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ORIGINAL RESEARCH

Clear graph of a ring

Shabir Ahmad Mir · Cihat Abdioğlu · Nadeem ur Rehman · Mohd Nazim · Muhammed Akkafa · Ece Yetkin Çelikel

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Abstract This research article introduces the concept of the clear graph associated with a ring $\mathcal{R}$ with identity, denoted as $Cr(\mathcal{R})$. This graph comprises vertices of the form $\{(x, u) : x$ is a unit regular element of R and u is a unit of $\mathcal{R}\}$ and two distinct vertices (x, u) and (y, v) are adjacent if and only if either $xy = yx = 0$ or $uv = vu = 1$. This research article also focuses on a specific subgraph of $Cr(\mathcal{R})$ denoted as $Cr_2(\mathcal{R})$, which is formed by vertices $\{(x, u) : x$ is a nonzero unit regular element of $R\}$. The significance of $Cr_2(\mathcal{R})$ within the context of $Cr(\mathcal{R})$ is explored in the article. Taken $Cr_2(\mathcal{R})$ into consideration, we found connectedness, regularity, planarity, and outer planarity. Moreover, we characterized the ring $\mathcal{R}$ for which $Cr_2(\mathcal{R})$ is unicyclic, a tree and a split graph. Finally, we have found genus one of $Cr_2(\mathcal{R})$.

Keywords Clear graph · Clean graph · Genus of a graph · Planar graph

Mathematics Subject Classification 05C10 · 05C12 · 05C25

1 Introduction

The investigation of graphs associated with algebraic structures is a rapidly growing field, with a focus on classifying the graphs of algebraic structures and vice versa. Researchers are particularly interested in under-

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S. A. Mir · N. ur Rehman (✉)
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail: mirshabir967@gmail.com
E-mail: nu.rehman.mm@amu.ac.in

C. Abdioğlu
Department of Mathematics and Science Education, Faculty of Education, Karamanoğlu Mehmetbey University, Karaman, Turkey
E-mail: cihat.abdioglu@gmail.com

M. Nazim
School of Basic and Applied Sciences, Faculty of Science and Technology, JSPM University, Pune 412207, India
E-mail: mnazim1882@gmail.com

M. Akkafa
Gaduate School of Natural and Applied Sciences, Karamanoğlu Mehmetbey University, Karaman, Turkey
E-mail: muhammedakkafa@gmail.com

E. Y. Çelikel
Department of Basic Sciences, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep, Turkey
E-mail: ece.celikel@hku.edu.tr
E-mail: yetkinece@gmail.com

standing the relationship between the algebraic structure of a known object and the graph-theoretic properties of a similar graph. When a combinatorial object is assigned to an algebraic structure, it often leads to intriguing problems in both algebra and combinatorics. Currently, one of the most active areas of research in this field is the study of graphs associated with commutative rings. Commutative rings are a fundamental algebraic structure in mathematics and have many applications in diverse areas such as coding theory, cryptography, and algebraic geometry.

We define a simple graph as $G = (\mathcal{V}, E)$, with $\mathcal{V}(G)$ representing the vertex set and $E(G)$ representing the edge set. The sizes of the vertex and edge sets are denoted as $|\mathcal{V}(G)|$ and $|E(G)|$ respectively. A graph is deemed connected when a path exists between any pair of vertices; otherwise, it is considered disconnected. $\omega(G)$, $\chi(G)$, and diam(G) denote the clique number, vertex chromatic number, and diameter of graph G respectively. P_n and C_n signify paths and cycles containing n vertices. K_n and $K_{m,n}$ represent complete graphs and complete bipartite graphs respectively. An induced subgraph H of graph G is one where if two distinct vertices x and y of H share an edge in G, they must also share an edge in H.

The direct sum of two graphs G_1 and G_2, denoted as $G_1 \oplus G_2$, encompasses all vertices from both G_1 and G_2. An edge exists in $G_1 \oplus G_2$ if and only if it exists in either G_1 or G_2, but not both. For further insights into graph theory concepts and terms, consult [8]. Recalling some common terms and notations will help us to understand how they are used in this research article. $\mathbb{Z}$ refers to the set of all integers, while $\mathbb{Z}_n$ represents the set of integers taken modulo n. An element e_i in $\mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$ is represented as $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ for $1 \leq i \leq n$. An element e in the ring $\mathcal{R}$ is idempotent if $e^2 = e$. In $\mathcal{R}$, r is a unit element if an element s in $\mathcal{R}$ satisfies $sr = 1$ and $rs = 1$. A unit regular element x in $\mathcal{R}$ fulfills $x = xux$, where u is a unit in $\mathcal{R}$. This means that x can be expressed as the product of an idempotent and a unit. The sets $Id(\mathcal{R})$, $U(\mathcal{R})$, and $U_{reg}(\mathcal{R})$ refer to idempotents, units, and unit regular elements of $\mathcal{R}$ respectively. Additionally, $U^{'}(\mathcal{R}) = \{u \in U(\mathcal{R}) : u^2 = 1\}$ and $U^{''}(\mathcal{R}) = U(\mathcal{R}) \setminus U^{'}(\mathcal{R})$.

