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ORIGINAL RESEARCH

A note on the energy critical inhomogeneous Hartree equation

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Abstract This note studies the inhomogeneous generalized Hartree equation

$$i\dot u + \Delta u = \pm |x|^{-\rho}|u|^{p-2}(J_\gamma * |\cdot|^{-\rho}|u|^p)u, \quad \rho > 0,\ p > 2.$$

The goal of this work is two-fold. First, one obtains the existence of a local solution in $C_T(H^{s_c})$, where the critical Sobolev exponent is given by the equality $\lambda^{\frac{2-2\rho+\gamma}{2(p-1)}}\|u_0(\lambda\cdot)\|_{\dot H^{s_c}} = \|u_0\|_{\dot H^{s_c}}$. Second, one investigates the uniqueness of critical solutions in $C_T(H^{s_c})$. Indeed, since one uses a fixed point argument in some Strichartz spaces, the uniqueness in the energy space is not trivial. In fact, the technique used in order to obtain the existence of a local sub-critical solution, which consists to divide the integrals on the unit centered ball of $\mathbb{R}^N$ and it's complementary is no more sufficient to conclude. To overcome this difficulty, one uses a fixed point argument with Strichartz estimates in some suitable Lorentz spaces, which enables us to handle the inhomogeneous term by the fact that $|x|^{-\rho} \in L^{\frac{N}{\rho},\infty}$.

Keywords Hartree equation · Nonlinear equations · Energy critical · Local existence · Uniqueness

Mathematics Subject Classification 35Q55

1 Introduction

This work considers the Cauchy problem for a generalized Hartree equation

$$\begin{cases} i\dot u + \Delta u = \mu|x|^{-\rho}|u|^{p-2}(J_\gamma * |\cdot|^{-\rho}|u|^p)u; \\ u_{|t=0} = u_0. \end{cases} \tag{1.1}$$

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Here and hereafter u is a complex valued function of the variable $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, for some integer $N\geq 1$. The inhomogeneous decaying term is $|x|^{-\rho}$, where $\rho>0$. The Riesz-potential is defined on $\mathbb{R}^N$ by

$$J_\gamma:=\frac{\Gamma(\frac{N-\gamma}{2})}{\Gamma(\frac{\gamma}{2})\pi^{\frac{N}{2}}2^\gamma|\cdot|^{N-\gamma}},\quad 0<\gamma<N.$$

The above equation is a generalization of the Hartree problem, which corresponds to $p=\gamma=2$, $\rho=0$ and $N=3$. The nonlinear equations of Hartree type model the mean-field limit of some quantum systems [12]. It arises also in Bose-Einstein condensates of a gas of bosonic particles [2,22]. The stationary equation was introduced by Ph. Choquard in order to describe the one-component plasma [20].

If u resolves the generalized Hartree equation (1.1), then also does the family

$$u_\lambda=\lambda^{\frac{2-2\rho+\gamma}{2(p-1)}}u(\lambda^2\cdot,\lambda\cdot),\quad \lambda>0.$$

The identity $\|u_\lambda(t)\|_{\dot H^s}=\lambda^{s-(\frac{N}{2}-\frac{2-2\rho+\gamma}{2(p-1)})}\|u(\lambda^2t)\|_{\dot H^s}$, gives the only one homogeneous Sobolev norm invariant under the above scaling, $H^{s_c}:=H^{\frac{N}{2}-\frac{2-2\rho+\gamma}{2(p-1)}}$. Two particular situations were widely investigated in the literature. The first one , called mass-critical regime, corresponds to $s_c=0$ and with an equivalent way $p=p_c:=\frac{\gamma+2-2\rho}{N}+1$. This case is related to the following mass conservation law

$$M[u(t)]:=\int_{\mathbb{R}^N}|u(t,x)|^2dx=M[u_0].$$

The second one, called energy-critical regime, corresponds to $s_c=1$ and with an equivalent way $p=p^c:=\frac{\gamma+2-2\rho}{N-2}+1$. This case is related to the following energy conservation law

$$E[u(t)]:=\int_{\mathbb{R}^N}\Big(|\nabla u(t,x)|^2+\frac{\mu}{p}|x|^{-\rho}|u(t,x)|^p(J_\gamma*|\cdot|^{-\rho}|u(t)|^p)\Big)dx=E[u_0].$$

Similarly, the $\dot H^s$-critical regime corresponds to $s_c=s$ and with an equivalent way

$$p=p_s^c:=\frac{\gamma+2-2\rho}{N-2s}+1. \tag{1.2}$$

In this note, one considers (1.1) with the Duhamel integral formulation

$$u=e^{i\cdot\Delta}u_0-i\mu\int_0^\cdot e^{i(\cdot-\tau)\Delta}\Big[|x|^{-\rho}|u|^{p-2}(J_\gamma*|\cdot|^{-\rho}|u|^p)u\Big]ds.$$

To the authors knowledge, the inhomogeneous Hartree equation was investigated in few papers. The first work dealing with this topic seems to be [13], where the instability of standing waves was obtained for small frequency under the conditions: $p=2$, $N\geq 3$, $0<\rho<2$ and $N-4+2\rho<\gamma<N$. Later on the inhomogeneous generalized Hartree equation was considered by the first author [1], where a sharp threshold of global well-posedness versus finite time blow-up of energy solutions was obtained via a Gagliardo-Nirenberg type estimate and the existence of ground states. The scattering of the spherically symmetric focusing global solutions, in the inter-critical regime, under the ground state threshold was proved by the first author in [25], using a new approach due to Dodson et al. [11]. The radial assumption was removed in [26]. The scattering of the defocusing global solutions in the inter-critical regime, was obtained by the authors in [24]. Recently, the well-posedness and scattering for the critical inhomogeneous generalized Hartree problem was treated in [4,6].

It is the aim of this paper to develop a local theory of the inhomogeneous generalized Hartree equation (1.1), in the energy critical regime. Indeed, the method used in [1, Theorem 5.1, Theorem 5.2] is no more applicable in the critical case. To the authors knowledge, there is two methods used in the literature to overcome this difficulty. The first one is to use a fixed point argument in some weighted Sobolev spaces [16,18]. This way was used recently in [4,17]. The second method uses a fixed point argument in Lorentz spaces [3,15]. In this manuscript, one follows the last method and one deals with the complications due to the non-local source term.

The rest of this paper is organized as follows. Section 2 contains the main result and some standard estimates. In section 3, one proves the main result.

Here and hereafter, one denotes, for simplicity, the Lebesgue and Sobolev spaces endowed with the usual norms

$$L^r := L^r(\mathbb{R}^N), \quad H^s := H^s(\mathbb{R}^N);$$
$$\|\cdot\|_r := \|\cdot\|_{L^r}, \quad \|\cdot\| := \|\cdot\|_2;$$
$$\|\cdot\|_{H^s} := \Big(\|\cdot\|^2 + \|(-\Delta)^{\frac{s}{2}}\cdot\|^2\Big)^{\frac{1}{2}}.$$

Moreover, for $\rho, c \geq 0$ and $u, v \in \mathbb{C}$, one defines the complex quantities

$$\mathcal{N}^{\rho,c}(u,v) := |x|^{-\rho}(J_\gamma * |\cdot|^{-c}|u|^p)|u|^{p-2}v;$$
$$\mathcal{N}^{\rho,c}(u) := \mathcal{N}^{\rho,c}(u,u);$$
$$\mathcal{N}(u,v) := \mathcal{N}^{\rho,\rho}(u,v);$$
$$\mathcal{N}(u) := \mathcal{N}(u,u).$$

Finally, for an eventual solution to (1.1), $T^* > 0$ denotes it's lifespan.

2 Main result and useful estimates

This section contains the main result and some estimates needed in the sequel.

2.1 Main result

The novelty of this note is to obtain the local existence and uniqueness of energy critical solutions to (1.1).

Theorem 2.1 *Let* $1 \leq N \leq 3$, $\max\{0, N-2s-2\} < \gamma < N$, $s \geq 0$ *and* $\rho > 0$. *Take* $\frac{N-(2-2\rho+\gamma)}{2} < s_c \leq s$ *and* $u_0 \in H^s$.

1. *Assume that* $s \leq 1$, $s < \frac{N}{2}$, $\gamma < N-2s$ *and* $\gamma < 2\rho < \min\{2s+\gamma, N-2s+\gamma\}$. *Then, there is* $T > 0$ *and a unique local solution of* (1.1), *in the space*

$$C_T(H^s) \cap L^q_T(W^{s,r}_2),$$

where (q, r) *is an admissible pair in the meaning of Proposition 2.9. Moreover, the mass* (*and energy if* $s = 1$) *is conserved. Furthermore, if* $s = s_c$, *the solution is global and scatters in* H^s, *for small data.*

2. *Assume that* $N \in \{2, 3\}$, $s \geq 1$, $\gamma < 2\rho < \min\{2+\gamma, 4\}$ *and* (3.12) *is satisfied. Then, there is at most one solution of* (1.1), *in*

$$C_T(H^s).$$

In view of the results stated in the above theorem, some comments are in order.

