_2] + (\beta_1 a + \beta_2)[\xi_1, \xi_2]^2 b + [\xi_1, \xi_2]qa[\xi_1, \xi_2] + [\xi_1, \xi_2]q[\xi_1, \xi_2]b - a[\xi_1, \xi_2]^2 - [\xi_1, \xi_2]b[\xi_1, \xi_2] + [\xi_1, \xi_2]c[\xi_1, \xi_2] + [\xi_1, \xi_2]^2 m, [\xi_1, \xi_2]\Big]_n = 0 \in T. \tag{6}$$

This implies $a \in C$ and hence (6) reduces to

$$[\xi_1, \xi_2]\Big[[\xi_1, \xi_2](\beta_1 a + \beta_2)b + qa[\xi_1, \xi_2] + q[\xi_1, \xi_2]b + (c - b)[\xi_1, \xi_2] + [\xi_1, \xi_2]m, [\xi_1, \xi_2]\Big]_n = 0 \in T. \tag{7}$$

This implies $\{m, b, 1\}$ is linearly C-dependent, because if they are linearly C-independent, then

$$[x\xi_1, \xi_2]^{n+1}[\xi_1, \xi_2]m = 0 \in T \tag{8}$$

implying $m = 0$, a contradiction. Thus there exists $\beta_1', \beta_2', \beta_3' \in C$ such that $\beta_1' m + \beta_2' b + \beta_3' = 0$. If $\beta_1' = 0$, then $b \in C$ and hence conclusion (1) is obtained. Let $\beta_1' \neq 0$. Then $m = \gamma' b + \gamma''$ for some $\gamma', \gamma'' \in C$. By (7)

$$[\xi_1, \xi_2]\Big[[\xi_1, \xi_2](\beta_1 a + \beta_2)b + qa[\xi_1, \xi_2] + q[\xi_1, \xi_2]b + (c - b)[\xi_1, \xi_2] + [\xi_1, \xi_2](\gamma' b + \gamma''), [\xi_1, \xi_2]\Big]_n = 0 \in T.$$

This implies either $b \in C$ or

$$[\xi_1, \xi_2]^{n+1}(\beta_1 a + \beta_2 + q + \gamma')[\xi_1, \xi_2]b = 0 \in T$$

implying $q = -\beta_1 a - \beta_2 - \gamma' \in C$. In any cases, we have conclusions. □

Lemma 2.2 *Let $R = M_k(K)$, $k \geq 3$, be the algebra of all $k \times k$-matrices over the field K with characteristic different from 2 and $a, b, c, p, q, m \in R$. If R satisfies*

$$\Big[p[\xi_1,\xi_2]a[\xi_1,\xi_2] + p[\xi_1,\xi_2]^2 b + [\xi_1,\xi_2]qa[\xi_1,\xi_2] + [\xi_1,\xi_2]q[\xi_1,\xi_2]b$$
$$-a[\xi_1,\xi_2]^2 - [\xi_1,\xi_2]b[\xi_1,\xi_2] + [\xi_1,\xi_2]c[\xi_1,\xi_2] + [\xi_1,\xi_2]^2 m, [\xi_1,\xi_2]\Big]_n = 0$$

then one of the following holds:

(1) $a, b \in K \cdot I_k$;
(2) $p, q \in K \cdot I_k$;
(3) $p, b \in K \cdot I_k$.

Proof We denote by e_{ij} the usual unit matrix with 1 in the (i, j)-th entry and zero elsewhere. Let $a = (a_{ij})_{k\times k}$, $b = (b_{ij})_{k\times k}$, $c = (c_{ij})_{k\times k}$, $p = (p_{ij})_{k\times k}$, $q = (q_{ij})_{k\times k}$, $m = (m_{ij})_{k\times k}$.

Let i, j, r be three different indices. Now we consider $\xi_1 = e_{ij}$ and $\xi_2 = e_{ji}$ and then $[\xi_1, \xi_2] = [e_{ij}, e_{ji}] = e_{ii} - e_{jj}$. By hypothesis, we get

$$\begin{aligned}&[p(e_{ii}-e_{jj})a(e_{ii}-e_{jj}) + p(e_{ii}+e_{jj})b + (e_{ii}-e_{jj})qa(e_{ii}-e_{jj})\\ &\quad +(e_{ii}-e_{jj})q(e_{ii}-e_{jj})b - a(e_{ii}+e_{jj}) - (e_{ii}-e_{jj})b(e_{ii}-e_{jj})\\ &\quad +(e_{ii}-e_{jj})c(e_{ii}-e_{jj}) + (e_{ii}+e_{jj})m, (e_{ii}-e_{jj})]_n = 0.\end{aligned} \tag{9}$$

Left multiplying by e_{kk} and right multiplying by e_{jj}, we get

$$e_{kk}\Big\{p(e_{ii}-e_{jj})a(-1)^{n+1}e_{jj} + p(e_{ii}+e_{jj})b(-1)^n e_{jj} - a(-1)^n e_{jj}\Big\} = 0$$

that is

$$p_{ki}(-a_{ij}+b_{ij}) + p_{kj}(a_{jj}+b_{jj}) - a_{kj} = 0. \tag{10}$$

Let $\phi \in Aut(M_k(K))$ such that $\phi(x) = (1+e_{ik})x(1-e_{ik})$ for all $x \in R$. Let $\phi(a) = (a'_{ij})_{k\times k}$, $\phi(b) = (b'_{ij})_{k\times k}$, $\phi(p) = (p'_{ij})_{k\times k}$.

Since $\phi(a), \phi(b), \phi(q)$ satisfy the same condition as a, b, q do, we may replace a_{ij}, b_{ij}, q_{ij} with $a'_{ij}, b'_{ij}, q'_{ij}$ in (22). Then we have

$$p'_{ki}(-a'_{ij}+b'_{ij}) + p'_{kj}(a'_{jj}+b'_{jj}) - a'_{kj} = 0. \tag{11}$$

This implies

$$p_{ki}(-a_{ij}+b'_{ij}-a_{kj}+b_{kj}) + p_{kj}(a_{jj}+b_{jj}) - a_{kj} = 0. \tag{12}$$

Together (10) and (12) implies that

$$p_{ki}(b-a)_{kj} = 0 \tag{13}$$

We consider the automorphism $\psi(x) = (1+e_{ij})x(1-e_{ij})$ for all $x \in R$ and if we denote $\psi(a) = (a''_{ij})_{k\times k}$, $\psi(b) = (b''_{ij})_{k\times k}$, $\psi(p) = (p''_{ij})_{k\times k}$ and using above argument, we have

$$p''_{ki}(b''_{kj} - a''_{kj}) = 0. \tag{14}$$

This implies

$$p_{ki}\{(b-a)_{kj} - (b-a)_{ki}\} = 0. \tag{15}$$

Thus we have

$$p_{ki}(b-a)_{ki} = 0. \tag{16}$$

Then by [9, Proposition 1], either $p \in K \cdot I_k$ or $b - a \in K \cdot I_k$.