One of the remarkable and groundbreaking concepts was the exploration of zero divisor graph in rings. Consider $\mathcal{R}$, a commutative ring with unity $1 \neq \{0\}$, $Z(\mathcal{R})$ denotes the set of zero-divisors of $\mathcal{R}$ and $Z^*(\mathcal{R}) = Z(\mathcal{R}) \setminus \{0\}$. Beck [4] was the first to introduce the zero-divisor graph of $\mathcal{R}$ where he was interested about the coloring. In his work, the elements of $\mathcal{R}$ were considered to be the vertices of graph and the edge exists between two distinct vertices u and v if and only if $uv = 0$. The adjustment were made by Anderson and Livingston [1], by considering the graph $\Gamma(\mathcal{R})$ with vertex set as the set of nonzero zero-divisors of $\mathcal{R}$, and adjacency relation is followed by two distinct vertices u and v if and only if $uv = 0$. Zero-divisor graphs of $\mathcal{R}$ have been studied by several authors. More literature and contribution on zero-divisor graphs can be seen in [2,3,5–7]. The association of a ring to a graph in the study of zero-divisor graphs provides insight into the algebraic characteristics of rings, particularly the structure of the set of zero-divisors. Akbari et al. in [14] introduce the idempotent graph, $I(\mathcal{R})$, with non trivial idempotents of $\mathcal{R}$ as vertices, and the edge is defined between two vertices x and y if and only if $xy = yx = 0$. Habibi et al. [10] introduced the innovative notion of a clean graph denoted as $Cl(\mathcal{R})$ for a given ring $\mathcal{R}$. This unique graph involves vertex pairs (x, u), where x symbolizes an idempotent element, and u represents a unit element within the ring $\mathcal{R}$. The graph's edges are formed exclusively between pairs of vertices, (x, u) and (y, v), satisfying either the conditions $xy = yx = 0$ or $uv = vu = 1$. This intricate structure of the clean graph, $Cl(\mathcal{R})$, can be deconstructed into two distinct subgraphs: $Cl_1(\mathcal{R})$ and $Cl_2(\mathcal{R})$. The vertices constituting $Cl_1(\mathcal{R})$ are specifically defined as $(0, u) : u$ represents a unit element within $\mathcal{R}$, and the vertices comprising $Cl_2(\mathcal{R})$ are of the form (e, u), where e stands for a nonzero idempotent element of the ring $\mathcal{R}$, and u denotes a unit element from the same ring.

This article delves into the exploration of $Cr_2(\mathcal{R})$, which is a subgraph of the larger graph $Cr(\mathcal{R})$ and is generated by the collection of vertices $(x, u) : x$ corresponds to a nonzero unit regular element and u corresponds to a unit element within $\mathcal{R}$. Notably, $Cr_2(\mathcal{R})$ plays a pivotal role within the context of $Cr(\mathcal{R})$. The study encompasses the analysis of various graph-theoretic properties of $Cr_2(\mathcal{R})$, encompassing connectedness, regularity, planarity, as well as outer planarity. Additionally, the article undertakes the task of characterizing the specific characteristics of the ring $\mathcal{R}$ for which $Cr_2(\mathcal{R})$ takes the form of an unicyclic graph, a tree, or a split graph. The investigation concludes by determining the genus one of the graph $Cr_2(\mathcal{R})$.

2 Preliminaries

Definition 2.1 In the context of ring theory, an element within a ring $\mathcal{R}$ is referred as *clear* if it can be expressed as the sum of two components: a *unit-regular* element and a *unit* element. In simpler terms, an element a within the ring $\mathcal{R}$ is labeled as *clear* if it can be written in the form $a = r + u$, where r represents a *unit-regular* element and u is a unit element. Moreover, if r is both non-zero and not part of the set of units $U(\mathcal{R})$, then the *clear element a* is categorized as nontrivial. A ring $\mathcal{R}$ is defined as a *clear ring* when this notion of being *clear* applies to all elements within the ring. It is important to note that the elements 0 and 1 are straightforward examples of *clear elements*. Additional insights and specifics can be explored in the reference provided as [16]. In the realm of rings, a significant subgroup is that of *unit-regular* rings, where every individual element qualifies as a unit-regular element. This subgroup encompasses idempotent elements as well. Notably, any *clean* element, which is conceptually related to *clear* elements, also falls under the category of *clear* elements. Consequently, any clean ring is clear. Additionally, an element categorized as unit-regular is, by nature, also considered *clear*. However, it is crucial to acknowledge that these relationships do not hold in reverse. As an example, the matrix $\begin{bmatrix} 12 & 5 \\ 0 & 0 \end{bmatrix} \in \mathbb{Z}^{2\times 2}$ qualifies as a *clear* element while not being classified as *clean*. Similarly, the ring $\mathbb{Z}_4$ is a representative instance of a *clear ring* that lacks unit-regularity. For a deeper comprehension and additional instances of unit regularity, further details can be referenced in the works [9] and [11].

