Indian J Pure Appl Math (2026) 57:500–510
https://doi.org/10.1007/s13226-024-00535-1

ORIGINAL RESEARCH

Distinguishing labelling of sets under the wreath product action of $\overrightarrow{A_n}$

Madhu Dadhwal · Pankaj

Received: 27 July 2023 / Accepted: 4 January 2024 / Published online: 27 January 2024

Abstract In this article, it is proved that the distinguishing number for the action of $\overrightarrow{A_n}$ on the set $[2n] = \{1, 2, \ldots, 2n\}$ is the n^{th} term of the sequence "1, 2, 3, 3, 4, 4, 4, 5, ... (n appears $n - 1$ times prepended with 1)". An optimal iterative algorithm to establish a closed formula to compute a distinguishing labelling of $[2n]$ under the natural action of $\overrightarrow{A_n}$ is provided. Also, a closed formula to compute the distinguishing number for the above said action is established, in the sequel, it is observed that finding the distinguishing number for the action of $\overrightarrow{A_n}$ on the set $[2n]$ is equivalent to answering a combinatorial problem which states that " what is the minimum number of boxes required to arrange n distinct pair of identical balls in such a way that exactly one box can contain only one pair of identical balls, but two pairs of identical balls can not be completely contained in two boxes with one pair in each box".

Keywords Cycle decomposition · Distinguishing group actions · Distinguishing labelling of set

Mathematics Subject Classification 20D60 · 05C25 · 20B35

1 Introduction

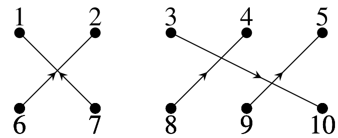
In 1998, Parvathi and Kamaraj [5], introduced the concept of signed Brauer algebras $\overrightarrow{B_n}(x)$ over the field of complex numbers and this algebra has a basis consisting of signed diagrams. The set of signed Brauer diagrams contains the set of Brauer diagrams. This inclusion can be realized by the fact that a Brauer diagram can be regarded as a signed Brauer diagram by marking each of its edges positive. For more information regarding Brauer and signed Brauer algebras one can refer to the papers [6,9,10]. Moreover, a signed Brauer diagram without any horizontal edge is a signed graph $\overrightarrow{d}$, containing two rows both with n vertices (dots). The n vertices in the first row are labelled as $\{1, 2, \ldots, n\}$ while in the second row as $\{n + 1, n + 2, \ldots, 2n\}$. Also, each vertex in the first row is joined to exactly one vertex in the second row with a directed edge. Furthermore, an edge in a signed Brauer diagram $\overrightarrow{d}$ is positive ($\downarrow$), if it originate from a vertex in the above row to a vertex in the row below it, while an edge in $\overrightarrow{d}$ is a negative edge ($\uparrow$), if it points towards a vertex in the above row originating from

Communicated by Bakshi Gurmeet Kaur.

Madhu Dadhwal and Pankaj have contributed equally to this work.

M. Dadhwal (✉) · Pankaj
Department of Mathematics and Statistics, Himachal Pradesh University, Summer Hill, Shimla, Himachal Pradesh 171005, India
E-mail: mpatial.math@gmail.com
Pankaj
E-mail: pankajratramath@gmail.com

a vertex in the row below it. An example of a signed Brauer diagram without any horizontal edge is graphically represented as follows:



The set of signed Brauer diagrams without any horizontal edge forms a group denoted by $\overrightarrow{S_n}$ and this group $\overrightarrow{S_n}$ corresponds to the wreath product of the groups $\mathbb{Z}_2$ by S_n (see [3]). The identification of $\overrightarrow{S_n}$ as a subgroup of the symmetric group S_{2n} can be found in [7], with this identification, the signed Brauer diagram discussed above can be identified as a permutation in S_{10}, which is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 6 & 5 & 8 & 9 & 2 & 1 & 10 & 3 & 4 \end{pmatrix}$$

with the cycle decomposition $(1\ 7)(2\ 6)(3\ 5\ 9)(4\ 8\ 10)$.

In this article, we compute the distinguishing number and a distinguishing labelling for the natural action of a special class of signed Brauer diagrams, i.e., the collection of signed Brauer diagrams without any horizontal edge and they correspond to even permutations in $\overrightarrow{S_n}$ (this class of signed Brauer diagrams is denoted by $\overrightarrow{A_n}$), on the set of its vertices. Sharma and Parmar [7] have given a characterization of even permutations in $\overrightarrow{S_n}$ in terms of the parity of edges (i.e., positive or negative) in the corresponding Brauer diagram. They have proved that an element of $\overrightarrow{S_n}$ corresponds to an even permutation or equivalently belongs to $\overrightarrow{A_n}$, if its corresponding Brauer diagram consists of an even number of negative edges. In addition, it is an index 2 subgroup of $\overrightarrow{S_n}$ and therefore a normal subgroup of order $2^{n-1}n!$. We are interested in finding the distinguishing number and a distinguishing labelling for the natural action of $\overrightarrow{A_n}$ on the set $[2n] = \{1, 2, \ldots, 2n\}$.

According to Tymoczko [11], a map $\phi : X \longrightarrow \{1, 2, \ldots, k\}$ is called a labelling for the action of a group on a set X. Moreover, a labelling ϕ is called a k-distinguishing labelling, if it is preserved only by the elements of the group which fixes every element of the set X (i.e., $Stab_G(X)$). The minimum k for which the action of G on a set X has a k-distinguishing labelling is the distinguishing number for the group action and is denoted by $D_G(X)$.

