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ORIGINAL RESEARCH

On $L(2, 1)$-labeling of zero-divisor graphs of finite commutative rings

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Abstract For a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, an $L(2, 1)$-labeling is an assignment of non-negative integer labels to vertices of $\mathcal{G}$. An $L(2, 1)$-labeling of $\mathcal{G}$ must satisfy two conditions: adjacent vertices in $\mathcal{G}$ should get labels which differ by at least two, and vertices at a distance of two from each other should get distinct labels. The λ-number of $\mathcal{G}$, denoted by $\lambda(\mathcal{G})$, represents the smallest positive integer ℓ for which an $L(2, 1)$-labeling exists, the vertices of $\mathcal{G}$ are provided labels from the set $\{0, 1, \ldots, \ell\}$. Let $\Gamma(R)$ be a zero-divisor graph of a finite commutative ring R with unity. In $\Gamma(R)$, vertices represent zero-divisors of R, and two vertices x and y are adjacent if and only if $xy = 0$ in R. The methodology of the research involves a detailed investigation into the structural aspects of zero-divisor graphs associated with specific classes of local and mixed rings, such as $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$, and $\mathbb{F}_q \times \mathbb{Z}_{p^n}$. This exploration leads us to compute the exact value of $L(2, 1)$-labeling number of these graphs.

Keywords Zero-divisor graph · $L(2, 1)$-labeling · λ-number · Partite truncation

Mathematics Subject Classification 05C15 · 05C25 · 13A99

1 Introduction

An $L(j, k)$-labeling of a finite simple undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where j and k are positive integers, is a function $f : \mathcal{V} \longrightarrow \mathbb{Z}_{\geq 0}$ such that the absolute difference between the labels of any two adjacent vertices must be at least j, and if two vertices are at a distance of two apart, the absolute difference between their labels must be at least k. The span of an $L(j, k)$-labeling f is defined as the difference between the maximum and minimum values of f, denoted by $span(f)$. Without loss of generality, we assume that the minimum value of any $L(j, k)$-labeling is 0 and the span its maximum value. The $L(j, k)$-labeling number of a graph $\mathcal{G}$, denoted by $\lambda_{j,k}(\mathcal{G})$, is the minimum span among all $L(j, k)$-labelings of $\mathcal{G}$. A minimal labeling refers to an $L(j, k)$-labeling f of $\mathcal{G}$ where $span(f) = \lambda_{j,k}(\mathcal{G})$. In particular, when $j = 2$ and $k = 1$, the $L(2, 1)$-labeling number is referred to as the λ-number of a graph, denoted by $\lambda(\mathcal{G})$.

The primary purpose of defining graphs on algebraic structures is to explore the relationship between algebraic invariants of the structure and combinatorial invariants of the associated graph. Various types of graphs, including Cayley graphs [7], power graphs [13], zero-divisor graphs [4], group-annihilator graphs [15], and torsion graphs [22], arise from algebraic structures such as groups, rings, and modules. For a commutative ring R with unity,

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Beck [5] introduced the concept of zero-divisor graphs associated with R. This graph is, denoted by $\Gamma'(R)$, and is called the Beck's zero-divisor graph, elements of R serve as the vertices of $\Gamma'(R)$, two distinct vertices u and v are adjacent if and only if $uv = 0$. Anderson and Livingston [4] modified Beck's zero-divisor graph by considering only non-trivial zero-divisors as vertices. They introduced the zero-divisor graph, denoted by $\Gamma(R)$, where two vertices are adjacent if and only if their product is zero [4]. For more information on zero-divisor graphs, see [1,3,16–19,21].

A ring R is called a *local* ring if it has a unique maximal ideal. The decomposition $R = R_1 \times \cdots \times R_r \times F_1 \times \cdots \times F_s$ represents the Artinian decomposition of R, where each R_i is a commutative local ring with unity, and each F_j is a field. A *mixed* ring is characterized by either $r \geq 1$ and $s \geq 1$ or $r \geq 2$ and $s = 0$ in its Artinian decomposition. A *reduced* ring is defined as a ring with $r = 0$ and $s \geq 1$ in its Artinian decomposition.

The need for efficiently assigning radio channels to transmitters without interference drives the study of the $L(j,k)$-labeling problem for graphs. Roberts [20] proposed the challenge of assigning non-negative integer channels to transmitters at different sites, ensuring that adjacent locations have distinct channels and nearby locations have channels that differ by at least two. The concept of $L(2,1)$-labeling has been studied extensively in the literature with numerous investigations exploring the algorithmic perspective of the $L(2,1)$-labeling problem [6,12]. The problem has been conclusively established as NP-hard for general graphs [10] and continues to exhibit NP-hardness for specific classes of graphs, including planar graphs, bipartite graphs, and chordal graphs [6]. A linear time algorithm for $L(2,1)$-labeling of trees has been given in [11]. Furthermore, $L(2,1)$-labeling has been examined in the context of graphs arising from algebraic structures, such as power graphs [14], Cayley graphs [23], and zero-divisor graphs of reduced rings [2]. In the present study, we delve deeper into the investigation of $L(2,1)$-labeling for zero-divisor graphs of local and mixed rings.

The structure of this research article is as follows: Section 2 provides essential definitions and preliminary results concerning $L(2,1)$-labeling of graphs. In Section 3, we compute the λ-numbers of zero-divisor graphs linked to specific types of rings. These include local rings such as $\mathbb{Z}_{p^n}$ (p is a prime number and n a positive integer), as well as mixed rings of the form $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ (q is a prime number and m a positive integer), and $\mathbb{F}_q \times \mathbb{Z}_{p^n}$, where $\mathbb{F}_q$ denotes a finite field of order q.

2 Preliminaries

In this section, we present several findings concerning $L(2,1)$-labeling of specific classes of graphs.

