Indian J Pure Appl Math (2026) 57:511–521
https://doi.org/10.1007/s13226-024-00558-8

ORIGINAL RESEARCH

Harmonic projections on Besov spaces in the real ball

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Received: 31 July 2023 / Accepted: 31 January 2024 / Published online: 25 February 2024

Abstract As is well known, Bergman projection operator continuously maps weighted Bergman and some more general spaces onto their holomorphic or harmonic subspaces. In this paper, we turn to Besov spaces and define three-parameter Besov spaces $\Lambda_\alpha^{p,q}$ of smooth functions over the unit ball in $\mathbb{R}^n$. A family of Bergman type operators is constructed whose members continuously project the Besov space $\Lambda_\alpha^{p,q}$ onto its harmonic subspace $h\Lambda_\alpha^{p,q}$. A generalization with more general indices is also given.

Keywords Harmonic Besov space · Bergman projection · Fractional derivative · Poisson–Bergman kernel

Mathematics Subject Classification 31B05 · 46E15 · 47B32

1 Introduction and main result

In this paper, we continue and develop our study [3,5] on some Bergman type projection operators acting on weighted spaces of harmonic functions over the real ball. Most of our notation is adopted from [1,3,5].

For an integer $n \geq 2$, let $B = B^n$ be the open unit ball in $\mathbb{R}^n$ and $S = \partial B$ be its boundary, the unit sphere. The pth mean values of a function $f(x) = f(r\zeta)$ defined on B are denoted by

$$M_p(f;r) := \left\| f(r\cdot) \right\|_{L^p(S;d\sigma)}, \qquad 0 \leq r < 1,\ 0 < p \leq \infty,$$

where $d\sigma$ is the $(n-1)$-dimensional area-surface measure on S normalized so that $\sigma(S) = 1$. The set of all (real) harmonic functions in the unit ball B is denoted by $h(B)$. The notation dV means the Lebesgue volume measure on B normalized to have total mass 1. In the polar coordinates, we have $dV(x) = n\, r^{n-1} dr d\sigma(\zeta)$.

The mixed norm space, denoted by $L(p,q,\alpha)\big(0 < p, q \leq \infty, \alpha \in \mathbb{R}\big)$, is defined to be the collection of those functions f measurable in the unit ball B, for which the following quasi-norm is finite:

$$\|f\|_{L(p,q,\alpha)} := \left(\int_0^1 (1-r)^{\alpha q - 1} M_p^q(f;r) dr \right)^{1/q}, \qquad 0 < q < \infty.$$

In the case $q = \infty$ the quasi-norm is to be interpreted as a supremum

$$\|f\|_{L(p,\infty,\alpha)} := \operatorname*{ess\,sup}_{0 \leq r < 1} (1-r)^\alpha M_p(f;r).$$

Communicated by E. K. Narayanan.

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For the subspace of $L(p,q,\alpha)$ consisting of harmonic functions, we set $h(p,q,\alpha) := h(B) \cap L(p,q,\alpha)$. The mixed norm spaces $h(p,q,\alpha)$ and their analogues consisting of holomorphic, harmonic or $\mathcal{M}$-harmonic functions in the disc, the ball in $\mathbb{C}^n$ or $\mathbb{R}^n$ have been extensively discussed in the past three decades. The mixed norm spaces of holomorphic functions in the unit disc were introduced by Hardy and Littlewood [17,18], and developed later by Flett [14]. For $p = q < \infty$ the spaces $h(p,q,\alpha)$ reduce to the weighted Bergman spaces, see monographs [12,29]. A lot of papers since 1980s are devoted to harmonic spaces $h(p,p,\alpha)$, $h(p,q,\alpha)$ on the unit ball in $\mathbb{R}^n$, see [6–8,10,19,23–26] and elsewhere, while the spaces $h(p,q,\alpha)$ consisting of n-harmonic functions on a polydisc in $\mathbb{C}^n$ have been studied in [1,2]. Throughout the paper, we use conventional notations. For typical points in $\mathbb{R}^n$ we always write $x = r\zeta,\ y = \rho\eta$ or $x = rx',\ y = \rho y'$ with $|x| = r,\ |y| = \rho$ and $\zeta,\ \eta,\ x',\ y' \in S$. The letters $C(\alpha,\beta,\dots)$, C_α etc. stand for various positive constants depending only on the parameters indicated. Sometimes we will not express the dependence on the parameters explicitly. Denote by $\mathbb{N}$ the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In the sequel, $[\alpha]$ stands for the largest integer less than or equal to $\alpha \in \mathbb{R}$. In what follows, we write $T : X \longrightarrow Y$ if T is a bounded operator mapping X to Y, i.e. $\|Tf\|_Y \le C\|f\|_X\ \ \forall f \in X$.

We define three-parameter Besov space $\Lambda^{p,q}_\alpha$ $(1 \le p,q \le \infty,\ \alpha > 0)$ to be a class of functions f whose certain (fractional) derivative $\mathcal{D}^\delta f$ is in the mixed norm space. The exact definition of the Besov spaces is given in Section 4.

Theorem A *Let $\alpha > 0$. If either $0 < p,q \le \infty,\ \beta > \max\{\alpha + (n-1)(1/p - 1),\ \alpha\}$, or $1 \le p \le \infty,\ 0 < q \le 1,\ \beta \ge \alpha$, then for all $u \in h(p,q,\alpha)$ we have*

$$u(x) = \frac{2}{n\,\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1}\, \mathcal{P}_\beta(x,y)\, u(y)\, dV(y), \qquad x \in B, \tag{1.1}$$

where $\mathcal{P}_\beta$ is the reproducing kernel of Poisson–Bergman type defined in Section 3.

Reproducing integral formula (1.1) is familiar for spaces $h(1,1,\alpha)$ and $h(2,2,\alpha)$, see, for example, [12,19, 24]. For a wide range of parameters and mixed norm space $h(p,q,\alpha)$, formula (1.1) is obtained in the recent paper [7]. Integral formula (1.1) induces a family of Bergman projection operators

$$T_\beta(u)(x) := \frac{2}{n\,\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1}\, \mathcal{P}_\beta(x,y)\, u(y)\, dV(y), \qquad x \in B.$$

Theorem A asserts that operator T_β is the identity map on $h(p,q,\alpha)$, that is, $T_\beta(u) = u,\ \forall\, u \in h(p,q,\alpha)$ for suitable parameters. Holomorphic counterparts of operators T_β are studied, for example, in [10,12,29], while their harmonic analogues appear in [8,10,12,19,20,23,26,27]. It is well known that operator T_β continuously projects mixed norm space $L(p,q,\alpha)$ onto its harmonic subspace $h(p,q,\alpha)$ (see, for example, [3,23]),

$$T_\beta : L(p,q,\alpha) \xrightarrow{\text{onto}} h(p,q,\alpha), \qquad 1 \le p,q \le \infty,\ \beta > \alpha > 0. \tag{1.2}$$

We are interested in finding a projection result like (1.2) for other function spaces, just for Besov spaces $\Lambda^{p,q}_\alpha$. In the next main theorem, a family of bounded harmonic projections in Besov spaces is found.