- The condition $\frac{N-(2-2\rho+\gamma)}{2} \leq s \leq 1$ gives the restriction on the spatial dimension $N \leq 2s+2-2\rho+\gamma < 2(1+s) < 4$;
- in the proof, one needs the restriction $\gamma < 2\rho$;
- the global existence and scattering for small data follow with classical arguments;
- the novelty here is the investigation of the critical case $s = s_c$. This extends [1] which treats the inter-critical regime;
- the essential ingredient here is the use of Lorentz spaces;
- the assumption $\frac{N-(2-2\rho+\gamma)}{2} < s_c$ is equivalent to $p > 2$, which is due to the singularity of $|u|^{p-2}$ for $p < 2$;
- the assumptions on s_c can be written using p as follows

$$2 \leq p \leq 1 + \frac{2+\gamma-2\rho}{N-2s};$$

- the above result covers the energy-critical regime $s_c = 1$, contrarily to [4]. It complements [15] to the space dimensions $N = 1, 2$;

- the problem(1.1) was treated in [17], using some weighted Strichartz type estimates;
- the first point in the above Theorem gives the uniqueness of solutions to (1.1) in the energy space intersected with a Strichartz one;
- the existence and uniqueness of solutions to (1.1) in $C_T(H^s)$ follows for $s = 1$ and $2 \leq N \leq 3$;
- the condition (3.12) is possible at least for $s \to 1$, $\gamma, \rho \to 0$ and $N = 3$;
- one denotes the Sobolev-Lorentz spaces [19],

$$W_q^{s,r} := [(I - \Delta)^{\frac{s}{2}}]^{-1} L^{r,q} \quad \text{and} \quad \dot{W}_q^{s,r} := (-\Delta)^{\frac{-s}{2}} L^{r,q};$$

- the scattering in H^s for small data means that there is $\varepsilon > 0$ such that if $\|u_0\|_{H^s} < \varepsilon$, then, there exists $u_+ \in H^s$ such that $\lim_{t\to+\infty} \|u(t) - e^{-it\Delta}u_+\|_{H^s} = 0$;
- the uniqueness of $\dot{H}^s$-critical solutions follows the method used in [7, Proposition 4.2.5];
- Taking account of the property $|x|^{-\rho} \in L^{\frac{N}{\rho},\infty}$, Lorentz spaces were used in order to obtain the local existence of energy-critical solutions to NLS with an ihomogeneous local source term [3], see also [5] for the fractional INLS. The presence of a non-local source term in (1.1) gives serious technical complications;
- in a paper in progress, the authors treat the ground state threshold of global existence and scattering versus finite time blow-up of solutions in the spirit of [8].

2.2 Useful estimates

Let us start with giving some properties of Lorentz spaces. If $f : \mathbb{R}^N \to \mathbb{C}$ is a measurable function, one denotes the associated Schwarz rearrangement by

$$f^*(s) := \inf\{\lambda > 0,\ |\{x \in \mathbb{R}^N,\ |f(x)| > \lambda\}| \leq s\}.$$

For $0 < p < \infty$ and $0 < q \leq \infty$, one denotes the Lorentz norm

$$\|f\|_{p,q} := \begin{cases} \left(\frac{q}{p}\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < p, q < \infty; \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t). \end{cases}$$

The Lorentz space is

$$L^{p,q}(\mathbb{R}^N) := \{f \text{ measurable }, \|f\|_{p,q} < \infty\}.$$

It is a quasi-Banach space endowed with the above quasi-norm. Moreover,

$$\begin{cases} L^{p,p} = L^p, \quad L^{\infty,\infty} = L^\infty\,(convention); \\ q \mapsto L^{p,q} \nearrow; \\ \||\cdot|^\gamma\|_{p,q} = \|\cdot\|^\gamma_{p\gamma,q\gamma}, \quad \forall \gamma > 0; \\ |\cdot|^{-\rho} \in L^{\frac{N}{\rho},\infty}. \end{cases}$$

In what follows, one gives the so-called generalized Hölder's estimate [23].

Lemma 2.2 *Let $1 < p_1, p_2, p < \infty$ and $1 \leq q_1, q_2, q \leq \infty$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad and \quad \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}.$$

Then, there is $C := C_{N,p_i,q_i}$ such that

$$\|uv\|_{p,q} \leq C\|u\|_{p_1,q_1}\|v\|_{p_2,q_2}.$$

Recall also some convolution properties [23].

Lemma 2.3 *Let $1 < p_i, p < \infty$, $1 \leq q, q_i \leq \infty$ and $1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}$. Thus,*

$$\|f * g\|_{p,q} \lesssim \|f\|_{p_1,q_1}\|g\|_{p_2,q_2}.$$

Now, one recalls a Hardy-Littlewood-Sobolev inequality [21].

Lemma 2.4 *Let* $0 < \gamma < N \geq 1$. *Then,*

1. $\|J_\gamma * f\|_{a,q} \lesssim \|J_\gamma\|_{\frac{N}{N-\gamma},\infty}\|f\|_{\frac{Na}{N+a\gamma},q_2} \lesssim \|f\|_{\frac{Na}{N+a\gamma},q_2}$, *if* $1 \leq q_2 \leq q \leq \infty$ *and* $\frac{N}{N-\gamma} < a < \infty$;
2. $\|(J_\gamma * f)g\|_{a,q} \lesssim \|f\|_{\frac{Na_1}{N+a_1\gamma},q_1}\|g\|_{a_2,q_2}$, *if* $0 \leq \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2} \leq 2$, $1 < a, a_2 < \infty$, $\frac{N}{N-\gamma} < a_1 < \infty$ *and* $\frac{1}{a} = \frac{1}{a_1} + \frac{1}{a_2}$.

Recall also some classical Sobolev embeddings [19].

Lemma 2.5 *Let* $1 < r < \infty$, $1 \leq q \leq \infty$ *and* $0 < s < \frac{N}{r}$. *Then,*

$$\dot{W}^{s,r}_q \hookrightarrow L^{1/(\frac{1}{r}-\frac{s}{N}),q}.$$

The next fractional chain rule [9,10] will be useful.

Lemma 2.6 *Let* $N \geq 1$, $0 < \beta \leq 1$, $1 < p, p_1, p_2 < \infty$ *and* $1 \leq q, q_i \leq \infty$, $i \in \{1, 2, 3, 4\}$. *Assume that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}$ *and* $F \in C^1(\mathbb{C})$. *Then,*

$$\|(-\Delta)^{\frac{\beta}{2}}F(u)\|_{p,q} \lesssim \|(-\Delta)^{\frac{\beta}{2}}u\|_{p_1,q_1}\|F'(u)\|_{p_2,q_2}, \quad \text{if} \quad q, q_1, q_2 < \infty; \tag{2.3}$$

$$\|(-\Delta)^{\frac{\beta}{2}}(uv)\|_{p,q} \lesssim \|(-\Delta)^{\frac{\beta}{2}}u\|_{p_1,q_1}\|v\|_{p_2,q_2} + \|(-\Delta)^{\frac{\beta}{2}}v\|_{p_3,q_3}\|u\|_{p_4,q_4}. \tag{2.4}$$

Let us give a Bootstrap result.

Lemma 2.7 *Let* $\kappa > 1$, $b > 0$ *and* $0 < a < (\frac{\kappa-1}{\kappa})(\kappa b)^{\frac{-1}{\kappa-1}}$. *Take* $X \in C([0, T], \mathbb{R}_+)$ *satisfying*

$$X(t) \leq a + b\Big(X(t)\Big)^\kappa, \quad \forall t \in [0, T];$$
$$X(0) \leq (\kappa b)^{\frac{1}{1-\kappa}}.$$

Thus,

$$X(t) \leq \frac{\kappa}{\kappa-1}a, \quad \forall t \in [0, T].$$

Proof The function $f(x) := bx^\kappa - x + a$ is decreasing on $[0, (b\kappa)^{\frac{1}{1-\kappa}}]$ and increasing on $[(b\kappa)^{\frac{1}{1-\kappa}}, \infty)$. The assumptions imply that $f((b\kappa)^{\frac{1}{1-\kappa}}) < 0$ and $f(\frac{\kappa}{\kappa-1}a) \leq 0$. As $f(X(t)) \geq 0$, $f(0) > 0$ and $X(0) \leq (b\kappa)^{\frac{1}{1-\kappa}}$, we conclude the proof by a continuity argument.

Let us recall the so-called Strichartz admissible couples.

Definition 2.8 A pair of positive real numbers (q, r) is said admissible if

$$N(\frac{1}{2} - \frac{1}{r}) = \frac{2}{q}, \tag{2.5}$$

and

$$2 \leq q, r \leq \frac{2N}{N-2} \quad \text{if} \quad N \geq 3; \tag{2.6}$$

$$2 \leq q, r < \infty \quad \text{if} \quad N = 2;\, ; \tag{2.7}$$

$$2 \leq q, r \leq \infty \quad \text{if} \quad N = 1. \tag{2.8}$$

For short, one denotes by Γ the set of admissible pairs.

Finally, one recalls a standard tool to control solutions of (1.1), namely the Strichartz estimate [14, Theorem 10.1].

Proposition 2.9 *Let* $N \geq 1$. *Then, there exists* $C > 0$ *such that*

1. $\|e^{i\cdot\Delta}u\|_{L^q(L^{r,2})} \leq C\|u\|$;
2. $\|\int_0^\cdot e^{i(\cdot-\tau)\Delta}f(\tau)\,d\tau\|_{L^q(L^{r,2})} \leq C\|f\|_{L^{\tilde{q}'}(L^{\tilde{r}',2})}$.

Here, (q, r) *and* $(\tilde{q}, \tilde{r})$ *are admissible in the meaning:* $q, \tilde{q} \geq 2$ *and*

$$N(\frac{1}{2} - \frac{1}{r}) = \frac{2}{q}, \quad N(\frac{1}{2} - \frac{1}{\tilde{r}}) = \frac{2}{\tilde{q}}.$$

3 Proof of Theorem 2.1

In this section, one establishes the main result of this note.

