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ORIGINAL RESEARCH

Intersection results for general classes of maps

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Abstract In this paper we use new coincidence theorems of the author to obtain a variety of Ky Fan matching type theorems for open coverings related to the map or maps. To establish our new matching results, we consider maps which are of KKM or BPK type (these include the Kakutani maps, the acyclic maps and more generally the admissible maps of Gorniewicz) together with maps which generate $HLPY$ type maps.

Keywords Coincidence point · Fixed point · Set–valued map

Mathematics Subject Classification 47H10 · 54H25

1 Introduction

Using recent coincidence theory of the author (see [9,11]), in this paper we present a variety of new Ky Fan matching type theorems for general classes of maps (these maps include BPK (see [12,13]) maps and $HLPY$ maps). Our results include both the campact and coercive type situation. In addition, we consider matching type results for a collection of maps and also we will reformulate our results to generate new intersection type theorems. In particular, we obtain existence results between a convex hull and an intersection and the basic approach is to consider maps whose convex hull is a $HLPY$ type map together with BPK type maps. Then to obtain our matching or intersection results we use a coincidence theorem.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$), if the following two conditions are satisfied: (i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic; and (ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note that for each $x \in X$ we assume that $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ

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(written $(p, q) \subset \phi$) if the following two conditions hold: (i). p is a Vietoris map; and (ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for all $x \in X$.

Now we define the admissible maps of Gorniewicz (see [6]). An upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. Examples of admissible maps are the Kakutani maps and the acyclic maps. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (see [6]) (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y. An upper semicontinuous map $\phi : X \to 2^Y$ (nonempty subsets of Y) is said to be acyclic (see [6]) (and we write $\phi \in AC(X, Y)$) if ϕ has compact, acyclic values.

We also discuss the following classes of maps in this paper. Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z, W)$ (see [8]) if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for all $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $\bigcup\{\,int\,S^{-1}(w) : \ w \in W\} = Z$; here $S^{-1}(w) = \{z \in Z : \ w \in S(z)\}$ and note that $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\,S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. These maps are related to the DKT maps in the literature and $F \in DKT(Z, W)$ (see [5]) if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for all $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note that these maps were motivated from the $\Phi^\star$ maps. We say $G \in \Phi^\star(Z, W)$ (see [2]) if W is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq G(x)$ for all $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$.

Next we describe a class of maps more general than the Ad maps in our setting. Let X be a convex subset of a Hausdorff topological vector space and Y a Hausdorff topological space. If $S,\ T : X \to 2^Y$ are two set–valued maps such that $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X then we call S a generalized KKM mappping w.r.t. T. Now the set–valued map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM map $S : X \to 2^Y$ w.r.t. T the family $\{\,\overline{S(x)} : \ x \in X\}$ has the finite intersection property (the intersection of each of its finite subfamily is nonempty). We let

$$KKM(X, Y) = \{T : X \to 2^Y \mid T \text{ has the } KKM \text{ property}\}.$$

Note that $Ad(X, Y) \subset KKM(X, Y)$ (see [3,4,13]). Next we recall the following result (see [4]).

Theorem 1.1 *Let X be a convex subset of a Hausdorff topological vector space and Y, Z be Hausdorff topological spaces.*

(i). $T \in KKM(X, Y)$ if and only if $T|_\triangle \in KKM(\triangle, Y)$ for each polytope $\triangle$ in X;
(ii). If $T \in KKM(X, Y)$ and $f \in C(Y, Z)$ then $f\,T \in KKM(X, Z)$;
(iii). If Y is a normal space, $\triangle$ a polytope of X and if $T : \triangle \to 2^Y$ is a set–valued map such that for each $f \in C(Y, \triangle)$ we have that $f\,T$ has a fixed point in $\triangle$, then $T \in KKM(\triangle, Y)$.

Also we recall (see [10]) the following two properties. Let C and X be convex subsets of a Hausdorff topological vector space E with $C \subseteq X$ and Y a Hausdorff topological space.

(i). If $T \in KKM(X, Y)$ then $G \equiv T|_C \in KKM(C, Y)$.
(ii). If $T \in KKM(X, Y)$, $T(X) \subseteq Z \subseteq Y$ and Z is closed in Y then $T \in KKM(X, Z)$.

Next we describe the better admissible class of maps BPK due to Park (see [13]). Let X be a convex subset of a Hausdorff topological vector space and Y a Hausdorff topological space. Now $F \in BPK(X, Y)$ if $F : X \to 2^Y$ and for any polytope P in X and any continuous map $f : F(P) \to P$ we have that $f\,(F|_P) : P \to 2^P$ has a fixed point.

From Theorem 1.1 (parts (i) and (iii)) note that $BPK(X, Y) \subseteq KKM(X, Y)$ when Y is normal. Also (see [13]) note that if $F \in KKM(X, Y)$ is closed then $F \in BPK(X, Y)$ and so the two classes coincide for closed compact maps.

We also note the following properties.

Let C and X be convex subsets of a Hausdorff topological vector space E with $C \subseteq X$ and Y a Hausdorff topological space.

(i). If $F \in BPK(X, Y)$ then $G \equiv F|_C \in BPK(C, Y)$.
Consider any polytope P in C and any continuous map $f : G(P) \to P$. Now since P is a polytope in X and (note that $G(P) = F(P)$ since $P \subseteq C$) $f : F(P) \to P$ is a continuous map then since $F \in BPK(X, Y)$ there exists a $x \in P$ with $x \in f\,F|_P\,(x)$ i.e. $x \in f\,G(x)$ since $x \in P \subseteq C$. Thus $G = F|_C \in BPK(C, Y)$.

(ii). If $F \in BPK(X, Y)$ with $F(X) \subseteq Z \subseteq Y$ then $F \in BPK(X, Z)$.
Note that $F : X \to 2^Y$ and let $G : X \to 2^Z$ be the map obtained by restricting the image of F. Consider any polytope P in X and any continuous map $f : G(P) \to P$. Now since $G(P) = F(P)$ then $f : F(P) \to P$ is continuous and since $F \in BPK(X, Y)$ there exists a $x \in P$ with $x \in f\,F|_P\,(x) = f\,G|_P\,(x)$. Thus $F \in BPK(X, Z)$.

(iii). If $F \in BPK(X, Y)$ and $f \in C(Y, X)$ then $f\,F \in BPK(X, X)$.