Theorem 1 *For any $1 \le p,q \le \infty,\ 0 < \alpha < \delta \le [\alpha]+1,\ 0 \le \lambda < \alpha + \beta - \delta,\ s \ge 1 - \frac{n}{2}$, the operator*

$$\Phi_{\beta,\lambda,\delta}(f)(x) := \frac{2}{n\,\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1}\, \mathcal{P}_\lambda(x,y)\, \mathcal{D}^\delta_{n,s} f(y)\, dV(y)$$

continuously maps the Besov space $\Lambda^{p,q}_\alpha$ of sufficiently smooth functions onto the harmonic space $h\Lambda^{p,q}_{\alpha+\beta-\delta-\lambda}$, with the norm inequality

$$\left\|\Phi_{\beta,\lambda,\delta}(f)\right\|_{h\Lambda^{p,q}_{\alpha+\beta-\delta-\lambda}} \le C(p,q,\alpha,\beta,\delta,\lambda,n)\, \|f\|_{\Lambda^{p,q}_\alpha}.$$

In particular, for $\beta = \delta + \lambda$ and $s = \lambda$, the operator $\Phi_{\beta,\lambda,\beta-\lambda}$ is a continuous projection of the Besov space $\Lambda^{p,q}_\alpha$ onto its harmonic subspace, $\Phi_{\beta,\lambda,\beta-\lambda} : \Lambda^{p,q}_\alpha \xrightarrow{\text{onto}} h\Lambda^{p,q}_\alpha$.

Remark 1 Theorem 1 can be viewed as a generalization and improvement of [5, Thm 1.3] and [1, Thm 5]. On the other hand, in the recent papers [16, Thm 1.4] and [13], for the special case $p = q \in [1,\infty]$ and some Bergman type operators similar to T_β, the authors obtained the boundedness of type $L(p,p,\alpha) \longrightarrow h\Lambda^{p,p}_\alpha$ that map a weighted Lebesgue space onto a two-parameter harmonic Besov space. However, these operators from [13,16] are in general not true projections onto a subspace, see [16, Remark 11.2]. In contrast to [13,16], our operators $\Phi_{\beta,\lambda,\beta-\lambda}$ as already mentioned, are true projections mapping the Besov space of smooth functions $\Lambda^{p,q}_\alpha$ onto its harmonic subspace $h\Lambda^{p,q}_\alpha$ for all $1 \le p,q \le \infty,\ \alpha > 0$.

2 Fractional integro-differentiation in $\mathbb{R}^n$

In this section we give the definitions of the well-known Riemann-Liouville fractional integro-differentiation operator D^α and its modifications denoted by $\mathcal{D}^\alpha$ and $\mathcal{D}^\alpha_{n,\lambda}$ introduced in [4,7].

Definition 1 (Classical Riemann–Liouville fractional integral and derivative)
For a function $f(r)$ of a single variable $r \in [0, 1)$, define the operator D^α,

$$D^{-\alpha} f(r) := \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} f(t)\, dt = \frac{r^\alpha}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tr)\, dt,$$
$$D^m f(r) := \left(\frac{d}{dr}\right)^m f(r), \qquad D^\alpha f(r) := D^{-(m-\alpha)} D^m f(r), \qquad D^0 f := f,$$

where $\alpha > 0,\ m \in \mathbb{N},\ m-1 < \alpha \le m.$

Definition 2 (Fractional integral and derivative over $\mathbb{R}^n,\ n \ge 2$)
Given a function $f(x)$ in the unit ball B and $\alpha > 0,\ \lambda \in \mathbb{R}$, let

$$\mathcal{D}^{-\alpha}_{n,\lambda} f(x) := r^{-(\alpha+\lambda+n/2-1)} D^{-\alpha} \left\{ r^{\lambda+n/2-1} f(x) \right\} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tx)\, t^{\lambda+n/2-1}\, dt,$$
$$\mathcal{D}^{\alpha}_{n,\lambda} f(x) := r^{-(\lambda+n/2-1)} D^{\alpha} \left\{ r^{\alpha+\lambda+n/2-1} f(x) \right\}, \qquad \mathcal{D}^0_{n,\lambda} f := f, \qquad r = |x|.$$

Actually the role of the subscript λ is not essential. For $\lambda = 0$, we will write simply $\mathcal{D}^\beta := \mathcal{D}^\beta_{n,0},\ \beta \in \mathbb{R}$. Obviously, $\mathcal{D}^{-\alpha}_{n,\lambda}|f(x)| \le \mathcal{D}^{-\alpha}|f(x)|$ if $\lambda \ge 0$. The general formula

$$\mathcal{D}^m_{n,\lambda} f(x) = r^{-(\lambda+n/2-1)} \frac{\partial^m}{\partial r^m} \left[r^{m+\lambda+n/2-1} f(x) \right], \qquad m \in \mathbb{N},$$

in particular, implies explicit form for fractional derivatives of lower orders

$$\mathcal{D}^1_{n,\lambda} f(x) = \left(\lambda + \frac{n}{2}\right) f + r\,\frac{\partial f}{\partial r} = \lambda f(x) + \mathcal{D}^1 f(x), \tag{2.1}$$

$$\begin{aligned}\mathcal{D}^2_{n,\lambda} f(x) &= \left(\lambda + 1 + \frac{n}{2}\right)\left(\lambda + \frac{n}{2}\right) f(x) + 2\left(\lambda + 1 + \frac{n}{2}\right) r\,\frac{\partial f}{\partial r} + r^2\,\frac{\partial^2 f}{\partial r^2} \\ &= \lambda(\lambda+1) f(x) + 2\lambda\, \mathcal{D}^1 f(x) + \mathcal{D}^2 f(x).\end{aligned} \tag{2.2}$$

Definition 2 leads to the inversion formula for sufficiently smooth functions $f(x)$

$$\mathcal{D}^\alpha_{n,\lambda} \mathcal{D}^{-\alpha}_{n,\lambda} f(x) = \mathcal{D}^{-\alpha}_{n,\lambda} \mathcal{D}^\alpha_{n,\lambda} f(x) = f(x), \qquad x \in B, \quad \alpha > 0, \quad \lambda \ge 1 - \frac{n}{2}. \tag{2.3}$$

Some other basic properties of the fractional operators such as semigroup and commutation relations are established in [4,7] and also in the next two lemmas.