3.1 Local existence of solutions

In this subsection, one proves Theorem 2.1 about the existence of a local solution to (1.1). Take an admissible pair $(q, r) \in \Gamma$ to be fixed later. First, by Lemma 2.4 and Lemma 2.2, one writes

$$\begin{aligned}
\|\mathcal{N}(u, v)\|_{r',2} &\lesssim \||x|^{-\rho}u^p\|_{1/(\frac{1}{a_1}+\frac{\gamma}{N}),\,\frac{2(2p-1)}{p}}\||x|^{-\rho}u^{p-2}v\|_{a_2,\,\frac{2(2p-1)}{p-1}}\\
&\lesssim \||x|^{-\rho}\|^2_{\frac{N}{\rho},\infty}\|u^p\|_{1/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}),\,\frac{2(2p-1)}{p}}\|u^{p-2}v\|_{1/(\frac{1}{a_2}-\frac{\rho}{N}),\,\frac{2(2p-1)}{p-1}}\\
&\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}),2(2p-1)}\|u^{p-2}v\|_{1/(\frac{1}{a_2}-\frac{\rho}{N}),\,\frac{2(2p-1)}{p-1}}\\
&\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}),2(2p-1)}\|u^{p-2}\|_{1/(\frac{1}{a_2}-\frac{\rho}{N}-\frac{1}{r}),\,\frac{2(2p-1)}{p-2}}\|v\|_{r,2(2p-1)}\\
&\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}),2(2p-1)}\|u\|^{p-2}_{(p-2)/(\frac{1}{a_2}-\frac{\rho}{N}-\frac{1}{r}),2(2p-1)}\|v\|_{r,2(2p-1)}\\
&\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}),2(2p-1)}\|u\|^{p-2}_{(p-2)/(\frac{1}{a_2}-\frac{\rho}{N}-\frac{1}{r}),2}\|v\|_{r,2}.
\end{aligned}$$

Here,

$$\begin{cases}
1=\frac{1}{r}+\frac{1}{a_1}+\frac{1}{a_2};\\
\frac{1}{p}(\frac{1}{a_1}+\frac{\gamma-\rho}{N})=\frac{1}{r}-\frac{s}{N}=\frac{1}{p-2}(\frac{1}{a_2}-\frac{\rho}{N}-\frac{1}{r});\\
0<\frac{1}{a_1}+\frac{\gamma-\rho}{N}<1;\\
0<\frac{1}{a_2}-\frac{\rho}{N}<1;\\
0<\frac{1}{a_2}-\frac{\rho}{N}-\frac{1}{r}<1.
\end{cases}$$

A direct computation gives

$$\begin{cases}
r=\frac{2Np}{2s(p-1)+\gamma+N-2\rho};\\
a_1=\frac{2N}{N-2s-\gamma};\\
a_2=\frac{2Np}{(p-1)(N+\gamma)+2(\rho+s)};\\
0<\frac{1}{a_1}+\frac{\gamma-\rho}{N}<1;\\
\frac{1}{r}<\frac{1}{a_2}-\frac{\rho}{N}<1.
\end{cases}$$

Here, r is well defined because $2s+\gamma>2\rho$ and one needs to check that

$$r\geq 2,\quad a_1>\frac{N}{N-\gamma}\quad\text{and}\quad a_2>1.$$

This is equivalent to

$$\begin{cases}
N>\gamma+2s,\ p\geq 1+\frac{\gamma-2\rho}{N-2s};\\
p>1-2\frac{N-\rho-s}{N-\gamma}.
\end{cases}$$

This is satisfied because $N>\gamma+2s$. Indeed, $p\geq 2$ and $\gamma<2\rho<2s+\gamma$ guarantee the other inequalities. In fact, for the last estimate, it is sufficient to have $2>1-2\frac{N-\rho-s}{N-\gamma}$, which reads $3N>\gamma+2\rho+2s$. This holds because $\gamma+2\rho+2s<2\gamma+4s<2N$. Now, one checks that

$$\begin{cases}
0<\frac{1}{a_1}+\frac{\gamma-\rho}{N}=\frac{N-2s+\gamma-2\rho}{2N}<1;\\
\frac{1}{r}=\frac{2s(p-1)+\gamma+N-2\rho}{2Np}<\frac{1}{a_2}-\frac{\rho}{N}=\frac{(p-1)(N+\gamma)+2(\rho+s)-2p\rho}{2Np}<1.
\end{cases}$$

This reads

$$\begin{cases} 0 < N - 2s + \gamma - 2\rho < 2N; \\ 2s(p-1) + \gamma + N - 2\rho < (p-1)(N+\gamma) + 2(\rho + s) - 2p\rho < 2Np. \end{cases}$$

This is satisfied if

$$\begin{cases} 0 < 2\rho < N - 2s + \gamma; \\ 2s(p-1) + \gamma + N - 2\rho < (p-1)(N+\gamma) + 2(\rho + s) - 2p\rho < 2Np. \end{cases}$$

The above system reads

$$\begin{cases} 0 < 2\rho < N - 2s + \gamma; \\ \gamma + N - 2(s+\rho) < (p-1)\Big(\gamma + N - 2(s+\rho)\Big); \\ 2(s-N) < 2(p-1)(N + 2\rho - \gamma). \end{cases}$$

So, because $s \leq 1 \leq N$ and $0 < \gamma < N$, it is sufficient to write

$$p > 2, \quad 0 < 2\rho < N - 2s + \gamma.$$

Now, with Sobolev embedding in Lemma 2.5, one has

$$\begin{aligned} \|\mathcal{N}(u,v)\|_{r',2} &\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}),2} \|u\|^{p-2}_{(p-2)/(\frac{1}{a_2}-\frac{\rho}{N}-\frac{1}{r}),2} \|v\|_{r,2} \\ &\lesssim \|u\|^{2(p-1)}_{\dot{W}^{s,r}_2} \|v\|_{r,2}. \end{aligned}$$

Let $q := \frac{4p}{(N-2s)(p-1)-\gamma+2\rho}$ be such that $(q,r) \in \Gamma$. Indeed, $2\rho > \gamma$ and $(N-2s)(p-1) \leq 2 + \gamma - 2\rho$ give $q \geq 2$. Thus, by Hölder estimate

$$\|\mathcal{N}(u,v)\|_{L^{q'}_T(L^{r',2})} \lesssim T^{1-\frac{2p}{q}} \|u\|^{2(p-1)}_{L^q_T(\dot{W}^{s,r}_2)} \|v\|_{L^q_T(L^{r,2})}. \tag{3.9}$$

Moreover, $1 - \frac{2p}{q} = \frac{2+\gamma-2\rho-(p-1)(N-2s)}{2} \geq 0$ and $1 - \frac{2p}{q} = 0$ if and only if $p = p_s^c := 1 + \frac{2+\gamma-2\rho}{N-2s}$. Now, let us estimate the term

$$\mathcal{M}(u,w) := (J_\gamma * [|\cdot|^{-\rho}|u|^{p-1}w])|x|^{-\rho}|u|^{p-1}.$$

Using Lemma 2.4, one writes

$$\begin{aligned} \|\mathcal{M}(u,w)\|_{r',2} &\lesssim \||x|^{-\rho}u^{p-1}w\|_{1/(\frac{1}{a_1}+\frac{\gamma}{N}),\frac{2(2p-1)}{p}} \||x|^{-\rho}u^{p-1}\|_{a_2,\frac{2(2p-1)}{p-1}} \\ &\lesssim \||x|^{-\rho}\|^2_{\frac{N}{\rho},\infty} \|u^{p-1}w\|_{1/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}),\frac{2(2p-1)}{p}} \|u^{p-1}\|_{1/(\frac{1}{a_2}-\frac{\rho}{N}),\frac{2(2p-1)}{p-1}} \\ &\lesssim \|u\|^{p-1}_{(p-1)/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}-\frac{1}{r}),2(2p-1)} \|w\|_{r,2(2p-1)} \|u\|^{p-1}_{(p-1)/(\frac{1}{a_2}-\frac{\rho}{N}),2(2p-1)} \\ &\lesssim \|u\|^{p-1}_{(p-1)/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}-\frac{1}{r}),2} \|u\|^{p-1}_{(p-1)/(\frac{1}{a_2}-\frac{\rho}{N}),2} \|w\|_{r,2}. \end{aligned}$$

Here,

$$\begin{cases} 1 = \frac{1}{r} + \frac{1}{a_1} + \frac{1}{a_2}; \\ \frac{1}{p-1}(\frac{1}{a_1} + \frac{\gamma-\rho}{N} - \frac{1}{r}) = \frac{1}{r} - \frac{s}{N} = \frac{1}{p-1}(\frac{1}{a_2} - \frac{\rho}{N}); \\ \frac{1}{r} < \frac{1}{a_1} + \frac{\gamma-\rho}{N} < 1; \\ 0 < \frac{1}{a_2} - \frac{\rho}{N} < 1 \end{cases}$$

A direct computation gives

$$\begin{cases} r = \frac{2Np}{2s(p-1)+\gamma+N-2\rho}; \\ a_1 = \frac{2N}{N-\gamma}; \\ a_2 = \frac{2Np}{(p-1)(N+\gamma-2s)+2\rho}. \end{cases}$$

Here, one needs to check that

$$r \geq 2, \quad a_1 > \frac{N}{N-\gamma} \quad \text{and} \quad a_2 > 1.$$

This is satisfied by previous assumptions. Now, one checks that

$$\begin{cases} \frac{1}{r} = \frac{2s(p-1)+\gamma+N-2\rho}{2Np} < \frac{1}{a_1} + \frac{\gamma-\rho}{N} = \frac{N+\gamma-2\rho}{2N} < 1; \\ 0 < \frac{1}{a_2} - \frac{\rho}{N} = \frac{(p-1)(N+\gamma-2s)+2\rho-2\rho p}{2Np} < 1. \end{cases}$$