Sharma et al. [8] had not only provided the complete set of distinguishing numbers for the actions of $\overrightarrow{S_2}$ & $\overrightarrow{A_2}$ on an arbitrary set X, but also computed a distinguishing labelling for the action of $\overrightarrow{S_3}$ & $\overrightarrow{A_3}$ on the set $\{1, 2, \ldots, 6\}$. But the problem of finding a distinguishing labelling for the action of $\overrightarrow{S_n}$ and $\overrightarrow{A_n}$ on the set $[2n]$ for an arbitrary n still remained unsolved. In this direction, the authors of the present article had given closed formulas to compute the distinguishing number and a distinguishing labelling for the action of $\overrightarrow{S_n}$ on $[2n]$, for every n in [2]. Here, in this article we are comprehensively providing closed formulas to compute the distinguishing number and a distinguishing labelling for the action of $\overrightarrow{A_n}$ on $[2n]$, for every n, as some more restrictions on the cycle decomposition of the elements of $\overrightarrow{A_n}$ has been found in comparison with that of $\overrightarrow{S_n}$. This forces the cycle decomposition of the elements of $\overrightarrow{A_n}$ to possess a specific structure than that of $\overrightarrow{S_n}$, which has been discussed in Section 2. *Consequently, the action of* $\overrightarrow{A_n}$ *on* $[2n]$ *can not be distinguished directly as a result of distinguishing the action of* $\overrightarrow{S_n}$*, while recognizing* $\overrightarrow{A_n}$ *as a subgroup of* $\overrightarrow{S_n}$*.*

The aforementioned difficulty gives rise to the following questions:

1. Since $\overrightarrow{A_n} \subset \overrightarrow{S_n} \subset S_{2n}$, so one of the most relevant problem is to answer the question: " What are the restrictions on the cycle decomposition of the elements of $\overrightarrow{A_n}$ in comparison to that of $\overrightarrow{S_n}$ when they are identified with elements of S_{2n}?"
2. What is the structure of the minimum possible elements in $\overrightarrow{A_n}$ which guarantees the invariance of each partitioning component in a partition of X under the action of non identity elements of $\overrightarrow{A_n}$, provided each partitioning component is invariant under the action of these elements?
3. Whether one can provide a closed formula to compute the distinguishing number and a corresponding distinguishing labelling for the action of $\overrightarrow{A_n}$ on $[2n]$, for arbitrary n?

Our main aspiration in this article is to answer the aforesaid questions and to provide a closed formula to find a distinguishing labelling for the action of $\overrightarrow{A_n}$ on $[2n]$, for arbitrary n using a different approach (i.e., by studying the cycle decomposition of the elements of $\overrightarrow{A_n}$). Also, an optimal algorithm to distinguish the action of $\overrightarrow{A_n}$ on $[2n]$ is supplied.

Section 2 describes the structure of disjoint cycles in the cycle decomposition of the elements of $\overrightarrow{A_n}$, which plays a significant role for its complete understanding. The invariance of the partitioning components in a partition of the set $[2n]$ under the action of the elements in $\overrightarrow{A_n}$ is discussed in Section 3. It is also observed that if all the partitioning subsets are invariant under a particular type of elements in $\overrightarrow{A_n}$, then they are also invariant under the action of every element in $\overrightarrow{A_n}$. In the end of this section, the distinguishing number as well as an optimal iterative algorithm to obtain a distinguishing labelling using the minimum possible labels for the action of $\overrightarrow{A_n}$ on the set $[2n]$ for arbitrary n is provided. In fact, we give a closed formula to obtain a distinguishing labelling for the natural action of $\overrightarrow{A_n}$ on the set $[2n]$ for arbitrary n.

2 Cycle decomposition of the elements in $\vec{A}_n$

The purpose of this section is to gather some significant results concerning the cycle decomposition of the elements of $\overrightarrow{A_n}$ which plays a vital role in the development of this article and throughout the notations are followed as in [2]. From this point on, unless specified otherwise $\sigma_1\sigma_2...\sigma_s$ represents the cycle decomposition of an element $\sigma \in \overrightarrow{A_n}$ and σ_j is a cycle in the cycle decomposition of σ. For convenience, we write

- σ_j **a cycle of type I :** if $\sigma_j = (a_1a_2...a_m)$ for some odd integer m, where $a_i \in \{1, 2, \dots, 2n\}$ satisfying $a_k^* \neq a_l$ for $k, l \in \{1, 2, \dots, m\}$;
- σ_j **a cycle of type II :** if $\sigma_j = (a_1a_2...a_m)$ for some even integer m, where $a_i \in \{1, 2, \dots, 2n\}$ satisfying $a_k^* \neq a_l$ for $k, l \in \{1, 2, \dots, m\}$; In the above two cases, the cycle $\sigma_j{}' = (a_1^*a_2^*...a_m^*)$ represents **the cycle complementary to** σ_j, which is again of the same type.
- σ_j **a cycle of type III :** if $\sigma_j = (a_1a_2...a_qa_1^*a_2^*...a_q^*)$, where $a_i \in \{1, 2, \dots, 2n\}$.

As an application of Theorem 2 in [2], the next theorem contributes as a complete characterization of the elements of $\overrightarrow{A_n}$ in terms of their cycle decomposition.

Theorem 1 *Let $\sigma \in \overrightarrow{A_n}$. Then a disjoint cycle in its cycle decomposition is either a cycle of type I, type II or type III.*
Furthermore, the cycle decomposition of σ is in fact one of the following form:

(i) *the finite product of cycles of type I along with their complementary cycles;*
(ii) *the finite product of cycles of type II along with their complementary cycles;*
(iii) *the product of an even number of cycles of type III;*
(iii) *the finite product of cycles of the form (i), (ii) or (iii).*

The following result is immediate from Theorem 3 of [2], as $\overrightarrow{A_n}$ is a subgroup of $\overrightarrow{S_n}$.

Theorem 2 *If $\sigma \in \overrightarrow{A_n}$, then the number of cycles of odd length (i.e., of type I) in the cycle decomposition of σ is always even.*

Next, we turn our attention to analyze the cycles of even lengths in the cycle decomposition of elements in $\overrightarrow{A_n}$.

Theorem 3 *The number of cycles of type II in the cycle decomposition of σ is always even.*

Proof Suppose that $\sigma = \sigma_1\sigma_2...\sigma_s$ is the cycle decomposition of an element $\sigma \in \overrightarrow{A_n}$ and it contains a cycle σ_j of type II. Now, since σ is also an element of $\overrightarrow{S_n}$, therefore it follows from Theorem 2 in [2] that there exists another cycle $\sigma_{j'}$, complimentary to σ_j, for some $j' \in \{1, 2, \dots, s\}$, $j \neq j'$, which is again a cycle of type II. Thus, we conclude that the cycles of type II always exist in a pair in the cycle decomposition of σ. Hence, the number of cycles of type II in the cycle decomposition of σ is always even. □

Note that a cycle of odd length represents an even permutation, however Theorem 2 infers that the number of cycles of odd length in the cycle decomposition of σ is always even, so the combination of all the cycles of odd length constitute an even permutation. On the other hand, a cycle τ of even length in the cycle decomposition of σ satisfies the property that either for at least one $x \in supp(\tau)$, $x^* \in supp(\tau)$ or for all $x \in supp(\tau)$, $x^* \notin supp(\tau)$. Accordingly, in view of Theorem 2 in [2], a cycle of even length in the cycle decomposition of σ is either of type II or type III. Moreover, Theorem 3 implies that the number of cycles of type II in the cycle decomposition is always even. So, the combination of type II cycles which in the cycle decomposition of σ also contributes an even permutation.