Note that $f\,F : X \to 2^X$. Consider any polytope P in X and any continuous map $g : f\,F(P) \to P$. We must show there exists a $x \in P$ with $x \in g\,(f\,F)|_P(x)$. To see this note that $g\,f\,F|_P = h\,F|_P : P \to 2^P$ where $h = g\,f : F(P) \to P$ is a continuous map (note that $h(F(P)) = g(f\,F(P)) \subseteq P$). Since $F \in BPK(X, Y)$ there exists a $x \in P$ with $x \in h\,F|_P\,(x)$ i.e. $x \in (g\,f)\,F|_P(x) = g(f\,F)|_P(x)$. Thus $f\,F \in BPK(X, X)$.

Finally we recall some coincidence results of the author (see [11]).

Theorem 1.2 *Let X be a convex subset of a Hausdorff topological vector space E and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and $G \in DKT(Y, X)$. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Theorem 1.3 *Let X be a convex subset of a Hausdorff topological vector space E and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and $G \in DKT(K, X)$ with $K = \overline{F(X)}$. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Theorem 1.4 *Let X be a convex subset of a Hausdorff topological vector space E and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and $G \in HLPY(Y, X)$. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Theorem 1.5 *Let X be a convex subset of a Hausdorff topological vector space E and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and $G \in HLPY(K, X)$ with $K = \overline{F(X)}$. Then there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Theorem 1.6 *Let X and Y be subsets of a Hausdorff topological vector space E with X convex and let $F \in BPK(X, Y)$ be an upper semicontinuous map with compact values. Let $G \in DKT(Y, X)$ and suppose there exists a map $T : Y \to X$ with $co\,(T(y)) \subseteq G(y)$ for all $y \in Y$, $T(y) \neq \emptyset$ and has convex values for each $y \in Y$, $T^{-1}(x)$ is open (in Y) for each $x \in X$ and there is a compact subset K of Y and a convex compact subset C of X with $T(y) \cap C \neq \emptyset$ (or, alternatively $co\,(T(y)) \cap C \neq \emptyset$) for $y \in Y \backslash K$. Then there exists an $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Theorem 1.7 *Let X and Y be subsets of a Hausdorff topological vector space E with X convex and let $F \in BPK(X, Y)$ be an upper semicontinuous map with compact values. Let $G \in HLPY(Y, X)$ and suppose there exists a map $T : Y \to X$ with $co\,(T(y)) \subseteq G(y)$ for all $y \in Y$, $\bigcup\{int\,T^{-1}(w) :\ w \in X\} = Y$ and there is a compact subset K of Y and a convex compact subset C of X with $R(y) \cap C \neq \emptyset$ (or, alternatively $co\,(R(y)) \cap C \neq \emptyset$) for $y \in Y \backslash K$ where $R : Y \to X$ is given by $R(y) = \{z :\ y \in int\,T^{-1}(z)\}$ for $y \in Y$. Then there exists an $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$.*

Remark 1.8 Next we note how the map R relates to the map T in Theorem 1.7. Let $y \in Y$ be fixed. Then if $z \in R(y)$ then $y \in int\,T^{-1}(z) \subseteq T^{-1}(z) = \{w \in Y : z \in T(w)\}$ so $z \in T(y)$. Consequently $R(y) \subseteq T(y)$ for all $y \in Y$.

As a special case, if say $T^{-1}(z)$ is open (in Y) for $z \in T(Y \backslash K)$ then $R(y) = T(y)$ for $y \in Y \backslash K$. To see this fix $y \in Y \backslash K$. Now if $z \in T(y)$ then $y \in T^{-1}(z) = int\,T^{-1}(z)$ so $z \in R(y)$. Thus $T(y) \subseteq R(y)$ for $y \in Y \backslash K$.

Remark 1.9 Now we present a condition to guarantee that $R(y) \cap C \neq \emptyset$ for $y \in Y \backslash K$ in Theorem 1.7. Let X, Y, E and $F \in BPK(X, Y)$ be as in Theorem 1.7. Let $G \in HLPY(Y, X)$ and suppose there exists a map $T : Y \to X$ with $co\,(T(y)) \subseteq G(y)$ for $y \in Y$ and $\bigcup\{int\,T^{-1}(w) :\ w \in X\} = Y$. Assume that there is a compact subset K of Y and a convex compact subset C of X with $\cap_{x \in C}\,\left(Y \backslash int\,T^{-1}(x)\right) \subseteq K$.

Let $y \in Y \backslash K$. Then since

$$\bigcap_{x \in C} \left(Y \backslash int\,T^{-1}(x)\right) = Y \backslash \left(\bigcup_{x \in C} int\,T^{-1}(x)\right) \subseteq K$$

we have

$$y \in Y \backslash K \subseteq Y \backslash \left(Y \backslash \left(\bigcup_{x \in C} int\, T^{-1}(x) \right) \right) = \bigcup_{x \in C} int\, T^{-1}(x).$$

Thus there exists a $x \in C$ with $y \in int\, T^{-1}(x)$ i.e. $x \in C$ and $x \in R(y)$ (here $R : Y \to X$ is given by $R(y) = \{z : \ y \in int\, T^{-1}(z)\}$ for $y \in Y$ as in Theorem 1.7) i.e. $R(y) \cap C \neq \emptyset$. Thus if $y \in Y \backslash K$ then $R(y) \cap C \neq \emptyset$.

2 Main results

In this section using our coincidence theory we present some Ky Fan matching type theorems.