This is satisfied because

$$2\rho < N + \gamma - 2s.$$

Thus, by Sobolev injections

$$\|\mathcal{M}(u,w)\|_{L_T^{q'}(L^{r',2})} \lesssim T^{1-\frac{2p}{q}} \|u\|^{2(p-1)}_{L_T^q(\dot W_2^{s,r})} \|w\|_{L_T^q(L^{r,2})}. \tag{3.10}$$

Let us estimate the next quantities

$$\begin{aligned} (I_1) &:= \|(J_\gamma * |\cdot|^{-\rho-s}|u|^p)|x|^{-\rho}|u|^{p-1}\|_{r',2}; \\ (I_2) &:= \|(J_\gamma * |\cdot|^{-\rho}|u|^{p-1}(-\Delta)^{\frac{s}{2}}u)|x|^{-\rho}|u|^{p-1}\|_{r',2}; \\ (II_1) &:= \|(J_\gamma * |\cdot|^{-\rho}|u|^p)|x|^{-\rho-s}|u|^{p-1}\|_{r',2}; \\ (II_2) &:= \|(J_\gamma * |\cdot|^{-\rho}|u|^p)|x|^{-\rho}|u|^{p-2}(-\Delta)^{\frac{s}{2}}u\|_{r',2}. \end{aligned}$$

The second and last terms can be controlled as previously

$$(I_2) + (II_2) \lesssim T^{1-\frac{2p}{q}} \|u\|^{2(p-1)}_{L_T^q(\dot W_2^{s,r})} \|(-\Delta)^{\frac{s}{2}} v\|_{L_T^q(L^{r,2})} \lesssim T^{1-\frac{2p}{q}} \|u\|^{2(p-1)}_{L_T^q(\dot W_2^{s,r})} \|v\|_{L_T^q(\dot W_2^{s,r})}.$$

One writes by Lemma 2.4,

$$\begin{aligned} (I_1) &\lesssim \||x|^{-\rho-s}u^p\|_{1/(\frac{1}{a_1}+\frac{\gamma}{N}),\frac{2(2p-1)}{p}} \||x|^{-\rho}u^{p-2}u\|_{a_2,\frac{2(2p-1)}{p-1}} \\ &\lesssim \||x|^{-\rho-s}\|_{\frac{N}{s+\rho},\infty} \|u^p\|_{1/(\frac{1}{a_1}+\frac{\gamma-s-\rho}{N}),\frac{2(2p-1)}{p}} \||x|^{-\rho}\|_{\frac{N}{\rho},\infty} \|u^{p-2}u\|_{1/(\frac{1}{a_2}-\frac{\rho}{N}),\frac{2(2p-1)}{p-1}} \\ &\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-s-\rho}{N}),2(2p-1)} \|u^{p-2}u\|_{1/(\frac{1}{a_2}-\frac{\rho}{N}),\frac{2(2p-1)}{p-1}} \\ &\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-s-\rho}{N}),2(2p-1)} \|u^{p-2}\|_{1/(\frac{1}{a_2}-\frac{\rho}{N}-\frac{1}{r}+\frac{s}{N}),\frac{2(2p-1)}{p-2}} \|u\|_{1/(\frac{1}{r}-\frac{s}{N}),2(2p-1)} \\ &\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-s-\rho}{N}),2(2p-1)} \|u\|^{p-2}_{(p-2)/(\frac{1}{a_2}-\frac{1}{r}+\frac{s-\rho}{N}),2(2p-1)} \|u\|_{1/(\frac{1}{r}-\frac{s}{N}),2(2p-1)} \\ &\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-s-\rho}{N}),2} \|u\|^{p-2}_{(p-2)/(\frac{1}{a_2}-\frac{1}{r}+\frac{s-\rho}{N}),2} \|u\|_{1/(\frac{1}{r}-\frac{s}{N}),2}. \end{aligned}$$

Here,

$$\begin{cases} 1 = \frac{1}{r} + \frac{1}{a_1} + \frac{1}{a_2}; \\ \frac{1}{p}(\frac{1}{a_1} + \frac{\gamma-\rho-s}{N}) = \frac{1}{r} - \frac{s}{N} = \frac{1}{p-2}(\frac{1}{a_2} - \frac{\rho-s}{N} - \frac{1}{r}); \\ \frac{s+\rho}{N} < \frac{1}{a_1} + \frac{\gamma}{N} < 1; \\ 0 < \frac{1}{r} - \frac{s}{N} < \frac{1}{a_2} - \frac{\rho}{N} < 1. \end{cases}$$

A direct computation gives

$$\begin{cases} r = \frac{2Np}{2s(p-1)+\gamma+N-2\rho}; \\ a_1 = \frac{2N}{N-\gamma}; \\ a_2 = \frac{2Np}{(p-1)(N+\gamma-2s)+2\rho}. \end{cases}$$

Here, one needs to check that

$$r \geq 2, \quad a_1 > \frac{N}{N-\gamma} \quad \text{and} \quad a_2 > 1.$$

This is satisfied sufficiently if

$$\begin{cases} N > 2s; \\ p \geq 1 + \frac{\gamma-2\rho}{N-2s}. \end{cases}$$

This is satisfied as previously. Now, let us chack that

$$\begin{cases} \frac{s+\rho}{N} < \frac{1}{a_1} + \frac{\gamma}{N} = \frac{N+\gamma}{2N} < 1; \\ 0 < \frac{1}{r} - \frac{s}{N} = \frac{2s(p-1)+\gamma+N-2\rho}{2Np} - \frac{s}{N} < \frac{(p-1)(N+\gamma-2s)+2\rho}{2Np} - \frac{\rho}{N} = \frac{1}{a_2} - \frac{\rho}{N} < 1. \end{cases}$$

The first line above is satisfied because

$$2\rho < N + \gamma - 2s.$$

The second line in the above system reads

$$0 < N + \gamma - 2\rho - 2s < (p-1)(N + \gamma - 2s - 2\rho) < 2Np,$$

which is satisfied because

$$p > 2.$$

Moreover, the term (II_1) can be estimated similarly and with Sobolev embedding, one has

$$\|(I_1) + (I_2) + (II_1) + (II_2)\|_{L_T^{q'}} \lesssim \| \|u\|^{2p-1}_{\dot{W}_2^{s,r}} \|_{L^{q'}(0,T)} \lesssim T^{1-\frac{2p}{q}} \|u\|^{2p-1}_{L_T^q(\dot{W}_2^{s,r})}.$$

Thanks to the fractional chain rule in Lemma 2.6, one gets

$$\|(-\Delta)^{\frac{s}{2}} \mathcal{N}(u)\|_{L_T^{q'}(L^{r',2})} \lesssim \| \|u\|^{2p-1}_{\dot{W}_2^{s,r}} \|_{L^{q'}(0,T)} \lesssim T^{1-\frac{2p}{q}} \|u\|^{2p-1}_{L_T^q(\dot{W}_2^{s,r})}. \tag{3.11}$$

Now, take the function

$$f(u) := e^{i\cdot\Delta} u_0 - i\mu \int_0^{\cdot} e^{i(\cdot-\tau)\Delta} \mathcal{N}(u)\, d\tau.$$

Take also for $R, T > 0$, the closed ceneterd ball of $L_T^q(W_2^{s,r})$ with radius R, denoted by

$$B_T(R) := \{u \in L_T^q(W_2^{s,r}), \quad \|u\|_{L_T^q(\dot{W}_2^{s,r})} \leq R\},$$

endowed with the complete distance

$$d(u, v) := \|u - v\|_{L_T^q(L^{r,2})}.$$

• First case $p < p_s^c$. Let us write the elementary identities

$$||u|^p - |v|^p| \lesssim |u - v|(|u|^{p-1} + |v|^{p-1});$$
$$||u|^{p-2}u - |v|^{p-2}v| \lesssim |u - v|(|u|^{p-2} + |v|^{p-2}),$$

Denote $w := u - v$. Then, using Strichartz estimates and (3.9)-(3.10), one has

$$\begin{aligned} d(f(u), f(v)) &\lesssim \|\mathcal{N}(u) - \mathcal{N}(v)\|_{L_T^{q'}(L^{r',2})} \\ &\lesssim \|(J_\gamma * [|\cdot|^{-\rho}(|u|^p - |v|^p)])|x|^{-\rho}|u|^{p-1}\|_{L_T^{q'}(L^{r',2})} \\ &+ \|(J_\gamma * |\cdot|^{-\rho}|v|^p)|x|^{-\rho}(|u|^{p-2}u - |v|^{p-2}v)\|_{L_T^{q'}(L^{r',2})} \end{aligned}$$

$$\begin{aligned}
&\lesssim \|(J_\gamma * [|\cdot|^{-\rho}(|u|^{p-1}+|v|^{p-1})w])|x|^{-\rho}|u|^{p-1}\|_{L_T^{q'}(L^{r',2})}\\
&+\|(J_\gamma * |\cdot|^{-\rho}|v|^{p})|x|^{-\rho}(|u|^{p-2}+|v|^{p-2})w\|_{L_T^{q'}(L^{r',2})}\\
&\lesssim T^{1-\frac{2p}{q}}(\|u\|^{2(p-1)}_{L_T^q(\dot W_2^{s,r})}+\|v\|^{2(p-1)}_{L_T^q(\dot W_2^{s,r})})\|w\|_{L_T^q(L^{r,2})}\\
&\lesssim T^{1-\frac{2p}{q}}R^{2(p-1)}d(u,v).
\end{aligned}$$

Moreover, Strichartz estimate with (3.11) give

$$\begin{aligned}
\|f(u)\|_{L_T^q(\dot W_2^{s,r})} &\lesssim \|u_0\|_{\dot H^s}+\|(-\Delta)^{\frac{s}{2}}\mathcal{N}(u)\|_{L_T^{q'}(L^{r',2})}\\
&\lesssim \|u_0\|_{\dot H^s}+T^{1-\frac{2p}{q}}\|u\|^{2p-1}_{L_T^q(\dot W_2^{s,r})}\\
&\le c\|u_0\|_{\dot H^s}+cT^{1-\frac{2p}{q}}R^{2p-1}.
\end{aligned}$$

Since $p<p_s^c$, one has $1-\frac{2p}{q}>0$. Thus, the existence of a local solution to (1.1) follows with a classical Picard argument by taking $c\|u_0\|_{\dot H^s}<\frac{R}{2}$ and $0<T<<1$.