Lemma 1 *For sufficiently smooth function $f(x)$ in the unit ball B, there hold the following semigroup and commutation relations ($\lambda, s \ge 1 - n/2$):*

$$\mathcal{D}^{-(\beta-\alpha)}_{n,\alpha+\lambda} f(x) = \mathcal{D}^{-\beta}_{n,\lambda} \mathcal{D}^\alpha_{n,\lambda} f(x), \qquad \beta > \alpha > 0, \tag{2.4}$$
$$\mathcal{D}^\beta_{n,s} \mathcal{D}^{-\alpha}_{n,\lambda} f(x) = \mathcal{D}^{-\alpha}_{n,\lambda} \mathcal{D}^\beta_{n,s} f(x), \qquad \alpha, \beta > 0, \tag{2.5}$$
in particular for $s = \lambda = 0$,
$$\mathcal{D}^\beta \mathcal{D}^{-\alpha} f(x) = \mathcal{D}^{-\alpha} \mathcal{D}^\beta f(x), \qquad \alpha, \beta > 0, \tag{2.6}$$
$$\mathcal{D}^\alpha_{n,\lambda} f(x) = \mathcal{D}^{-(\beta-\alpha)}_{n,\alpha+\lambda} \mathcal{D}^\beta_{n,\lambda} f(x), \qquad \beta > \alpha > 0, \tag{2.7}$$
$$\mathcal{D}^{\alpha+\beta}_{n,\lambda} f(x) = \mathcal{D}^\beta_{n,\alpha+\lambda} \mathcal{D}^\alpha_{n,\lambda} f(x), \qquad \alpha, \beta > 0. \tag{2.8}$$

Proof Definitions 1-2 together with the inversion formula (2.3) lead to

$$\begin{aligned}
\mathcal{D}^{-\beta}_{n,\lambda}\mathcal{D}^{\alpha}_{n,\lambda} f &= r^{-(\beta+\lambda+n/2-1)} D^{-\beta}\{r^{\lambda+n/2-1}\mathcal{D}^{\alpha}_{n,\lambda} f(x)\} \\
&= r^{-(\beta+\lambda+n/2-1)} D^{-(\beta-\alpha)}\Big[r^{\alpha+\lambda+n/2-1} r^{-(\alpha+\lambda+n/2-1)} D^{-\alpha}\{r^{\lambda+n/2-1}\mathcal{D}^{\alpha}_{n,\lambda} f(x)\}\Big] \\
&= r^{-(\beta+\lambda+n/2-1)} D^{-(\beta-\alpha)}\Big[r^{\alpha+\lambda+n/2-1}\mathcal{D}^{-\alpha}_{n,\lambda}\mathcal{D}^{\alpha}_{n,\lambda} f(x)\Big] \\
&= r^{-\alpha}\, r^{-(\beta-\alpha+\lambda+n/2-1)} D^{-(\beta-\alpha)}\Big[r^{\lambda+n/2-1} r^{\alpha} f(x)\Big] = r^{-\alpha}\,\mathcal{D}^{-(\beta-\alpha)}_{n,\lambda}\Big[r^{\alpha}\, f(x)\Big] \\
&= r^{-\alpha} r^{\alpha}\,\frac{1}{\Gamma(\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1} f(tx)\, t^{\alpha+\lambda+n/2-1}\, dt = \mathcal{D}^{-(\beta-\alpha)}_{n,\alpha+\lambda}\, f(x),
\end{aligned}$$

for $\beta > \alpha > 0$. This proves formula (2.4). Now, assume $\beta = m$ is an integer and show that

$$\mathcal{D}^{m}_{n,s}\mathcal{D}^{-\alpha}_{n,\lambda} f(x) = \mathcal{D}^{-\alpha}_{n,\lambda}\mathcal{D}^{m}_{n,s} f(x), \qquad m \in \mathbb{N}. \tag{2.9}$$

Indeed, first we show a simpler commutation formula

$$r^m D^m \mathcal{D}^{-\alpha}_{n,\lambda} f(x) = \mathcal{D}^{-\alpha}_{n,\lambda}\{r^m D^m f(x)\}, \qquad x = r\zeta. \tag{2.10}$$

To this end, we expand it by using the obvious formula $\frac{\partial^m}{\partial r^m} f(tr\zeta) = t^m\, \frac{\partial^m}{\partial (tr)^m} f(tr\zeta)$,

$$\begin{aligned}
r^m D^m \mathcal{D}^{-\alpha}_{n,\lambda} f(x) &= r^m \frac{\partial^m}{\partial r^m}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1} f(tx)\, t^{\lambda+n/2-1}\, dt\right] \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\left[(tr)^m \frac{\partial^m}{\partial (tr)^m} f(tx)\right] t^{\lambda+n/2-1}\, dt \\
&= \mathcal{D}^{-\alpha}_{n,\lambda}\{r^m D^m f(x)\},
\end{aligned}$$

which coincides with (2.10). For any $m \in \mathbb{N}$ there exist constants $c_k = c_k(m,n) > 0$ $(k = 0, 1, 2, \ldots, m-1)$ with $c_m = 1$ such that

$$\mathcal{D}^{m}_{n,s} f(x) = r^{-(s+n/2-1)} D^m\{r^{m+s+n/2-1}\, f(x)\} = \sum_{k=0}^{m} c_k\, r^k\, D^k f(x). \tag{2.11}$$

Then, by using (2.10) and (2.11), we obtain (2.9):

$$\mathcal{D}^{m}_{n,s}\mathcal{D}^{-\alpha}_{n,\lambda} f(x) = \sum_{k=0}^{n} c_k\, r^k\, D^k \mathcal{D}^{-\alpha}_{n,\lambda} f(x) = \mathcal{D}^{-\alpha}_{n,\lambda}\left\{\sum_{k=0}^{m} c_k\, r^k\, D^k f(x)\right\} = \mathcal{D}^{-\alpha}_{n,\lambda}\mathcal{D}^{m}_{n,s} f(x).$$

Now by using (2.4), (2.9) and Fubini's theorem we obtain, with $m > \beta$,

$$\mathcal{D}^{-\alpha}_{n,\lambda}\,\mathcal{D}^{\beta}_{n,s} f(x) = \mathcal{D}^{-\alpha}_{n,\lambda}\,\mathcal{D}^{m}_{n,s}\,\mathcal{D}^{-(m-\beta)}_{n,\beta+s} f(x) = \mathcal{D}^{m}_{n,s}\,\mathcal{D}^{-(m-\beta)}_{n,\beta+s}\,\mathcal{D}^{-\alpha}_{n,\lambda} f(x) = \mathcal{D}^{\beta}_{n,s}\mathcal{D}^{-\alpha}_{n,\lambda} f(x).$$

This completes the proof of (2.5) and (2.6). Formulas (2.7) and (2.8) are combinations of (2.4) and (2.5) together with the inversion formula (2.3). □

Formula (2.8) together with (2.1) immediately implies the next recurrence relation for the fractional derivative.