Theorem 2.1 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and $T : X \to 2^Y$ is such that $\overline{F(X)} \subseteq T(X)$ and $T(x)$ is open (in Y) for each $x \in X$. Then there exists a finite set $M \subseteq X$ with*

$$F(co\,(M)) \bigcap \left(\bigcap_{x \in M} T(x) \right) \neq \emptyset.$$

Proof Let $G(w) = co\,(T^{-1}(w))$ for $w \in Y$. We claim that $G \in HLPY(K, X)$ where $K = \overline{F(X)}$. Note that $T^{-1} : K \to 2^X$ since if $y \in K$ then from $\overline{F(X)} \subseteq T(X)$ there exists a $w \in X$ with $y \in T(w)$ (i.e. $w \in T^{-1}(y)$). Thus $G : K \to 2^X$ and $G(w)$ is convex for each $w \in K$. If we show

$$\bigcup_{x \in X} int_K\, G^{-1}(x) = K \tag{2.1}$$

then $G \in HLPY(K, X)$. Note that $\bigcup_{x \in X} int_K\, G^{-1}(x) \subseteq K$ so we now need only show that $K \subseteq \bigcup_{x \in X} int_K\, G^{-1}(x)$. Let $y \in K$ and as above there exists a $w \in X$ with $y \in T(w)$. Fix $\eta \in T(w)$ and note that $w \in T^{-1}(\eta) \subseteq co\,(T^{-1}(\eta)) = G(\eta)$ (note that $G(\eta)$ is defined for $\eta \in Y$ and $T^{-1}(\eta)$ may be empty if $\eta \notin K$), so $w \in G(\eta)$ i.e. $\eta \in G^{-1}(w)$. Consequently $T(w) \subseteq G^{-1}(w)$ and since $T(w)$ is open we have $int\, T(w) = T(w)$. As a result from $y \in T(w)$ and the above we have $y \in int\, T(w) \subseteq G^{-1}(w)$, so $y \in int\, G^{-1}(w)$. Thus we have proved that $K \subseteq \bigcup_{x \in X} int\, G^{-1}(x)$. Next note that

$$K \subseteq K \bigcap \left(\bigcup_{x \in X} int\, G^{-1}(x) \right) = \bigcup_{x \in X} \left(K \bigcap int\, G^{-1}(x) \right)$$

and since for each $x \in X$ note $K \bigcap int\, G^{-1}(x)$ is open in K then

$$K \subseteq \bigcup_{x \in X} int_K\, G^{-1}(x).$$

Thus (2.1) holds so $G \in HLPY(K, X)$ (here the selection of G chosen is G itself). Now Theorem 1.5 guarantees the existence of a $x_0 \in X$ with $G^{-1}(x_0) \cap F(x_0) \neq \emptyset$. Let $y \in G^{-1}(x_0) \cap F(x_0)$. Then $x_0 \in G(y) = co\,(T^{-1}(y))$ so from the definition of co there exists a $M = \{z_1, \ldots, z_n\} \subseteq X$ with $M \subseteq T^{-1}(y)$ such that $x_0 \in co\,(M)$. Fix $i \in \{1, \ldots, n\}$ and note that $z_i \in T^{-1}(y)$ so $y \in T(z_i)$ i.e. $y \in T(z_i)$ for all $i \in \{1, \ldots, n\}$ so $y \in \bigcap_{i=1}^n T(z_i)$. Also from above $y \in F(x_0)$ and $x_0 \in co\,(M)$ so $y \in F(co\,(M))$. Combining gives $y \in F(co\,(M)) \bigcap \left(\bigcap_{i=1}^n T(z_i) \right)$. □

Theorem 2.2 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and $T : X \to 2^Y$ is such that $T(X) = Y$ and $T(x)$ is open (in Y) for each $x \in X$. Then there exists a finite set $M \subseteq X$ with*

$$F(co\,(M)) \bigcap \left(\bigcap_{x \in M} T(x) \right) \neq \emptyset.$$

Proof Let G be as in Theorem 2.1. Note that $T^{-1} : Y \to 2^X$ since if $y \in Y$ then from $T(X) = Y$ there exists a $w \in X$ with $y \in T(w)$ (i.e. $w \in T^{-1}(y)$). Also note that $\bigcup_{x \in X} \, int\, G^{-1}(x) = Y$ since if $y \in Y$ there exists a $w \in X$ with $y \in T(w)$ and as in Theorem 2.1 we have $y \in int\, T(w) \subseteq G^{-1}(w)$, so $y \in int\, G^{-1}(w)$. As a result $Y \subseteq \bigcup_{x \in X} \, int\, G^{-1}(x)$ i.e. $\bigcup_{x \in X} \, int\, G^{-1}(x) = Y$, so $G \in HLPY(Y, X)$. Now apply Theorem 1.4. □

Now we consider Theorem 1.7 instead of Theorem 1.4 in the proof of Theorem 2.2.

Theorem 2.3 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is an upper semicontinuous map with compact values and $T : X \to 2^Y$ is such that $T(X) = Y$ and $T(x)$ is open (in Y) for each $x \in X$. In addition assume that there is a compact subset K of Y and a convex compact subset C of X with $R(y) \cap C \neq \emptyset$ (or, alternatively $co\,(R(y)) \cap C \neq \emptyset$) for $y \in Y \backslash K$ where $R : Y \to X$ is given by $R(y) = \{z : \ y \in int\, G^{-1}(z)\}$ for $y \in Y$ where $G(w) = co\,(T^{-1}(w))$ for $w \in Y$. Then there exists a finite set $M \subseteq X$ with*

$$F(co\,(M)) \bigcap \left(\bigcap_{x \in M} T(x) \right) \neq \emptyset.$$

Proof As in Theorem 2.2 note that $\bigcup_{x \in X} \, int\, G^{-1}(x) = Y$ so $G \in HLPY(Y, X)$. Now Theorem 1.7 guarantees the existence of a $x_0 \in X$ with $G^{-1}(x_0) \cap F(x_0) \neq \emptyset$. A similar argument as in Theorem 2.1 completes the proof. □

In Theorem 2.1, Theorem 2.2 and Theorem 2.3 the map G was of $HLPY$ type. One could of course change the assumption on T so that G is of DKT or $\Phi^\star$ type. Let us first consider Theorem 2.3.

Theorem 2.4 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is an upper semicontinuous map with compact values and $T : X \to 2^Y$ is such that $T(X) = Y$. Let $G(w) = co\,(T^{-1}(w))$ for $w \in Y$ and assume that $G^{-1}(w)$ is open (in Y) for $w \in X$. In addition assume that there is a compact subset K of Y and a convex compact subset C of X with $G(y) \cap C \neq \emptyset$ for $y \in Y \backslash K$. Then there exists a finite set $M \subseteq X$ with*

$$F(co\,(M)) \bigcap \left(\bigcap_{x \in M} T(x) \right) \neq \emptyset.$$

Proof Note that $T^{-1} : Y \to 2^X$ and $G \in DTK(Y, X)$ (here the selection of G is G itself). Now Theorem 1.6 guarantees the existence of a $x_0 \in X$ with $G^{-1}(x_0) \cap F(x_0) \neq \emptyset$. A similar argument as in Theorem 2.1 completes the proof. □

Remark 2.5 Now we note a special case of G in Theorem 2.4. Note that if $(T^{-1})^{-1}(w)$ is open (in Y) for $w \in X$ then from [14, Lemma 5.1] we have that $G^{-1}(w)$ is open (in Y) for $w \in X$. Also note that if $T^{-1}(w)$ is convex for each $w \in Y$ then $G(w) = T^{-1}(w)$ for $w \in Y$

Also we immediately have the analogue of Theorem 2.2 (here we apply Theorem 1.2 instead of Theorem 1.4).