Theorem 4 *The number of cycles of type III in the cycle decomposition of σ is always even.*

Proof Suppose that the cycle decomposition of σ contains a cycle σ_j of type III. If possible, let the number of cycles of type III in the cycle decomposition of σ be odd. It is observed that the combination of the cycles of type III in the cycle decomposition of σ give rise to an odd permutation, because a cycle of even length is an odd permutation and by our assumption there exists an odd number of such cycles in the cycle decomposition of σ. Thus, in regards to the above discussion, the cycles in the cycle decomposition of σ are in total a product of two even permutations with one odd permutation. This concludes that the permutation σ is an odd permutation, which is a contradiction. Hence, we conclude that the number of cycles of type III in the cycle decomposition of σ is always even. □

By combining Theorem 3 and Theorem 4, we have

Theorem 5 *If $\sigma \in \overrightarrow{A_n}$, then the number of cycles of even length in the cycle decomposition of σ is always even.*

Combination of Theorem 2 and Theorem 5 leads to an important result, which describes the structure of the elements in $\overrightarrow{A_n}$.

Theorem 6 *Every permutation $\sigma \in \overrightarrow{A_n}$ is always a product of an even number of pairwise disjoint cycles i.e., the number of cycles in its cycle decomposition is always even.*

3 Distinguishing Labelling for the Action of $\vec{A}_n$ on [2n]

We first confine our attention to collect some basic results required to compute the distinguishing number for the action of $\overrightarrow{A_n}$ on $[2n]$.

Theorem 7 [2] *Let G be a group acting on a set X, then there exists a k-distinguishing labelling for the action of G on the set X if and only if the set X has a k-partition, say $X = \sqcup_{i=1}^{k} X_i$ such that for every $g \in G \setminus Stab_G(X)$, at least one partitioning component of X is not g-invariant.*

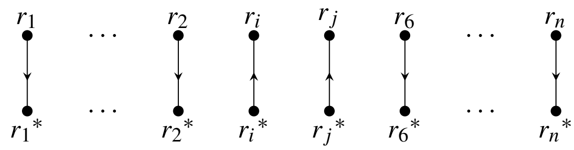
Recall that if a group G is acting on a set X and $g \in G$, then a subset $Y \subseteq X$ is called g-invariant, if the set $\{gy : y \in Y\} \subseteq Y$, under the action of G restricted to Y.

Next, we investigate some conditions on the types of elements in partitioning components which give rise to the existence of a non-identity element in $\overrightarrow{A_n}$ such that each partitioning component stays invariant under the action of this element. Now, onwards in the forthcoming results of this section X represents the set $[2n] = \{1, 2, \ldots, 2n\}$, where $n \in \mathbb{N}$. Also, in order to distinguish elements of the set X less than or equal to n and greater than n, we fix the notations $r_1, r_2, \ldots, r_n$ for the elements less than or equal to n and $s_1, s_2, \ldots, s_n$ for the elements greater than n.

Note that a single transposition of the form $(r_i r_i^*)$ represents an odd permutation, therefore all such transpositions belongs to $\overrightarrow{S_n} \setminus \overrightarrow{A_n}$ while $\overrightarrow{A_n}$ may contain a product of transpositions of the form $(r_i r_i^*)(r_j r_j^*)$. This implies that the existence of a pair of elements of the form r_i, r_i^* in a partitioning component of X ensures the invariance of each partitioning component under the action of the elements of $\overrightarrow{S_n}$ (see Proposition 19, [2]), but this is not the case with $\overrightarrow{A_n}$. However, a similar result holds for $\overrightarrow{A_n}$, if we extend the condition for two pairs of elements as demonstrated in the following proposition:

Proposition 8 *Let $\overrightarrow{A_n}$ acts on the set X and $X = \sqcup_{j=1}^{k} X_j$ be a partition of X. If one of the partitioning components contains elements r_i, $r_i{}^*$, r_j, $r_j{}^*$ for some $i, j \in \{1, 2, \ldots, n\}$, then there exists an element $\tau \in \overrightarrow{A_n}$ of the form $(r_i r_i{}^*)(r_j r_j{}^*)$ such that each partitioning component in the given partition is τ-invariant.*

Proof Let $\overrightarrow{A_n}$ acts on the set X and $X = \sqcup_{j=1}^{k} X_j$ be a partition of X. Assume that the partitioning component X_j for some $j \in \{1, 2, ...k\}$, contains elements $r_i, r_i{}^*, r_j$ and $r_j{}^*$, for some $i, j \in \{1, 2, \ldots, n\}$. Next, we examine a signed Brauer diagram without any horizontal edge which involves only vertical edges with exactly two negative edges between $r_i, r_i{}^*$ & $r_j, r_j{}^*$ and a positive edge between $r_k, r_k{}^*$ $\forall$ $k \in \{1, 2, \ldots, n\} \setminus \{i, j\}$, which can be represented graphically as follows:



The above diagram can be identified as a product of two transpositions $\tau = (r_i r_i{}^*)(r_j r_j{}^*)$ in S_{2n} which is an even permutation of $\overrightarrow{S_n}$, so it is indeed an element in $\overrightarrow{A_n}$. Clearly, the permutation τ interchanges the elements r_i with $r_i{}^*$ and r_j with $r_j{}^*$, while fixes all other elements in the set X. Hence, it follows that each partitioning component in the given partition of X is τ-invariant. □

Proposition 9 *Let $\overrightarrow{A_n}$ acts on the set X and $X = \sqcup_{j=1}^{r} X_j$ be a partition of X satisfying, for some $i, j \in \{1, 2, \ldots, n\}$ and $s, t \in \{1, 2, \ldots, r\}$ either*

(i) $r_i, r_j \in X_s$ *and* $r_i{}^*, r_j{}^* \in X_t$
or
(ii) $r_i, r_j{}^* \in X_s$ *and* $r_i{}^*, r_j \in X_t$
or
(iii) $r_i, r_i{}^* \in X_s$ *and* $r_j, r_j{}^* \in X_t$.