Our motivation to introduce a new graph structure arises from the concept of employing clear elements as primary vertices. This inspiration draws from the domains of clear rings, clean graphs, and idempotent graphs. The subsequent definition encapsulates the core essence of what we are aiming to define and investigate.

Definition 2.2 Let $\mathcal{R}$ denotes a ring with an identity element. We define the *clear graph* of $\mathcal{R}$ as a graphical structure characterized by vertices represented in the form of (x, u), where x signifies a unit-regular element, and u stands for a unit element within $\mathcal{R}$. The linkage between two distinct vertices (x, u) and (y, v) is established through an edge if and only if the conditions $xy = yx = 0$ or $uv = vu = 1$ are met. We denote *Clear graph* of $\mathcal{R}$ by $Cr(\mathcal{R})$ and $V(Cr(\mathcal{R}))$ represents vertex set of $Cr(\mathcal{R})$. The notion $(x, u) \sim (y, v)$ means (x, u) and (y, v) are adjacent vertices, that is $xy = yx = 0$ or $uv = vu = 1$. Since $(0, u)$ is adjacent to any vertex, we should omit this case. So, the vertex set may or should be as follows:
$V(Cr(\mathcal{R})) = \{(x, u) : x \text{ is nonzero unit regular and } u \text{ is unit of } \mathcal{R}\}$.

Our aim is to introduce and study some basic properties of Clear graph denoted by $Cr(\mathcal{R})$, some of its subgraphs $Cr_1(\mathcal{R})$ and $Cr_2(\mathcal{R})$, where

$$V(Cr_1(\mathcal{R})) = \{(0, u) : u \ \text{ is unit of } \ \mathcal{R}\}$$
$$and$$
$$V(Cr_2(\mathcal{R})) = \{(e, u) : 0 \neq e \in U_{reg}(\mathcal{R}) \ \text{ and } \ u \in U(\mathcal{R})\}$$

Evidently, $Cr_1(\mathcal{R})$ forms a complete subgraph within the structure of $Cr(\mathcal{R})$. Consequently, the investigation of fundamental attributes pertaining to $Cr(\mathcal{R})$ primarily hinges on the properties of $Cr_2(\mathcal{R})$. First, we construct some examples of $Cr_2(\mathcal{R})$ which then motivate us to identify when the clear graph is isomorphic to some well-known graphs.

Example 2.1

(1) $Cr_2(\mathbb{Z}_3) = K_2 \oplus K_2$.
(2) Let $\mathcal{R} = \mathbb{Z}_3$. Then $Id(\mathcal{R}) = \{0, 1\}, U(\mathcal{R}) = \{1, 2\}$ and $U_{reg}(\mathcal{R}) = \{0, 1, 2\}$. Then, $V(Cl_2(\mathcal{R})) = \{(1, 1), (1, 2)\}$ and $V(Cr_2(\mathcal{R})) = \{v_1 = (1, 1), v_2 = (1, 2), v_3 = (2, 1), v_4 = (2, 2)\}$. So, $Cl_2(\mathcal{R})$ is a null graph with two vertices and $Cr_2(\mathcal{R}) = K_2 \oplus K_2$:
(3) Let $\mathcal{R} = \mathbb{Z}_5$. Then, $V(Cl_2(\mathcal{R})) = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ and $V(Cr_2(\mathcal{R}))\{(1, 1), (1, 2), (1, 3), (1, 4), \ldots, (4, 1), (4, 2), (4, 3), (4, 4)\}$. So, $Cl_2(\mathcal{R})$ is K_2 [(1, 2) is adjacent to (1, 3)].
However, $Cr_2(\mathcal{R}) = K_4 \oplus K_{4,4} \oplus K_4$, shown in Figure 2. where $v_{i,j}$ means the vertex (i, j).
(4) More generally, we have
   - $Cr_2(\mathbb{Z}_2) \cong K_1$.
   - $Cr_2(\mathbb{Z}_3) \cong K_2 \oplus K_2$.
   - $Cr_2(\mathbb{Z}_5) \cong K_4 \oplus K_{4,4} \oplus K_4$.
   - $Cr_2(\mathbb{Z}_7) \cong K_6 \oplus 2K_{6,6} \oplus K_{6,6}$.
   - $Cr_2(\mathbb{Z}_{11}) \cong K_{10} \oplus 4K_{10,10} \oplus K_{10}$.
   - $Cr_2(\mathbb{Z}_{13}) \cong K_{12} \oplus 5K_{12,12} \oplus K_{12}$.