Theorem 2.6 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and $T : X \to 2^Y$ is such that $T(X) = Y$. Assume that $G^{-1}(w)$ is open (in Y) for $w \in X$ where $G(z) = co\,(T^{-1}(z))$ for $z \in Y$. Then there exists a finite set $M \subseteq X$ with*

$$F(co\,(M)) \bigcap \left(\bigcap_{x \in M} T(x) \right) \neq \emptyset.$$

Now we consider the analogue of Theorem 2.1.

Theorem 2.7 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and $T : X \to 2^Y$ is such that $\overline{F(X)} \subseteq T(X)$. Assume that $G^{-1}(w)$ is open (in Y) for $w \in X$ where $G(z) = co\,(T^{-1}(z))$ for $z \in Y$. Then there exists a finite set $M \subseteq X$ with*

$$F(co\,(M)) \bigcap \left(\bigcap_{x \in M} T(x) \right) \neq \emptyset.$$

Proof Let $K = \overline{F(X)}$ and note (as in Theorem 2.1) that $T^{-1} : K \to 2^X$. Let $G^\star$ denote the restriction of G to K i.e $G^\star = G|_K$. Then $(G^\star)^{-1}(w)$ is open (in K) for $w \in X$ since for $w \in X$ note that

$$\begin{aligned}(G^\star)^{-1}(w) &= \{z \in K : w \in G^\star(z)\} = \{z \in K : w \in G(z)\}\\ &= K \cap \{z \in Y : w \in G(z)\} = K \cap G^{-1}(w)\end{aligned}$$

which is open in $K \cap Y = K$. Thus $G^\star \in DKT(K, X)$. Now Theorem 1.3 guarantees the existence of a $x_0 \in X$ with $(G^\star)^{-1}(x_0) \cap F(x_0) \neq \emptyset$. Let $y \in (G^\star)^{-1}(x_0) \cap F(x_0)$ and note that $y \in K$ since $K = \overline{F(X)}$. Thus $x_0 \in G^\star(y) = G(y) = co\,(T^{-1}(y))$ and now the argument in Theorem 2.1 completes the proof. □

We can extend our results to collections. We illustrate this with Theorem 2.1 (and Theorem 2.2).

Theorem 2.8 *Let $\{X_i\}_{i=1}^n$ be a family of convex sets in Hausdorff topological vector spaces and $\{Y\}_{i=1}^n$ a family of sets in Hausdorff topological spaces. For each $i \in \{1, \dots, n\}$ let $F_i \in Ad(X_i, Y_i)$ be a compact map and $T_i : X_i \to 2^{Y_i}$ with $T_i(z)$ open (in Y_i) for each $z \in X_i$. In addition assume that $\overline{F_i(X_i)} \subseteq T_i(X_i)$ for each $i \in \{1, \dots, n\}$ (or, alternatively $T_i(X_i) = Y_i$ for each $i \in \{1, \dots, n\}$). Then there exists a finite set $M \subseteq X = \prod_{i=1}^n X_i$ with*

$$\bigcup_{y=(y_i)_1^n \in co\,(M)} \prod_{i=1}^n F_i(y_i) \bigcap \left(\bigcap_{x=(x_i)_1^n \in M} \prod_{i=1}^n T_i(x_i) \right) \neq \emptyset.$$

Proof Let $F(x) = \prod_{i=1}^n F_i(x_i)$ and $T(x) = \prod_{i=1}^n T_i(x_i)$ for $x = (x_1, \dots, x_n) \in X$. Note (see [7]) that the product of admissible with respect to Gorniewicz maps is admissible with respect to Gorniewicz (follows from Künneth's theorem and the product of upper semicontinuous maps with compact values is upper semicontinous with compact values (see [1])) so $F \in Ad(X, Y)$ where $Y = \prod_{i=1}^n Y_i$. Since $Ad(X, Y) \subseteq BPK(X, Y)$ then $F \in BPK(X, Y)$ is a compact map. Also note that (finite product here) $T(x) = \prod_{i=1}^n T_i(x_i)$ is open (in Y) for each $x = (x_i)_1^n \in X$.

If we assume that $\overline{F_i(X_i)} \subseteq T_i(X_i)$ for each $i \in \{1, \dots, n\}$ then note that

$$\overline{F(X)} = \overline{\prod_{i=1}^n F_i(X_i)} = \prod_{i=1}^n \overline{F_i(X_i)} \subseteq \prod_{i=1}^n T_i(X_i) = T(X)$$

(or, alternatively if we assume that $T_i(X_i) = Y_i$ for each $i \in \{1, \dots, n\}$ then $T(X) = Y$) and Theorem 2.1 (or, alternatively Theorem 2.2) guarantees that there exists a finite set $M \subseteq X$ with $F(\,co\,(M)) \bigcap \left(\bigcap_{x\in M} T(x)\right) \neq \emptyset$. □

The above theorems have contrapositive reformulations (we will just consider Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.8).