• Second case $p=p_s^c$. In such a case $1-\frac{2p}{q}=0$. Taking $0<R<c^{-\frac{1}{2(p-1)}}$, one has the contraction

$$d(f(u),f(v))\le cR^{2(p-1)}d(u,v).$$

Moreover, by Strichartz estimate, there is $\epsilon(T)\to 0$ as $T\to 0$ such that

$$\begin{aligned}
\|f(u)\|_{L_T^q(\dot W_2^{s,r})} &\le \|e^{i\cdot\Delta}u_0\|_{L_T^q(\dot W_2^{s,r})}+c\|(-\Delta)^{\frac{s}{2}}\mathcal{N}(u)\|_{L_T^{q'}(L^{r',2})}\\
&\le \|e^{i\cdot\Delta}u_0\|_{L_T^q(\dot W_2^{s,r})}+c\|u\|^{2p-1}_{L_T^q(\dot W_2^{s,r})}\\
&\le \epsilon(T)+cR^{2p-1}.
\end{aligned}$$

Now, choosing $0<T<<1$, one obtains a contraction of $B_T(R)$. The existence of a local solution to the energy critical problem (1.1) follows with a standard Picard fixed point argument.

3.2 Small data theory

Since $s=s_c$, the above calculus give

$$\begin{cases} d(f(u),f(v))\le cR^{2(p-1)}d(u,v);\\ \|f(u)\|_{L_T^q(\dot W_2^{s,r})}\le c\|u_0\|_{\dot H^s}+cR^{2p-1}.\end{cases}$$

Thus, denoting by $\varepsilon:=\|u_0\|_{\dot H^s}$ and taking $R:=2c\varepsilon$, one has for $0<\varepsilon<(2c)^{-\frac{1}{2(p-1)}-1}$,

$$c\varepsilon+cR^{2p-1}<R \quad\text{and}\quad cR^{2(p-1)}<R.$$

Thus, with a standard fix point argument, there exists a global solution for (1.1).
Now, take $0<t,t'<\infty$ and using Strichartz estimate and Duhamel integral formula via previous calculus

$$\begin{aligned}
\|e^{it\Delta}u(t)-e^{it'\Delta}u(t')\|_{H^s} &\lesssim \|\mathcal{N}\|_{L^{q'}((t,t'),W_2^{s,r'})}\\
&\lesssim \|u\|^{2p-1}_{L^q((t,t'),W_2^{s,r})}.
\end{aligned}$$

Taking account of the above calculus with $f(u)=u$, via Lemma 2.7, one has

$$\|u\|_{L^q((t,t'),W_2^{s,r})}\lesssim \varepsilon+\|u\|^{2p-1}_{L^q((t,t'),W_2^{s,r})}\lesssim\varepsilon.$$

Thus, when $t \to \infty$,

$$e^{it\Delta}u(t) \to u_+, \quad \text{in} \quad H^s.$$

Then, when $t \to \infty$,

$$u(t) \to e^{-it\Delta}u_+, \quad \text{in} \quad H^s.$$

This finishes the proof.

3.3 Uniqueness of energy solutions

In this subsection, one proves the existence of at most one solution to (1.1) in the energy space. Take $T > 0$ and two solutions of (1.1), denoted by $u, v \in C_T(H^s)$ and $w := u - v$. Taking account of the integral formula and Strichartz estimates, one writes

$$\|w\|_{L^q_T(L^{r,2})} \lesssim \|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^{q_1'}_T(L^{r_1',2})},$$

where $(q_1, r_1) \in \Gamma$ to be fixed later and

$$(q, r) := \Big(\frac{4p}{(N-2s)(p-1) - \gamma + 2\rho}, \frac{2Np}{2s(p-1) + \gamma + N - 2\rho}\Big) \in \Gamma.$$

Arguing as previously, one writes

$$\begin{aligned}
\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^{r_1',2}} &\lesssim \|(J_\gamma * [|\cdot|^{-\rho}(|u|^p - |v|^p)])|x|^{-\rho}|u|^{p-1}\|_{L^{q_1'}_T(L^{r_1',2})} \\
&\quad + \|(J_\gamma * |\cdot|^{-\rho}|v|^p)|x|^{-\rho}(|u|^{p-2}u - |v|^{p-2}v)\|_{L^{q_1'}_T(L^{r_1',2})} \\
&\lesssim \|(J_\gamma * [|\cdot|^{-\rho}(|u|^{p-1} + |v|^{p-1})w])|x|^{-\rho}|u|^{p-1}\|_{L^{q_1'}_T(L^{r_1',2})} \\
&\quad + \|(J_\gamma * |\cdot|^{-\rho}|v|^p)|x|^{-\rho}(|u|^{p-2} + |v|^{p-2})w\|_{L^{q_1'}_T(L^{r_1',2})}.
\end{aligned}$$

So, it is sufficient to control the terms

$$\|\mathcal{N}(u, w)\|_{L^{q_1'}_T(L^{r_1',2})} \quad \text{and} \quad \|\mathcal{M}(u, w)\|_{L^{q_1'}_T(L^{r_1',2})}.$$

By Lemma 2.4, one writes

$$\begin{aligned}
\|\mathcal{N}(u, w)\|_{r_1',2} &\lesssim \||x|^{-\rho}u^p\|_{1/(\frac{1}{a_1}+\frac{\gamma}{N}), \frac{2(2p-1)}{p}} \||x|^{-\rho}u^{p-2}w\|_{a_2, \frac{2(2p-1)}{p-1}} \\
&\lesssim \||x|^{-\rho}\|^2_{\frac{N}{\rho},\infty}\|u^p\|_{1/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}), \frac{2(2p-1)}{p}} \|u^{p-2}w\|_{1/(\frac{1}{a_2}-\frac{\rho}{N}), \frac{2(2p-1)}{p-1}} \\
&\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}), 2(2p-1)}\|u^{p-2}w\|_{1/(\frac{1}{a_2}-\frac{\rho}{N}), \frac{2(2p-1)}{p-1}} \\
&\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}), 2(2p-1)}\|u^{p-2}\|_{1/(\frac{1}{a_2}-\frac{\rho}{N}-\frac{1}{r}), \frac{2(2p-1)}{p-2}} \|w\|_{r,2(2p-1)} \\
&\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}), 2(2p-1)}\|u\|^{p-2}_{(p-2)/(\frac{1}{a_2}-\frac{\rho}{N}-\frac{1}{r}), 2(2p-1)} \|w\|_{r,2(2p-1)} \\
&\lesssim \|u\|^p_{p/(\frac{1}{a_1}+\frac{\gamma-\rho}{N}), 2}\|u\|^{p-2}_{(p-2)/(\frac{1}{a_2}-\frac{\rho}{N}-\frac{1}{r}), 2} \|w\|_{r,2}.
\end{aligned}$$

Here, one assumes that

$$\begin{cases}
1 = \frac{1}{r_1} + \frac{1}{a_1} + \frac{1}{a_2}; \\
\frac{1}{2} - \frac{s}{N} \le \frac{1}{p}(\frac{1}{a_1} + \frac{\gamma-\rho}{N}), \ \frac{1}{p-2}(\frac{1}{a_2} - \frac{\rho}{N} - \frac{1}{r}) \le \frac{1}{2}; \\
2 \le r, \quad a_1 > \frac{N}{N-\gamma}, \quad a_2 > 1.
\end{cases}$$

Let us start with a_1, which satisfies

$$\begin{cases} p(\frac{1}{2}-\frac{s}{N})-\frac{\gamma-\rho}{N}\leq\frac{1}{a_1}\leq\frac{p}{2}-\frac{\gamma-\rho}{N}.; \\ \frac{1}{a_1}<\frac{N-\gamma}{N}. \end{cases}$$

This gives necessarily

$$p(\frac{1}{2}-\frac{s}{N})-\frac{\gamma-\rho}{N}<\frac{N-\gamma}{N}.$$

Thus,

$$p<1+\frac{N-2\rho+2s}{N-2s}.$$

This is satisfied if $p_s^c<1+\frac{N-2\rho+2s}{N-2s}$. This is true because $s\geq 1$ gives

$$2+\gamma<N+2s.$$

Since $r=\frac{2Np}{2s(p-1)+\gamma+N-2\rho}$, let us pick a_2 such that

$$\begin{cases} \frac{\rho}{N}+(p-2)(\frac{1}{2}-\frac{s}{N})+\frac{2s(p-1)+\gamma+N-2\rho}{2Np}\leq\frac{1}{a_2}\leq\frac{\rho}{N}+\frac{p-2}{2}+\frac{2s(p-1)+\gamma+N-2\rho}{2Np}; \\ a_2>1. \end{cases}$$

This is possible if

$$\frac{\rho}{N}+(p-2)(\frac{1}{2}-\frac{s}{N})+\frac{2s(p-1)+\gamma+N-2\rho}{2Np}<1.$$