Proposition 2.1 $Cr_2(\mathbb{Z}_p) \cong K_{p-1} \oplus \underbrace{K_{p-1,p-1} \oplus \cdots \oplus K_{p-1,p-1}}_{\frac{p-3}{2} \text{ times}} \oplus K_{p-1}$, *where $p > 3$ is prime.*

Fig. 1 .

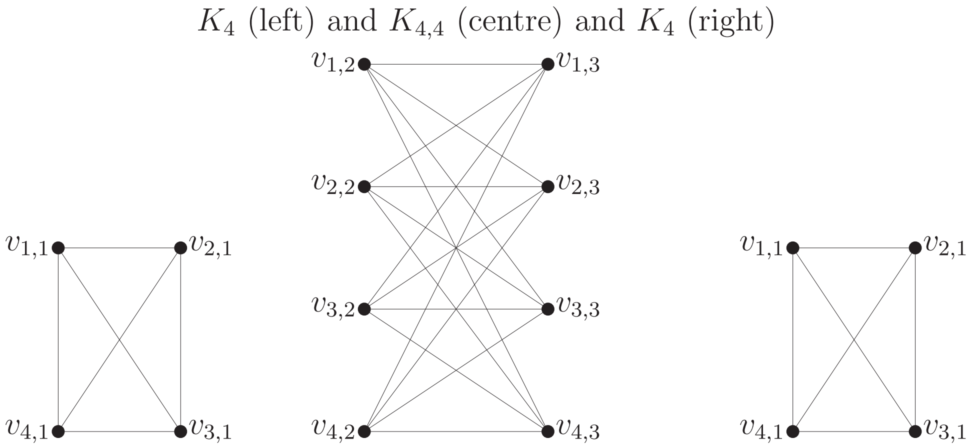


Fig. 2 .

Proof Since $\mathbb{Z}_p$ is a field, there is no zero-divisor and clearly $U(\mathbb{Z}_p) = \mathbb{Z}_p \setminus \{0\}$. Now, we have two cases:

Case I. If $u^2 = 1$ (i.e, $u = u^{-1}$), then $u^2 - 1 = (u-1)(u+1) \equiv 0 \pmod p$, which implies that either $u = \overline{1}$ or $u = \overline{p-1}$. Observe that all vertices of the form $(x, \overline{1})$, where $x \in \mathbb{Z}_p \setminus \{0\}$, forms a complete graph K_{p-1}.

Case II. Suppose that $u^2 \neq 1$. Note that the number of elements satisfying $u^2 \neq 1$ is clearly $p-3$. If u is such a unit element, then (x, u) and (y, u^{-1}) are adjacent, where $x, y \in Z_p \setminus \{0\}$, not necessarily distinct. Thus, these vertices form a bipartite graph $K_{p-1,p-1}$.

Therefore, we conclude that $Cr_2(\mathbb{Z}_p) \cong K_{p-1} \oplus \underbrace{K_{p-1,p-1} \oplus \cdots \oplus K_{p-1,p-1}}_{\frac{p-3}{2} \text{ times}} \oplus K_{p-1}$. □

If $\mathbb{F}$ denotes a finite field, as is commonly understood, then $|\mathbb{F}|p^n$ for some prime p and positive integer n. So, we can conclude the following result.

Corollary 2.1 *Let $\mathbb{F}$ be a finite field. Then,*

$$Cr(\mathbb{F}) \cong K_{|\mathbb{F}|-1} \oplus \underbrace{K_{|\mathbb{F}|-1,|\mathbb{F}|-1} \oplus \cdots \oplus K_{|\mathbb{F}|-1,|\mathbb{F}|-1}}_{\frac{|\mathbb{F}|-3}{2} \text{ times}} \oplus K_{|\mathbb{F}|-1}.$$

3 Connectedness of clear graph

This section starts with a study of the fundamental characteristics of $Cr(\mathcal{R})$, such as connectedness, diameter and regularity, which then inspires us to define subsequent sections. We start by stating the following remark:

Remark 3.1 Clearly $Cr(\mathcal{R}) = Cr_1(\mathcal{R}) \vee Cr_2(\mathcal{R})$, where $Cr_1(\mathcal{R})$ is a complete subgraph of $Cr(\mathcal{R})$. As a result, studying $Cr_2(\mathcal{R})$ seems to be necessary before exploring the primary characteristics of $Cr(\mathcal{R})$.