Now we consider $\chi \in Aut(M_k(K))$ such that $\chi(x) = (1+e_{jk})x(1-e_{jk})$ for all $x \in R$. Let $\chi(a) = (a'''_{ij})_{k\times k}$, $\chi(b) = (b'''_{ij})_{k\times k}$, $\chi(p) = (p'''_{ij})_{k\times k}$. Again using above relation, it follows that

$$p'''_{ki}(-a'''_{ij} + b'''_{ij}) + p'''_{kj}(a'''_{jj} + b'''_{jj}) - a'''_{kj} = 0, \tag{17}$$

which gives

$$p_{ki}(-a_{ij} + b_{ij}) + p_{kj}(a_{jj} + b_{jj} + a_{kj} + b_{kj}) - a_{kj} = 0. \tag{18}$$

Again, together (10) and (18) implies that

$$p_{kj}(a+b)_{kj} = 0. \tag{19}$$

Then again by [9, Proposition 1], either $p \in K \cdot I_k$ or $a + b \in K \cdot I_k$.

If $p \notin K \cdot I_k$, then $a+b, a-b \in K \cdot I_k$ implying $a, b \in K \cdot I_k$. Therefore, either $p \in K \cdot I_k$ or $a, b \in K \cdot I_k$.

Let $p \in K \cdot I_k$. Then by hypothesis

$$\Big[[\xi_1, \xi_2](ap + qa - b + c)[\xi_1, \xi_2] + [\xi_1, \xi_2]^2(bp + m) + [\xi_1, \xi_2]q[\xi_1, \xi_2]b$$
$$-a[\xi_1, \xi_2]^2, [\xi_1, \xi_2]\Big]_n = 0.$$

Let $u = ap + qa - b + c$ and $v = bp + m$. Then

$$\Big[[\xi_1, \xi_2]u[\xi_1, \xi_2] + [\xi_1, \xi_2]^2 v + [\xi_1, \xi_2]q[\xi_1, \xi_2]b$$
$$-a[\xi_1, \xi_2]^2, [\xi_1, \xi_2]\Big]_n = 0.$$

Replace $[\xi_1, \xi_2] = e_{ii} - e_{jj}$, and obtain

$$\Big[(e_{ii} - e_{jj})u(e_{ii} - e_{jj}) + (e_{ii} + e_{jj})v + (e_{ii} - e_{jj})q(e_{ii} - e_{jj})b$$
$$-a(e_{ii} + e_{jj}), (e_{ii} - e_{jj})\Big]_n = 0.$$

Left multiplying by e_{jj} and right multiplying by e_{kk}, we get

$$(-1)^n e_{jj}\Big\{(e_{ii} + e_{jj})v + (e_{ii} - e_{jj})q(e_{ii} - e_{jj})b\Big\}e_{kk} = 0.$$

which gives

$$v_{jk} - q_{ji}b_{ik} + q_{jj}b_{jk} = 0. \tag{20}$$

Now we consider $\overline{\varphi} \in Aut(M_k(K))$ such that $\overline{\varphi}(x) = (1 + e_{kj})x(1 - e_{kj})$ for all $x \in R$. Denote $\overline{\varphi}(v) = (v''''_{ij})_{k\times k}$, $\overline{\varphi}(b) = (b''''_{ij})_{k\times k}$, $\overline{\varphi}(q) = (q''''_{ij})_{k\times k}$. Then, again using above relation, it follows that

$$v''''_{jk} - q''''_{ji}b''''_{ik} + q''''_{jj}b''''_{jk} = 0.$$

This implies

$$v_{jk} - q_{ji}b_{ik} + (q_{jj} - q_{jk})b_{jk} = 0. \tag{21}$$

The relations (20) and (21) together implies $q_{jk}b_{jk} = 0$. Then by [9, Proposition 1], either $b \in K \cdot I_k$ or $q \in K \cdot I_k$. $\square$

By the same manner, one can prove the following.

Corollary 2.3 *Let $R = M_k(K)$, $k \geq 3$, be the algebra of all $k \times k$-matrices over the field K with characteristic different from 2 and $a, b, c, q, p, u, m \in R$. If R satisfies*

$$\Big[p[\xi_1,\xi_2]a[\xi_1,\xi_2] + p[\xi_1,\xi_2]^2b + [\xi_1,\xi_2]u[\xi_1,\xi_2] + [\xi_1,\xi_2]q[\xi_1,\xi_2]b$$
$$-a[\xi_1,\xi_2]^2 - [\xi_1,\xi_2]b[\xi_1,\xi_2] + [\xi_1,\xi_2]c[\xi_1,\xi_2] + [\xi_1,\xi_2]^2m, [\xi_1,\xi_2]\Big]_n = 0$$

then one of the following holds:

(1) $a, b \in K \cdot I_k$;
(2) $p, q \in K \cdot I_k$;
(3) $p, b \in K \cdot I_k$.

Lemma 2.4 [10, Lemma 3] *Let R be a prime ring of char $(R) \neq 2$ and $a, b, c, q \in U$. If*

$$[a[\xi_1,\xi_2] + [\xi_1,\xi_2]b, [\xi_1,\xi_2]]_k[\xi_1,\xi_2] - [\xi_1,\xi_2][c[\xi_1,\xi_2] + [\xi_1,\xi_2]q, [\xi_1,\xi_2]]_k = 0$$

for all $\xi_1, \xi_2 \in R$, then one of the following holds:

(1) $a, q, b - c \in C$;
(2) R satisfies s_4 and $a - b + c - q \in C$.

Lemma 2.5 *Let R be a prime ring of char $(R) \neq 2$ and $a, b, c, q, p, m \in R$. If $a, b \in C$ and R satisfies*

$$\Big[p[\xi_1,\xi_2]a[\xi_1,\xi_2] + p[\xi_1,\xi_2]^2b + [\xi_1,\xi_2]qa[\xi_1,\xi_2] + [\xi_1,\xi_2]q[\xi_1,\xi_2]b$$
$$-a[\xi_1,\xi_2]^2 - [\xi_1,\xi_2]b[\xi_1,\xi_2] + [\xi_1,\xi_2]c[\xi_1,\xi_2] + [\xi_1,\xi_2]^2m, [\xi_1,\xi_2]\Big]_n = 0$$

then one of the following holds:

(1) $(a + b)p - a, (a + b)q - b + c, m \in C$;
(2) R satisfies s_4.

Proof By the hypothesis, R satisfies

$$\Big[(ap + bp - a)[\xi_1,\xi_2]^2 + [\xi_1,\xi_2](aq + bq - b + c)[\xi_1,\xi_2] + [\xi_1,\xi_2]^2m, [\xi_1,\xi_2]\Big]_n = 0.$$

By Lemma 2.4, one of the following holds:

(i) $(a + b)p - a, (a + b)q - b + c, m \in C$;
(ii) R satisfies s_4.

□

Lemma 2.6 *Let R be a prime ring of char $(R) \neq 2$ and $a, b, c, q, p, m \in R$. If $p, q \in C$ and R satisfies*

$$\Big[p[\xi_1,\xi_2]a[\xi_1,\xi_2] + p[\xi_1,\xi_2]^2b + [\xi_1,\xi_2]qa[\xi_1,\xi_2] + [\xi_1,\xi_2]q[\xi_1,\xi_2]b$$
$$-a[\xi_1,\xi_2]^2 - [\xi_1,\xi_2]b[\xi_1,\xi_2] + [\xi_1,\xi_2]c[\xi_1,\xi_2] + [\xi_1,\xi_2]^2m, [\xi_1,\xi_2]\Big]_n = 0$$

then one of the following holds:

(1) $a(p + q) + c - b, b(p + q) + m, a \in C$;
(2) R satisfies s_4.