Then there exists an element τ which is a product of two transpositions of the form $(r_i r_j)(r_i{}^* r_j{}^*)$ or $(r_i r_j{}^*)(r_i{}^* r_j)$ or $(r_i r_i{}^*)(r_j r_j{}^*) \in \overrightarrow{A_n}$ respectively, such that each partitioning component in the given partition of X is τ-invariant.

Proof Let $\overrightarrow{S_n}$ acts on the set X and $X = \sqcup_{i=1}^{r} X_i$ be a partition of X such that for some $i, j \in \{1, 2, \ldots, n\}$ and $s, t \in \{1, 2, \ldots, r\}$ either condition (i), (ii) or (iii) is satisfied. First, we discuss the proof when condition (i) or (ii) is satisfied, which will be followed immediately as an application of Proposition 21 of [2] applied for the subgroup $\overrightarrow{A_n}$ of $\overrightarrow{S_n}$. On the other hand, assume that for $i, j \in \{1, 2, \ldots, n\}$, we have $r_i, r_i{}^* \in X_s$ and $r_j, r_j{}^* \in X_t$, where $s, t \in \{1, 2, \ldots, r\}$. We again inspect the same signed Brauer diagram as examined in the proof of the previous proposition that was identified as an element τ of the form $(r_i r_j)(r_i{}^* \, r_j{}^*)$ in S_{2n}, which implies that τ is surly an element in $\overrightarrow{A_n}$. Obviously, the element τ interchanges the pairs of elements (r_i, r_j), $(r_i{}^*, r_j{}^*)$ and fixes every other element in the set X. Hence, it follows that each partitioning component in the given partition of X is invariant under the action of τ. □

The next result describes a condition on the cardinality of a partitioning component in a partition of the set X, for the invariance of each partitioning component under the action of an element of the form $(r_i r_i{}^*)(r_j r_j{}^*)$ in $\overrightarrow{A_n}$.

Theorem 10 *If $\overrightarrow{A_n}$ acts on the set X and $X = \sqcup_{j=1}^{r} X_j$ is a partition of X such that $|X_k| \geq n + 2$, for some $k \in \{1, 2, \ldots, r\}$, then there exists a permutation τ of the form $(r_i r_i{}^*)(r_j r_j{}^*)$ in $\overrightarrow{A_n}$ such that each partitioning component in the given partition is τ-invariant.*

Proof Assume that the group $\overrightarrow{A_n}$ acts on the set X and $X = \sqcup_{j=1}^{r} X_j$ is a partition of X such that $|X_k| \geq n + 2$, for some $k \in \{1, 2, \ldots, r\}$. We claim that the subset X_k contains at least two pairs of elements $r_i, s_i = r_i{}^*$ and $r_j, s_j = r_j{}^*$ for some $i, j \in \{1, 2, \ldots, n\}$. If not, then the subset X_k consists at most one pair of elements of the form $r_i, s_i = r_i{}^*$. Thus, for every $x \in X_k \setminus \{r_i, r_i^*\}$, we have $x^* \notin X_k$, so $x^* \in \sqcup_{j=1, j\neq k}^{r} X_j$, gives that $\sum_{j=1, j\neq k}^{r} |X_j| \geq n$. But in this case, $|X| = |X_k| + \sum_{j=1, j\neq k}^{r} |X_j| \geq (n+2) + n = 2n + 2$, which is absurd. Thus, the subset X_k contains at least two pairs of elements $r_i, s_i = r_i{}^*$ and $r_j, s_j = r_j{}^*$ for some $i, j \in \{1, 2, \ldots, n\}$. Hence, the result follows by using Proposition 8. □

Next, we recall an important result, i.e., Theorem 22 of [2], concerning the invariant subsets of a set X under the action of elements in the group $\overrightarrow{S_n}$. Note that this result can be applied for $\overrightarrow{A_n}$ using the fact that $\overrightarrow{A_n} \subset \overrightarrow{S_n}$. So, we have

Proposition 11 [2] *If $\overrightarrow{A_n}$ is acting on the set $X = \sqcup_{i=1}^{k} X_i$ and $\sigma_1\sigma_2...\sigma_s$ is the cycle decomposition of an element $\sigma \in \overrightarrow{A_n}$, then the following statements are equivalent:*

(i) *Each partitioning component of X is σ-invariant;*
(ii) *Each partitioning component of X is σ_j-invariant, for all $j \in \{1, 2, \dots, s\}$;*
(iii) *For every $j \in \{1, 2, \dots, s\}$, $supp(\sigma_j) \subseteq X_i$, for some i.*

The next theorem guarantees that the invariance of each partitioning component under the action of the elements of the form $(r_i r_i^*)(r_j r_j^*)$ or $(r_i r_j)(r_i^* r_j^*)$ or $(r_i r_j^*)(r_i^* r_j)$ in $\overrightarrow{A_n}$ is necessary, otherwise at least one partitioning component of X could not be invariant under the action of a non-identity element in $\overrightarrow{A_n}$.

Theorem 12 *If $\overrightarrow{A_n}$ acts on the set X, with $X = \sqcup_{i=1}^{k} X_i$ and $e \neq \sigma \in \overrightarrow{A_n}$ such that each partitioning component X_i is invariant under σ, then there exists an element $\tau \in \overrightarrow{A_n}$ of the form $(r_i r_i^*)(r_j r_j^*)$, $(r_i r_j)(r_i^* r_j^*)$ or $(r_i r_j^*)(r_i^* r_j)$ such that each X_i is τ-invariant.*

Proof Assume that $e \neq \sigma$ is an element in $\overrightarrow{A_n}$ such that each partitioning component X_i is invariant under $\sigma = \sigma_1\sigma_2...\sigma_s$ in S_{2n}. It is concluded from Proposition 11 that for every $j \in \{1, 2, \dots, s\}$, $supp(\sigma_j)$ is entirely contained in exactly one partitioning component of X.