Lemma 3.1 *If $x \sim y$ in $Cl_2(\mathcal{R})$, then $x \sim y$ in $Cr_2(\mathcal{R})$.*

Proof Assume $x = (e_1, u_1)$ and $y = (e_2, u_2)$ be two vertices of $Cl_2(\mathcal{R})$ such that $x \sim y$, implies either $e_1e_2 = 0$ or $u_1u_2 = 1$, where e_1, e_2 are idempotent elements of $\mathcal{R}$. Since every idempotent element is unit regular. Hence, e_1, e_2 are unit regular elements of $\mathcal{R}$. Therefore, $x \sim y$ in $Cr_2(\mathcal{R})$. □

Theorem 3.1 *Let $\mathcal{R}$ be a ring with $|U(\mathcal{R})| \geq 2$, then $Cr_2(\mathcal{R})$ is connected if and only if $\mathcal{R}$ has a non-trivial idempotent.*

Proof Suppose that $\mathcal{R}$ has a non-trivial idempotent and let $x = (s, a)$ and $y = (r, b)$ denotes two vertices of $Cr_2(R)$. It's evident that if either $sr = 0$ or $ab = 1$, then $d(x, y) = 1$, else, following cases can be considered:
Case (1) If $s = r$, then the subsequent subcases occur:

(i) If $s = r = 1$, the path $(1, a) \sim (q, a^{-1}) \sim (1 - q, b^{-1}) \sim (1, b)$ exists, where q is non-trivial idempotent of $\mathcal{R}$.
(ii) If $s, r \neq 1$, the path $(s, a) \sim (1 - s, b) \sim (r, b)$ leads to $d(x, y) = 2$.

Case (2) If $s \neq r$, then we have

(i) If s or r is equal to 1, assuming $s = 1$ for instance, the path $(1, a) \sim (1 - r, a^{-1}) \sim (r, b)$ is of length 2.
(ii) If $s, r \neq 1$, the path $(s, a) \sim (1 - s, b^{-1}) \sim (r, b)$ also has a length of 2.

After analyzing various instances, we determine that $Cr_2(\mathcal{R})$ is connected with $diam(Cr_2(\mathcal{R})) \leq 3$.

Conversely, given $Cr_2(\mathcal{R})$ is connected. Assume that $\mathcal{R}$ lacks non trivial idempotent, then $V(Cr_2(\mathcal{R})) = \{(a, b); a, b \in U(\mathcal{R})\}$. Now, we can encounter through the following cases:

Case (i) When $1 \neq a \in U(\mathcal{R})$, such that $a^2 = 1$, define $S_i = \{(a, a_i) : a \in U(\mathcal{R})\}$. Then $Cr_2(\mathcal{R})$ has complete subgraphs induced by each S_i. However, S_i and S_j are disconnected for $i \neq j$, leading to a contradiction.
Case (ii) Suppose, there exists $1 \neq a \in U(\mathcal{R})$ and $b \in U(\mathcal{R})$, such that $ab = ba = 1$. Consider, $P_1 : \{(1, 1) \sim (a, 1) \sim (b, 1)\} \in Cr_2(\mathcal{R})$, $P_2 : \{(1, a) \sim (1, b) \sim (a, a) \sim (a, b) \sim (b, a) \sim (b, b)\} \in Cr_2(\mathcal{R})$ are two paths in $Cr_2(\mathcal{R})$, but there is no path between the vertices of P_1 and P_2, leading to a contradiction. Thus, the proof is complete.

□

Lemma 3.2 *Let $\mathcal{R}$ be a ring, then $Cr_2(\mathcal{R})$ has a triangle if $\mathcal{R}$ has a non trivial idempotent.*

Proof Suppose x is non trivial idempotent of $\mathcal{R}$, the cycle in $Cr_2(\mathcal{R})$ of the form $(1, u) \sim (x, u^{-1}) \sim (1 - x, u^{-1}) \sim (1, u)$ is a triangle. Hence, we are done. □

Theorem 3.2 *Let $\mathcal{R}$ be a ring then the following holds:*

(1). If $u^2 = 1 \;\forall\, u \in U(\mathcal{R})$, then $Cr_2(\mathcal{R})$ is regular if and only if $\mathcal{R}$ has no non trivial idempotent.
(2). If for all $1 \neq u \in U(\mathcal{R})$ there exists $1 \neq v \in U(\mathcal{R})$ such that $u \neq v$, $uv = 1$, then $Cr_2(\mathcal{R})$ is regular if and only if $\mathcal{R}$ has no non trivial idempotent.

Proof

(1) We know that $U(\mathcal{R}) = U^{'}(\mathcal{R}) \bigcup U^{''}(\mathcal{R})$ where $U^{'}(\mathcal{R}) = \{u \in U(\mathcal{R}); u^2 = 1\}$ and $U^{''}(\mathcal{R}) = \{u \in U(\mathcal{R});\ uv = 1,\ v \in U(\mathcal{R})\}$. Assume $u \in U^{'}(\mathcal{R})$ and $\mathcal{R}$ has no non trivial idempotent. Then, $V(Cr_2(\mathcal{R})) = \{(e, u);\ \ e \in U_{reg}(\mathcal{R}),\ \ \ u \in U^{'}(\mathcal{R})\}$. Clearly, $(e, u) \sim (f, u)$ for every $e, f \in U_{reg}(\mathcal{R})$ and $u \in U^{'}(\mathcal{R})$, and hence $Cr_2(\mathcal{R}) \cong \{K_{|U_{reg}(\mathcal{R})|} \oplus K_{|U_{reg}(\mathcal{R})|} \oplus \cdots \oplus K_{|U_{reg}(\mathcal{R})|}\}|U_{reg}(\mathcal{R})|$times. This implies $Cr_2(\mathcal{R})$ is a regular graph.
Conversely, given $u^2 = 1 \;\forall\, u \in U(\mathcal{R})$ and $Cr_2(\mathcal{R})$ is regular. Suppose $\mathcal{R}$ has a non trivial idempotent then for $\{(1, u), (e, u), (1 - e, v)\} \in V(Cr_2(\mathcal{R}))$, we have if $(1 - e, u) \sim (1, u)$ then $(1 - e, u) \sim (e, u)$ and $(1 - e, v) \sim (e, u)$ but $(1 - e, v) \nsim (1, u)$, shows that $deg(1, u) < deg(e, u)$, a contradiction.