Proof Under the above condition R satisfies

$$\Big[[\xi_1,\xi_2](ap + aq + c - b)[\xi_1,\xi_2] + [\xi_1,\xi_2]^2(bp + bq + m) - a[\xi_1,\xi_2]^2, [\xi_1,\xi_2]\Big]_n = 0$$

Then conclusion follows by Lemma 2.4. □

Lemma 2.7 *Let R be a prime ring of char $(R) \neq 2$ and $a, b, c, q, p, m \in R$. If $p, b \in C$ and R satisfies*

$$\Big[p[\xi_1,\xi_2]a[\xi_1,\xi_2] + p[\xi_1,\xi_2]^2 b + [\xi_1,\xi_2]qa[\xi_1,\xi_2] + [\xi_1,\xi_2]q[\xi_1,\xi_2]b$$
$$-a[\xi_1,\xi_2]^2 - [\xi_1,\xi_2]b[\xi_1,\xi_2] + [\xi_1,\xi_2]c[\xi_1,\xi_2] + [\xi_1,\xi_2]^2 m, [\xi_1,\xi_2]\Big]_n = 0$$

then one of the following holds:

(1) $ap + qa + bq + c, a + b, m \in C$;
(2) R satisfies s_4.

Proof In this case R satisfies

$$\Big[[\xi_1,\xi_2](ap + qa + bq + c)[\xi_1,\xi_2] + (bp - a - b)[\xi_1,\xi_2]^2 + [\xi_1,\xi_2]^2 m, [\xi_1,\xi_2]\Big]_n = 0$$

By Lemma 2.4, one of the following holds:
(i) $ap + qa + bq + c, bp - a - b, m \in C$. Since $p, b \in C$, $a + b \in C$.
(ii) R satisfies s_4. □

Proposition 2.8 *Let R be a prime ring of char $R \neq 2$, U its Utumi quotient ring, C the extended centroid of R and L a noncentral Lie ideal of R. If $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ are three inner generalized derivations of R and $n \geq 1$ a fixed integer such that*

$$[\mathcal{T}_2(u)\mathcal{T}_1(u) - \mathcal{T}_1(u)u + u\mathcal{T}_3(u), u]_n = 0$$

for all $u \in L$, then one of the following holds:

(1) there exists $\mu \in C$ such that $\mathcal{T}_1 = 0$, $\mathcal{T}_2$ is any map, $\mathcal{T}_3(x) = \mu x$ for all $x \in R$;
(2) there exist $\lambda \in C$ and $b, c \in U$ such that $\mathcal{T}_1(x) = \lambda x$, $\mathcal{T}_2(x) = xb$ and $\mathcal{T}_3(x) = cx$ for all $x \in R$ with $\lambda b + c \in C$;
(3) there exist $\gamma \in C$ and $a, c, m \in U$ such that $\mathcal{T}_1(x) = xa$, $\mathcal{T}_2(x) = \gamma x$ and $\mathcal{T}_3(x) = cx + xm$ for all $x \in R$ with $c - a, \gamma a + m \in C$;
(4) R satisfies s_4, the standard identity of degree 4.

Proof Let $\mathcal{T}_1(x) = ax + xb$, $\mathcal{T}_2(x) = px + xq$ and $\mathcal{T}_3(x) = cx + xm$ for all $x \in R$ and for some fixed $a, b, c, p, q, m \in U$. Since $\mathcal{L}$ is a noncentral Lie ideal of R and char $(R) \neq 2$, by [3, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq \mathcal{L}$. Therefore, $[I, I] \subseteq \mathcal{L}$. Moreover I, R and U satisfy the same GPIs (see [19]). Thus U satisfies

$$\Big[p[\xi_1,\xi_2]a[\xi_1,\xi_2] + p[\xi_1,\xi_2]^2 b + [\xi_1,\xi_2]qa[\xi_1,\xi_2] + [\xi_1,\xi_2]q[\xi_1,\xi_2]b$$
$$-a[\xi_1,\xi_2]^2 - [\xi_1,\xi_2]b[\xi_1,\xi_2] + [\xi_1,\xi_2]c[\xi_1,\xi_2] + [\xi_1,\xi_2]^2 m, [\xi_1,\xi_2]\Big]_n = 0.$$

If (22) is a trivial generalized polynomial identity for R, then by Lemma 2.1, we have $a, b \in C$ or $p, q \in C$ or $p, b \in C$.

Now we assume that (22) is nontrivial GPI for U. Now $U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C, satisfies (22), when C is infinite and $U, U \otimes_C \overline{C}$ are prime and centrally closed by [12, Theorem 2.5 and Theorem 3.5]. So we may replace R by U and $U \otimes_C \overline{C}$ according as C is finite or infinite. Therefore, we conclude that R is centrally closed over C and C is either finite or algebraically closed. In view of Martindale's theorem [21], R is then a primitive ring with a nonzero socle $H = soc(R)$ and C as its associated division ring. In the light of Jacobson's theorem [15, P-75], R is isomorphic to the dense ring of linear transformations on a vector space V over C.

If $\dim_C(V) = m$ is finite, then $R \cong M_m(C)$. If $m = 2$, then R satisfies s_4, which is our last conclusion. Thus we assume that $m \geq 3$. By Lemma 2.2, we get either $a, b \in C$ or $p, q \in C$ or $p, b \in C$.

Next we may assume that $dim_C(V) = \infty$. In this case also, we want to prove that $a, b \in C$ or $p, q \in C$ or $p, b \in C$. On contrary, we assume that

- $a \notin C$ or $b \notin C$;
- $p \notin C$ or $q \notin C$;
- $p \notin C$ or $b \notin C$.

Since $dim_C(V) = \infty$, R and so $H = soc(R)$ do not satisfy any polynomial identities. We know that if for some $v \in U$, $[v, H] = (0)$, then $v \in C$. Therefore, by assumption, there exists $t_1, t_2, t_3, t_4, t_5, t_6, t_7 \in H$ such that

(1) either $[a, t_1] \neq 0$ or $[b, t_2] \neq 0$;
(2) either $[p, t_3] \neq 0$ or $[q, t_4] \neq 0$;
(3) either $[p, t_5] \neq 0$ or $[b, t_6] \neq 0$;
(4) $s_4(t_7, t_8, t_9, t_{10}) \neq 0$.

By Litoff's theorem [13], there exists idempotent $e \in H$ such that

(1) $t_1, \ldots, t_{10} \in eRe$;
(2) $at_i, t_ia, bt_i, t_ib, pt_i, t_ip, qt_i, t_iq \in eRe$ for all $i = 1, \ldots, 10$.

Moreover, $eRe \cong M_k(C)$, the ring of all $k \times k$ matrices over C.