The concept of holes, multiplicities, and gaps in an $L(2,1)$-labeling was introduced by the authors in [9]. Consider an $L(2,1)$-labeling f of a graph $\mathcal{G}$. For a positive integer h satisfying $0 < h < \text{span}(f)$, let $f_h = \{v \in V(\mathcal{G}) : f(v) = h\}$. If f_h is empty, we refer to h as a "hole" in f. On the other hand, if $|f_h| \geq 2$, we term h a "multiplicity" in f. Additionally, h is classified as a "gap" if it represents a hole and satisfies $|f_{h-1}| = |f_{h+1}| = 1$, along with the condition that $(v^{h-1}, v^{h+1}) \in E(\mathcal{G})$, where $f(v^{h-1}) = h - 1$ and $f(v^{h+1}) = h + 1$. We denote the collections of holes, multiplicities, and gaps in f as $H(f)$, $M(f)$, and $G(f)$, respectively. Moreover, let $h(f)$ and $g(f)$ denote the cardinalities of $H(f)$ and $G(f)$.

The function f is referred to as the "minimum $L(2,1)$-labeling" of graph $\mathcal{G}$ if it has the minimum number of holes among the set $\Lambda(\mathcal{G})$, where $\Lambda(\mathcal{G})$ denotes the collection of all minimal $L(2,1)$-labelings of $\mathcal{G}$. It is important to note that while a minimal $L(2,1)$-labeling of a graph $\mathcal{G}$ may not be unique, the minimum $L(2,1)$-labeling of a graph $\mathcal{G}$ is unique up to some permutation of the labels provided by it.

A p-group is defined as a group in which the order of every element is a power of a prime number p. In other words, a finite group is classified as a p-group if and only if its order is a power of a prime p. The power graph, denoted by Γ_G, of a finite group G was introduced in [8,13]. It is a simple graph with vertices as elements of G. Two vertices in Γ_G are adjacent if and only if one vertex is a power of the other. The computation of an exact λ-number for power graphs of dihedral group D_{2n}, p-groups, and the generalized quaternion group Q_{4n} was conducted in [14]. For certain classes of groups of order n, an upper bound related to the λ-number of a power graph was established.

The following result discussed in [14], provides some insights of the λ-number in power graphs.

Theorem 1 *([14], Theorem 4.1) If G is a group of order n, where n is not a prime power, then $\lambda(\Gamma_G) \leq 2n - 4$. This upper bound is achieved if and only if G is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_{2q}$, where q is a prime greater than* 2.

In addition, the $L(j,k)$-labeling of Cayley graphs has been investigated in [23].

The operation of partite truncation in graphs was introduced and studied by the authors in [2]. They investigated shift in the λ-number of a graph resulting from partite truncation, and determined the λ-number for zero-divisor graphs associated with certain classes of reduced rings. The authors proved the following significant result for graphs with diameter less or equal to two.

Proposition 2 *([2], Proposition 10) Let $\mathcal{G}$ be a graph with $diam(\mathcal{G}) < 3$, and let the minimum labeling of $\mathcal{G}$ have s many holes. If $\mathcal{G}_{m+1}$ is the graph obtained from $\mathcal{G}$ when m isolated vertices, and subsequently a dominant vertex are added to it, then the following holds,*

$$\lambda(\mathcal{G}_{m+1}) = \begin{cases} \lambda(\mathcal{G}) + m - s + 2, & m > s, \\ \lambda(\mathcal{G}) + 2, & m \leq s. \end{cases}$$

Proof Let $u_1, u_2, \ldots, u_m$ denote the isolated vertices, and u_0 be the dominant vertex. Also, let f be the minimum labeling of $\mathcal{G}$ with holes $h_1, h_2, \ldots, h_s$. We define $L(2, 1)$-labeling g on $\mathcal{G}_{m+1}$ as, $g(u_0) = 0,\ g(v) = f(v) + 2,$ for all $v \in V(\mathcal{G})$ and $g(u_i) = h_i + 2,$ for all $i,\ 1 \leq i \leq s$. For $m > s,\ g(u_{s+j}) = \lambda(\mathcal{G}) + j + 2,$ for all $j,\ 1 \leq j \leq m - s$.

Therefore,

$$span(g) = \begin{cases} \lambda(\mathcal{G}) + m - s + 2, & m > s, \\ \lambda(\mathcal{G}) + 2, & m \leq s. \end{cases}$$

Since $diam(\mathcal{G}) < 3$, therefore f must be one-to-one. This implies, g is well defined $L(2, 1)$-labeling of $\mathcal{G}_{m+1}$. Thus, $\lambda(\mathcal{G}_{m+1}) \leq span(g)$. If we assume that there exist another $L(2, 1)$-labeling h of $\mathcal{G}_{m+1}$ with $span(h) < span(g)$, then the restriction $h \restriction_{V(\mathcal{G})}$ is a well defined $L(2, 1)$-labeling of $\mathcal{G}$ with $span(h \restriction_{V(\mathcal{G})}) < span(f)$, a contradiction.

3 λ-number of Local and Mixed Rings

This section is devoted to study λ-numbers of zero-divisor graphs associated with some class of local and mixed rings. We provide an algorithm related to a minimal $L(2, 1)$-labeling of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^n})$, and determine its λ-number.

Theorem 3 *For any prime p, the λ-number of $\Gamma(\mathbb{Z}_{p^n})$ is the following,*

$$\lambda(\Gamma(\mathbb{Z}_{p^n})) = \begin{cases} p^{n-1} + p - 3, & \text{if } n \geq 3, \\ 2p - 4, & \text{if } n = 2. \end{cases}$$

Proof The set of non-trivial zero-divisors of $\mathbb{Z}_{p^n}$ and, consequently, the vertices of $\Gamma(\mathbb{Z}_{p^n})$ are given as,

$$Z^*(\mathbb{Z}_{p^n}) = \left\{\alpha_{i,l}\, p^i : \gcd(\alpha_{i,l}, p) = 1,\ 1 \leq l \leq \phi(p^{n-i}),\ \text{and } 1 \leq i \leq n - 1\right\},$$

where ϕ is Euler's totient function. For each i, denote the set of vertices corresponding to zero-divisors $\alpha_{i,l}\, p^i$ by V^i. Note that, $\alpha_{i,l}\, p^i.\alpha_{j,l}\, p^j = 0$ if and only if $i + j \geq n$. Now, we consider the following cases.

Case-I: $n \geq 3$. Consider the case when n is even.