(2) Let $1 \neq u \in U(\mathcal{R})$ there exists $1 \neq v \in U(\mathcal{R})$ such that $u \neq v$, $uv = 1$, and $\mathcal{R}$ has no non trivial idempotent, then $V(Cr_2(\mathcal{R})) = \{(e, u);\ \ e \in U_{reg}(\mathcal{R}),\ \ \ u \in U^{''}(\mathcal{R})\}$. Clearly, $(e, u) \sim (f, v)$ for every $e, f \in U_{reg}(\mathcal{R}), u, v \in U^{''}(\mathcal{R})$, and hence $Cr_2(\mathcal{R}) \cong \{K_{|U_{reg}(\mathcal{R})|,|U_{reg}(\mathcal{R})|} \oplus K_{|U_{reg}(\mathcal{R})|,|U_{reg}(\mathcal{R})|} \oplus \cdots \oplus K_{|U_{reg}(\mathcal{R})|,|U_{reg}(\mathcal{R})|}\}\frac{|U_{reg}(\mathcal{R})|}{2}$times, which is a regular graph.
Conversely, for given regular graph $Cr_2(\mathcal{R})$ with $1 \neq u \in U(\mathcal{R})$ there exists $1 \neq v \in U(\mathcal{R})$ such that $u \neq v$, $uv = 1$. Suppose $\mathcal{R}$ has a non trivial idempotent. Let $\{(u, v), (1, u), (e, u), (1 - e, u)\} \in V(Cr_2(\mathcal{R}))$. If $(u, v) \sim (1, u)$, then $(u, v) \sim (e, u)$. Also, $(1 - e, u) \sim (e, u)$ but $(1 - e, u) \nsim (1, u)$, this implies $deg(1, u) < deg(e, u)$, a contradiction. Thus, our supposition is wrong. Hence, the proof is complete.

□

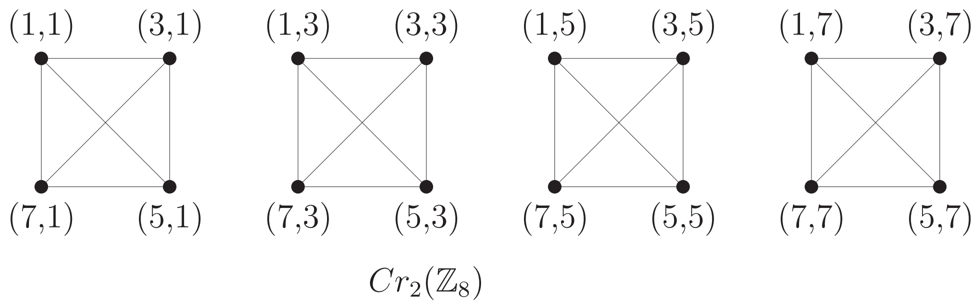


Fig. 3 .