Corollary 2.9 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and $S : X \to 2^Y$ is such that (a). $F(co\,(A)) \subseteq S(A)$ for each finite subset A of X, and (b). $S(x)$ is closed (in Y) for each $x \in X$. Then $\overline{F(X)} \bigcap \left(\bigcap_{x\in X} S(x)\right) \neq \emptyset$.*

Proof To see this let $T(x) = Y \backslash S(x)$ for $x \in X$ and assume that $\overline{F(X)} \bigcap \left(\bigcap_{x\in X} S(x)\right) = \emptyset$ (i.e. assume that $\overline{F(X)} \subseteq T(X)$ since $Y \backslash \left(\bigcap_{x\in X} S(x)\right) = \bigcup_{x\in X} (Y \backslash S(x)) = \bigcup_{x\in X} T(x) = T(X)$). Now (b) and Theorem 2.1 guarantee the existence of a finite set $M \subseteq X$ with $F(\,co\,(M)) \bigcap \left(\bigcap_{x\in M} T(x)\right) \neq \emptyset$ i.e. (a) is not true when $A = M$ since $\bigcap_{x\in M} T(x) = \bigcap_{x\in M} (Y \backslash S(x)) = Y \backslash \left(\bigcup_{x\in M} S(x)\right) = Y \backslash S(M)$. □

Corollary 2.10 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and $S : X \to 2^Y$ is such that (a). $F(co\,(A)) \subseteq S(A)$ for each finite subset A of X, and (b). $S(x)$ is closed (in Y) for each $x \in X$. Then $\bigcap_{x\in X} S(x) \neq \emptyset$.*

Proof Let $T(x) = Y \backslash S(x)$ for $x \in X$ and assume that $\bigcap_{x\in X} S(x) = \emptyset$ (i.e. assume that $T(X) = Y$ since $Y = Y \backslash \left(\bigcap_{x\in X} S(x)\right) = \bigcup_{x\in X} (Y \backslash S(x)) = \bigcup_{x\in X} T(x) = T(X)$). Now (b) and Theorem 2.2 guarantee the existence of a finite set $M \subseteq X$ with $F(\,co\,(M)) \bigcap \left(\bigcap_{x\in M} T(x)\right) \neq \emptyset$, and this contradicts (a). □

Corollary 2.11 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is an upper semicontinuous map with compact values and $S : X \to 2^Y$ is such that (a). $F(co\,(A)) \subseteq S(A)$ for each finite subset A of X, and (b). $S(x)$ is closed (in Y) for each $x \in X$. In addition assume that there is a compact subset K of Y and a convex compact subset C of X with $R(y) \cap C \neq \emptyset$ (or, alternatively $co\,(R(y)) \cap C \neq \emptyset$) for $y \in Y \backslash K$ where $R : Y \to X$ is given by $R(y) = \{z : \ y \in int\, G^{-1}(z)\}$ for $y \in Y$ where $G(w) = co\,(T^{-1}(w))$ for $w \in Y$; here $T(x) = Y \backslash S(x)$ for $x \in X$. Then $\bigcap_{x\in X} S(x) \neq \emptyset$.*

Proof Let $T(x) = Y\backslash S(x)$ for $x \in X$ and assume that $\bigcap_{x\in X} S(x) = \emptyset$ (i.e. assume $T(X) = Y$). Now Theorem 2.3 guarantee the existence of a finite set $M \subseteq X$ with $F(\,co\,(M)) \bigcap \left(\bigcap_{x\in M} T(x)\right) \neq \emptyset$, and this contradicts (a). □

Corollary 2.12 *Let $\{X_i\}_{i=1}^n$ be a family of convex sets in Hausdorff topological vector spaces and $\{Y\}_{i=1}^n$ a family of sets in Hausdorff topological spaces. For each $i \in \{1, \ldots, n\}$ let $F_i \in Ad(X_i, Y_i)$ be a compact map and $S_i : X_i \to 2^{Y_i}$ is such that:*

(a). for each finite subset A of $X = \prod_{i=1}^n X_i$ there exists at least one $j \in \{1, \ldots, n\}$ with

$$\bigcup_{y=(y_i)_1^n \in co\,(A)} F_j(y_j) \subseteq \bigcup_{x=(x_i)_1^n \in A} S_j(x_j)$$

(or, alternatively

$$\bigcup_{y=(y_i)_1^n \in co\,(A)} \prod_{i=1}^n F_i(y_i) \subseteq \prod_{i=1}^n \bigcup_{x=(x_i)_1^n \in A} S_i(x_i)$$

for each finite subset A of X),

(b). $S_i(z)$ closed (in Y_i) for each $z \in X_i$.

Then there exists a $i \in \{1, \ldots, n\}$ with $\overline{F_i(X_i)} \bigcap \left(\bigcap_{w\in X_i} S_i(w)\right) \neq \emptyset$.

Proof For $i \in \{1, \ldots, n\}$ let $T_i(z_i) = Y_i\backslash S_i(z_i)$ for $z_i \in X_i$. Assume that there does not exist a $i \in \{1, \ldots, n\}$ with $\overline{F_i(X_i)} \bigcap \left(\bigcap_{w\in X_i} S_i(w)\right) \neq \emptyset$ (i.e. assume that $\overline{F_i(X_i)} \bigcap \left(\bigcap_{w\in X_i} S_i(w)\right) = \emptyset$ for all $i \in \{1, \ldots, n\}$). Note we are assuming that $\overline{F_i(X_i)} \subseteq T_i(X_i)$ for $i \in \{1, \ldots, n\}$ since $Y_i \backslash \left(\bigcap_{w\in X_i} S_i(w)\right) = \bigcup_{w\in X_i} (Y_i\backslash S_i(w)) = \bigcup_{w\in X_i} T_i(w) = T_i(X_i)$. Now (b) and Theorem 2.8 guarantee the existence of a finite set $M \subseteq X = \prod_{i=1}^n X_i$ with

$$\bigcup_{y=(y_i)_1^n \in co\,(M)} \prod_{i=1}^n F_i(y_i) \bigcap \left(\bigcap_{x=(x_i)_1^n \in M} \prod_{i=1}^n T_i(x_i) \right) \neq \emptyset,$$

so (a) is not true since we note that

$$\bigcap_{x=(x_i)_1^n \in M} \prod_{i=1}^n T_i(x_i) = \prod_{i=1}^n \bigcap_{x=(x_i)_1^n \in M} T_i(x_i) = \prod_{i=1}^n \bigcap_{x=(x_i)_1^n \in M} (Y_i\backslash S_i(x_i))$$
$$= \prod_{i=1}^n \left(Y_i \ \backslash \bigcup_{x=(x_i)_1^n \in M} S_i(x_i) \right).$$

□

Next we will relax the compactness of F in Corollary 2.9 and Theorem 2.1 (see also Theorem 2.3 and Theorem 2.4).