Thus, the following two possibilities arise:

Possibility 1: If the cycle decomposition of σ contains a cycle $\sigma_j = (a_1 a_2 ... a_m)$ of odd length $m \geq 3$, where $a_i \in \{1, 2, \dots, 2n\}$, then by Theorem 2 there exists another m-cycle of the form $(a_1^* a_2^* ... a_m^*)$ in the cycle decomposition of σ. Choose $i, j \in \{1, 2, \dots, m\}$, $i \neq j$, corresponding to our choice, we have an element $\tau = (a_i a_j)(a_i^* a_j^*) \in \overrightarrow{A_n}$. The element $\tau = (a_i a_j)(a_i^* a_j^*)$ is indeed an element in $\overrightarrow{A_n}$ is immediate by using Proposition 9 and the fact that τ is an even permutation.

Furthermore, because the supports of two transpositions $(a_i a_j)$ and $(a_i^* a_j^*)$ are disjoint and each of them is completely contained in exactly one partitioning component of X, which infers that each partitioning component of X is τ-invariant.

Possibility 2: If the cycle decomposition of σ contains a cycle $\sigma_j = (a_1 a_2 ... a_m)$ of even length $2q = m \geq 4$, where $a_i \in \{1, 2, \dots, 2n\}$, then by using Theorem 2 in [2], the cycle σ_j is either of the form $(a_1 a_2 ... a_q a_1^* a_2^* ... a_q^*)$ or $a_l^* \notin supp(\sigma_j)$, for all $l \in \{1, 2, \dots, m\}$.

Case 1. Let $\sigma_j = (a_1 a_2 ... a_q a_1^* a_2^* ... a_q^*)$. Then by Theorem 4, σ contains another cycle of the form $\sigma_k = (b_1 b_2 ... b_r b_1^* b_2^* ... b_r^*)$ in its cycle decomposition. In this case, choose two elements one from each set $\{a_1, a_2, \dots, a_q\}$ and $\{b_1, b_2, \dots, b_r\}$, then corresponding to this choice, we have an element $\tau = (a_s a_s^*)(b_t b_t^*) \in \overrightarrow{A_n}$ such that each partitioning component is invariant under τ, because both the sets $supp(\sigma_j)$ and $supp(\sigma_k)$ should be completely contained in one of the partitioning components of X and so the sets $\{a_s, a_s^*\}$, $\{b_t, b_t^*\}$ are entirely contained in exactly one of the partitioning components of X.

Case 2. If $\sigma_j = (a_1 a_2 ... a_m)$ satisfies the condition that $a_l^* \notin supp(\sigma_j)$, for all $l \in \{1, 2, \dots, m\}$, then again by using Theorem 2 in [2], it is concluded that the permutation σ contains another m-cycle $\sigma_{j'}$ of the form $(a_1^* a_2^* ... a_m^*)$, for some $j' \in \{1, 2, \dots, s\}$ with $j' \neq j$. Now, similar to the possibility 1, if we choose two distinct numbers i, j in $\{1, 2, \dots, m\}$, then correspondingly there exists an element $\tau = (a_i a_j)(a_i^* a_j^*) \in \overrightarrow{A_n}$ with the property that each partitioning component of X is τ-invariant. This completes the proof. □

Theorem 13 *The distinguishing number for the action of $\overrightarrow{A_n}$ on the set X is at most n i.e., $D_{\overrightarrow{A_n}}(X) \leq n$.*

Proof In order to prove that $D_{\overrightarrow{S_n}}(X) \leq n$, it is sufficient to show that there exists an n-distinguishing labelling for the action of $\overrightarrow{A_n}$ on the set X. Further, in view of Theorem 7 and Theorem 12, it is enough to produce an n-partition of the set X satisfying the condition that each partitioning component is not invariant under the action of those elements of $\overrightarrow{A_n}$ which are precisely of the form $(r_i r_j)(r_i^* r_j^*)$ or $(r_i r_j^*)(r_i^* r_j)$ or $(r_i r_i^*)(r_j r_j^*)$, for some $r_i, r_j \in \{1, 2, \dots, n\}$. Consider an n-partition of the set X with partitioning components $\{1, 2, \dots, n, n+1\}$, $\{n+2\}, \dots, \{2n-1\}$ and $\{2n\}$. Clearly, each partitioning component in the above partition of X satisfies the required property and hence the result. □

The sequence "n appears $n-1$ times" is a sequence of integers given by 2,** 3, 3, 4, 4, 4, 5, ...**, which can be obtained by substituting** $t = 2, 3, 4, ...$ **respectively in** $f(t) = \left\lceil \frac{\sqrt{8t-7}+1}{2} \right\rceil = \left\lfloor \sqrt{2(t-1)} + \frac{1}{2} \right\rfloor + 1$ ***(see [4]***).***

Note that the value of $f(t)$ for $t = 1$ is $= 1$. Thus, for a fixed $n \geq 1$, we denote the n^{th} term of the sequence 1, 2, 3, 3, 4, 4, 4, 5, ..., obtained by prepending 1 in the beginning of the above sequence by ζ and is given by $\zeta = \left\lceil \frac{\sqrt{8n-7}+1}{2} \right\rceil = \left\lfloor \sqrt{2(n-1)} + \frac{1}{2} \right\rfloor + 1$.

It is easy to see that $\zeta = n$ if and only if $n = 1, 2, 3$. This follows from the fact that $n > \zeta$ if $n > 3$.

Next, we attempt to relate the order of a partitioning component with the number of partitioning components in a partition of X corresponding to a distinguishing labelling for the action of $\overrightarrow{A_n}$ on X.

Suppose that $X = \sqcup_{j=1}^{k} X_j$ is a k-partition of X corresponding to a k-distinguishing labelling for the action of $\overrightarrow{A_n}$ on X. By Theorem 7, at least one partitioning component of X is not g-invariant, for every $e \neq g \in \overrightarrow{A_n}$. Note that in view of Theorem 2 and Theorem 6, $\overrightarrow{A_n}$ does not comprise any element which is specifically a transposition or a 3-cycle. So, a partitioning component of X can contain a pair of elements of the form r_i, r_i^*, but Proposition 9 constraint it for only one partitioning component of X. Also, Proposition 8 implies that a partitioning cannot contain two pairs of elements of the form r_i, r_i^*. Thus, at most one partitioning component of X can contain at best one pair of elements of the form r_i, r_i^*, otherwise there exists an element $e \neq g \in \overrightarrow{A_n}$ of the form $(r_i r_i^*)(r_j r_j^*)$ or $(r_i r_j)(r_i^* r_j^*)$ or $(r_i r_j^*)(r_i^* r_j)$, such that each partitioning component of X is g-invariant.