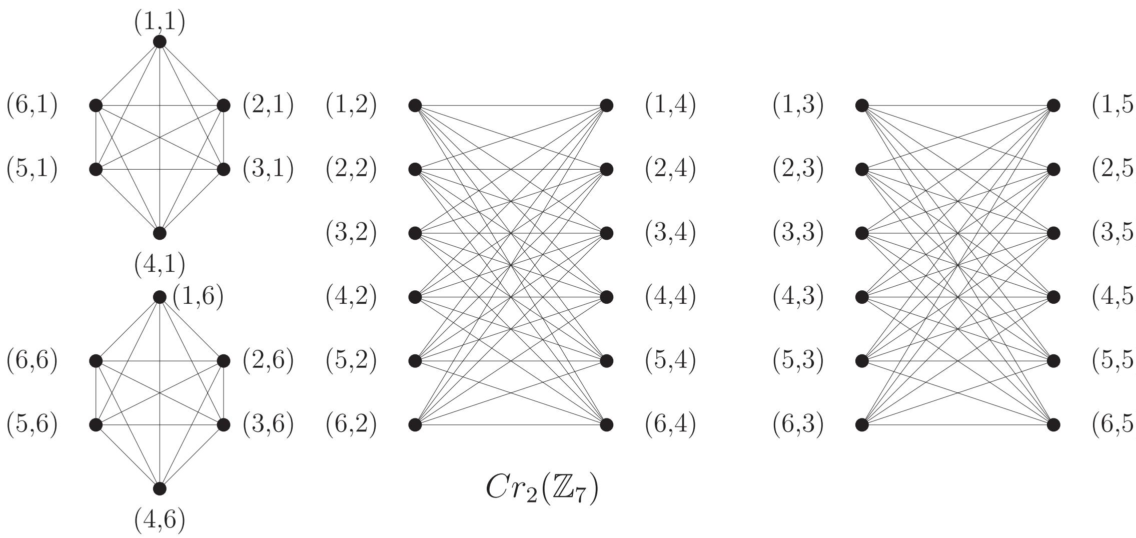


Fig. 4 .

Remark 3.2 It is to be noted that in mixed case that is when both $1 \neq u, v \in U(\mathcal{R})$, such that $u^2 = v^2 = 1$ or $uv = vu = 1$. Then $Cr_2(\mathcal{R})$ is not regular shown in Example 3.2. Also, Example 3.1 is an example of regular graph.

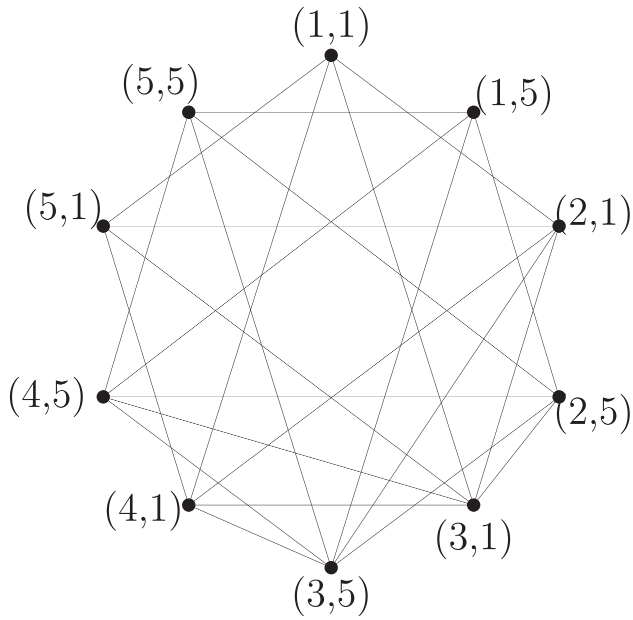
Example 3.1 Let $\mathcal{R} = \mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, then $Id(\mathbb{Z}_8) = \{0, 1\}$ and $U(\mathbb{Z}_8) = \{1, 3, 5, 7\}$ implies $U_{reg}(\mathbb{Z}_8) = \{1, 3, 5, 7\}$. Then $V(Cr_2(\mathbb{Z}_8)) = \{(1, 1), (1, 3), (1, 5), (1, 7), (3, 1), (3, 3), (3, 5), (3, 7), (5, 1), (5, 3), (5, 5), (5, 7), (7, 1), (7, 3), (7, 5), (7, 7)\}$. Then $Cr_2(\mathbb{Z}_8)$ is shown in Figure 3. which is a regular graph.

Example 3.2 Let $\mathcal{R} = \mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, then $Id(\mathbb{Z}_7) = \{0, 1\}$ and $U(\mathbb{Z}_7) = \{1, 2, 3, 4, 5, 6\}$ implies $U_{reg}(\mathbb{Z}_7) = \{1, 2, 3, 4, 5, 6\}$. Then $V(Cr_2(\mathbb{Z}_7)) = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$. Then $Cr_2(\mathbb{Z}_7)$ is shown in Figure 4. which is not a regular graph.

Remark 3.3 In the above theorem, the condition of non trivial idempotentness cannot be relaxed, because consider the ring: $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, thus $Id(\mathbb{Z}_6) = \{0, 1, 3, 4\}$, $U(\mathbb{Z}_6) = \{1, 5\}$, and $U_{reg}(\mathbb{Z}_6) = \{1, 2, 3, 4, 5\}$. Therefore, $V(Cr_2(\mathbb{Z}_6)) = \{(1, 1), (1, 5), (2, 1), (2, 5), (3, 1), (3, 5), (4, 1), (4, 5), (5, 1), (5, 5)\}$. Also, $U(\mathbb{Z}_6)$ satisfies $u^2 = 1$ and $\mathbb{Z}_6$ has non trivial idempotents, but $Cr_2(\mathbb{Z}_6)$ is not regular as shown in Figure 5.

4 Planarity of clear graphs

Here, we examine the finite commutative ring with unity that satisfies the condition where $Cr_2(\mathcal{R})$ demonstrates properties of being planar, unicyclic, a tree, an outer planar, or a split graph.



$Cr_2(\mathbb{Z}_6)$

Fig. 5 .