Theorem 2.13 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is an upper semicontinuous map with compact values and $S : X \to 2^Y$ is such that (a). $S(x)$ is closed (in Y) for each $x \in X$, (b). $F(co\,(A)) \subseteq S(A)$ for each finite subset A of X, and (c). there exists a compact subset K of Y and a convex compact subset C of X with $\bigcap_{x\in C} S(x) \subseteq K$. Then $\overline{F(X)} \bigcap \left(\bigcap_{x\in X} S(x)\right) \neq \emptyset$.*

Proof Suppose $\overline{F(X)} \bigcap \left(\bigcap_{x\in X} S(x)\right) = \emptyset$ (i.e. $\cap_{x\in X} S(x) \subseteq Y\backslash\overline{F(X)}$). Now

$$K \subseteq \left(\bigcap_{x\in X} S(x)\right) \bigcup \left(Y\backslash \bigcap_{x\in X} S(x)\right) \subseteq \left(Y\backslash\overline{F(X)}\right) \bigcup \left(\bigcup_{x\in X}(Y\backslash S(x))\right)$$

and note that $Y\backslash\overline{F(X)}$ and $Y\backslash S(x)$ (for $x \in X$) are open in Y. Then since K is compact there exists a finite subset $\{x_1, \ldots, x_n\}$ of X with

$$K \subseteq \left(Y\backslash\overline{F(X)}\right) \bigcup \left(\bigcup_{i=1}^{n}(Y\backslash S(x_i))\right).$$

Let $Z = co\,(C \cup \{x_1, \ldots, x_n\})$ and note that Z is a convex compact subset (see [1]) of X. Now since $\bigcap_{x\in C} S(x) \subseteq K$ we have

$$Y\backslash K \subseteq Y\backslash \left(\bigcap_{x\in C} S(x)\right) = \bigcup_{x\in C} (Y\backslash S(x)).$$

Note that

$$Y = K \cup (Y\backslash K) \subseteq \left(Y\backslash\overline{F(X)}\right) \bigcup \left(\bigcup_{i=1}^{n}(Y\backslash S(x_i))\right) \bigcup \left(\bigcup_{x\in C} (Y\backslash S(x))\right)$$

so

$$Y = \left(Y\backslash\overline{F(X)}\right) \bigcup \left(\bigcup_{x\in Z} (Y\backslash S(x))\right).$$

As a result

$$\emptyset = Y\backslash Y = \left[Y\backslash \left(Y\backslash\overline{F(X)}\right)\right] \bigcap \left[Y\backslash \left(\bigcup_{x\in Z} (Y\backslash S(x))\right)\right]$$
$$= \overline{F(X)}] \bigcap \left[Y\backslash \left(Y\backslash \left(\bigcap_{x\in Z} S(x)\right)\right)\right] = \overline{F(X)} \bigcap \left(\bigcap_{x\in Z} S(x)\right)$$

i.e.

$$\overline{F(X)} \bigcap \left(\bigcap_{x\in Z} S(x)\right) = \emptyset. \tag{2.2}$$

Consider a map Q given by $Q(x) = S(x) \cap \overline{F(Z)}$ for $x \in Z$. Note that $Q : Z \to 2^Y$ (for a fixed $x \in Z$ consider assumption (b) with $A = \{x\}$ i.e. $F(x) \subseteq S(x)$ so $\emptyset \neq F(x) = S(x) \cap F(x) \subseteq S(x) \cap \overline{F(Z)}$). Note from assumption (a) that $Q(x)$ is closed (in Y) for each $x \in Z$. Also for each finite subset A of Z note that $F(co\,(A)) \subseteq S(A)$. Now since Z is convex we have $co\,(A) \subseteq Z$ so $F(co\,(A)) \subseteq F(Z) \subseteq \overline{F(Z)}$. Thus $F(co\,(A)) \subseteq S(A) \cap \overline{F(Z)} = Q(A)$. Also $F \in BPK(Z, Y)$ and since F is upper semicontinuous with compact values and Z is compact then $F \in BPK(Z, Y)$ is a compact map. Now Corollary 2.9 (with $X = Z$, $Y = Y$, $F = F$ and $S = Q$) guarantees that $\overline{F(Z)} \bigcap \left(\bigcap_{x\in Z} Q(x)\right) \neq \emptyset$. Thus there exists a $w \in \overline{F(Z)}$ with $w \in \bigcap_{x\in Z} Q(x)$. Now since for $x \in Z$ we have $Q(x) = S(x) \cap \overline{F(Z)} \subseteq S(x)$ we have $w \in \bigcap_{x\in Z} Q(x) \subseteq \bigcap_{x\in Z} S(x)$, and also since $Z \subseteq X$ we have $w \in \overline{F(X)}$. Thus $\overline{F(X)} \bigcap \left(\bigcap_{x\in Z} S(x)\right) \neq \emptyset$, which contradicts (2.2). □

Remark 2.14 We can observe more in Theorem 2.13. In fact $\overline{F(X)} \bigcap K \bigcap \left(\bigcap_{x\in X} S(x)\right) \neq \emptyset$ since $K \supseteq \bigcap_{x\in C} S(x)$ and $\bigcap_{x\in X} S(x) \subseteq \bigcap_{x\in C} S(x)$ yields

$$\overline{F(X)} \bigcap K \bigcap \left(\bigcap_{x\in X} S(x)\right) \supseteq \overline{F(X)} \bigcap \left(\bigcap_{x\in C} S(x)\right) \bigcap \left(\bigcap_{x\in X} S(x)\right) = \overline{F(X)} \bigcap \left(\bigcap_{x\in X} S(x)\right) \neq \emptyset.$$

Remark 2.15 Note that the assumption that $F \in BPK(X, Y)$ is upper semicontinuous with compact values can be replaced by $F \in BPK(Z, Y)$ is a compact map in Theorem 2.13 (here Z is as in the proof of Theorem 2.13).

Remark 2.16 Theorem 2.13 extends and generalises Theorem 8 in [4] (see also [3]).

Remark 2.17 We now present an observation about condition (c) in Theorem 2.13. Since $\bigcap_{x\in C} S(x) \subseteq K$ we have $Y\backslash K \subseteq Y\backslash \left(\bigcap_{x\in C} S(x)\right) = \bigcup_{x\in C} (Y\backslash S(x))$. Thus if $z \in Y\backslash K$ then there exists a $x \in C$ with $z \in Y\backslash S(x) \equiv T(x)$ (here $T(w) = Y\backslash S(w)$ for $w \in X$). As a result $x \in C$ and $z \in T(x)$ i.e. $x \in C$ and $x \in T^{-1}(z)$ i.e. $T^{-1}(z) \cap C \neq \emptyset$. Thus if $z \in Y\backslash K$ then $T^{-1}(z) \bigcap C \neq \emptyset$.