Subcase - I: $n = 2k$. The set of vertices in $\bigcup_{i=k}^{n-1} V^i$ induce a clique in $\Gamma(\mathbb{Z}_{p^n})$ of order $\sum_{i=k}^{n-1} \phi(p^{n-i}) = p^k - 1$, denoted by $\mathbb{K}_{p^k-1}$. For each $i,\ i < k,\ \alpha_{i,l}\, p^i.\alpha_{j,l}\, p^j = 0$ if and only if $j \in \{n - 1, n - 2, \ldots n - i\}$. This implies, for each $i,\ i < k$, the vertices in V^i are independent, and are adjacent to vertices in V^j, for all $j \in \{n - 1, n - 2, \ldots, n - i\}$. A rough sketch of $\Gamma(\mathbb{Z}_{p^n})$ is given in Figure 1.

Define an $L(2, 1)$-labeling $f : Z^*(\mathbb{Z}_{p^n}) \longrightarrow \mathbb{Z}_{\geq 0}$ by,

$$f(\alpha_{n-1,l}\, p^{n-1}) = 2l - 2,\ \text{for all } l,\ 1 \leq l \leq \phi(p)$$

for $i \leq k$,

$$f(\alpha_{n-i,l}\, p^{n-i}) = \max_{v \in V^{n-(i-1)}} \left\{f(v)\right\} + 2l,$$

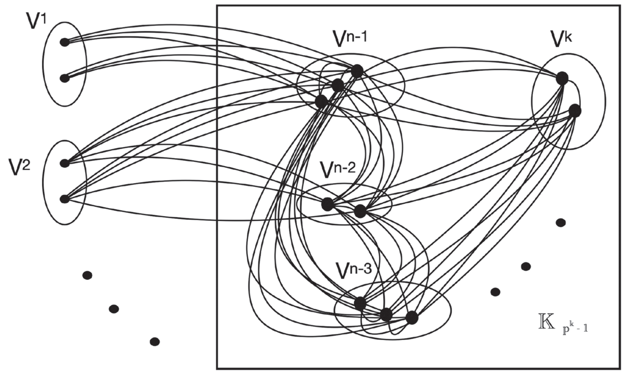

Fig. 1 A Rough sketch of $\Gamma(\mathbb{Z}_{p^n})$

$$= 2p^{i-1} + 2l - 4.$$

Where, $\max\limits_{v \in V^{n-(i-1)}} \{f(v)\} = 2p^{i-1} - 4$. Note that, $\max\limits_{v \in V^k} \{f(v)\} = 2p^k - 4$. This completes the labeling of the clique $\mathbb{K}_{p^k-1}$. Furthermore, for $i < k$, we label the vertices in sets V^i as follows,

$$f(\alpha_{1,l}p) = 2p + 2l - 3, \text{ for all } l, \ 1 \le l \le p^k - p,$$

and

$$f(\alpha_{1,l}p) = p^k + l + p - 3, \text{ for all } l, \ p^k - p < l \le \phi(p^{n-1}).$$

For $2 \le i < k$,

$$f(\alpha_{i,l}p^i) = p^{n-1} + p^k + p - p^{n-i} + l - 3.$$

Therefore, $\text{span}(f) = p^{n-1} + p - 3$. The labeling f is well defined, since any two vertices in sets V^i, $i \ge k$ are adjacent to each other, and their labeling differs by at least 2. Also, any two vertices in V^i with $i < k$ are at distance two from each other, and their labeling differs by at least one. Consequently, $\lambda(\Gamma) \le p^{n-1} + p - 3$.
Subcase - II : $n = 2k + 1$.

For all $i, j > k$, $\alpha_{i,l}p^i.\alpha_{j,l}p^j = 0$ and $\alpha_{i,l}p^i.\alpha_{k,l}p^k = 0$. This implies, vertices in $\bigcup\limits_{i=k+1}^{n-1} V^i \cup \{p^k\}$ induce a clique in $\Gamma(\mathbb{Z}_{p^n})$ of order p^k, denoted by $\mathbb{K}_{p^k}$. For each i, $1 \le i \le k$, vertices in V^i are independent and are adjacent to vertices in V^j, for all $j, n-i \le j \le n-1$. For vertices in $\bigcup\limits_{i=k+1}^{n-1} V^i \cup \{p^k\}$, define an $L(2, 1)$-labeling f in the same manner as we did in subcase-I. Therefore, $f(p^k) = 2p^k - 2$. This completes the labeling of the clique $\mathbb{K}_{p^k}$.

Label the vertices in V^1 as follows,

$$f(\alpha_{1,l}p) = 2p + 2l - 3, \text{ for all } l, 1 \le l \le p^k - p + 1,$$

and

$$f(\alpha_{1,l}p) = p^k + l + p - 2, \text{ for all } l, \ p^k - p + 1 < l \le \phi(p^{n-1}).$$

The remaining vertices in sets V^i, $1 < i \le k - 1$ are labeled in the same manner as in subcase-I. Thus, $f(\alpha_{k-1,\phi(p^k)}p^{k-1}) = p^{n-1} + p^k + p - p^{k+1} - 2$.

For the set $V^k \setminus \{p^k\}$ define,

$$f(\alpha_{k,l}p^k) = p^{n-1} + p^k + p - p^{k+1} + l - 3, \text{ for all } l, 2 \le l \le \phi(p^{k+1}).$$

Thus, we have $\text{span}(f) = p^{n-1} + p - 3$. On similar lines as in Subcase I, it can be verified that f is a well-defined $L(2, 1)$-labeling of $\Gamma(\mathbb{Z}_{p^n})$, and $\lambda(\Gamma(\mathbb{Z}_{p^n})) \le p^{n-1} + p - 3$.