This is equivalent to

$$Q(p):=(N-2s)p^2-2(2N-\rho-3s)p+N+\gamma-2s-2\rho<0.$$

The above polynomial function has two different roots because $N>2s$ and $Q(1)<0$. Thus, the previous estimate reads

$$Q(p_s^c)<0.$$

This is equivalent to

$$(N-2s)(\frac{2+\gamma-2\rho}{N-2s})^2-2(N-\rho-s)\frac{2+\gamma-2\rho}{N-2s}-2N+\gamma+2s<0.$$

This is equivalent to

$$(2+\gamma-2\rho)^2-2(N-\rho-s)(2+\gamma-2\rho)+(N-2s)(-2N+\gamma+2s)<0. \tag{3.12}$$

Finally, one needs to have an admissible pair

$$\frac{1}{2}\leq 1-\frac{1}{r_1}=\frac{1}{a_1}+\frac{1}{a_2}\leq\frac{p}{2}+\frac{\rho-\gamma}{N}+\frac{\rho}{N}+\frac{p-2}{2}+\frac{1}{r}.$$

This is satisfied because $2\rho-\gamma\geq 0$. Thus, with Sobolev injections and Hölder inequality

$$\|\mathcal{N}(u,w)\|_{L_T^{q_1'}(L^{r_1',2})}\lesssim T^{\frac{1}{q_1'}-\frac{1}{q}}\|u\|_{L_T^\infty(H^s)}^{2(p-1)}\|w\|_{L_T^q(L^{r,2})}.$$

Now, the inequality $q_1'<q$ reads

$$N(\frac{1}{2}-\frac{1}{r})=\frac{2}{q}<2-\frac{2}{q_1}=2-N(\frac{1}{2}-\frac{1}{r_1}).$$

Thus,

$$\frac{1}{r} > 1 - \frac{1}{r_1} - \frac{2}{N} = \frac{1}{a_1} + \frac{1}{a_2} - \frac{2}{N} > 2(p-1)(\frac{1}{2} - \frac{s}{N}) + \frac{2\rho - \gamma}{N} + \frac{1}{r} - \frac{2}{N}.$$

This is satisfied if $p \leq p_s^c$. Moreover, $\frac{1}{q_1'} - \frac{1}{q} \geq 0$ and $\frac{1}{q_1'} - \frac{1}{q} = 0$ if and only if $p = p_s^c$. Now, let $(q_2, r_2) \in \Gamma$ to be fixed later and the term

$$\|\mathcal{M}(u, w)\|_{L_T^{q_2'}(L^{r_2',2})}.$$

Using Lemma 2.4, one writes

$$\begin{aligned}
\|\mathcal{M}(u, w)\|_{r_2',2} &\lesssim \||x|^{-\rho} u^{p-1} w\|_{1/(\frac{1}{c_1} + \frac{\gamma}{N}), \frac{2(2p-1)}{p}} \||x|^{-\rho} u^{p-1}\|_{c_2, \frac{2(2p-1)}{p-1}} \\
&\lesssim \||x|^{-\rho}\|^2_{\frac{N}{\rho}, \infty} \|u^{p-1} w\|_{1/(\frac{1}{c_1} + \frac{\gamma-\rho}{N}), \frac{2(2p-1)}{p}} \|u^{p-1}\|_{1/(\frac{1}{c_2} - \frac{\rho}{N}), \frac{2(2p-1)}{p-1}} \\
&\lesssim \|u\|^{p-1}_{(p-1)/(\frac{1}{c_1} + \frac{\gamma-\rho}{N} - \frac{1}{r}), 2(2p-1)} \|w\|_{r, 2(2p-1)} \|u^{p-1}\|_{1/(\frac{1}{c_2} - \frac{\rho}{N}), \frac{2(2p-1)}{p-1}} \\
&\lesssim \|u\|^{p-1}_{(p-1)/(\frac{1}{c_1} + \frac{\gamma-\rho}{N} - \frac{1}{r}), 2(2p-1)} \|u\|_{1/(\frac{1}{c_2} - \frac{\rho}{N}), 2(2p-1)} \|w\|_{r, 2(2p-1)} \\
&\lesssim \|u\|^{p-1}_{(p-1)/(\frac{1}{c_1} + \frac{\gamma-\rho}{N} - \frac{1}{r}), 2} \|u\|_{1/(\frac{1}{c_2} - \frac{\rho}{N}), 2} \|w\|_{r, 2}.
\end{aligned}$$

Here, one assumes that

$$\begin{cases}
1 = \frac{1}{r_2} + \frac{1}{c_1} + \frac{1}{c_2}; \\
\frac{1}{2} - \frac{s}{N} \leq \frac{1}{p-1}(\frac{1}{c_1} + \frac{\gamma-\rho}{N} - \frac{1}{r}), \ \frac{1}{p-1}(\frac{1}{c_2} - \frac{\rho}{N}) \leq \frac{1}{2}; \\
r_2 \geq 2, \ c_1 > \frac{N}{N-\gamma}, \ c_2 > 1.
\end{cases}$$

Take the real number c_2 satisfying

$$(p-1)(\frac{1}{2} - \frac{s}{N}) + \frac{\rho}{N} \leq \frac{1}{c_2} < \min\{\frac{p-1}{2} + \frac{\rho}{N}, 1\}.$$

This implies that

$$(p-1)(\frac{1}{2} - \frac{s}{N}) + \frac{\rho}{N} < 1.$$

This is satisfied because $p \leq p_s^c < 1 + \frac{2(N-\rho)}{N-2s}$, under the assumption $2 < 2 + \gamma < 2N$, which holds because $2 \leq N \leq 3$ and $\gamma < N$. Now, since $r = \frac{2Np}{2s(p-1)+\gamma+N-2\rho}$, let us pick c_1 such that

$$\begin{cases}
\frac{\rho-\gamma}{N} + (p-1)(\frac{1}{2} - \frac{s}{N}) + \frac{2s(p-1)+\gamma+N-2\rho}{2Np} \leq \frac{1}{c_1} \leq \frac{\rho-\gamma}{N} + \frac{p-1}{2} + \frac{2s(p-1)+\gamma+N-2\rho}{2Np}; \\
c_1 > \frac{N}{N-\gamma}.
\end{cases}$$

This is possible if

$$\frac{\rho - \gamma}{N} + (p-1)(\frac{1}{2} - \frac{s}{N}) + \frac{2s(p-1) + \gamma + N - 2\rho}{2Np} < \frac{N - \gamma}{N}.$$

The above estimate reads

$$\frac{\rho}{N} + (p-1)(\frac{1}{2} - \frac{s}{N}) + \frac{2s(p-1) + \gamma + N - 2\rho}{2Np} < 1.$$

This equation is weaker than (3.12) and so is satisfied. Finally, let us check that

$$\begin{cases}
\frac{1}{2} \leq 1 - \frac{1}{r_2} = \frac{1}{c_1} + \frac{1}{c_2} \leq \frac{p-1}{2} + \frac{\rho-\gamma}{N} + \frac{1}{r} + \frac{\rho}{N} + \frac{p-1}{2}; \\
\qquad \leq \frac{N-\gamma}{N} + 1; \\
\qquad \leq \frac{N-\gamma}{N} + \frac{p-1}{2} + \frac{\rho}{N}; \\
\qquad \leq \frac{p-1}{2} + \frac{\rho-\gamma}{N} + \frac{1}{r} + 1.
\end{cases}$$

This is satisfied because $2\rho-\gamma\geq 0$. Thus,

$$\|\mathcal{M}(u,w)\|_{L_T^{q_2'}(L^{r_2',2})}\lesssim T^{\frac{1}{q_2'}-\frac{1}{q}}\|u\|^{2(p-1)}_{L_T^\infty(H^s)}\|w\|_{L_T^q(L^{r,2})}.$$

Now, let us check that $q_2'\leq q$, which reads

$$N(\frac{1}{2}-\frac{1}{r})=\frac{2}{q}\leq 2-\frac{2}{q_2}=2-N(\frac{1}{2}-\frac{1}{r_2}).$$

Thus,

$$\frac{1}{r}\geq 1-\frac{1}{r_2}-\frac{2}{N}=\frac{1}{c_1}+\frac{1}{c_2}-\frac{2}{N}\geq 2(p-1)(\frac{1}{2}-\frac{s}{N})+\frac{2\rho-\gamma}{N}+\frac{1}{r}-\frac{2}{N}.$$

This is satisfied under the above assumption and gives the existence of $c>0$ such that

$$\|\mathcal{M}(u,w)\|_{L_T^{q_1'}(L^{r_1',2})}\lesssim T^{c\delta_p^{p_s^c}}\|u\|^{2(p-1)}_{L_T^\infty(H^s)}\|w\|_{L_T^q(L^{r,2})}.$$

Finally, with Strichartz estimates

$$\|w\|_{L_T^q(L^{r,2})}\lesssim T^{c\delta_p^{p_s^c}}\|u\|^{2(p-1)}_{L_T^\infty(H^s)}\|w\|_{L_T^q(L^{r,2})}.$$

• Uniqueness in the energy sub-critical regime $2\leq p<p_s^c$.
The uniqueness follows with the above estimate and a translation argument.
• Uniqueness in the energy critical regime $p=p_s^c$.
In such a case the above calculus is no more sufficient to conclude. Let us denote for $T, A>0$, the interval $I:=[0,T]$ and the quantities

$$\mathcal{N}^A:=(\mathcal{N}^{0,\rho}(u)-\mathcal{N}^{0,\rho}(v))\chi_{\{|u|+|v|>A\}};$$
$$\mathcal{N}_A:=(\mathcal{N}^{0,\rho}(u)-\mathcal{N}^{0,\rho}(v))\chi_{\{|u|+|v|\leq A\}}.$$