Further, let $X_k = \{a_1, a_2, \ldots, a_p\}$ be a partitioning component of X with $|X_k| = p$. Now, if X_k already contains a pair of elements of the form a_i, a_i^*, then by above discussion, the $p-2$ elements in the set $\{a_1^*, a_2^*, \ldots, a_p^*\} \setminus \{a_i, a_i^*\}$ must lie in distinct partitioning components other than X_k. This infers that the number of partitioning components are at least $p-1$.

On the other hand, if the elements in the partitioning component X_k satisfies the condition that $a_i \neq a_j^*$ for $i, j \in \{1, 2, \ldots, p\}$, then each element in the set $\{a_1^*, a_2^*, \ldots, a_p^*\}$ must be contained in distinct partitioning components of X, including or excluding the partitioning component X_k depending upon the situation that whether the partitioning component X_k can contain a pair of elements x, x^* or not (the case X_k could not contain any pair of elements x, x^* can occur, when a partitioning component other than X_k already contains the pair of elements $x, x^* \in X$, as only one partitioning component of X can do so). Accordingly, this forces the number of partitioning components of X to be at least p or $p+1$ respectively.

Finally, we summarize the above discussion in the following proposition.

Proposition 14 *Suppose that $X = \sqcup_{j=1}^{r} X_j$ is a k-partition of X corresponding to a k-distinguishing labelling for the action of $\overrightarrow{A_n}$ on X. If a partitioning component $X_k = \{a_1, a_2, \ldots, a_p\}$ in a partition of the set X contains p elements, then the number of partitioning components of X is*

$$\geq \begin{cases} p-1, & \textit{if there already exist elements } a_i, a_j \in X_k \textit{ such that } a_i^* = a_j, \\ p, & \textit{if } \forall\, a_i \in X_k, a_i^* \notin X_k, \textit{ but } a_i^* \textit{ for some } i \textit{ can be adjoined to } X_k, \\ p+1, & \textit{if } \forall\, a_i \in X_k, a_i^* \notin X_k, \textit{ but } a_i^* \textit{ for some } i \textit{ cannot be adjoined to } X_k. \end{cases}$$

Remark 15 The above discussion demonstrated that a partitioning component with the maximum number of elements in it must contains a pair of elements of the form r_i, r_j with $r_i^* = r_j$, otherwise, it will lead to the existence of an extra component in the partition of X and in view of Theorem 7, our purpose of minimizing the number of partitioning components in order to obtain the distinguishing number may not be optimally achieved.

Remark 16 The above proposition describes that if we increase the number of elements in a partitioning component by 1, it may also increase the distinguishing number by 1. This indicates that the number of elements in each partitioning component must be close enough, i.e., the difference between the number of elements in any two partitioning components should not be very large.

Now, we have all the necessary information to provide a lower bound for the action of $\overrightarrow{A_n}$ on X.

Theorem 17 *The distinguishing number for the action of $\overrightarrow{A_n}$ on the set X is at least ζ, i.e., $D_{\overrightarrow{A_n}}(X) \geq \zeta$.*

Proof If possible, assume that the distinguishing number for the action of $\overrightarrow{A_n}$ on the set X is $\zeta - 1$. Then by definition there exists a $(\zeta - 1)$-distinguishing labelling for the action of $\overrightarrow{A_n}$ on the set X. Also, by Theorem 7, the set X can be partitioned as a disjoint union of $(\zeta - 1)$ number of subsets such that for every $e \neq \sigma \in \overrightarrow{A_n}$, at least one partitioning component is not σ-invariant. Furthermore, in view of Proposition 8, Proposition 9 and Remark 15, exactly one partitioning component of X, X_t (say) for some $t \in \{1, 2, \ldots, r\}$, can contain a pair of elements r_i, r_i^* for some $i \in \{1, 2, \ldots, n\}$. Now, in order to get a better estimate of the elements in the partitioning components, we omit this pair of elements from X_j, so that $|X|$ becomes $2n-2$. Thus, Proposition 14 infers that the order of each partitioning component of X can not exceed $\zeta - 2$, as there are $\zeta - 1$ partitioning components and each partitioning component can not contain a pair of elements of the form r_j, r_j^*, for some $j \in \{1, 2, \ldots, n\} \setminus \{i\}$. However, if all the $\zeta - 1$ partitioning components have their maximum possible order (i.e., $|X_i| = \zeta - 2$, for all i), then

$$\begin{aligned}|X| &= |X_1| + |X_2| + \ldots + |X_\gamma|\\ 2n-2 &= (\zeta-2) + (\zeta-2) + \ldots + (\zeta-2)\\ &= (\zeta-1)(\zeta-2)\\ &= \zeta^2 - 3\zeta + 2\end{aligned}$$

which is not possible, because next we shall show that for all $n \geq 1$, $2n-2 > \zeta^2 - 3\zeta + 2$.

For this, we know that $x - 1 < \lfloor x \rfloor \leq x$, for all $x \in \mathbb{R}$. This implies that

$$\left(\sqrt{2n-2} + \frac{1}{2}\right) - 1 < \left\lfloor \sqrt{2n-2} + \frac{1}{2} \right\rfloor \leq \left(\sqrt{2n-2} + \frac{1}{2}\right), \text{ for } x = \sqrt{2n-2} + \frac{1}{2}.$$

Adding 1 to the above inequality, we get

$$\sqrt{2n-2} + \frac{1}{2} < \zeta \leq \sqrt{2n-2} + \frac{3}{2}, \text{ as } \zeta = \left\lfloor \sqrt{2n-2} + \frac{1}{2} \right\rfloor + 1. \tag{1}$$

Multiplying equation (1) by ζ, we have

$$\zeta^2 \leq \left(\sqrt{2n-2} + \frac{3}{2}\right)\zeta, \text{ as } \zeta \geq 1.$$