The contrapositive reformulation of Theorem 2.13 is as folows (see Theorem 2.1).

Corollary 2.18 *Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is an upper semicontinuous map with compact values and $T : X \to 2^Y$ is such that (a). $T(x)$ is open (in Y) for each $x \in X$, (b). $\overline{F(X)} \subseteq T(X)$, and (c). there exists a compact subset K of Y and a convex compact subset C of X with $Y\backslash \left(\bigcup_{x\in C} T(x)\right) \subseteq K$. Then there exists a finite set $M \subseteq X$ with*

$$F(\,co\,(M)) \bigcap \left(\bigcap_{x\in M} T(x)\right) \neq \emptyset.$$

Proof Let $S(x) = Y\backslash T(x)$ for $x \in X$ and note that condition (a) in Theorem 2.13 is satisfied. Suppose for each finite set $A \subseteq X$ we have $F(\,co\,(A)) \bigcap \left(\bigcap_{x\in A} T(x)\right) = \emptyset$. Then

$$F(co\,(A)) \subseteq Y\backslash \left(\bigcap_{x\in A} T(x)\right) = \bigcup_{x\in A} (Y\backslash T(x)) = \bigcup_{x\in A} S(x) = S(A),$$

so condition (b) in Theorem 2.13 is satisfied. Also notice

$$\bigcap_{x\in C} S(x) = \bigcap_{x\in C} (Y\backslash T(x)) = Y\backslash \left(\bigcup_{x\in C} T(x)\right) \subseteq K,$$

so condition (c) in Theorem 2.13 is satisfied. Now Theorem 2.13 implies $\overline{F(X)} \bigcap \left(\bigcap_{x\in X} S(x)\right) \neq \emptyset$, which contradicts assumption (b) in Theorem 2.18 since

$$\bigcap_{x\in X} S(x) = \bigcap_{x\in X} (Y\backslash T(x)) = Y\backslash \left(\bigcup_{x\in X} T(x)\right) = Y\backslash T(X).$$

□

Remark 2.19 Now we make an observation about condition (c) in Corollary 2.18. Since $Y\backslash \left(\bigcup_{x\in C} T(x)\right) \subseteq K$ we have $Y\backslash K \subseteq Y\backslash \left(Y\backslash \left(\bigcup_{x\in C} T(x)\right)\right) = \bigcup_{x\in C} T(x)$. Thus if $z \in Y\backslash K$ then there exists a $x \in C$ with $z \in T(x)$ so $T^{-1}(z) \cap C \neq \emptyset$.

Remark 2.20 We now make an observation about a coincidence. Note from Corollary 2.18 that there exists a finite set $M \subseteq X$ and a $z \in F(\,co\,(M)) \bigcap \left(\bigcap_{x\in M} T(x)\right)$. Therefore $z \in F(w)$ for some $w \in co\,(M)$ and $z \in \bigcap_{x\in M} T(x)$. Note that $z \in T(x)$ for all $x \in M$ i.e. $x \in T^{-1}(z)$ for all $x \in M$ so $w \in co(M) \subseteq co(T^{-1}(z))$ i.e. $z \in F(w)$ and $w \in co(T^{-1}(z))$.

We now extend Theorem 2.13 to collections (see Theorem 2.8).

Theorem 2.21 *Let $\{X_i\}_{i=1}^n$ be a family of convex sets in Hausdorff topological vector spaces and $\{Y\}_{i=1}^n$ a family of sets in Hausdorff topological spaces. For each $i \in \{1, \dots, n\}$ let $F_i \in Ad(X_i, Y_i)$ and $T_i : X_i \to 2^{Y_i}$ is such that (a). $T_i(z)$ is open (in Y_i) for each $z \in X_i$, (b). $\overline{F_i(X_i)} \subseteq T_i(X_i)$, and (c). there exists a compact subset K_i of Y_i and a convex compact subset C_i of X_i with $Y_i\backslash \left(\bigcup_{w\in C_i} T_i(w)\right) \subseteq K_i$. Then there exists a finite set $M \subseteq X = \prod_{i=1}^n X_i$ with*

$$\bigcup_{y=(y_i)_1^n\in co\,(M)} \prod_{i=1}^n F_i(y_i) \bigcap \left(\bigcap_{x=(x_i)_1^n\in M} \prod_{i=1}^n T_i(x_i)\right) \neq \emptyset.$$

Proof Let $F(x) = \prod_{i=1}^{n} F_i(x_i)$, $T(x) = \prod_{i=1}^{n} T_i(x_i)$ for $x = (x_1, \dots, x_n) \in X$, $C = \prod_{i=1}^{n} C_i$ and $K = \prod_{i=1}^{n} K_i$. The argument in Theorem 2.8 guarantees that $F \in Ad(X, Y)$ (here $Y = \prod_{i=1}^{n} Y_i$), $T(x)$ is open (in Y) for each $x \in X$, and $\overline{F(X)} \subseteq T(X)$. Finally notice

$$Y \backslash \bigcup_{w=(w_i)_1^n \in C} T(w) = \bigcap_{w=(w_i)_1^n \in C} (Y \backslash T(w)) = \bigcap_{w=(w_i)_1^n \in C} \prod_{i=1}^{n} (Y_i \backslash T_i(w_i))$$
$$= \prod_{i=1}^{n} \bigcap_{w=(w_i)_1^n \in C} (Y_i \backslash T_i(w_i)) = \prod_{i=1}^{n} \left(Y_i \backslash \left(\bigcup_{w=(w_i)_1^n \in C} T_i(w_i) \right) \right) \subseteq \prod_{i=1}^{n} K_i = K.$$

Now Corollary 2.18 guarantees the existence of a finite set $M \subseteq X$ with $F(co\,(M)) \cap (\cap_{x \in M} T(x)) \neq \emptyset$. □

We finally note that the results in this paper can be used to generate variational and minimax type inequalities. To illustrate this we will consider Corollary 2.10 and we will take one real valued function to initiate the idea.

Example 2.22 Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in BPK(X, Y)$ is a compact map and assume that there exists a $\phi : X \times Y \to \mathbf{R}$ with (a). for each fixed $x \in X$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on Y, and (b). $F(co\,(A)) \subseteq S(A)$ for each finite subset A of X where $S(x) = \{y \in Y : \phi(x, y) \leq 0\}$ for $x \in X$. Then there exists a $y_0 \in Y$ such that $\phi(x, y) \leq 0$ for each $x \in X$.