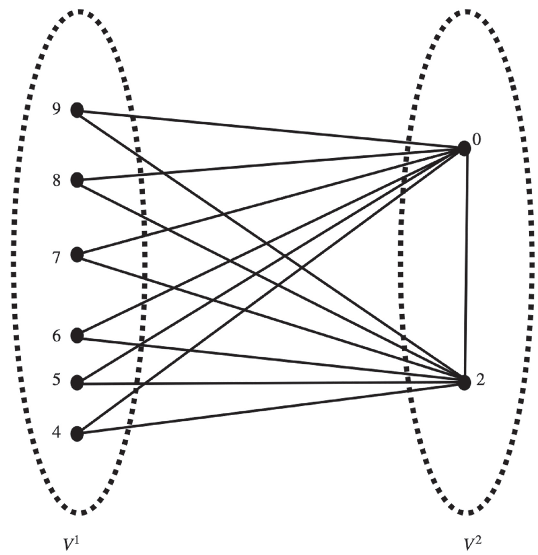

Fig. 2 $\Gamma(\mathbb{Z}_{27})$

Since $diam(\Gamma(\mathbb{Z}_{p^n})) = 2$, any $L(2, 1)$-labeling of $\Gamma(\mathbb{Z}_{p^n})$ must be injective. Also, each vertex in V^i, for all i, $1 \leq i \leq n-2$, is adjacent to every vertex in V^{n-1}, and any two vertices in V^{n-1} are adjacent to each other. It follows that any $L(2, 1)$-labeling of $\Gamma(\mathbb{Z}_{p^n})$ must have $p-1$ many holes. Therefore, $\lambda(\Gamma(\mathbb{Z}_{p^n})) \geq p^{n-1}+p-3$, and consequently, $\lambda(\Gamma(\mathbb{Z}_{p^n})) = p^{n-1} + p - 3$.

Case-II: $n = 2$.

The set of non-trivial zero-divisors $\mathbb{Z}_{p^2}$ and hence the vertices of $\Gamma(\mathbb{Z}_{p^2})$ is given as, $Z^*(\mathbb{Z}_{p^2}) = \{\alpha_i p : (\alpha_i, p) = 1, \text{ and } 1 \leq i \leq p-1\}$. Therefore, $\Gamma(\mathbb{Z}_{p^2}) \cong \mathbb{K}_{p-1}$, which implies $\lambda(\Gamma(\mathbb{Z}_{p^2})) = 2p - 4$.

The example below demonstrates the special case of Theorem 3 when $p = 3$ and $n = 3$.

Example 4 The zero-divisor graph $\Gamma(\mathbb{Z}_{27})$ is shown in the Figure 2 below, featuring the partition and labeling of vertices as explained in Theorem 3.

As can bee seen from Theorem 3 and the Figure 2, $\lambda(\Gamma(\mathbb{Z}_{27})) = 9$.

The following example illustrates Proposition 2 for $\Gamma(\mathbb{Z}_{p^n})$ and provides λ-number of the graph $\Gamma'(\mathbb{Z}_{p^n})$.

Example 5 The vertex p^{n-1} in the graph $\Gamma(\mathbb{Z}_{p^n})$ is clearly adjacent to all other vertices in the graph. Consequently, we deduce that $diam(\Gamma(\mathbb{Z}_{p^n})) < 3$. Furthermore, the order of $\Gamma(\mathbb{Z}_{p^n})$ is $p^{n-1} - 1$, it has $\phi(p)$ many holes for $n \geq 3$, and $\phi(p) - 1$ many holes for $n = 2$. Thus, by Theorem 3, $\lambda(\Gamma'(\mathbb{Z}_{p^n})) = p^n$ for all $n \geq 2$.

Below, we determine the exact value of λ-number of the zero-divisor graph associated with a mixed ring of the form $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$, where p and q are primes and $m, n \geq 3$.

Recall, the *tensor product* of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, denoted by $\mathcal{G}_1 \otimes \mathcal{G}_2$, has a vertex set $V(\mathcal{G}_1 \otimes \mathcal{G}_2) = V(\mathcal{G}_1) \times V(\mathcal{G}_2)$ with (a, b) adjacent to (c, d) in $\mathcal{G}_1 \otimes \mathcal{G}_2$ if and only if a is adjacent to c in $\mathcal{G}_1$ and b is adjacent to d in $\mathcal{G}_2$.

Theorem 6 *Let $R \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$, where p and q are primes, and $m, n \geq 3$ with $p^n \leq q^m$. Then the following hold,*

$$\lambda(\Gamma(R)) = p^{n-1}q^m + pq - 3.$$

Proof Let $U(\mathbb{Z}_{p^n})$ and $U(\mathbb{Z}_{q^m})$ as the sets of units in the rings $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{q^m}$, respectively. Also, let $Z^*(\mathbb{Z}_{p^n})$ and $Z^*(\mathbb{Z}_{q^m})$ denote the sets of zero-divisors in $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{q^m}$, respectively. The non-trivial zero-divisors of R and, consequently, the vertices of $\Gamma(R)$ are partitioned as follows:

$$V_1 = \{(r_p, 0) : r_p \in Z^*(\mathbb{Z}_{p^n})\},$$
$$V_2 = \{(0, r_q) : r_q \in Z^*(\mathbb{Z}_{q^m})\},$$
$$U_1 = \{(u_p, 0) : u_p \in U(\mathbb{Z}_{p^n})\},$$

$$U_2 = \{(0, u_q) : u_q \in U(\mathbb{Z}_{q^m})\},$$
$$W = \{(r_p, r_q) : r_p \in Z^*(\mathbb{Z}_{p^n}) \text{ and } r_q \in Z^*(\mathbb{Z}_{q^m})\},$$
$$W_1 = \{(u_p, r_q) : u_p \in U(\mathbb{Z}_{p^n}) \text{ and } r_q \in Z^*(\mathbb{Z}_{q^m})\},$$
$$W_2 = \{(r_p, u_q) : r_p \in Z^*(\mathbb{Z}_{p^n}) \text{ and } u_q \in U(\mathbb{Z}_{q^m})\}.$$

This partition is structured based on whether the first and second coordinate of the zero-divisor are a unit, zero, or a zero-divisor in their respective rings. Consequently, we have the following cardinalities:

$$|V_1| = p^{n-1} - 1, \quad |V_2| = q^{m-1} - 1,$$
$$|U_1| = p^n - p^{n-1}, \quad |U_2| = q^m - q^{m-1},$$
$$|W| = (p^{n-1} - 1)(q^{m-1} - 1),$$
$$|W_1| = (p^n - p^{n-1})(q^{m-1} - 1),$$
$$|W_2| = (p^{n-1} - 1)(q^m - q^{m-1}).$$

The order of $\Gamma(R)$ is given by $|V(\Gamma(R))| = p^n q^{m-1} + p^{n-1} q^m - p^{n-1} q^{m-1} - 1$. As discussed above the non-trivial zero-divisors of $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{q^m}$ are represented by $\alpha_{i,l} p^i$ and $\beta_{j,k} q^j$ respectively, where $1 \le i \le n-1$, $1 \le j \le m-1$, $1 \le l \le \phi(p^{n-i})$, $1 \le k \le \phi(q^{m-j})$, and $\gcd(\alpha_{i,l}, p^{n-i}) = \gcd(\beta_{j,k}, q^{m-j}) = 1$.