This leads to the inequality,

$$\begin{aligned}\zeta^2 - 3\zeta &\leq \left(\sqrt{2n-2} + \frac{3}{2}\right)\zeta - 3\zeta\\ &= \zeta\left(\sqrt{2n-2} + \frac{3}{2} - 3\right)\\ &= \zeta\left(\sqrt{2n-2} - \frac{3}{2}\right)\\ &\leq \left(\sqrt{2n-2} + \frac{3}{2}\right)\left(\sqrt{2n-2} - \frac{3}{2}\right), \text{ (using (1))}\\ &= 2n - 2 - \frac{9}{4}.\end{aligned}$$

Therefore,

$$\begin{aligned}\zeta^2 - 3\zeta + 2 &\leq 2n - 2 - \frac{9}{4} + 2\\ &< 2n - 2 - \frac{1}{4}.\end{aligned}$$

Thus, we deduce that $2n-2 > \zeta^2 - 3\zeta + 2$, for $n \geq 1$. Hence, the distinguishing number for the action of $\overrightarrow{A_n}$ on the set X is at least ζ. □

Note that for $n = 1$, group $\overrightarrow{A_1}$ is the trivial group. Further, in view of Theorem 7, $X = X_1 \cup X_2$, where $X_1 = \{1, 1^*, 2\}$, $X_2 = \{2^*\}$ is a partition of $X = \{1, 1^*, 2, 2^*\}$ with the minimum number of partitioning components such that these partitioning components are not σ-invariant for every $e \neq \sigma \in \overrightarrow{A_2}$. Thus, the partition $X = X_1 \sqcup X_2$ gives rise to a 2-distinguishing labelling, along with 2 is the minimum possible number for which there exists a distinguishing labelling.

Now, we provide a recursive algorithm to construct a ζ-partition of the set X in such a way that the partitioning subsets does not satisfy the hypothesis in Proposition 8 and 9.

Construction 18 1. Initialize $i = 2$, so that $X = \{1, 1^*, 2, 2^*\}$. Set $X_1 = \{1^*, 1, 2\}$ and $X_2 = \{2^*\}$.
2. While $i < n$, do
   (a) Increment i by 1, so that $X = \{1, \ldots, (i+1), 1^*, \ldots, (i+1)^*\}$.
   (b) If all the partitioning components in the previous stage except X_1 have the same cardinality which is two less than the cardinality of X_1, then generate a new partitioning component, adjoin $(i+1)^*$ to it and $i+1$ to X_2.
   (c) If all the partitioning components in the previous stage except X_1 do not have the same cardinality, then collect those partitioning components of X which have cardinality strictly less than the cardinality of X_1. Arrange these partitioning components in the decreasing order of their cardinality in such a way that in case two partitioning components have the same cardinality, then the partitioning component with smaller suffix will come first. Then
      (i) if the cardinality of all partitioning components in this arrangement except the last partitioning component, is equal to $|X_2|$, while the last partitioning component has its cardinality one less than that of X_2, in this case, adjoin $(i+1)$ to X_1 and $(i+1)^*$ to the last partitioning component in the above specified arrangement.
      (ii) if at least two partitioning components in this arrangement have their cardinality exactly one less than $|X_2|$, then adjoin $(i+1)$ to the first partitioning component in the above said order which have its cardinality equal to $|X_2|-1$ and $(i+1)^*$ to the last partitioning component in the above specified arrangement.
3. Go back to 2.

Next, we step by step illustrate the above algorithm for $n = 7$. In this case, the algorithm terminates after five iterations.

$\{1^*, 1, 2\}$	$\{1^*, 1, 2\}$	$\{1^*, 1, 2, 4\}$	$\{1^*, 1, 2, 4\}$	$\{1^*, 1, 2, 4\}$	$\{1^*, 1, 2, 4, 7\}$
$\{2^*\}$	$\{2^*, 3\}$	$\{2^*, 3\}$	$\{2^*, 3, 5\}$	$\{2^*, 3, 5\}$	$\{2^*, 3, 5\}$
	$\{3^*\}$	$\{3^*, 4^*\}$	$\{3^*, 4^*\}$	$\{3^*, 4^*, 6\}$	$\{3^*, 4^*, 6\}$
			$\{5^*\}$	$\{5^*, 6^*\}$	$\{5^*, 6^*, 7^*\}$
Initial Stage	First Iteration	Second Iteration	Third Iteration	Fourth Iteration	Fifth Iteration

As an application of the above construction, we have

Proposition 19 *Construction 18 terminates after $(n-2)$ iterations and at that stage there are ζ number of partitioning components of X.*

Proof Initially, there are two partitioning subsets containing four elements in total and after this, with each iteration the order of the set X is decreased by 2. The total number of elements in the set X is $2n$, so for a given n, the construction stops after $(n-2)$ iterations.

By construction 18, the number of partitioning components, when $2i$ elements namely $1, 2, \ldots, i, 1^*, 2^*, \ldots, i^*$ are already arranged in the partitioning components, forms the sequence 1, 2, 3, 3, 4, 4, 4, 5..., of numbers whose n^{th} term is given by ζ. This follows from the fact that during the construction, a new partitioning component is introduced, only if in the previous iteration $(m-1)(m-2)+2$ elements namely; $1, 2, \ldots, (\frac{(m-1)(m-2)}{2}+1)$, $1^*, 2^*, \ldots, (\frac{(m-1)(m-2)}{2}+1)^*$ are arranged such that all the partitioning components except X_1 have their cardinality $|X_1|-2$. This gives that the number of partitioning subset of X is the integer $k = max(A)$; where $A = \{m \in \mathbb{Z} : \frac{(m-1)(m-2)}{2}+1 < n\}$.

Next, we claim that $k = \zeta$. For this, it is enough to show that the integer $\zeta \in A$, while $\zeta + 1 \notin A$. Obviously, $\zeta \in A$, as $2n-2 > (\zeta-1)(\zeta-2) = \zeta^2 - 3\zeta + 2$, so it remains to show that $\zeta + 1 \notin A$, i.e., $\zeta(\zeta-1) \geq 2n-2$.