By (22) we have

$$\Big[epe[\xi_1, \xi_2]eae[\xi_1, \xi_2] + epe[\xi_1, \xi_2]^2ebe + [\xi_1, \xi_2]e(qa)e[\xi_1, \xi_2] + [\xi_1, \xi_2]eqe[\xi_1, \xi_2]ebe$$
$$-eae[\xi_1, \xi_2]^2 - [\xi_1, \xi_2]ebe[\xi_1, \xi_2] + [\xi_1, \xi_2]ece[\xi_1, \xi_2] + [\xi_1, \xi_2]^2eme, [\xi_1, \xi_2]\Big]_n = 0,$$

is a GPI for eRe. Then by Corollary 2.3 case one of the following holds:

(1) $eae, ebe \in Ce$, which contradicts with the choices of t_1 and t_2 in H;
(2) $epe, eqe \in Ce$, which contradicts with the choices of t_3 and t_4 in H;
(3) $epe, ebe \in Ce$, which contradicts with the choices of t_5 and t_6 in H;
(4) eRe satisfies s_4, which contradicts with the choices of t_7, t_8, t_9, t_{10} in H.

Therefore, whether $dim_C(V)$ is finite or infinite, in any cases we have $a, b \in C$ or $p, q \in C$ or $p, b \in C$. We consider these situations in the following cases:

Case-1. *When* $a, b \in C$.

Then by Lemma 2.5, $(a+b)p-a$, $(a+b)q-b+c$, $m \in C$. Now $(a+b)p-a \in C$ implies either $a+b = 0$ or $p \in C$. If $a + b = 0$, then $c \in C$ and so $\mathcal{T}_1(x) = (a + b)x = 0$, $\mathcal{T}_2(x) = px + xq$ and $\mathcal{T}_3(x) = (c + m)x$ for all $x \in R$. This gives conclusion (1).

Moreover, if $a + b \neq 0$, then $p \in C$ and hence $\mathcal{T}_1(x) = (a + b)x$, $\mathcal{T}_2(x) = x(p + q)$ and $\mathcal{T}_3(x) = (c + m)x$ for all $x \in R$ with $(a + b)(p + q) + (c + m) \in C$. This gives conclusion (2).

Case-2. *When* $p, q \in C$.

Then by Lemma 2.6, $a(p+q)+c-b$, $b(p+q)+m$, $a \in C$. Therefore, $c-(a+b) \in C$ and $(a+b)(p+q)+m \in C$. Thus $\mathcal{T}_1(x) = x(a + b)$, $\mathcal{T}_2(x) = (p + q)x$ and $\mathcal{T}_3(x) = cx + xm$ for all $x \in R$. This gives conclusion (3).

Case-3. *When* $p, b \in C$.

Then by Lemma 2.7, $ap + qa + bq + c$, $a + b$, $m \in C$. In this case also $(a + b)(p + q) + (c + m) \in C$. Thus $\mathcal{T}_1(x) = (a + b)x$, $\mathcal{T}_2(x) = x(p + q)$ and $\mathcal{T}_3(x) = (c + m)x$ for all $x \in R$, which is conclusion (2). □

3 Proof of the Theorem 1.1

Since char $R \neq 2$ and $\mathcal{L}$ is a noncentral Lie ideal of R, by [3, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq \mathcal{L}$. Therefore, $[I, I] \subseteq \mathcal{L}$. Hence by our assumption we have $[\mathcal{T}_2([\xi_1, \xi_2])\mathcal{T}_1([\xi_1, \xi_2]) - \mathcal{T}_1([\xi_1, \xi_2])[\xi_1, \xi_2] + [\xi_1, \xi_2]\mathcal{T}_3([\xi_1, \xi_2]), [\xi_1, \xi_2]]_n = 0$ for all $\xi_1, \xi_2 \in I$.

By [18, Theorem 3], there exist $a, p, c \in U$ and derivations d, g and δ over U such that $\mathcal{T}_1(x) = ax + d(x)$, $\mathcal{T}_2(x) = px + g(x)$ and $\mathcal{T}_3(x) = cx + \delta(x)$.

Since I, R and U satisfy the same GPIs (see [19]) as well as same differential identities (see [19]), then without loss of any generality we may assume that U satisfies

$$\Big[(p[\xi_1, \xi_2] + g([\xi_1, \xi_2]))(a[\xi_1, \xi_2] + d([\xi_1, \xi_2])) - (a[\xi_1, \xi_2] + d([\xi_1, \xi_2]))[\xi_1, \xi_2]$$
$$+[\xi_1, \xi_2](c[\xi_1, \xi_2] + \delta([\xi_1, \xi_2])), [\xi_1, \xi_2]\Big]_n = 0. \quad (22)$$

If R satisfies s_4, then we have our conclusion. Thus we assume that R does not satisfy s_4. If d, g and δ all are inner derivations then conclusion follows by Proposition 2.8. Thus to prove our Theorem, we consider the following conditions:

- d, g are inner and δ is outer.
- d, δ are inner and g is outer.
- δ, g are inner and d is outer.
- d is inner and g, δ are outer.
- g is inner and d, δ are outer.
- δ is inner and d, g are outer.
- d, g and δ all are outer.

We consider above situations into the following cases.

Case 1: *d, g are inner and $\delta$is outer.*

Let $d(x) = [b, x]$ and $g(x) = [q, x]$ for all $x \in R$ and for some $b, q \in U$. Then $\mathcal{T}_1(x) = ax + d(x) = (a + b)x - xb$ and $\mathcal{T}_2(x) = (p + q)x - xq$ for all $x \in R$. By hypothesis, U satisfies

$$\Big[((p+q)[\xi_1,\xi_2]-[\xi_1,\xi_2]q)((a+b)[\xi_1,\xi_2]-[\xi_1,\xi_2]b) \\ -((a+b)[\xi_1,\xi_2]-[\xi_1,\xi_2]b)[\xi_1,\xi_2] \\ +[\xi_1,\xi_2](c[\xi_1,\xi_2]+\delta([\xi_1,\xi_2])),[\xi_1,\xi_2]\Big]_n = 0. \tag{23}$$

By Kharchenko's Theorem [16], U satisfies

$$\Big[((p+q)[\xi_1,\xi_2]-[\xi_1,\xi_2]q)((a+b)[\xi_1,\xi_2]-[\xi_1,\xi_2]b) \\ -((a+b)[\xi_1,\xi_2]-[\xi_1,\xi_2]b)[\xi_1,\xi_2] \\ +[\xi_1,\xi_2](c[\xi_1,\xi_2]+[s_1,\xi_2]+[\xi_1,s_2]),[\xi_1,\xi_2]\Big]_n = 0. \tag{24}$$

In particular, U satisfies blended components

$$\Big[[\xi_1,\xi_2]([s_1,\xi_2]+[\xi_1,s_2]),[\xi_1,\xi_2]\Big]_n = 0. \tag{25}$$

This is a polynomial identity (PI). Hence by [17, Lemma 2] there exists a field K such that $U \subseteq M_t(K)$, $t > 1$ and $M_t(K)$ satisfies

$$[\xi_1,\xi_2]\Big[([s_1,\xi_2]+[\xi_1,s_2]),[\xi_1,\xi_2]\Big]_n = 0. \tag{26}$$

But for $\xi_1 = e_{12}, \xi_2 = e_{21}, s_1 = e_{11}, s_2 = 0$, we have $0 = [\xi_1,\xi_2]\Big[([s_1,\xi_2]+[\xi_1,s_2]),[\xi_1,\xi_2]\Big]_n = 2^n e_{21}$, a contradiction.