Note that the vertices in sets U_1, U_2, W_1, and W_2 are independent. Furthermore, vertices in sets V_1 and V_2 induce subgraphs $\Gamma(\mathbb{Z}_{p^n})$ and $\Gamma(\mathbb{Z}_{q^m})$ respectively in $\Gamma(R)$. Additionally, the vertices in W induce a subgraph $\Gamma(\mathbb{Z}_{p^n}) \otimes \Gamma(\mathbb{Z}_{q^m})$ within $\Gamma(R)$. We further partition W by considering the following cases.

Case-I: $n = 2a$ and $m = 2b$. Then,

$$S_1 = \left\{(\alpha_{i,l} p^i, \beta_{j,k} q^j) : i \ge a,\ j \ge b\right\},$$
$$S_2 = \left\{(\alpha_{i,l} p^i, \beta_{j,k} q^j) : i \ge a,\ j < b\right\},$$
$$S_3 = \left\{(\alpha_{i,l} p^i, \beta_{j,k} q^j) : i < a,\ j \ge b\right\},$$
$$S_4 = \left\{(\alpha_{i,l} p^i, \beta_{j,k} q^j) : i < a,\ j < b\right\}.$$

It follows that any two vertices in S_1 are adjacent to each other, and hence induce the clique $\mathbb{K}_{(p^a-1)(q^b-1)}$ in $\Gamma(R)$. The vertices in S_2, S_3 and S_4 are independent, and no vertex in S_4 is adjacent to a vertex in S_2 or S_3. However, there are adjacencies between the vertices in S_4 and S_1 which are described below.
Each vertex $(\alpha_{i,l} p^i, \beta_{j,k} q^j) \in S_4$ is adjacent to vertices in the following subset A_1 of S_1,

$$A_1 = \left\{(\alpha_{i',l} p^{i'}, \beta_{j',k} q^{j'}) : n - i \le i' \le n-1 \text{ and } m - j \le j' \le m-1\right\}, \text{ where } |A_1| = \big(p^i - 1\big)\big(q^j - 1\big).$$

Each vertex $(\alpha_{i,l} p^i, \beta_{j,k} q^j) \in S_2$ is adjacent to vertices in the following subset A_2 of S_1,

$$A_2 = \left\{(\alpha_{i',l} p^{i'}, \beta_{j',k} q^{j'}) : a \le i' \le n-1 \text{ and } m - j \le j' \le m-1\right\}, \text{ where } |A_2| = \big(p^k - 1\big)\big(q^j - 1\big).$$

Each vertex $(\alpha_{i,l} p^i, \beta_{j,k} q^j) \in S_3$ is adjacent to vertices in the following subset A_3 of S_1,

$$A_3 = \left\{(\alpha_{i',l} p^{i'}, \beta_{j',k} q^{j'}) : n - i \le i' \le n-1 \text{ and } b \le j' \le m-1\right\}, \text{ where } |A_3| = \big(p^i - 1\big)\big(q^l - 1\big).$$

Each vertex $(\alpha_{i,l} p^i, \beta_{j,k} q^j) \in S_2$ is adjacent to vertices in the following subset A_4 of S_3,

$$A_4 = \left\{(\alpha_{i',l} p^{i'}, \beta_{j',k} q^{j'}) : n - i \le i' \le a-1 \text{ and } m - j \le j' \le m-1\right\}, \text{ where } |A_4| = \big(p^i - p^k\big)\big(q^j - 1\big).$$

Each vertex $(\alpha_{i,l} p^i, \beta_{j,k} q^j) \in W$ is adjacent to vertices in the following subsets B_1, B_2 of V_1 and V_2 respectively,

$$B_1 = \left\{(\alpha_{i',l} p^{i'}, 0) : n - i \le i' \le n-1\right\},$$
$$B_2 = \left\{(0, \beta_{j',k} q^{j'}) : m - j \le j' \le m-1\right\},$$

where $|B_1| = \big(p^i - 1\big)$, and $|B_2| = \big(q^j - 1\big)$.

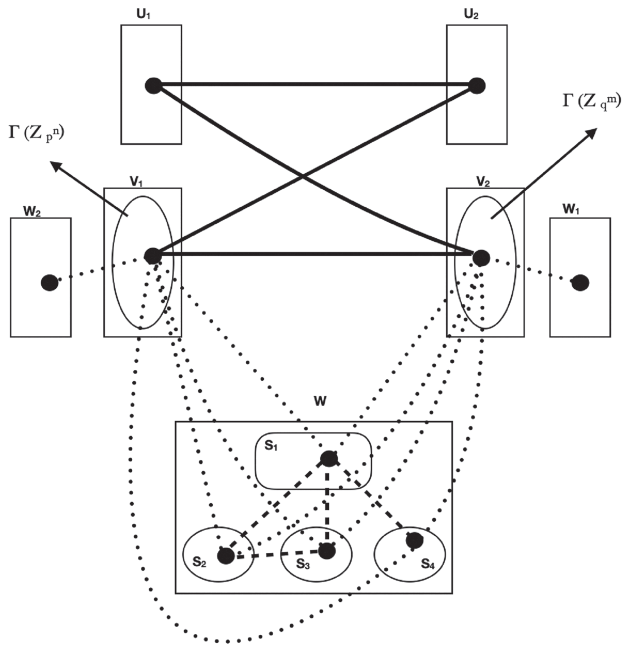

Fig. 3 A Rough sketch of $\Gamma(\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m})$

Each vertex $(\alpha_{i,l}p^i, u_q) \in W_2$, and each vertex $(u_p, \beta_{j,k}q^j) \in W_1$ is adjacent to vertices in B_1 and B_2 respectively. Also, each vertex in V_1, V_2, S_2, S_3 and S_4 is adjacent to vertices in the subset $C_1 = \{(\alpha_{n-1,l}p^{n-1}, \beta_{m-1,k}q^{m-1}) : 1 \le l \le p-1 \text{ and } 1 \le k \le q-1\}$ of S_1, where $|C_1| = (p-1)(q-1)$.