To see this notice (a) guarantees that $S(x)$ is closed in Y for each $x \in X$, so Corollary 2.10 guarantees that $\bigcap_{x \in X} S(x) \neq \emptyset$. Take $y_0 \in \bigcap_{x \in X} S(x)$, so $\phi(x, y_0) \leq 0$ for each $x \in X$.

Remark 2.23 We now give an example of a map $F \in BPK(X, Y)$ in Example 2.22. Assume that there exists a $\psi : X \times Y \to \mathbf{R}$ with for each fixed $x \in X$, $\{y \in Y : \psi(x, y) \leq 0\}$ is nonempty, compact and acyclic. Let $F : X \to 2^Y$ be given by $F(x) = \{y \in Y : \psi(x, y) \leq 0\}$ for all $x \in X$ and assume F is upper semicontinuous. Then $F \in AC(X, Y)$ so $F \in BPK(X, Y)$.

Example 2.24 Let X be a convex subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Assume that there exist maps $\phi,\ \psi : X \times Y \to \mathbf{R}$ with $\phi(x, y) \leq \psi(x, y)$ for $(x, y) \in X \times Y$, and for each fixed $x \in X$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on Y. We will consider the minimax inequality:

$$\inf_{y \in Y} \sup_{x \in X} \phi(x, y) \leq \sup_{x \in X} \inf_{y \in Y} \psi(x, y). \tag{2.3}$$

If $\sup_{x \in X} \inf_{y \in Y} \psi(x, y) = \infty$, then (2.3) holds. We will now assume $\sup_{x \in X} \inf_{y \in Y} \psi(x, y) < \infty$, and in this case for each $\alpha > \sup_{x \in X} \inf_{y \in Y} \psi(x, y)$ (or alternatively, for each $\alpha \in \mathbf{R}$) let $F_\alpha(x) = \{y \in Y : \psi(x, y) \leq \alpha\}$, $S_\alpha(x) = \{y \in Y : \phi(x, y) \leq \alpha\}$ for $x \in X$, and assume if $F_\alpha(x)$ has nonempty values for each $x \in X$ then $F_\alpha \in BPK(X, Y)$ and it is a compact map, and $F_\alpha(co\,(A)) \subseteq S_\alpha(A)$ for each finite subset A of X. Then (2.3) holds.

To see this suppose (2.3) is false. Then there exists a $\alpha \in \mathbf{R}$ with

$$\sup_{x \in X} \inf_{y \in Y} \psi(x, y) < \alpha < \inf_{y \in Y} \sup_{x \in X} \phi(x, y). \tag{2.4}$$

Now either there exists a $x_0 \in X$ with $F_\alpha(x_0) = \emptyset$ or $F_\alpha(x) \neq \emptyset$ for each $x \in X$. Suppose first there exists a $x_0 \in X$ with $F_\alpha(x_0) = \emptyset$. Then $\psi(x_0, y) > \alpha$ for each $y \in Y$, so $\inf_{y \in Y} \psi(x_0, y) \geq \alpha$, and as a result $\sup_{x \in X} \inf_{y \in Y} \psi(x, y) \geq \alpha$, a contractiction (see (2.4)). It remains to consider the case when $F_\alpha(x) \neq \emptyset$ for each $x \in X$. Then $F_\alpha \in BPK(X, Y)$ and Corollary 2.10 guarantees $\bigcap_{x \in X} S(x) \neq \emptyset$, and let $y_0 \in \bigcap_{x \in X} S(x)$. Thus $\phi(x, y_0) \leq \alpha$ for each $x \in X$, so $\sup_{x \in X} \phi(x, y_0) \leq \alpha$, and as a result $\inf_{y \in Y} \sup_{x \in X} \phi(x, y) \leq \alpha$, a contractiction (see (2.4)). Thus (2.3) holds.

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References

1. C.D. Aliprantis and K.C. Border, Infinite dimensional analysis, *Springer Verlag*, Berlin, 1994.
2. H. Ben-El-Mechaiekh, P. Deguire and A. Granas, Points fixes et coincidences pour les applications multivoques II (Applications de type Φ and $\Phi^{\star}$), *C.R. Acad. Sc.*, **295**(1982), 381–384.
3. T.H. Chang, Y.Y. Huang, J.C. Jeng and T.H. Chang and K.H. Kuo, On S–KKM property and related topics, *Jour. Math. Anal. Appl.,* **229**(1999), 212–227.
4. T.H. Chang and C.L. Yen, KKM property and fixed point theorems, *Jour. Math. Anal. Appl.,* **203**(1996), 224–235.
5. X.P. Ding, W.K. Kim and K.K. Tan, A selection theorem and its applications, *Bulletin Australian Math. Soc.,* **46**(1992), 205–212.
6. L. Gorniewicz, Topological fixed point theory of multivalued mappings, *Kluwer Acad. Publishers*, Dordrecht, 1991.
7. L. Gorniewicz and M. Slosarski, Topological essentiality and differential inclusions, *Bull. Austral. Math. Soc.,* **45**(1992), 177–193.
8. L.J. Lim, S. Park and Z.T. Yu, Remarks on fixed points, maximal elements and equilibria of generalized games, *Jour. Math. Anal. Appl.,* **233**(1999), 581–596.
9. D. O'Regan, Coincidence results and Leray–Schauder alternatives between multivalued maps with continuous selections and admissible maps, *Topology and its Applications,* **284**(2020), Art. No. 107368, 6pp.
10. D. O'Regan, KKM type maps and collectively coincidence theory, submitted.
11. D. O'Regan, Coincidence theory for better admissible multifunctions, *Journal of Analysis*, to appear.
12. D.O'Regan and J. Peran, Fixed points for better admissible multifunctions on proximity spaces, *Jour. Math. Anal. Appl.,* **380**(2011), 882–887.
13. S. Park, Coincidence theorems for the better admissible multimaps and their applications, *Nonlinear Anal. ,* **30**(1997), 4183–4191.
14. N.C. Yanelis and N.D. Prabhakar, Existence of maximal elements and equlibria in linear topological spaces, *J. Math. Econom.,* **12**(1983), 233–245.