Now, from (1) we have

$$\sqrt{2n-2} + \frac{1}{2} < \zeta \leq \sqrt{2n-2} + \frac{3}{2}, \text{ as } \zeta = \left\lfloor \sqrt{2n-2} + \frac{1}{2} \right\rfloor + 1.$$

On multiplying both sides by ζ, we have

$$\zeta^2 > \left(\sqrt{2n-2} + \frac{1}{2}\right)\zeta, \text{ as } \zeta \geq 1.$$

Now, adding $-\zeta$ on both sides, we get

$$\begin{aligned}\zeta^2-\zeta &> \Big(\sqrt{2n-2}+\frac{1}{2}\Big)\zeta-\zeta\\ \zeta(\zeta-1) &> \zeta\Big(\sqrt{2n-2}+\frac{1}{2}-1\Big)\\ &= \zeta\Big(\sqrt{2n-2}-\frac{1}{2}\Big)\\ &> \Big(\sqrt{2n-2}+\frac{1}{2}\Big)\Big(\sqrt{2n-2}-\frac{1}{2}\Big)\\ &= 2n-2-\frac{1}{4}.\end{aligned}$$

Since, $\zeta(\zeta-1)$ is always an integer and it is greater than $2n-2-\frac{1}{4}$. Therefore, we have $\zeta(\zeta-1)\geq 2n-2>0$. Hence, $\zeta+1\notin A$, as required. □

The next proposition is an application of Construction 18 that guarantees the existence of a ζ-distinguishing labelling for the action of $\overrightarrow{A_n}$ on X.

Proposition 20 *Construction 18 produces a ζ-distinguishing labelling for the action of $\overrightarrow{A_n}$ on X.*

Proof Let $\overrightarrow{A_n}$ acts on the set X. Clearly, by Proposition 19, at the end of the construction there are ζ number of partitioning components, say $X_1, X_2, \ldots, X_\zeta$. Define a labelling $\phi : X \longrightarrow \{1,2,\ldots,\zeta\}$ by $\phi(x)=i$, if $x\in X_i$, for $i=1,2,\ldots,\zeta$. Now, in view of Theorem 7, for ϕ to be a ζ-distinguishing labelling we need to show that for every $e\neq\sigma\in\overrightarrow{A_n}$, at least one partitioning component is not σ-invariant. Further, by using Theorem 12, it is enough to show that at least one partitioning component of X is not σ-invariant, for all $\sigma\in\overrightarrow{A_n}$ which are literally of the form $(r_ir_i^*)(r_jr_j^*)$ or $(r_ir_j)(r_i^*r_j^*)$ or $(r_ir_j^*)(r_i^*r_j)$ in $\overrightarrow{A_n}$.

Clearly, by the Construction 18, neither a partitioning component contains the set $\{r_i, r_j, r_i^*, r_j^*\}$ nor any two partitioning components completely contain two pair of elements (r_i, r_i^*) & (r_j, r_j^*) with one pair in each partitioning component, for some $i, j\in\{1,2,\ldots,n\}$. Hence, the result follows by using Proposition 8 and Proposition 9. □

It is important to note here that Theorem 17 and Proposition 20 confirms the optimality of the above algorithm for the construction of a distinguishing labelling using the minimum possible labels. Also, by combining Theorem 17 and Proposition 20, we obtain a closed formula to calculate the distinguishing number for the action of $\overrightarrow{A_n}$ on the set X, for arbitrary n.

In [1], Chan has given an explicit formula (see Corollary 2.4) and one can use this to calculate the distinguishing number for the action of $\overrightarrow{A_n}$ on the set X, which in general requires more computation work and is cumbersome to compute for higher values of n. We have thus simplified this computational complexity, by providing the following simple formula in a closed form.

Theorem 21 *$D_{\overrightarrow{A_n}}(X)=\zeta$, where $\zeta=\lfloor\sqrt{2n-2}+\frac{1}{2}\rfloor+1$.*

Finally, Construction 18 and Proposition 20 establish a closed formula to obtain a ζ-distinguishing labelling for the action of $\overrightarrow{A_n}$ on the set X.

Theorem 22 *If $\overrightarrow{A_n}$ acts on the set X, then the map $\phi : X\longrightarrow\{1,2,\ldots,\zeta\}$ defined by*

$$\phi(x)=\begin{cases} x-\frac{p(p+1)}{2}; & \text{if } 1\leq x\leq n\\ q; & \text{if } n+1\leq x\leq 2n,\end{cases}$$

where $p=\left\lfloor\frac{(\sqrt{8x-7}-1)}{2}\right\rfloor$ and $q=\left\lceil\frac{\sqrt{8(x-n)-7}+1}{2}\right\rceil$, is a ζ-distinguishing labelling of X under the action of $\overrightarrow{A_n}$.

Acknowledgements The second author gratefully acknowledges the financial assistance of CSIR-HRDG for SRF.

References

1. Chan, M.: The distinguishing number of the direct product and wreath product action, J. Algebr Comb 24, 331-345 (2006)
2. Dadhwal, M., Pankaj.: On distinguishing labelling of sets under the wreath product action, Indian J. Pure Appl. Math. (2022) https://doi.org/10.1007/s13226-022-00314-w
3. Musli, C.: Representations of finite groups, Hindustan book agency, (1993)
4. OEIS Foundation Inc: The On-line Encyclopedia of Integer Sequences, https://oeis.org/A003057
5. Parvathi, M., Kamaraj, M.: Signed Brauer's algebras, Comm. in Algebra 26(3), 839-855 (1998)
6. Parvathi, M., Selvaraj, C.: Signed Brauer's algebras as Centralizer algebras, Comm. in Algebra 27(12), 5985-5998 (1999)
7. Sharma, R. P., Parmar, R.: Some Subgroups of Symmetric Groups Determined Using Signed Brauer Diagrams, J. Algebra Appl. 18(8), 1950152 (12 pages) (2019), https://doi.org/10.1142/S0219498819501524
8. Sharma, R. P., Parmar, R., Kapil, V. S.: Labeling of Sets Under the Action of $\overrightarrow{S_n}$ and $\overrightarrow{A_n}$, Algebra and its Applications, Springer Proceedings in Mathematics & Statistics. https://doi.org/10.1007/978-981-10-1651-6
9. Sharma, R. P., Kapil, V. S.: Irreducible S_n-Modules and A cellular structure of the Signed Brauer Algebra, Southeast Asian Bulletin of Mathematics 35, 497-522 (2011)
10. Sharma, R. P., Kapil, V. S.: Generic irreducibles of the Brauer algebras, Contemporary Mathematics (AMS) 456, 189-204 (2008)
11. Tymoczko, J.: Distinguishing Number for Graph and Groups, Elect. J. Comb., #R63 (2004)