A rough sketch of $\Gamma(R)$ is shown in Figure 3, with a solid line indicating that every vertex in one set is adjacent to every vertex in the other, whereas dotted and dashed lines indicating that not every vertex in one set is adjacent to every vertex in the other.

Let f and g be minimal labelings of $\Gamma(\mathbb{Z}_{p^n})$ and $\Gamma(\mathbb{Z}_{q^m})$ as defined in Theorem 3. We define an $L(2,1)$-labeling $h : V(\Gamma(R)) \longrightarrow \mathbb{Z}_{\ge 0}$ as follows,

$$\begin{aligned}
h(v_2) &= g(v_2), \text{ for all } v_2 \in V_2,\\
h(u_{i_2}) &= \lambda(\Gamma(\mathbb{Z}_{q^m})) + i_2, \text{ for all } u_{i_2} \in U_2, \text{ and } 1 \le i_2 \le |U_2|,\\
&= q^{m-1} + q + i_2 - 3,\\
h(v_1) &= f(v_1) + q^m + q - 1, \text{ for all } v_1 \in V_1,\\
h(u_{i_1}) &= p^{n-1} + q^m + p + q + i_1 - 4, \text{ for all } u_{i_1} \in U_1, \text{ and } 1 \le i_1 \le |U_1|,\\
h(s_{j_1}) &= p^n + q^m + p + q + 2j_1 - 5, \text{ for all } s_{j_1} \in S_1, \text{ and } 1 \le j_1 \le |S_1|.
\end{aligned}$$

Therefore, $\max_{s \in C_1}\{h(s)\} = p^n + q^m + p + q + 2(p-1)(q-1) - 5$. Note that, while labeling vertices in S_1, we begin with vertices in $C_1 \subset S_1$.

Consider the subset $C_2 = \{(\alpha_{1,l}p, \beta_{1,k}q) : 1 \le l \le \phi(p^{n-1} \text{ and } 1 \le k \le \phi(q^{m-1}\}$ of S_4 with $|C_2| = (p^{n-1} - p^{n-2})(q^{m-1} - q^{m-2})$. For $s_{j_4} \in C_2$, define,

$$h(s_{j_4}) = p^n + q^m + p + q + 2(p-1)(q-1) + 2j_4 - 4, \text{ for all } j_4, 1 \le j_4 \le (p^k - 1)(q^l - 1) - (p-1)(q-1),$$

and

$$h(s_{j_4}) = p^n + q^m + pq + (p^k - 1)(q^l - 1) + j_4 - 3, \text{ for all } j_4, (p^k - 1)(q^l - 1) - (p-1)(q-1) < j_4 \le |C_2|.$$

Therefore, $\max_{v \in C_2}\{h(v)\} = p^n + q^m + pq + (p^k - 1)(q^l - 1) + (p^{n-1} - p^{n-2})(q^{m-1} - q^{m-2}) - 3 = \delta_1$ (say). For $s_{j_3} \in S_3$, define,

$$h(s_{j_3}) = \delta_1 + j_3, \text{ for all } j_3, 1 \le j_3 \le |S_3|.$$

Therefore, $\max_{v\in S_3}\{h(v)\} = p^n + q^m + pq - q^l + p^{n-2}q^{m-2}(pq - p - q + 1) + p^{n-1}(q^l - 1) - 2 = \delta_2$ (say). For $s_{j_4} \in S_4 \setminus C_2$, define,

$$h(s_{j_4}) = \delta_2 + j_4 - |C_2|, \text{ for all } j_4, |C_2| < j_4 \le |S_4|.$$

Therefore, $\max_{v\in S_4}\{h(v)\} = p^n + q^m + pq + q^l(p^k - 1) + q^{m-1}(p^{n-1} - p^k) - p^{n-1} - 2 = \delta_3$ (say). For $s_{j_2} \in S_2$, define,

$$h(s_{j_2}) = \delta_3 + j_2, \text{ for all } j_2, 1 \le j_2 \le |S_2|.$$

Therefore, $\max_{v\in S_2}\{h(v)\} = p^n + q^m + pq + q^{m-1}p^{n-1} - p^{n-1} - q^{m-1} - 2 = \delta_4$ (say).

Note that $d(w_{t_1}, u_{i_2}) = d(w'_{t_2}, u'_{i_1}) = d(w_{t_1}, w'_{t_2}) = 3$, where $w_{t_1} \in W_1, u_{i_2} \in U_2, w'_{t_2} \in W_2$ and $u'_{i_1} \in U_1$. Furthermore, we define,

$$\begin{aligned} h(w'_{t_2}) &= h(u'_{i_1}), \text{ for all } i_1, 1 \le i_1 \le |U_1|,\\ h(w'_{(|U_1|+1)}) &= q^m + q - 2,\\ h(w'_{t_2}) &= \delta_4 + t_2 - |U_1| - 1, \text{ for all } t_2, |U_1| + 1 < t_2 \le |W_2|. \end{aligned}$$

Therefore, $\max_{v\in W_2}\{h(v)\} = p^{n-1}q^m + pq - 3$.