Case 2: *$d, \delta$are inner and $g$is outer.*

Let $d(x) = [b, x]$ and $\delta(x) = [m, x]$ for all $x \in R$ and for some $b, m \in U$. Then $\mathcal{T}_1(x) = ax + d(x) = (a + b)x - xb$ and $\mathcal{T}_3(x) = (c + m)x - xm$ for all $x \in R$. By hypothesis, U satisfies

$$\Big[(p[\xi_1,\xi_2]+g([\xi_1,\xi_2]))((a+b)[\xi_1,\xi_2]-[\xi_1,\xi_2]b) \\ -((a+b)[\xi_1,\xi_2]-[\xi_1,\xi_2]b)[\xi_1,\xi_2] \\ +[\xi_1,\xi_2]((c+m)[\xi_1,\xi_2]-[\xi_1,\xi_2]m),[\xi_1,\xi_2]\Big]_n = 0. \tag{27}$$

Since g is outer, by Kharchenko's Theorem [16], U satisfies

$$\Big[(p[\xi_1,\xi_2]+[s_1,\xi_2]+[\xi_1,s_2])((a+b)[\xi_1,\xi_2]-[\xi_1,\xi_2]b)\\ -((a+b)[\xi_1,\xi_2]-[\xi_1,\xi_2]b)[\xi_1,\xi_2]\\ +[\xi_1,\xi_2]((c+m)[\xi_1,\xi_2]-[\xi_1,\xi_2]m),[\xi_1,\xi_2]\Big]_n=0. \tag{28}$$

In particular, U satisfies blended components

$$\Big[([s_1,\xi_2]+[\xi_1,s_2])((a+b)[\xi_1,\xi_2]-[\xi_1,\xi_2]b),[\xi_1,\xi_2]\Big]_n=0. \tag{29}$$

For $s_2=0$ and $s_1=\xi_1$, we have

$$[\xi_1,\xi_2]\Big[a[\xi_1,\xi_2]+[b,[\xi_1,\xi_2]],[\xi_1,\xi_2]\Big]_n=0. \tag{30}$$

Since we assumed that R does not satisfy s_4, by Lemma 2.4, $a,b\in C$. By (29), U satisfies

$$\Big[([s_1,\xi_2]+[\xi_1,s_2]),[\xi_1,\xi_2]\Big]_n a[\xi_1,\xi_2]=0. \tag{31}$$

Replacing s_1 with $[a',\xi_1]$ and s_2 with $[a',\xi_2]$ for some $a'\notin C$, we have

$$\Big[[a',[\xi_1,\xi_2]],[\xi_1,\xi_2]\Big]_n a[\xi_1,\xi_2]=0. \tag{32}$$

By Lemma 2.4, we have $a=0$. Thus $\mathcal{T}_1(x)=0$ and $\mathcal{T}_2(x)=px+g(x)$ and hence by hypothesis U satisfies $[[\xi_1,\xi_2]((c+m)[\xi_1,\xi_2]-[\xi_1,\xi_2]m),[\xi_1,\xi_2]]_n=0$. Again by Lemma 2.4, $c,m\in C$, i.e., $\mathcal{T}_3(x)=cx$. This gives conclusion (1).

Case 3: δ, g *are inner and dis outer.*

Let $g(x)=[q,x]$ and $\delta(x)=[m,x]$ for all $x\in R$ and for some $q,m\in U$. Then $\mathcal{T}_2(x)=(p+q)x-xq$ and $\mathcal{T}_3(x)=(c+m)x-xm$ for all $x\in R$. By hypothesis, U satisfies

$$\Big[((p+q)[\xi_1,\xi_2]-[\xi_1,\xi_2]q)(a[\xi_1,\xi_2]+d([\xi_1,\xi_2]))\\ -(a[\xi_1,\xi_2]+d([\xi_1,\xi_2]))[\xi_1,\xi_2]\\ +[\xi_1,\xi_2]((c+m)[\xi_1,\xi_2]-[\xi_1,\xi_2]m),[\xi_1,\xi_2]\Big]_n=0. \tag{33}$$

Since d is outer, by Kharchenko's Theorem [16], U satisfies

$$\Big[((p+q)[\xi_1,\xi_2]-[\xi_1,\xi_2]q)(a[\xi_1,\xi_2]+[s_1,\xi_2]+[\xi_1,s_2])\\ -(a[\xi_1,\xi_2]+[s_1,\xi_2]+[\xi_1,s_2])[\xi_1,\xi_2]\\ +[\xi_1,\xi_2]((c+m)[\xi_1,\xi_2]-[\xi_1,\xi_2]m),[\xi_1,\xi_2]\Big]_n=0 \tag{34}$$

and hence U satisfies blended component

$$\Big[((p+q)[\xi_1,\xi_2]-[\xi_1,\xi_2]q)([s_1,\xi_2]+[\xi_1,s_2])\\ -([s_1,\xi_2]+[\xi_1,s_2])[\xi_1,\xi_2],[\xi_1,\xi_2]\Big]_n=0. \tag{35}$$

Assuming $s_1 = \xi_1$ and $s_2 = 0$ and then using Lemma 2.4, we have $p, q \in C$. Thus (35) reduces to

$$\Big[p[\xi_1, \xi_2]([s_1, \xi_2] + [\xi_1, s_2]) - ([s_1, \xi_2] + [\xi_1, s_2])[\xi_1, \xi_2], [\xi_1, \xi_2]\Big]_n = 0. \tag{36}$$

Replacing s_1 with $[p', \xi_1]$ and s_2 with $[p', \xi_2]$ in (36) for some $p' \notin C$, we have

$$\Big[[\xi_1, \xi_2][pp', [\xi_1, \xi_2]] - [p', [\xi_1, \xi_2]][\xi_1, \xi_2], [\xi_1, \xi_2]\Big]_n = 0. \tag{37}$$

By Lemma 2.4, $p', pp' \in C$, a contradiction.

Case 4: *$d$is inner and $g, \delta$are outer.*

Let $d(x) = [b, x]$ for all $x \in R$ and for some $b \in U$. Then $\mathcal{T}_1(x) = (a + b)x - xb$ for all $x \in R$. By hypothesis, U satisfies

$$\begin{aligned}\Big[&(p[\xi_1, \xi_2] + g([\xi_1, \xi_2]))((a + b)[\xi_1, \xi_2] - [\xi_1, \xi_2]b)\\ &-((a + b)[\xi_1, \xi_2] - [\xi_1, \xi_2]b)[\xi_1, \xi_2]\\ &+[\xi_1, \xi_2](c[\xi_1, \xi_2] + \delta([\xi_1, \xi_2])), [\xi_1, \xi_2]\Big]_n = 0.\end{aligned} \tag{38}$$

If g and δ are linearly C-independent, by Kharchenko's Theorem [16], U satisfies

$$\begin{aligned}\Big[&(p[\xi_1, \xi_2] + [s_1, \xi_2] + [\xi_1, s_2])((a + b)[\xi_1, \xi_2] - [\xi_1, \xi_2]b)\\ &-((a + b)[\xi_1, \xi_2] - [\xi_1, \xi_2]b)[\xi_1, \xi_2]\\ &+[\xi_1, \xi_2](c[\xi_1, \xi_2] + [t_1, \xi_2] + [\xi_1, t_2]), [\xi_1, \xi_2]\Big]_n = 0.\end{aligned} \tag{39}$$

In particular, U satisfies blended component

$$\Big[[\xi_1, \xi_2]([t_1, \xi_2] + [\xi_1, t_2]), [\xi_1, \xi_2]\Big]_n = 0. \tag{40}$$

This is same as (26) and hence by same argument it leads to a contradiction.