Lemma 2 *There holds the recurrence relation*

$$\mathcal{D}^{m+1} f = \big(m + \mathcal{D}^1\big)\mathcal{D}^m f = \mathcal{D}^m\big(m + \mathcal{D}^1\big) f = \mathcal{D}^1_{n,m}\mathcal{D}^m f = \mathcal{D}^m\mathcal{D}^1_{n,m} f, \tag{2.12}$$

for $m \in \mathbb{N}_0$ and all sufficiently smooth functions.

Notice formula (2.12) is a generalization of that with the Poisson kernel $\mathcal{P}_0$ in place of the general function f, see [26, p.91], [27, p.238]. We now need some standard inequalities:

$$\int_S \frac{d\sigma(\xi)}{|\xi - x|^{\alpha+n-1}} \le C(\alpha, n)\frac{1}{(1-|x|)^{\alpha}}, \qquad x \in B, \tag{2.13}$$

$$\int_0^1 \frac{(1-t)^{\alpha-1}}{(1-rt)^{\beta}}dt \le C(\alpha, \beta)\frac{1}{(1-r)^{\beta-\alpha}}, \qquad 0 < r < 1, \tag{2.14}$$

$$\int_0^1 \frac{(1-t)^{\alpha-1}}{|\eta - tx|^{\beta}}dt \le C(\alpha, \beta, n)\frac{1}{|\eta - x|^{\beta-\alpha}}, \qquad \eta \in S,\ x \in B, \tag{2.15}$$

for $\beta > \alpha > 0$. The estimates (2.13)-(2.15) are well known and can be found, for example, in [8,19,24,26,27].

3 Poisson–Bergman Kernel $\mathcal{P}_\alpha$ and upper estimates

The extended Poisson kernel over the unit ball is given by

$$\mathcal{P}(x, y) \equiv \mathcal{P}_0(x, y) := \frac{1 - |x|^2|y|^2}{(1 - 2x \cdot y + |x|^2|y|^2)^{n/2}} = \frac{1 - |x|^2|y|^2}{[x, y]^n}, \qquad x \in B,\ y \in \overline{B}.$$

Here and afterward, we use the notation $[x, y] := \sqrt{1 - 2x \cdot y + |x|^2|y|^2}$, where $x \cdot y$ stands for the dot product in $\mathbb{R}^n$.

Definition 3 (Harmonic Poisson–Bergman kernel in B, [7])

$$\mathcal{P}_\alpha(x, y) := \mathcal{D}^\alpha \mathcal{P}(x, y), \quad x, y \in B, \quad \alpha \ge 0,$$

where the differentiation $\mathcal{D}^\alpha$ is understood with respect to either of x or y since $\mathcal{D}_x^\alpha \mathcal{P}(x, y) = \mathcal{D}_y^\alpha \mathcal{P}(x, y)$.

Equivalent or similar kernels mostly defined by means of series expansion in zonal and spherical harmonics can be found in [10] (for integer α), [7,8,12,15,16,19,23,24,26,27]. Note that Poisson–Bergman type kernel $\mathcal{P}_\alpha(x, y)$ is a harmonic function in B with respect to both x and y, $\mathcal{P}_\alpha(x, y) = \mathcal{P}_\alpha(y, x)$, and also continuously extendable to $\overline{B}$ in one variable whenever the second is fixed in B. A derivation of the first order Poisson–Bergman kernel $\mathcal{P}_1 = \mathcal{D}^1\mathcal{P}_0$ by a direct differentiation via (2.1) gives us a closed explicit form of $\mathcal{P}_1$,

$$\mathcal{P}_1(x, y) = \frac{n}{2}\mathcal{P}_0 + r\frac{\partial \mathcal{P}_0}{\partial r} = \frac{1}{2\,[x, y]^n}\left[\frac{n\left(1 - |x|^2|y|^2\right)^2}{[x, y]^2} - 4|x|^2|y|^2\right].$$

An application of formulas (2.7)-(2.8) in Lemma 1 to the extended Poisson kernel $\mathcal{P}_0$ leads to the useful formulas for the kernels $\mathcal{P}_\alpha = \mathcal{D}^\alpha \mathcal{P}_0$:

$$\mathcal{P}_\alpha(x, y) = \mathcal{D}_{n,\alpha}^{-(m-\alpha)}\mathcal{P}_m(x, y), \qquad m > \alpha > 0,\ m \in \mathbb{N}, \tag{3.1}$$

$$\mathcal{P}_{\alpha+\beta}(x, y) = \mathcal{D}_{n,\alpha}^{\beta}\mathcal{P}_\alpha(x, y), \qquad \alpha, \beta > 0. \tag{3.2}$$

The growth rate of the Poisson-Bergman kernel and its derivatives/gradients are well known since 1980s even for general bounded smooth domains, see [9,10,12,19,21,24,27],

$$|\mathcal{P}_\alpha(x, y)| \le \frac{C(\alpha, n)}{[x, y]^{\alpha+n-1}} \quad \text{and} \quad \left|\nabla_x \mathcal{P}_\alpha(x, y)\right| \le \frac{C(\alpha, n)}{[x, y]^{\alpha+n}}, \qquad x, y \in B. \tag{3.3}$$

Although estimates (3.3) are well known, we give below a direct proof for them including higher-order gradients and fractional derivatives based on our Definitions 1-3.

Lemma 3 *For* $k > 0,\ m \in \mathbb{N},\ x, y \in B,$

$$\left|\frac{\partial}{\partial r}[x, y]^{-k}\right| \le \frac{k}{[x, y]^{k+1}} \quad \text{and} \quad \left|\left(\frac{\partial}{\partial r}\right)^m [x, y]^{-k}\right| \le \frac{C(k, m)}{[x, y]^{k+m}}. \tag{3.4}$$

Proof Since $[x, y] = [y, x] = \left|y' - x|y|\right| = \left|x' - y|x|\right|,\ x, y \neq 0$, first we compute

$$\frac{\partial [x, y]^2}{\partial r} = \frac{\partial}{\partial r}\left|x' - yr\right|^2 = -2y \cdot (x' - yr),$$
$$\frac{\partial}{\partial r}[x, y]^{-k} = -\frac{k}{2}\left(\left|x' - yr\right|^2\right)^{-k/2-1}\frac{\partial}{\partial r}\left|x' - yr\right|^2 = \frac{k\, y \cdot (x' - yr)}{\left|x' - yr\right|^{k+2}}.$$