Finally,

$$h(w_{t_1}) = h(u_{i_2}), \text{ for all } t_1, 1 \le t_1 \le |U_2|.$$

Since $p^n \le q^m$, therefore, $|W_1| \le |W_2|$. The remaining vertices of W_1 are labeled same as the vertices w'_{t_2} of W_2, where $|U_1| + 1 < t_2 \le |W_2|$. This concludes the labeling of all the vertices in $\Gamma(R)$. It is evident from the structure of $\Gamma(R)$ that h is a well-defined $L(2, 1)$-labeling, and consequently, $\lambda(\Gamma(R)) \le p^{n-1}q^m + pq - 3$.

The vertices in sets $\{(\alpha_{n-1,l}p^{n-1}, 0) : 1 \le l \le p-1\}, \{(0, \beta_{m-1,k}q^{m-1}) : 1 \le k \le q-1\}$, and $\{(\alpha_{n-1,l}p^{n-1}, \beta_{m-1,k}q^{m-1}) : 1 \le l \le p-1 \text{ and } 1 \le k \le q-1\}$ induce the complete subgraph $\mathbb{K}_{pq-1}$ in $\Gamma(R)$, and every other vertex in $\Gamma(R)$ is adjacent to vertices in $\mathbb{K}_{pq-1}$. This implies, any minimal labeling of $\Gamma(R)$ must have at least $pq - 2$ many holes. Therefore, h is minimal, and we have, $\lambda(\Gamma(R)) = p^{n-1}q^m + pq - 3$.

For the remaining cases: $n = 2a, m = 2b + 1; n = 2a + 1, m = 2b + 1; n = 2a + 1, m = 2b$, we follow the same approach as in case-I with the exception that vertices in W are classified in such a way that S_1 has maximum cardinality and vertices in S_1 induce a clique in $\Gamma(R)$.

Remark 7 For cases: $m = 2 = n;\ m \ge 3, n = 2;\ m = 2, n \ge 3$, the λ-number of the graph $\Gamma(\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m})$ is $pq^2 + pq - 5$, $pq^m + pq - 4$, and $p^{n-1}q^2 + pq - 4$ respectively. These values can be obtained by using case-II of Theorem 3 and case-I of Theorem 6.

In the following result, we obtain λ-number of the zero-divisor graph associated with a mixed rings of the form $\mathbb{F}_q \times \mathbb{Z}_{p^n}$.

Theorem 8 *Let $n > 1$ be a positive integer, and let $R \cong \mathbb{F}_q \times \mathbb{Z}_{p^n}$, where $\mathbb{F}_q$ is a finite field of order q and p is prime. Then the following holds,*

$$\lambda(\Gamma(R)) = \begin{cases} qp^{n-1} + p - 3, & \text{if } n \ge 3 \text{ and } q > \frac{p^n-1}{p^{n-1}-1},\\ p^n + p + q - 3, & \text{if } n \ge 3 \text{ and } q \le \frac{p^n-1}{p^{n-1}-1},\\ p^2 + p + q - 4, & \text{if } n = 2 \text{ and } q < p + 1,\\ pq + p - 3, & \text{if } n = 2 \text{ and } q \ge p + 1. \end{cases}$$

Proof Recall that the non-trivial zero-divisors of $\mathbb{Z}_{p^n}$ are,

$$Z^*(\mathbb{Z}_{p^n}) = \left\{\alpha_{i,l}p^i : \gcd(\alpha_{i,l}, p) = 1,\ 1 \le l \le \phi(p^{n-i}),\ \text{and } 1 \le i \le n-1\right\}.$$

We partition the vertices of $\Gamma(R)$ as follows,

$$V^1 = \left\{(a, 0) : a \in \mathbb{F}_q \setminus \{0\}\right\},$$

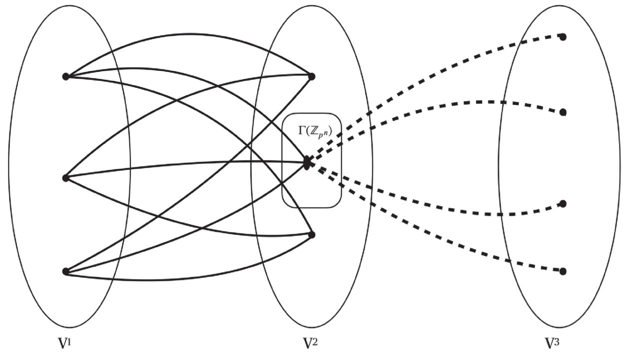

Fig. 4 A Rough sketch of $\Gamma(\mathbb{F}_q \times \mathbb{Z}_{p^n})$

$$V^2 = \{(0, b) : b \in \mathbb{Z}_{p^n} \setminus \{0\}\},$$
$$V^3 = \{(a, \alpha_{i,l} p^i) : a \in \mathbb{F}_q \setminus \{0\} \text{ and } \alpha_{i,l} p^i \in Z^*(\mathbb{Z}_{p^n})\},$$

where $|V^1| = q-1$, $|V^2| = p^n - 1$, $|V^3| = (q-1)(p^{n-1}-1)$. Therefore, the order of $\Gamma(R)$ is $p^{n-1}(p+q-1)$. Notice that, there are no adjacencies between vertices in V^1 and V^3 which implies, $V^1 \cup V^3$ is an independent set of vertices. However, $W = \{(0, \alpha_{i,l} p^i) : \alpha_{i,l} p^i \in Z^*(\mathbb{Z}_{p^n})\} \subset V^2$ induces the subgraph $\Gamma(\mathbb{Z}_{p^n})$ in $\Gamma(R)$. In addition, each vertex of V^1 is adjacent to every vertex of V^2, whereas each vertex of V^3 is adjacent to some vertices in V^2. A rough sketch of $\Gamma(R)$ is shown in Figure 4. We consider the following cases.

Case-1: $n \geq 3$.

By case-I of Theorem 3, $\lambda(\Gamma(\mathbb{Z}_{p^n})) = p^{n-1} + p - 3$, and let f be its corresponding minimal labeling. Define an $L(2,1)$-labeling $g : V(\Gamma(R)) \longrightarrow \mathbb{Z}_{\geq 0}$ as,

$$g(v) = f(v), \text{ for all } v \in W.$$

For $v_{i_3} \in V^3$, define,

$$g(v_{i_3}) = \lambda(\Gamma(\mathbb{Z}_{p^n})) + i_3, \text{ for all } i_3, 1 \leq i_3 \leq |V^3|.$$

Therefore, $\max_{v \in V^3}\{g(v)\} = qp^{n-1} + p - q - 2$. Note that, while labeling vertices in V^3, we begin with vertices of the form $(a, \alpha_{1,l} p)$.