If g and δ are linearly C-dependent, there exists $0 \neq \lambda \in C$ such that $g(x) = \lambda\delta(x) + [p', x]$ for all $x \in U$ and for some $p' \in U$. Then U satisfies

$$\begin{aligned}\Big[&(p[\xi_1, \xi_2] + \lambda\delta([\xi_1, \xi_2]) + [p', [\xi_1, \xi_2]])((a + b)[\xi_1, \xi_2] - [\xi_1, \xi_2]b)\\ &-((a + b)[\xi_1, \xi_2] - [\xi_1, \xi_2]b)[\xi_1, \xi_2]\\ &+[\xi_1, \xi_2](c[\xi_1, \xi_2] + \delta([\xi_1, \xi_2])), [\xi_1, \xi_2]\Big]_n = 0.\end{aligned} \tag{41}$$

By Kharchenko's Theorem [16], U satisfies

$$\begin{aligned}\Big[&(p[\xi_1, \xi_2] + \lambda([s_1, \xi_2] + [\xi_1, s_2]) + [p', [\xi_1, \xi_2]])((a + b)[\xi_1, \xi_2] - [\xi_1, \xi_2]b)\\ &-((a + b)[\xi_1, \xi_2] - [\xi_1, \xi_2]b)[\xi_1, \xi_2]\\ &+[\xi_1, \xi_2](c[\xi_1, \xi_2] + [s_1, \xi_2] + [\xi_1, s_2]), [\xi_1, \xi_2]\Big]_n = 0.\end{aligned} \tag{42}$$

In particular, U satisfies blended component

$$\begin{aligned}\Big[&\lambda([s_1, \xi_2] + [\xi_1, s_2])((a + b)[\xi] - [\xi_1, \xi_2]b)\\ &+[\xi_1, \xi_2]([s_1, \xi_2] + [\xi_1, s_2]), [\xi_1, \xi_2]\Big]_n = 0.\end{aligned} \tag{43}$$

Assuming $s_1 = \xi_1$ and $s_2 = 0$, we have from above

$$[\xi_1, \xi_2]\Big[\lambda(a+b)[\xi_1, \xi_2] - [\xi_1, \xi_2]\lambda b, [\xi_1, \xi_2]\Big]_n = 0. \tag{44}$$

By Lemma 2.4, λb, $\lambda(a+b) \in C$. Since $\lambda \neq 0$, we have $a, b \in C$ and hence from above U satisfies

$$\Big[(\lambda a)([s_1, \xi_2] + [\xi_1, s_2])[\xi_1, \xi_2] + [\xi_1, \xi_2]([s_1, \xi_2] + [\xi_1, s_2]), [\xi_1, \xi_2]\Big]_n = 0. \tag{45}$$

Replacing s_1 with $[p', \xi_1]$ and s_2 with $[p', \xi_2]$ in above relation for some $p' \notin C$, we have

$$\Big[[\lambda a p', [\xi_1, \xi_2]][\xi_1, \xi_2] + [\xi_1, \xi_2][p', [\xi_1, \xi_2]], [\xi_1, \xi_2]\Big]_n = 0 \tag{46}$$

By Lemma 2.4, $p' \in C$, a contradiction.

Case 5: *$g$is inner and d, $\delta$are outer.*

Let $g(x) = [q, x]$ for all $x \in R$ and for some $q \in U$. Then $\mathcal{T}_2(x) = (p+q)x - xq$ for all $x \in R$. By hypothesis, U satisfies

$$\begin{aligned}&\Big[((p+q)[\xi_1, \xi_2] - [\xi_1, \xi_2]q)(a[\xi_1, \xi_2] + d([\xi_1, \xi_2])) - (a[\xi_1, \xi_2] + d([\xi_1, \xi_2]))[\xi_1, \xi_2]\\ &+[\xi_1, \xi_2](c[\xi_1, \xi_2] + \delta([\xi_1, \xi_2])), [\xi_1, \xi_2]\Big]_n = 0.\end{aligned} \tag{47}$$

If d and δ are linearly C-independent, by Kharchenko's Theorem [16], U satisfies

$$\begin{aligned}&\Big[((p+q)[\xi_1, \xi_2] - [\xi_1, \xi_2]q)(a[\xi_1, \xi_2] + [s_1, \xi_2] + [\xi_1, s_2])\\ &-(a[\xi_1, \xi_2] + [s_1, \xi_2] + [\xi_1, s_2])[\xi_1, \xi_2]\\ &+[\xi_1, \xi_2](c[\xi_1, \xi_2] + [t_1, \xi_2] + [\xi_1, t_2]), [\xi_1, \xi_2]\Big]_n = 0.\end{aligned} \tag{48}$$

In particular, U satisfies blended component

$$\Big[[\xi_1, \xi_2]([t_1, \xi_2] + [\xi_1, t_2]), [\xi_1, \xi_2]\Big]_n = 0. \tag{49}$$

This is same as (26) and hence by same argument it leads to a contradiction.

If d and δ are linearly C-dependent, there exists $0 \neq \lambda \in C$ such that $d(x) = \lambda\delta(x) + [p', x]$ for all $x \in U$ and for some $p' \in U$. Then U satisfies

$$\begin{aligned}&\Bigg[\Big((p+q)[\xi_1, \xi_2] - [\xi_1, \xi_2]q\Big)\Big(a[\xi_1, \xi_2] + \lambda\delta([\xi_1, \xi_2]) + [p', [\xi_1, \xi_2]]\Big)\\ &-\Big(a[\xi_1, \xi_2] + \lambda\delta([\xi_1, \xi_2]) + [p', [\xi_1, \xi_2]]\Big)[\xi_1, \xi_2]\\ &+[\xi_1, \xi_2]\Big(c[\xi_1, \xi_2] + \delta([\xi_1, \xi_2])\Big), [\xi_1, \xi_2]\Bigg]_n = 0.\end{aligned} \tag{50}$$

By Kharchenko's Theorem [16], U satisfies

$$\begin{aligned}&\Bigg[\Big((p+q)[\xi_1, \xi_2] - [\xi_1, \xi_2]q\Big)\Big(a[\xi_1, \xi_2] + \lambda([s_1, \xi_2] + [\xi_1, s_2]) + [p', [\xi_1, \xi_2]]\Big)\\ &-\Big(a[\xi_1, \xi_2] + \lambda([s_1, \xi_2] + [\xi_1, s_2]) + [p', [\xi_1, \xi_2]]\Big)[\xi_1, \xi_2]\\ &+[\xi_1, \xi_2]\Big(c[\xi_1, \xi_2] + ([s_1, \xi_2] + [\xi_1, s_2])\Big), [\xi_1, \xi_2]\Bigg]_n = 0.\end{aligned} \tag{51}$$

In particular, U satisfies blended component

$$\Big[((p+q)[\xi_1,\xi_2]-[\xi_1,\xi_2]q)(\lambda([s_1,\xi_2]+[\xi_1,s_2]))-\lambda([s_1,\xi_2]+[\xi_1,s_2])[\xi_1,\xi_2] \\ +[\xi_1,\xi_2]([s_1,\xi_2]+[\xi_1,s_2]),[\xi_1,\xi_2]\Big]_n=0. \tag{52}$$