Then, for $w:=u-v$,

$$\begin{aligned}
|\mathcal{N}_A| &\leq |(J_\gamma*[|\cdot|^{-\rho}(|u|^p-|v|^p)\chi_{\{|u|+|v|\leq A\}}])|u|^{p-1}|\chi_{\{|u|+|v|\leq A\}}\\
&+ |(J_\gamma*[|\cdot|^{-\rho}(|u|^p-|v|^p)\chi_{\{|u|+|v|\geq A\}}])|u|^{p-1}|\chi_{\{|u|+|v|\leq A\}}\\
&+ |(J_\gamma*[|\cdot|^{-\rho}|v|^p\chi_{\{|u|+|v|\leq A\}}])(|u|^{p-2}u-|v|^{p-2}v)|\chi_{\{|u|+|v|\leq A\}}\\
&+ |(J_\gamma*[|\cdot|^{-\rho}|v|^p\chi_{\{|u|+|v|\geq A\}}])(|u|^{p-2}u-|v|^{p-2}v)|\chi_{\{|u|+|v|\leq A\}}\\
&\lesssim A^{2(p-1)}(J_\gamma*[|\cdot|^{-\rho}|w|])\\
&+ A^{p-2}|(J_\gamma*[|\cdot|^{-\rho}|v|^p\chi_{\{|u|+|v|\leq A\}}])|w|\chi_{\{|u|+|v|\leq A\}}\\
&+ |(J_\gamma*[|\cdot|^{-\rho}(|u|^{p-1}+|v|^{p-1})|w|\chi_{\{|u|+|v|\geq A\}}])|u|^{p-1}|\chi_{\{|u|+|v|\leq A\}}\\
&+ |(J_\gamma*[|\cdot|^{-\rho}|v|^p\chi_{\{|u|+|v|\geq A\}}])(|u|^{p-2}+|v|^{p-2})w|\chi_{\{|u|+|v|\leq A\}}\\
&:=\mathcal{N}_A^1+\mathcal{N}_A^2+\mathcal{N}_A^3+\mathcal{N}_A^4.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\mathcal{N}^A| &\leq |(J_\gamma*[|\cdot|^{-\rho}(|u|^p-|v|^p)])|u|^{p-1}|\chi_{\{|u|+|v|>A\}}\\
&+ |(J_\gamma*[|\cdot|^{-\rho}|v|^p])(|u|^{p-2}u-|v|^{p-2}v)|\chi_{\{|u|+|v|>A\}}\\
&\lesssim \Big((J_\gamma*[|\cdot|^{-\rho}(|u|^{p-1}+|v|^{p-1})|w|])|u|^{p-1}+(J_\gamma*[|\cdot|^{-\rho}|v|^p])(|u|^{p-2}+|v|^{p-2})|w|\Big)\chi_{\{|u|+|v|>A\}}.
\end{aligned}$$

Take two admissible pairs $(q_i,r_i)\in\Gamma$ and $t>0$ such that

$$\max\{\frac{N-2}{2N},\frac{N+2(\gamma-2\rho)}{2N}\}\leq\frac{1}{r_1}\leq\min\{\frac{1}{2},\frac{2+N+2(\gamma-2\rho)}{2N},\frac{N-\rho}{N}\}.$$

This is possible because

$$-2 \le \gamma - 2\rho \le 0 \quad \text{gives} \quad 2\rho < 2 + N.$$

Let us pick

$$\begin{cases} 1 = \frac{\rho}{N} + \frac{1}{r_2} + \frac{1}{r_3}; \\ 1 = \frac{\gamma-2\rho}{N} + \frac{1}{r_2} + \frac{1}{r_1}. \end{cases}$$

Applying Strichartz estimates, one writes

$$\begin{aligned} \|w\|_{L_t^2(L^{\frac{2N}{N-2},2}) \cap L_t^{q_1}(L^{r_1,2})} &\lesssim \||x|^{-\rho}(\mathcal{N}_A^1 + \mathcal{N}_A^2)\|_{L_t^{q_2'}(L^{r_2',2})} + \||x|^{-\rho}(\mathcal{N}^A + \mathcal{N}_A^3 + \mathcal{N}_A^4)\|_{L_t^2(L^{\frac{2N}{N+2},2})} \\ &\lesssim \|\mathcal{N}_A^1 + \mathcal{N}_A^2\|_{L_t^{q_2'}(L^{r_3,2})} + \||x|^{-\rho}(\mathcal{N}^A + \mathcal{N}_A^3 + \mathcal{N}_A^4)\|_{L_t^2(L^{\frac{2N}{N+2},2})}. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\mathcal{N}_A^1 + \mathcal{N}_A^2\|_{L_t^{q_2'}(L^{r_3,2})} &\lesssim A^{2(p-1)}\|(J_\gamma * [|\cdot|^{-\rho}|w|])\chi_{\{|u|+|v|\le A\}}\|_{L_t^{q_2'}(L^{r_3,2})} \\ &\quad + A^{p-2}\|(J_\gamma * [|\cdot|^{-\rho}|v|^p\chi_{\{|u|+|v|\le A\}}])|w|\chi_{\{|u|+|v|\le A\}}\|_{L_t^{q_2'}(L^{r_3,2})}. \end{aligned} \tag{3.13}$$

Using Lemma 2.4, one has

$$\begin{aligned} \|(J_\gamma * [|\cdot|^{-\rho}|w|])\chi_{\{|u|+|v|\le A\}}\|_{r_3,2} &\lesssim \||\cdot|^{-\rho}|w|\|_{1/(\frac{1}{r_3}+\frac{\gamma}{N}),2} \\ &\lesssim \|w\|_{1/(\frac{1}{r_3}+\frac{\gamma-\rho}{N}),2} \\ &\lesssim \|w\|_{r_1,2}. \end{aligned} \tag{3.14}$$

Here,

$$\begin{cases} \frac{1}{r_1} = \frac{\gamma-\rho}{N} + \frac{1}{r_3} = \frac{\gamma-2\rho}{N} + 1 - \frac{1}{r_2}; \\ \frac{\rho-\gamma}{N} < \frac{1}{r_3} = \frac{1}{r_1} - \frac{\gamma-\rho}{N} < \frac{N-\gamma}{N}. \end{cases}$$

By the above assumptions, one gets $2 \le r_2 \le \frac{2N}{N-2}$. Moreover, with a similar way, using Lemma 2.4, one has

$$\begin{aligned} \|(J_\gamma * [|\cdot|^{-\rho}|v|^p])|w|\chi_{\{|u|+|v|\le A\}}\|_{L^{r_3,2}} &\lesssim \|(J_\gamma * |\cdot|^{-\rho})|w|\|_{L^{r_3,2}} \\ &\lesssim \|w\|_{1/(\frac{1}{r_3}+\frac{\gamma-\rho}{N}),2} \\ &\lesssim \|w\|_{r_1,2}. \end{aligned} \tag{3.15}$$

Thus, by (3.13), (3.14) and (3.15), one gets

$$\|\mathcal{N}_A^1 + \mathcal{N}_A^2\|_{L_t^{q_2'}(L^{r_3,2})} \lesssim (A^{2(p-1)} + A^{p-2})\|w\|_{L_t^{q_2'}(L^{r_1,2})}.$$

With triangular inequality, one has

$$\begin{aligned} &\||x|^{-\rho}(\mathcal{N}^A + \mathcal{N}_A^3 + \mathcal{N}_A^4)\|_{L^{\frac{2N}{2+N},2}} \\ \lesssim\ &\||x|^{-\rho}(J_\gamma * [|\cdot|^{-\rho}(|u|^{p-1} + |v|^{p-1})|w|])|u|^{p-1}\chi_{\{|u|+|v|>A\}}\|_{L^{\frac{2N}{2+N},2}} \\ +\ &\||x|^{-\rho}(J_\gamma * [|\cdot|^{-\rho}|v|^p])(|u|^{p-2} + |v|^{p-2})|w|\chi_{\{|u|+|v|>A\}}\|_{L^{\frac{2N}{2+N},2}} \\ +\ &\||x|^{-\rho}(J_\gamma * [|\cdot|^{-\rho}(|u|^{p-1} + |v|^{p-1})|w|\chi_{\{|u|+|v|\ge A\}}])|u|^{p-1}|\chi_{\{|u|+|v|\le A\}}\|_{L^{\frac{2N}{2+N},2}} \\ +\ &\||x|^{-\rho}(J_\gamma * [|\cdot|^{-\rho}|v|^p\chi_{\{|u|+|v|\ge A\}}])(|u|^{p-2} + |v|^{p-2})w|\chi_{\{|u|+|v|\le A\}}\|_{L^{\frac{2N}{2+N},2}} \\ :=\ &(I) + (II) + (III) + (IV). \end{aligned}$$

Now, one writes by Lemma 2.4, since $p=p_s^c$ and $2-N<\rho<2$,

$$\begin{aligned}(II)&\lesssim \|(J_\gamma*[|\cdot|^{-\rho}|v|^p])(|u|^{p-2}+|v|^{p-2})\chi_{\{|u|+|v|>A\}}\|_{\frac{N}{2-\rho},2}\|w\|_{L^{\frac{2N}{-2+N},2}}\\&\lesssim \||x|^{-\rho}|v|^p\|_{1/(\frac{2+\gamma-\rho}{N}-\frac{(p-2)(N-2s)}{2N}),2}(\|u\chi_{\{|u|+|v|>A\}}\|^{p-2}_{\frac{2N}{N-2s},\infty}+\|v\chi_{\{|u|+|v|>A\}}\|^{p-2}_{\frac{2N}{N-2s},\infty})\|w\|_{L^{\frac{2N}{-2+N},2}}\\&\lesssim \|v\|^p_{\frac{2N}{N-2s},2}\Big(\|u\chi_{\{|u|+|v|>A\}}\|_{\frac{2N}{N-2s},\infty}+\|v\chi_{\{|u|+|v|>A\}}\|_{\frac{2N}{N-2s},\infty}\Big)^{p-2}\|w\|_{L^{\frac{2N}{N-2}},2}.\end{aligned}$$