From this, the first estimate in (3.4) follows immediately. Assuming the last inequality in (3.4) holds for all derivatives of order $1, 2, \ldots, m-1$, the mth order derivative case follows by induction. Indeed, by the Leibniz rule

$$\left(\frac{\partial}{\partial r}\right)^m [x, y]^{-k} = \left(\frac{\partial}{\partial r}\right)^{m-1}\frac{\partial}{\partial r}[x, y]^{-k} = \left(\frac{\partial}{\partial r}\right)^{m-1}\frac{k\, y \cdot (x' - yr)}{[x, y]^{k+2}}$$
$$= k\, y \cdot (x' - yr)\left(\frac{\partial}{\partial r}\right)^{m-1}[x, y]^{-k-2} - (m-1)k|y|^2\left(\frac{\partial}{\partial r}\right)^{m-2}[x, y]^{-k-2}.$$

So, by the induction assumption,

$$\left|\left(\frac{\partial}{\partial r}\right)^m [x, y]^{-k}\right| \leq k\left|x' - yr\right|\frac{C(k, m)}{[x, y]^{k+m+1}} + \frac{C(k, m)}{[x, y]^{k+m}} = \frac{C(k, m)}{[x, y]^{k+m}},$$

as desired. □

Lemma 4 *Let $U = U^1$ denote either of the following four differential operators $\frac{\partial}{\partial r},\ r\frac{\partial}{\partial r},\ \nabla,\ \mathcal{D}^1$. Then for $x, y \in B,\ m \in \mathbb{N}_0$,*

$$\left|U^m \mathcal{P}_0(x, y)\right| \leq \frac{C(m, n)}{[x, y]^{m+n-1}} \quad \text{and} \quad \left|\mathcal{P}_m(x, y)\right| \leq \frac{C(m, n)}{[x, y]^{m+n-1}}. \tag{3.5}$$

In the latter inequality (3.5)*, actually we have taken $U^m = \mathcal{D}^m$. Moreover, for general $\alpha \geq 0,\ \gamma > 0,\ \lambda \geq 1 - n/2$,*

$$\left|\mathcal{P}_\alpha(x, y)\right| \leq \frac{C(\alpha, n)}{[x, y]^{\alpha+n-1}}, \tag{3.6}$$
$$\left|\mathcal{D}^\gamma_{n,\lambda}\mathcal{P}_\alpha(x, y)\right| \leq \frac{C(\alpha, \gamma, \lambda, n)}{[x, y]^{\alpha+\gamma+n-1}}, \qquad x, y \in B. \tag{3.7}$$

The differential operator in (3.7) *can be replaced by any higher-order gradient ∇^m.*

Proof By the Leibniz rule and Lemma 3,

$$\left(\frac{\partial}{\partial r}\right)^m \mathcal{P}_0(x, y) = (1 - |x|^2|y|^2)\left(\frac{\partial}{\partial r}\right)^m [x, y]^{-n} - 2m|y|^2 r\left(\frac{\partial}{\partial r}\right)^{m-1}[x, y]^{-n}$$
$$- \frac{m(m-1)}{2}2m|y|^2\left(\frac{\partial}{\partial r}\right)^{m-2}[x, y]^{-n}.$$

Applying the estimates in Lemma 3 we get

$$\left|\left(\frac{\partial}{\partial r}\right)^m \mathcal{P}_0(x, y)\right| \leq (1 - |x|^2|y|^2)\frac{C(m, n)}{[x, y]^{m+n}} + \frac{C(m, n)}{[x, y]^{m+n-1}} + \frac{C(m, n)}{[x, y]^{m+n-2}}$$
$$\leq \frac{C(m, n)}{[x, y]^{m+n-1}}.$$

Remaining three inequalities in (3.5) can be proved similarly taking into account the definition $\mathcal{P}_m = \mathcal{D}^m\mathcal{P}_0 = r^{-(n/2-1)}\left(\frac{\partial}{\partial r}\right)^m\left[r^{m+n/2-1}\,\mathcal{P}_0\right]$.

For non-integer $\alpha > 0,\ m-1 < \alpha < m\ (m \in \mathbb{N})$, by (3.1), (3.5), (2.15) and Definition 2, we get

$$\begin{aligned}
\left|\mathcal{P}_\alpha(x,y)\right| &= \left|\mathcal{D}_{n,\alpha}^{-(m-\alpha)}\mathcal{P}_m(x,y)\right| \\
&\le \frac{1}{\Gamma(m-\alpha)}\int_0^1 t^{\alpha+n/2-1}(1-t)^{m-\alpha-1}\left|\mathcal{P}_m(tx,y)\right| dt \\
&\le C(\alpha,m)\int_0^1 \frac{t^{\alpha+n/2-1}(1-t)^{m-\alpha-1}}{\left|x'-y|x|\right|^{m+n-1}}\,dt \le \frac{C(\alpha,n)}{\left|x'-y|x|\right|^{\alpha+n-1}},
\end{aligned}$$

which is precisely the inequality (3.6). Further, inequality (3.7) with $\lambda = \alpha$ is contained in (3.6) on account of (3.2), while for general λ or for ∇^m it can be proved in much the same way by using formulas (2.12), (3.2), (3.5). We omit the details. □

4 Besov spaces and Proof of Theorem 1

In this section, we define three-parameter Besov spaces for harmonic and non-harmonic smooth functions and prove the main theorem.

Definition 4 The function $f(x)$ smooth enough in B, is said to belong to (three-parameter) Besov space $\Lambda_\alpha^{p,q}$ $(1 \le p, q \le \infty, \alpha > 0)$ if $\mathcal{D}^{[\alpha]+1}f \in L(p,q,[\alpha]+1-\alpha)$, where $[\alpha]$ is the largest integer less than or equal to α. The Besov space $\Lambda_\alpha^{p,q}$ is equipped with a norm $\|f\|_{\Lambda_\alpha^{p,q}} := \|\mathcal{D}^{[\alpha]+1}f\|_{L(p,q,[\alpha]+1-\alpha)}$. Let $h\Lambda_\alpha^{p,q}$ be the subspace of $\Lambda_\alpha^{p,q}$ consisting of harmonic functions.

For a harmonic function $f \in h\Lambda_\alpha^{p,q}$, the derivative of order $[\alpha]+1$ may be replaced by a fractional derivative of arbitrary order $\gamma > \alpha$, and the corresponding norms are equivalent: $\|f\|_{\Lambda_\alpha^{p,q}} \approx \left\|\mathcal{D}_{n,\lambda}^{\gamma} f\right\|_{L(p,q,\gamma-\alpha)}$ (see, e.g., [28]). For $p = q = \infty$, we come to the ordinary Lipschitz space of order α.