Assuming $s_1=\xi_1$ and $s_2=0$, we have from above

$$\Big[(\lambda(p+q)[\xi_1,\xi_2]-[\xi_1,\xi_2]\lambda q)[\xi_1,\xi_2],[\xi_1,\xi_2]\Big]_n=0. \tag{53}$$

By Lemma 2.4, $\lambda q, \lambda(p+q)\in C$. Since $\lambda\neq 0$, we have $p,q\in C$ and hence from above U satisfies

$$\Big[\lambda p[\xi_1,\xi_2]([s_1,\xi_2]+[\xi_1,s_2])-\lambda([s_1,\xi_2]+[\xi_1,s_2])[\xi_1,\xi_2] \\ +[\xi_1,\xi_2]([s_1,\xi_2]+[\xi_1,s_2]),[\xi_1,\xi_2]\Big]_n=0. \tag{54}$$

Replacing s_1 with $[p'',\xi_1]$ and s_2 with $[p'',\xi_2]$ in above relation for some $p''\notin C$, we have

$$\Big[[\xi_1,\xi_2][\lambda pp''+p'',[\xi_1,\xi_2]]-[\lambda p'',[\xi_1,\xi_2]][\xi_1,\xi_2],[\xi_1,\xi_2]\Big]_n=0. \tag{55}$$

By Lemma 2.4, $\lambda p''\in C$. Since $\lambda\neq 0$, we must have $p''\in C$, a contradiction.

Case 6: δ*is inner and* d, g*are outer.*

Let $\delta(x)=[m,x]$ for all $x\in R$ and for some $m\in U$. Then $T_3(x)=(c+m)x-xm$ for all $x\in R$. By hypothesis, U satisfies

$$\Big[(p[\xi_1,\xi_2]+g([\xi_1,\xi_2]))(a[\xi_1,\xi_2]+d([\xi_1,\xi_2]))-(a[\xi_1,\xi_2]+d([\xi_1,\xi_2]))[\xi_1,\xi_2] \\ +[\xi_1,\xi_2]((c+m)[\xi_1,\xi_2]-[\xi_1,\xi_2]m),[\xi_1,\xi_2]\Big]_n=0. \tag{56}$$

If d and g are linearly C-independent, by Kharchenko's Theorem [16], U satisfies

$$\Big[(p[\xi_1,\xi_2]+[s_1,\xi_2]+[\xi_1,s_2])(a[\xi_1,\xi_2]+[t_1,\xi_2]+[\xi_1,t_2]) \\ -(a[\xi_1,\xi_2]+[t_1,\xi_2]+[\xi_1,t_2])[\xi_1,\xi_2] \\ +[\xi_1,\xi_2]((c+m)[\xi_1,\xi_2]-[\xi_1,\xi_2]m),[\xi_1,\xi_2]\Big]_n=0. \tag{57}$$

In particular, U satisfies blended component

$$\Big[([s_1,\xi_2]+[\xi_1,s_2])([t_1,\xi_2]+[\xi_1,t_2]),[\xi_1,\xi_2]\Big]_n=0. \tag{58}$$

This is a PI. Hence by above argument, there exists a field K such that $U\subseteq M_t(K)$, $t>1$ and the matrix ring $M_t(K)$ satisfies $\Big[([s_1,\xi_2]+[\xi_1,s_2])([t_1,\xi_2]+[\xi_1,t_2]),[\xi_1,\xi_2]\Big]_n=0$. Hence for $s_1=0$, $s_2=e_{22}$, $\xi_1=e_{12}$, $\xi_2=e_{21}$, $t_1=e_{12}+e_{21}$, $t_2=0$, we have $0=\Big[([s_1,\xi_2]+[\xi_1,s_2])([t_1,\xi_2]+[\xi_1,t_2]),[\xi_1,\xi_2]\Big]_n=(-1)^{n+1}2^ne_{12}$, a contradiction.

If d and g are linearly C-dependent, there exists $0 \neq \lambda \in C$ such that $d(x) = \lambda g(x) + [p', x]$ for all $x \in U$ and for some $p' \in U$. Then U satisfies

$$\Big[(p[\xi_1, \xi_2] + g([\xi_1, \xi_2]))(a[\xi_1, \xi_2] + \lambda g([\xi_1, \xi_2]) + [p', [\xi_1, \xi_2]]) - (a[\xi_1, \xi_2] + \lambda g([\xi_1, \xi_2]) + [p', [\xi_1, \xi_2]])[\xi_1, \xi_2] + [\xi_1, \xi_2]((c + m)[\xi_1, \xi_2] - [\xi_1, \xi_2]m), [\xi_1, \xi_2]\Big]_n = 0. \tag{59}$$

By Kharchenko's Theorem [16], U satisfies

$$\Big[(p[\xi_1, \xi_2] + ([s_1, \xi_2] + [\xi_1, s_2]))(a[\xi_1, \xi_2] + \lambda([s_1, \xi_2] + [\xi_1, s_2]) + [p', [\xi_1, \xi_2]]) - (a[\xi_1, \xi_2] + \lambda([s_1, \xi_2] + [\xi_1, s_2]) + [p', [\xi_1, \xi_2]])[\xi_1, \xi_2] + [\xi_1, \xi_2]((c + m)[\xi_1, \xi_2] - [\xi_1, \xi_2]m), [\xi_1, \xi_2]\Big]_n = 0. \tag{60}$$

In particular, U satisfies blended component

$$\Big[([s_1, \xi_2] + [\xi_1, s_2])(\lambda([s_1, \xi_2] + [\xi_1, s_2])), [\xi_1, \xi_2]\Big]_n = 0. \tag{61}$$

Since $\lambda \neq 0$, from above U satisfies

$$\Big[([s_1, \xi_2] + [\xi_1, s_2])^2, [\xi_1, \xi_2]\Big]_n = 0. \tag{62}$$

This is a PI. Hence by same argument, there exists a field K such that $U \subseteq M_k(K)$, $k > 1$ and the matrix ring $M_k(K)$ satisfies $\Big[([s_1, \xi_2] + [\xi_1, s_2])^2, [\xi_1, \xi_2]\Big]_n = 0$. Since R does not satisfy s_4, $k \geq 3$. Thus for $s_1 = e_{13}$, $s_2 = e_{31}$, $\xi_1 = e_{12}$, $\xi_2 = e_{21}$, we have $0 = \Big[([s_1, \xi_2] + [\xi_1, s_2])^2, [\xi_1, \xi_2]\Big]_n = 2^n e_{21}$, a contradiction.

Case 7: *d, gand δall are outer.*

Now two cases may arise here. Either d, g and δ are linearly C-independent or d, g and δ are linearly C-dependent. We consider these cases in the following subcases.