Moreover, similarly

$$\begin{aligned}(I)&=\||x|^{-\rho}(J_\gamma*[|\cdot|^{-\rho}(|u|^{p-1}+|v|^{p-1})|w|])|u|^{p-1}\chi_{\{|u|+|v|>A\}}\|_{L^{\frac{2N}{2+N},2}}\\&\lesssim \|(J_\gamma*[|\cdot|^{-\rho}(|u|^{p-1}+|v|^{p-1})|w|])|u|^{p-1}\chi_{\{|u|+|v|>A\}}\|_{L^{\frac{2N}{2+N-2\rho},2}}\\&\lesssim \|(J_\gamma*[|\cdot|^{-\rho}(|u|^{p-1}+|v|^{p-1})|w|])\|_{\frac{2N}{N-\gamma},2}\|u\chi_{\{|u|+|v|>A\}}\|^{p-1}_{L^{\frac{2N}{N-2s},2}}\\&\lesssim (\|u\|_{\frac{2N}{N-2s},2}+\|v\|_{\frac{2N}{N-2s},2})^{p-1}\|w\|_{\frac{2N}{N-2},2}\|u\chi_{\{|u|+|v|>A\}}\|^{p-1}_{L^{\frac{2N}{N-2s},\infty}}.\end{aligned}$$

Moreover, similarly

$$\begin{aligned}(III)&=\||x|^{-\rho}(J_\gamma*[|\cdot|^{-\rho}(|u|^{p-1}+|v|^{p-1})|w|\chi_{\{|u|+|v|\ge A\}}])|u|^{p-1}\|_{L^{\frac{2N}{2+N},2}}\\&\lesssim \|(J_\gamma*[|\cdot|^{-\rho}(|u|^{p-1}+|v|^{p-1})|w|\chi_{\{|u|+|v|\ge A\}}])|u|^{p-1}\|_{L^{\frac{2N}{2+N-2\rho},2}}\\&\lesssim \|(J_\gamma*[|\cdot|^{-\rho}(|u|^{p-1}+|v|^{p-1})|w|\chi_{\{|u|+|v|\ge A\}}])\|_{L^{\frac{2N}{N-\gamma},2}}\|u\|^{p-1}_{L^{\frac{2N}{N-2s},2}}\\&\lesssim (\|u\chi_{\{|u|+|v|>A\}}\|_{\frac{2N}{N-2s},\infty}+\|v\chi_{\{|u|+|v|>A\}}\|_{\frac{2N}{N-2s},\infty})^{p-1}\|w\|_{\frac{2N}{N-2},2}\|u\|^{p-1}_{L^{\frac{2N}{N-2s},2}}.\end{aligned}$$

With the same way

$$\begin{aligned}(IV)&\le \||x|^{-\rho}(J_\gamma*[|\cdot|^{-\rho}|v|^p\chi_{\{|u|+|v|\ge A\}}])(|u|^{p-2}+|v|^{p-2})w\|_{L^{\frac{2N}{2+N},2}}\\&\lesssim \|v\chi_{\{|u|+|v|>A\}}\|^p_{\frac{2N}{N-2s},\infty}\|w\|_{\frac{2N}{N-2},2}(\|u\|^{p-2}_{L^{\frac{2N}{N-2s},2}}+\|v\|^{p-2}_{L^{\frac{2N}{N-2s},2}}).\end{aligned}$$

Regrouping the above calculus, one gets

$$\begin{aligned}&\|w\|_{L_t^2(L^{\frac{2N}{N-2},2})\cap L_t^{q_1}(L^{r_1,2})}\\&\lesssim \||x|^{-\rho}(\mathcal{N}_A^1+\mathcal{N}_A^2)\|_{L_t^{q_2'}(L^{r_2',2})}+\||x|^{-\rho}(\mathcal{N}^A+\mathcal{N}_A^3+\mathcal{N}_A^4)\|_{L_t^2(L^{\frac{2N}{N+2},2})}\\&\lesssim (A^{2(p-1)}+A^{p-2})\|w\|_{L_t^{q_2'}(L^{r_1,2})}+\|(I)+(II)+(III)+(IV)\|_{L_t^2}\\&\lesssim (A^{2(p-1)}+A^{p-2})\|w\|_{L_t^{q_2'}(L^{r_1,2})}\\&+\|v\|^p_{L_t^\infty(L^{\frac{2N}{N-2s},2})}\Big(\|u\chi_{\{|u|+|v|>A\}}\|_{L_t^\infty(L^{\frac{2N}{N-2s},\infty})}+\|v\chi_{\{|u|+|v|>A\}}\|_{L_t^\infty(L^{\frac{2N}{N-2s},\infty})}\Big)^{p-2}\|w\|_{L_t^2(L^{\frac{2N}{N-2},2})}\\&+\Big(\|u\|_{L_t^\infty(L^{\frac{2N}{N-2s},2})}+\|v\|_{L_t^\infty(L^{\frac{2N}{N-2s},2})}\Big)^{p-1}\|w\|_{L_t^2(L^{\frac{2N}{N-2},2})}\|u\chi_{\{|u|+|v|>A\}}\|^{p-1}_{L_t^\infty(L^{\frac{2N}{N-2s},\infty})}\\&+\Big(\|u\chi_{\{|u|+|v|>A\}}\|_{L_t^\infty(L^{\frac{2N}{N-2s},\infty})}+\|v\chi_{\{|u|+|v|>A\}}\|_{L_t^\infty(L^{\frac{2N}{N-2s},\infty})}\Big)^{p-1}\|w\|_{L_t^2(L^{\frac{2N}{N-2},2})}\|u\|^{p-1}_{L_t^\infty(L^{\frac{2N}{N-2s},2})}\\&+\|v\chi_{\{|u|+|v|>A\}}\|^p_{L_t^\infty(L^{\frac{2N}{N-2s},\infty})}\|w\|_{L_t^2(L^{\frac{2N}{N-2},2})}\Big(\|u\|^{p-2}_{L_t^\infty(L^{\frac{2N}{N-2s},2})}+\|v\|^{p-2}_{L_t^\infty(L^{\frac{2N}{N-2s},2})}\Big).\end{aligned}$$

By dominated convergence Theorem, via Sobolev injections and a continuity argument and arguing as in [7, (4.2.13) page 87], one writes

$$\|u\chi_{\{|u|+|v|>A\}}\|_{L_t^\infty(L^{\frac{2N}{N-2s},\infty})}+\|v\chi_{\{|u|+|v|>A\}}\|_{L_t^\infty(L^{\frac{2N}{N-2s},\infty})}$$

$$\begin{aligned}
&\lesssim \|u\chi_{\{|u|+|v|>A\}}\|_{L_t^\infty(L^{\frac{2N}{N-2s}})} + \|v\chi_{\{|u|+|v|>A\}}\|_{L_t^\infty(L^{\frac{2N}{N-2s}})} \\
&\to 0, \quad \text{as} \quad A\to\infty. \qquad (3.16)
\end{aligned}$$

Thus, by Sobolev injections, for $A >> 1$, there is $0 < \epsilon(A) << 1$, such that

$$\|w\|_{L_t^2(L^{\frac{2N}{N-2},2})\cap L_t^{q_1}(L^{r_1,2})} \lesssim (A^{2(p-1)} + A^{p-2})\|w\|_{L_t^{q_2'}(L^{r_1,2})} + \epsilon(A)\|w\|_{L_t^2(L^{\frac{2N}{N-2},2})}.$$

So, for some $C > 0$,

$$\begin{aligned}
\|w\|_{L_t^{q_1}(L^{r_1,2})} &\lesssim \frac{A^{2(p-1)} + A^{p-2}}{1-\epsilon(A)}\|w\|_{L_t^{q_2'}(L^{r_1,2})} \\
&\le C\frac{A^{2(p-1)} + A^{p-2}}{1-\epsilon(A)} t^{\frac{1}{q_2'}-\frac{1}{q_1}}\|w\|_{L_t^{q_1}(L^{r_1,2})}. \qquad (3.17)
\end{aligned}$$

Since $\frac{1}{r_1} = \frac{\gamma-2\rho}{N} + \frac{1}{r_2'}$, one has

$$\begin{aligned}
\frac{2}{q_1} &= N(\frac{1}{2} - \frac{1}{r_1}) \\
&= N(\frac{1}{2} + \frac{\gamma-2\rho}{N} - \frac{1}{r_2'}) \\
&= \frac{N}{2} - 2\rho + \gamma - N(1 - \frac{1}{r_2}) \\
&= -\frac{N}{2} - 2\rho + \gamma + \frac{N}{r_2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{2}{q_1} &= -2\rho + \gamma - N(\frac{1}{2} - \frac{1}{r_2}) \\
&= -2\rho + \gamma - \frac{2}{q_2} \\
&= -2\rho + \gamma - 2(1 - \frac{1}{q_2'}) \\
&= -2\rho + \gamma - 2 + \frac{2}{q_2'}.
\end{aligned}$$

Since $2\rho > \gamma$, one has $\frac{1}{q_2'} - \frac{1}{q_1} > 0$. Now, we fix $A >> 1$ and then we take $0 < t << 1$ such $C\frac{A^{2(p-1)}+A^{p-2}}{1-\epsilon(A)}t^{\frac{1}{q_2'}-\frac{1}{q_1}} < 1$. So, by (3.17), the uniqueness follows for small time and so for all time with a classical time translation.

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Declarations

Competing interests On behalf of all authors, the corresponding author states that there is no conflict of interest. No data-sets were generated or analyzed during the current study.

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