We need the next well-known Hardy's inequalities, see, e.g., [14,28],

$$\int_0^1 (1-r)^{\beta-1}\left(\int_0^r h(t)\,dt\right)^q dr \le C(q,\beta)\int_0^1 (1-r)^{q+\beta-1}\,h^q(r)\,dr, \tag{4.1}$$

$$\int_0^1 (1-r)^{-\beta-1}\left(\int_r^1 h(t)\,dt\right)^q dr \le C(q,\beta)\int_0^1 (1-r)^{q-\beta-1}\,h^q(r)\,dr, \tag{4.2}$$

for $1 \le q < \infty,\ \beta > 0,\ h(r) \ge 0$.

We also need some modifications of Hardy and Littlewood's [17] fractional integration inequalities in weighted Lebesgue spaces, see [4]. We apply them to the mixed norm spaces $L(p,q,\alpha)$, $1 \le p, q \le \infty$, and adopt to our operators as follows:

$$\left\|\mathcal{D}_{n,\delta}^{-\beta} f\right\|_{L(p,q,\alpha-\beta)} \le C(q,\alpha,\beta,\delta)\|f\|_{L(p,q,\alpha)}, \qquad \alpha > \beta > 0, \quad \delta > \frac{1}{q} - \frac{n}{2}, \tag{4.3}$$

if the norm in the right-hand side is finite.

Proof of Theorem 1 Given a function $f(x) \in \Lambda_\alpha^{p,q}$, it suffices to prove

$$\left\|\mathcal{D}^\gamma \Phi_{\beta,\lambda,\delta}(f)\right\|_{L(p,q,\gamma-\alpha-\beta+\delta+\lambda)} \le C\left\|\mathcal{D}^{[\alpha]+1}f\right\|_{L(p,q,[\alpha]+1-\alpha)} \tag{4.4}$$

for some $\gamma > \alpha+\beta-\delta-\lambda > 0$. Differentiation of $\Phi_{\beta,\lambda,\delta}(f)$ with $\mathcal{D}^\gamma$ together with the kernel estimates (3.7) yields

$$\mathcal{D}^\gamma \Phi_{\beta,\lambda,\delta}(f)(x) = \frac{2}{n\,\Gamma(\beta)}\int_B (1-|y|^2)^{\beta-1}\,\mathcal{D}^\gamma \mathcal{P}_\lambda(x,y)\,\mathcal{D}_{n,s}^{\delta} f(y)\,dV(y),$$

$$\left|\mathcal{D}^\gamma \Phi_{\beta,\lambda,\delta}(f)(x)\right| \le C(\beta,\gamma,\lambda,n)\int_B \frac{(1-\rho^2)^{\beta-1}}{|\rho x-\eta|^{\gamma+\lambda+n-1}}\left|\mathcal{D}_{n,s}^{\delta} f(\rho\eta)\right|\rho^{n-1}d\rho\,d\sigma(\eta).$$

Replace here x by Qx, where Q is an arbitrary orthogonal linear transformation $Q:\mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, $|Qx|=|x|$ for all $x\in\mathbb{R}^n$. Making also the change $\eta\mapsto Q\eta$, we find that

$$\begin{aligned}\left|\mathcal{D}^\gamma\Phi_{\beta,\lambda,\delta}(f)(Qx)\right| &\le C\int_B \frac{(1-\rho^2)^{\beta-1}}{|\rho Qx-Q\eta|^{\gamma+\lambda+n-1}}\left|\mathcal{D}^\delta_{n,s}f(\rho Q\eta)\right|\rho^{n-1}d\rho\, d\sigma(\eta)\\ &= C\int_B \frac{(1-\rho^2)^{\beta-1}}{|\rho x-\eta|^{\gamma+\lambda+n-1}}\left|\mathcal{D}^\delta_{n,s}f(\rho Q\eta)\right|\rho^{n-1}d\rho\, d\sigma(\eta).\end{aligned}$$

Now integrate with respect to the Haar measure dQ on the orthogonal group, by Minkowski's inequality, estimates (2.13)-(2.15) and the identity $M_p(F;|z|)=\left(\int|F(Qz)|^p dQ\right)^{1/p}$, $z\in B$, where the integral is taken over the orthogonal group. Hence

$$\begin{aligned}&M_p\big(\mathcal{D}^\gamma\Phi_{\beta,\lambda,\delta}(f);r\big)\le C(\beta,\gamma,\lambda,n)\int_0^1\int_S \frac{(1-\rho^2)^{\beta-1}}{|\rho x-\eta|^{\gamma+\lambda+n-1}}\,M_p(\mathcal{D}^\delta_{n,s}f;\rho)\,d\rho\,d\sigma(\eta)\\ &\le C\left(\int_0^r+\int_r^1\right)\frac{(1-\rho^2)^{\beta-1}}{(1-r\rho)^{\gamma+\lambda}}\,M_p(\mathcal{D}^\delta_{n,s}f;\rho)\,d\rho\\ &\le C\int_0^r(1-\rho)^{\beta-\gamma-\lambda-1}M_p(\mathcal{D}^\delta_{n,s}f;\rho)\,d\rho+C\,(1-r)^{-\gamma-\lambda}\int_r^1(1-\rho)^{\beta-1}\,M_p(\mathcal{D}^\delta_{n,s}f;\rho)\,d\rho.\end{aligned}$$

Case $1\le q<\infty$. The convergence of the last integral is justified by Hölder's inequality ($1<q<\infty$):

$$\begin{aligned}&\int_0^1(1-\rho)^{\beta-1}\,M_p(\mathcal{D}^\delta_{n,s}f;\rho)\,d\rho=\int_0^1(1-\rho)^{\delta-\alpha}(1-\rho)^{\alpha+\beta-\delta}\,M_p(\mathcal{D}^\delta_{n,s}f;\rho)\,\frac{d\rho}{1-\rho}\\ &\le\left(\int_0^1(1-\rho)^{q'(\alpha+\beta-\delta)}\,\frac{d\rho}{1-\rho}\right)^{1/q'}\left(\int_0^1(1-\rho)^{(\delta-\alpha)q}\,M_p^q(\mathcal{D}^\delta_{n,s}f;\rho)\,\frac{d\rho}{1-\rho}\right)^{1/q}\\ &=C(\alpha,\beta,\delta,q)\left\|\mathcal{D}^\delta_{n,s}f\right\|_{L(p,q,\delta-\alpha)}.\end{aligned}$$