Sub-case-i: *d, gand $\delta$are linearly C-independent.*

Applying Kharchenko's Theorem [16] to (22), U satisfies

$$\Big[(p[\xi_1, \xi_2] + [s_1, \xi_2] + [\xi_1, s_2])(a[\xi_1, \xi_2] + [t_1, \xi_2] + [\xi_1, t_2]) - (a[\xi_1, \xi_2] + [t_1, \xi_2] + [\xi_1, t_2])[\xi_1, \xi_2] + [\xi_1, \xi_2](c[\xi_1, \xi_2] + [u_1, \xi_2] + [\xi_1, u_2]), [\xi_1, \xi_2]\Big]_n = 0. \tag{63}$$

In particular, U satisfies blended component

$$\Big[[\xi_1, \xi_2]([u_1, \xi_2] + [\xi_1, u_2]), [\xi_1, \xi_2]\Big]_n = 0. \tag{64}$$

This is same as (26) and hence by same argument it leads to a contradiction.

Sub-case-ii: *d, gand $\delta$are linearly C-dependent.*

There exist $\alpha', \beta', \gamma' \in C$ and $q' \in U$ such that $\alpha' g(x) + \beta' d(x) + \gamma' \delta(x) = [q', x]$ for all $x \in U$.

If $(\alpha', \beta') = (0, 0)$ then δ is inner, a contradiction. Thus we may assume that either $\alpha' \neq 0$ or $\beta' \neq 0$. Without loss of generality, we assume here that $\alpha' \neq 0$. Then we can write $g(x) = \beta d(x) + \gamma \delta(x) + [q, x]$ for all $x \in U$, for some $\beta, \gamma \in C$ and $q \in U$.

By hypothesis, U satisfies

$$\Bigg[\Big(p[\xi_1,\xi_2]+\beta d([\xi_1,\xi_2])+\gamma\delta([\xi_1,\xi_2])+[q,[\xi_1,\xi_2]]\Big)$$
$$.\Big(a[\xi_1,\xi_2]+d([\xi_1,\xi_2])\Big)$$
$$-(a[\xi_1,\xi_2]+d([\xi_1,\xi_2]))[\xi_1,\xi_2]$$
$$+[\xi_1,\xi_2](c[\xi_1,\xi_2]+\delta([\xi_1,\xi_2])),[\xi_1,\xi_2]\Bigg]_n=0. \tag{65}$$

If d and δ are linearly C-independent, then applying Kharchenko's Theorem [16], we can replace $d(\xi_1)$ with s_1, $d(\xi_2)$ with s_2, $\delta(\xi_1)$ with t_1, $\delta(\xi_2)$ with t_2 and then U satisfies

$$\Bigg[\Big(p[\xi_1,\xi_2]+\beta([s_1,\xi_2]+[\xi_1,s_2])+\gamma([t_1,\xi_2]+[\xi_1,t_2])+[q,[\xi_1,\xi_2]]\Big)$$
$$\Big(a[\xi_1,\xi_2]+([s_1,\xi_2]+[\xi_1,s_2])\Big)$$
$$-(a[\xi_1,x_2]+([s_1,\xi_2]+[\xi_1,s_2]))[\xi_1,\xi_2]$$
$$+[\xi_1,\xi_2](c[\xi_1,\xi_2]+[t_1,\xi_2]+[\xi_1,t_2]),[\xi_1,\xi_2]\Bigg]_n=0. \tag{66}$$

In particular, U satisfies blended components

$$\Bigg[\gamma([t_1,\xi_2]+[\xi_1,t_2])(a[\xi_1,\xi_2]+([s_1,\xi_2]+[\xi_1,s_2]))$$
$$+[\xi_1,\xi_2]([t_1,\xi_2]+[\xi_1,t_2]),[\xi_1,\xi_2]\Bigg]_n=0. \tag{67}$$

In particular, U satisfies (assuming $s_1=s_2=0$)

$$\Big[\gamma([t_1,\xi_2]+[\xi_1,t_2])([s_1,\xi_2]+[\xi_1,s_2]),[\xi_1,\xi_2]\Big]_n=0. \tag{68}$$

and hence

$$\gamma\Big[([t_1,\xi_2]+[\xi_1,t_2])^2,[\xi_1,\xi_2]\Big]_n=0. \tag{69}$$

If $\gamma\neq 0$, then $\Big[([t_1,\xi_2]+[\xi_1,t_2])^2,[\xi_1,\xi_2]\Big]_n=0$ which is same as (62) and then by same argument, it leads to a contradiction. Thus $\gamma=0$ and then by (67), U satisfies

$$[\xi_1,\xi_2]\Big[([t_1,\xi_2]+[\xi_1,t_2]),[\xi_1,\xi_2]\Big]_n=0. \tag{70}$$

This is same as (26) and hence by same argument it leads to a contradiction.

If d and δ are linearly C-dependent modulo inner, let $\delta(x)=\eta d(x)+[b',x]$ for all $x\in U$ and for some $b'\in U$. By (65), U satisfies

$$\Bigg[\Big(p[\xi_1,\xi_2]+\beta d([\xi_1,\xi_2])+\gamma\eta d([\xi_1,\xi_2])+[q+\gamma b',[\xi_1,\xi_2]]\Big)$$
$$.\Big(a[\xi_1,\xi_2]+d([\xi_1,\xi_2])\Big)$$
$$-(a[\xi_1,\xi_2]+d([\xi_1,\xi_2]))[\xi_1,\xi_2]$$
$$+[\xi_1,\xi_2](c[\xi_1,\xi_2]+\eta d([\xi_1,\xi_2])+[b',[\xi_1,\xi_2]]),[\xi_1,\xi_2]\Bigg]_n=0. \tag{71}$$

By Kharchenko's Theorem [16], we can replace $d(\xi_1)$ with s_1, $d(\xi_2)$ with s_2, and then U satisfies

$$\Bigg[\bigg(p[\xi_1,\xi_2]+(\beta+\gamma\eta)([s_1,\xi_2]+[\xi_1,s_2])+[q+\gamma b',[\xi_1,\xi_2]]\bigg)$$
$$.\bigg(a[\xi_1,\xi_2]+([s_1,\xi_2]+[\xi_1,s_2])\bigg)$$
$$-(a[\xi_1,\xi_2]+([s_1,\xi_2]+[\xi_1,s_2]))[\xi_1,\xi_2]$$
$$+[\xi_1,\xi_2](c[\xi_1,\xi_2]+\eta([s_1,\xi_2]+[\xi_1,s_2])+[b',[\xi_1,\xi_2]]),[\xi_1,\xi_2]\Bigg]_n=0. \tag{72}$$

In particular, U satisfies blended component

$$\Big[(\beta+\gamma\eta)([s_1,\xi_2]+[\xi_1,s_2])([s_1,\xi_2]+[\xi_1,s_2]),[\xi_1,\xi_2]\Big]_n=0 \tag{73}$$

that is

$$(\beta+\gamma\eta)\Big[([s_1,\xi_2]+[\xi_1,s_2])^2,[\xi_1,\xi_2]\Big]_n=0. \tag{74}$$

If $(\beta+\gamma\eta)\neq 0$, then $\Big[([s_1,\xi_2]+[\xi_1,s_2])^2,[\xi_1,\xi_2]\Big]_n=0$ which is same as (62) and then by same argument, it leads to a contradiction. Thus $(\beta+\gamma\eta)=0$. Then $g(x)=\beta d(x)+\gamma(\eta d(x)+[b',x])+[q,x]=(\beta+\gamma\eta)d(x)+[q+\gamma\eta b',x]=[q+\gamma\eta b',x]$ for all $x\in U$, which is inner derivation, a contradiction.

This completes the proof of the theorem.

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