For $v_{i_2} \in V^2 \setminus W$, define,

$$g(v_{i_2}) = \lambda(\Gamma(\mathbb{Z}_{p^n})) + i_2, \text{ for all } i_2, 1 \leq i_2 \leq |V^2 \setminus W|.$$

This implies, $\max_{v \in V^2 \setminus W}\{g(v)\} = p^n + p - 3$.

For labeling vertices in V^1, the following subcases are considered.

Subcase-1.1: $q > \frac{p^n-1}{p^{n-1}-1}$. Then, $|V^3| > |V^2 \setminus W|$. For $v_{i_1} \in V^1$, define,

$$g(v_{i_1}) = qp^{n-1} + p + i_1 - q - 2, \text{ for all } i_1, 1 \leq i_1 \leq |V^1|.$$

This completes the labeling of all vertices of $\Gamma(R)$ in this case, and therefore, $\max_{v \in V(\Gamma(R))} g(v) = qp^{n-1} + p - 3$.

Subcase-1.2: $q \leq \frac{p^n-1}{p^{n-1}-1}$.

Then, $|V^3| \leq |V^2 \setminus W|$. For $v_{i_1} \in V^1$, define,

$$g(v_{i_1}) = p^n + p + i_1 - 2, \text{ for all } i_1, 1 \leq i_1 \leq |V^1|.$$

This completes the labeling of all vertices of $\Gamma(R)$ in this case, and therefore, $\max_{v \in V(\Gamma(R))} g(v) = p^n + p + q - 3$.

Case-2: $n = 2$.

The subgraph induced by W is complete in $\Gamma(R)$ and each vertex in V^3 is adjacent to every vertex in W. By case-II of Theorem 3, $\lambda(\Gamma(\mathbb{Z}_{p^2})) = 2p - 4$, let f be its corresponding minimal labeling. Define an $L(2,1)$-labeling $g : V(\Gamma(R)) \longrightarrow \mathbb{Z}_{\geq 0}$ as,

$$g(v) = f(v), \text{ for all } v \in W.$$

For $v_{i_2} \in V^2 \setminus W$, define,

$$g(v_{i_2}) = \lambda(\Gamma(\mathbb{Z}_{p^2})) + i_2, \text{ for all } i_2, 1 \leq i_2 \leq |V^2 \setminus W|.$$

This implies, $\max_{v \in V^2 \setminus W} \{g(v)\} = p^2 + p - 4$.

For $v_{i_3} \in V^3$, define,

$$g(v_{i_3}) = \lambda(\Gamma(\mathbb{Z}_{p^2})) + i_3 + 1, \text{ for all } i_3, 1 \leq i_3 \leq |V^3|.$$

This implies, $\max_{v \in V^3}\{g(v)\} = p + qp - q - 2$. To label vertices in V^1, we consider the following subcases.

Subcase-2.1: $q < p + 1$.

Then, $|V^3| < |V^2 \setminus W|$. For $v_{i_1} \in V^1$, define,

$$g(v_{i_1}) = p^2 + p + i_1 - 3, \text{ for all } i_1, 1 \leq i_1 \leq |V^1|.$$

This completes the labeling of all vertices of $\Gamma(R)$ in this case, and therefore, $\max_{v \in V(\Gamma(R))} g(v) = p^2 + p + q - 4$.

Subcase-2.2: $q \geq p + 1$.

Then, $|V^3| \geq |V^2 \setminus W|$. For $v_{i_1} \in V^1$, define,

$$g(v_{i_1}) = p + qp + i_1 - q - 2 \text{ for all } i_1, 1 \leq i_1 \leq |V^1|.$$

This completes the labeling of all vertices of $\Gamma(R)$ in this case, and therefore, $\max_{v \in V(\Gamma(R))} g(v) = pq + p - 3$.

From the structure of $\Gamma(R)$, it is evident that g is a well defined $L(2,1)$-labeling of $\Gamma(R)$ in every case discussed above, and consequently,

$$\lambda(\Gamma(R)) \leq \begin{cases} qp^{n-1} + p - 3, & \text{if } n \geq 3 \text{ and } q > \frac{p^n - 1}{p^{n-1} - 1}, \\ p^n + p + q - 3, & \text{if } n \geq 3 \text{ and } q \leq \frac{p^n - 1}{p^{n-1} - 1}, \\ p^2 + p + q - 4, & \text{if } n = 2 \text{ and } q < p + 1, \\ pq + p - 3, & \text{if } n = 2 \text{ and } q \geq p + 1. \end{cases}$$

Since g admits maximum number of multiplicities, and least number of holes among all $L(2,1)$-labelings of $\Gamma(R)$. Therefore, g is minimal, and the result follows.

To determine λ-number of the zero-divisor graph of a ring R , when R is product of more than two local rings becomes challenging due to increasing complexity in the structure of the graph. As a natural extension and an intriguing avenue for future research, we pose the following open problem.

problem 1 *Determine λ-number of the zero-divisor graph $\Gamma(R)$ when $R \cong \mathbb{Z}p_1^{n_1} \times \mathbb{Z}p_2^{n_2} \times \mathbb{Z}_{p_k^{n_k}}$, where p_i is a prime number and n_i is a positive integer for each i, $1 \leq i \leq k$.*

Conclusion

In this research article, we discussed the $L(2,1)$-labeling problem in zero-divisor graphs associated with finite commutative rings. We computed exact values of the λ-number and corresponding minimal labeling for zero-divisor graphs realized by local and mixed rings. By studying these specific structures of zero-divisor graphs realized by these rings, we have provided valuable insights into the combinatorial properties of these graph structures in relation to ring theory. As a prospect for future research, we concluded the paper by presenting an open problem that invites further exploration in this direction.

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Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Competing interest There is no conflict of interest to declare.

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