Now, formula (2.7) in Lemma 1, $\mathcal{D}^\delta_{n,s}f(x)=\mathcal{D}^{-(m-\delta)}_{n,\delta+s}\,\mathcal{D}^m_{n,s}f(x)$, $m\ge\delta>0$, makes it possible to estimate in mixed norms by using (4.3):

$$\begin{aligned}\left\|\mathcal{D}^\delta_{n,s}f\right\|_{L(p,q,\delta-\alpha)}&=\left\|\mathcal{D}^{-(m-\delta)}_{n,\delta+s}\,\mathcal{D}^m_{n,s}f\right\|_{L(p,q,\delta-\alpha)}\\ &\le C\left\|\mathcal{D}^m_{n,s}f\right\|_{L(p,q,m-\alpha)}\le C\left\|\mathcal{D}^m f\right\|_{L(p,q,m-\alpha)}<+\infty,\end{aligned}\tag{4.5}$$

with $m:=[\alpha]+1$. A further estimation based on Hardy's inequalities (4.1)-(4.2) can be given as follows:

$$\begin{aligned}&\left\|\mathcal{D}^\gamma\Phi_{\beta,\lambda,\delta}(f)\right\|_{L(p,q,\gamma-\alpha-\beta+\delta+\lambda)}=\left\|(1-r)^{\gamma-\alpha-\beta+\delta+\lambda}M_p(\mathcal{D}^\gamma\Phi_{\beta,\lambda,\delta}(f);r)\right\|_{L^q\left(\frac{dr}{1-r}\right)}\\ &\le C\left\|(1-r)^{\gamma-\alpha-\beta+\delta+\lambda}\int_0^r(1-\rho)^{\beta-\gamma-\lambda-1}M_p(\mathcal{D}^\delta_{n,s}f;\rho)\,d\rho\right\|_{L^q\left(\frac{dr}{1-r}\right)}\\ &+C\left\|(1-r)^{\gamma-\alpha-\beta+\delta+\lambda}\,(1-r)^{-\gamma-\lambda}\int_r^1(1-\rho)^{\beta-1}\,M_p(\mathcal{D}^\delta_{n,s}f;\rho)\,d\rho\right\|_{L^q\left(\frac{dr}{1-r}\right)}\\ &\le C\left[\int_0^1(1-r)^{q+(\gamma-\alpha-\beta+\delta)q-1}\Big((1-r)^{\beta-\gamma-\lambda-1}M_p(\mathcal{D}^\delta_{n,s}f;r)\Big)^q dr\right]^{1/q}\\ &+C\left[\int_0^1(1-r)^{q+(\delta-\alpha-\beta)q-1}\,(1-r)^{q(\beta-1)}\,M_p^q(\mathcal{D}^\delta_{n,s}f;r)\,dr\right]^{1/q}\\ &=C\|\mathcal{D}^\delta_{n,s}f\|_{L(p,q,\delta-\alpha)}\le C\left\|\mathcal{D}^m f\right\|_{L(p,q,m-\alpha)},\end{aligned}\tag{4.6}$$

by virtue of (4.5). This leads to the required inequality (4.4).

Case $q=\infty$. Since $f(x)\in\Lambda^{p,\infty}_\alpha$, in view of (4.3) and (4.5), we have $\mathcal{D}^\delta_{n,s}f(x)\in L(p,\infty,\delta-\alpha)$ and

$$(1-r)^{\delta-\alpha}\,M_p(\mathcal{D}^\delta_{n,s}f;r)\le C\|\mathcal{D}^m f\|_{L(p,\infty,m-\alpha)},$$

with $m=[\alpha]+1$. Hence, for $\gamma>\alpha+\beta-\delta-\lambda>0$,

$$\begin{aligned}M_p\big(\mathcal{D}^\gamma\Phi_{\beta,\lambda,\delta}(f);r\big)&\le C(\beta,\gamma,\lambda,n)\int_0^1\frac{(1-\rho)^{\beta-1}}{(1-r\rho)^{\lambda+\gamma}}\,M_p(\mathcal{D}^\delta_{n,s}f;\rho)\,d\rho\\&\le C(\alpha,\beta,\gamma,\lambda,n)\int_0^1\frac{(1-\rho)^{\beta-1}}{(1-r\rho)^{\lambda+\gamma}}\,\frac{\|\mathcal{D}^m f\|_{L(p,\infty,m-\alpha)}}{(1-\rho)^{\delta-\alpha}}\,d\rho\\&\le C(\alpha,\beta,\gamma,\delta,\lambda,n)\,\frac{\|\mathcal{D}^m f\|_{L(p,\infty,m-\alpha)}}{(1-r)^{\lambda+\gamma-\alpha-\beta+\delta}},\end{aligned}$$

by estimate (2.14). Therefore,

$$\big\|\mathcal{D}^\gamma\Phi_{\beta,\lambda,\delta}(f)\big\|_{L(p,\infty,\gamma-\alpha-\beta+\delta+\lambda)}\le C\big\|\mathcal{D}^m f\big\|_{L(p,\infty,m-\alpha)},$$

that is, $\big\|\Phi_{\beta,\lambda,\delta}(f)\big\|_{\Lambda^{p,\infty}_{\alpha+\beta-\delta-\lambda}}\le C\|f\|_{\Lambda^{p,\infty}_\alpha}$. Thus, we have proved that the operator $\Phi_{\beta,\lambda,\delta}$ continuously maps $\Lambda^{p,q}_\alpha$ into $h\Lambda^{p,q}_{\alpha+\beta-\delta-\lambda}$, i.e. $\Phi_{\beta,\lambda,\delta}:\Lambda^{p,q}_\alpha\xrightarrow{\text{into}}h\Lambda^{p,q}_{\alpha+\beta-\delta-\lambda}$ for all $1\le p,q\le\infty$.

For the surjectivity of $\Phi_{\beta,\lambda,\delta}:\Lambda^{p,q}_\alpha\longrightarrow h\Lambda^{p,q}_{\alpha+\beta-\delta-\lambda}$, we take arbitrary harmonic function $g\in h\Lambda^{p,q}_{\alpha+\beta-\delta-\lambda}$, so $\mathcal{D}^{\beta-\lambda}_{n,\lambda}g\in h(p,q,\delta-\alpha)$. According to the continuous inclusions (ii),(vi) in [6, Thm 1.1],

$$\mathcal{D}^{\beta-\lambda}_{n,\lambda}g\in h(p,q,\delta-\alpha)\subset h(1,\infty,\delta-\alpha)\subset h(1,1,\beta-\lambda)\subset h(1,1,\beta),$$

since $\beta-\lambda>\delta-\alpha>0$. By Theorem A, $\mathcal{D}^{\beta-\lambda}_{n,\lambda}g(x)=T_\beta\big(\mathcal{D}^{\beta-\lambda}_{n,\lambda}g\big)(x)$, and then taking the inverse operator $\mathcal{D}^{-(\beta-\lambda)}_{n,\lambda}$, we obtain

$$\begin{aligned}g(x)&=\frac{2}{n\,\Gamma(\beta)}\int_B(1-|y|^2)^{\beta-1}\Big[\mathcal{D}^{-(\beta-\lambda)}_{n,\lambda}\mathcal{D}^\beta\mathcal{P}_0(x,y)\Big]\mathcal{D}^{\beta-\lambda}_{n,\lambda}g(y)\,dV(y)\\&=\frac{2}{n\,\Gamma(\beta)}\int_B(1-|y|^2)^{\beta-1}\,\mathcal{P}_\lambda(x,y)\,\mathcal{D}^\delta_{n,s}\Big[\mathcal{D}^{-\delta}_{n,s}\mathcal{D}^{\beta-\lambda}_{n,\lambda}g(y)\Big]dV(y)=:\Phi_{\beta,\lambda,\delta}(\psi)(x),\end{aligned}$$

where we have used (3.1), Lemma 1 and the inversion formula (2.3). Now it remains to show only the function $\psi:=\mathcal{D}^{-\delta}_{n,s}\mathcal{D}^{\beta-\lambda}_{n,\lambda}g$ is in $h\Lambda^{p,q}_\alpha$. For an integer $m\in\mathbb{N}$, $m\ge\delta>\alpha>0$, we apply Hardy–Littlewood type theorems on (fractional) integro-differentiation in harmonic spaces $h(p,q,\alpha)$ (see [25]). Also, by using the commutation formula (2.5), we consecutively obtain the following chain of implications

$$\begin{aligned}g\in h\Lambda^{p,q}_{\alpha+\beta-\delta-\lambda}\quad&\Longrightarrow\quad\mathcal{D}^{\beta-\lambda}_{n,\lambda}g\in h(p,q,\delta-\alpha)\\&\Longrightarrow\quad\mathcal{D}^m\mathcal{D}^{\beta-\lambda}_{n,\lambda}g\in h(p,q,\delta-\alpha+m)\\&\Longrightarrow\quad\mathcal{D}^{-\delta}_{n,s}\mathcal{D}^m\mathcal{D}^{\beta-\lambda}_{n,\lambda}g=\mathcal{D}^m\mathcal{D}^{-\delta}_{n,s}\mathcal{D}^{\beta-\lambda}_{n,\lambda}g\in h(p,q,m-\alpha)\\&\Longrightarrow\quad\mathcal{D}^m\psi=\mathcal{D}^m\big(\mathcal{D}^{-\delta}_{n,s}\mathcal{D}^{\beta-\lambda}_{n,\lambda}g\big)\in h(p,q,m-\alpha)\\&\Longrightarrow\quad\psi\in h\Lambda^{p,q}_\alpha.\end{aligned}$$

In particular, for $\beta=\delta+\lambda$ and $s=\lambda$, the operator $\Phi_{\beta,\lambda,\beta-\lambda}$ becomes a continuous projection from the Besov space $\Lambda^{p,q}_\alpha$ onto its harmonic subspace, $\Phi_{\beta,\lambda,\beta-\lambda}:\Lambda^{p,q}_\alpha\xrightarrow{\text{onto}}h\Lambda^{p,q}_\alpha$. Indeed, assume that $u\in h\Lambda^{p,q}_\alpha$, so $\mathcal{D}^\delta_{n,\lambda}u\in h(p,q,\delta-\alpha)\subset h(1,1,\beta)$ for $\beta\ge\delta>\alpha>0$. Arguing as above, we obtain $\mathcal{D}^\delta_{n,\lambda}u(x)=\big(T_\beta(\mathcal{D}^\delta_{n,\lambda}u)\big)(x)$, and

$$\begin{aligned}u(x)=\mathcal{D}^{-\delta}_{n,\lambda}\mathcal{D}^\delta_{n,\lambda}u(x)&=\frac{2}{n\,\Gamma(\beta)}\int_B(1-|y|^2)^{\beta-1}\,\mathcal{D}^{-\delta}_{n,\lambda}\mathcal{D}^\beta\mathcal{P}_0(x,y)\,\mathcal{D}^\delta_{n,\lambda}u(y)\,dV(y)\\&=\frac{2}{n\,\Gamma(\beta)}\int_B(1-|y|^2)^{\beta-1}\,\mathcal{P}_{\beta-\delta}(x,y)\,\mathcal{D}^\delta_{n,\lambda}u(y)\,dV(y)\\&=\Phi_{\beta,\beta-\delta,\delta}(u)(x)=\Phi_{\beta,\lambda,\beta-\lambda}(u)(x).\end{aligned}$$

Here we have used formula (2.7), $\mathcal{D}^{-\delta}_{n,\lambda}\mathcal{D}^\beta\mathcal{P}_0=\mathcal{D}^{\beta-\delta}\mathcal{P}_0=\mathcal{P}_{\beta-\delta}$. Thus, $\Phi:=\Phi_{\beta,\lambda,\beta-\lambda}$ (with $s=\lambda$) is the identity operator on $h\Lambda^{p,q}_\alpha$, $\Phi^2=\Phi$, and indeed a bounded projection from the Besov space $\Lambda^{p,q}_\alpha$ onto $h\Lambda^{p,q}_\alpha$.

This completes the proof of Theorem 1. □

As a corollary, for $p = q = \infty$, we get a bounded projection on Lipschitz spaces.

Corollary 1 *For any* $0 < \alpha < \beta - \lambda \leq [\alpha] + 1,\ s = \lambda \geq 0$, *the operator* $\Phi_{\beta,\lambda,\beta-\lambda}$ *continuously maps* $\Lambda_\alpha^{\infty,\infty}$ *onto* $h\Lambda_\alpha^{\infty,\infty}$, *that is, continuously projects Lipschitz space onto its harmonic subspace.*

Analogous boundedness of Bergman projection T_β on classical Lipschitz spaces $\mathrm{Lip}\,\alpha$ is obtained in [3].

Definition 5 A function $f(x)$ given in the unit ball B is said to belong to Lipschitz space $\mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$ if $|f(x) - f(y)| \leq C(\alpha, n)|x - y|^\alpha,\ x, y \in B$.

Theorem 2 ([3]) *For* $0 < \alpha < 1$, $\beta > 0$, *the operator* T_β *continuously projects* $\mathrm{Lip}\,\alpha$ *onto its harmonic subspace* $\mathrm{hLip}\,\alpha$, $T_\beta : \mathrm{Lip}\,\alpha \xrightarrow{\text{onto}} \mathrm{hLip}\,\alpha$, *with corresponding norm inequality.*

Remark 2 Theorem 2 asserts that the Bergman operator T_β acts as a bounded harmonic projection on Lipschitz spaces. Preservation of Lipschitz spaces under the Bergman projection was established earlier in [22] (for unweighted Bergman projection T_1) and [11] (for integer β).

Acknowledgements Supported by the YSU in the frames of the internal research project, and the Mathematical Studies Center at Yerevan State University.

Declarations

Conflict of interest This work does not have any conflict of interest.

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