Theorem 4.1 [12] *(Kuratowski's Theorem) A graph G is planar if and only if it does not contains subdivision of* K_5 *or* $K_{3,3}$.

Theorem 4.2 *Let* $\mathcal{R}$ *be a finite commutative ring with unity. Then* $Cr_2(\mathcal{R})$ *is planar if and only if* $\mathcal{R}$ *is isomorphic to one of the following rings:* $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}, \mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}, \frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2\rangle}, \frac{\mathbb{Z}_4[x]}{\langle 2x, x^2\rangle}, \frac{\mathbb{Z}_2[x,y]}{\langle x^2, xy, y^2\rangle}$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof Assume $Cr_2(\mathcal{R})$ is planar. Since $\mathcal{R}$ is finite, $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where $\mathcal{R}_i$ is a local ring for each i and $n \geq 2$. If $n \geq 3$, an examination reveals that the induced subgraph created by the collection $\{(e_1, u), (e_2, u), (e_3, u), (e_1 + e_2, u), (u, u)\}$, where $u = e_1 + e_2 + \cdots + e_n \in U(\mathcal{R})$, is K_5 in $Cr_2(\mathcal{R})$, leading to a contradiction by Theorem 4.1. Hence, $n \leq 2$.
Suppose $n = 2$. i,e., $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2$. If $|\mathcal{R}_i| \geq 3$ for atleast one i, let us assume $|\mathcal{R}_1| \geq 3$. Then the subgraph induced by the set $\{(e_1, v), (e, v), (e_2, v), (v, v), (f, v)\}$, where $v = e_1 + e_2$, $e = (a, 0)$, $f = (a, 1)$ and $1 \neq a \in U(\mathcal{R}_1)$, is K_5 in $Cr_2(\mathcal{R})$, a contradiction by Theorem 4.1. Hence, $|\mathcal{R}_i| = 2$ for $i = 1, 2$ implies $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Now, suppose $n = 1$ i.e., $\mathcal{R}$ is a local ring. If $\mathcal{R}$ is a field and $|\mathcal{R}| \geq 4$, then $|U(\mathcal{R})| \geq 3$. Let $1, u, u^{-1} \in U(\mathcal{R})$, then the subgraph induced by the set $\{(1, u), (u, u), (u^{-1}, u), (u^{-1}, u^{-1}), (1, u^{-1}), (u, u^{-1})\}$ is $K_{3,3}$ in $Cr_2(\mathcal{R})$, a contradiction by Theorem 4.1. Hence, $|\mathcal{R}| \leq 3$, this implies $\mathcal{R} \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.

Also, if $\mathcal{R}$ is a local ring which is not field and $|\mathcal{R}| \geq 9$ then $|U(\mathcal{R})| \geq 6$. Then $Cr_2(\mathcal{R})$ contains K_5 induced by the set $\{(u_1, 1), (u_2, 1), (u_3, 1), (u_4, 1), (1, 1)\}$, where $1 \neq u_i \in U(\mathcal{R})$, a contradiction. Hence, $|\mathcal{R}| \leq 8$, which Implies that $\mathcal{R} \cong \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}, \mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}, \frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2\rangle}, \frac{\mathbb{Z}_4[x]}{\langle 2x, x^2\rangle}$ or $\frac{\mathbb{Z}_2[x,y]}{\langle x^2, xy, y^2\rangle}$.
Conversely, if $\mathcal{R} \cong \mathbb{Z}_2$, then $Cr_2(\mathcal{R}) \cong K_1$ is a planar graph. If $\mathcal{R} \cong \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, then $Cr_2(\mathcal{R}) \cong K_2 \oplus K_2$ is a planar graph. If $\mathcal{R} \cong \mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}, \frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2\rangle}, \frac{\mathbb{Z}_4[x]}{\langle 2x, x^2\rangle}$ or $\frac{\mathbb{Z}_2[x,y]}{\langle x^2, xy, y^2\rangle}$, then $Cr_2(\mathcal{R}) \cong K_4 \oplus K_4 \oplus K_4 \oplus K_4$ is a planar graph. Finally, if $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $Cr_2(\mathcal{R}) \cong K_3$ is a planar graph. □

Theorem 4.3 *Let* $\mathcal{R}$ *be a finite commutative ring with unity. Then* $Cr_2(\mathcal{R})$ *is unicyclic if and only if* $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof Assume that $Cr_2(\mathcal{R})$ is a unicyclic graph. Since $\mathcal{R}$ is finite, $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where $\mathcal{R}_i$ is a local ring for each i and $n \geq 1$. Suppose $n \geq 3$, then $(e_1, u) \sim (e_2, u) \sim (e_3, u) \sim (e_1, u)$ and $(e_1, u) \sim (u, u) \sim (e_2, u) \sim (e_1, u)$, where $u = e_1 + e_2 + \cdots + e_n \in U(\mathcal{R})$, are two different cycles in $Cr_2(\mathcal{R})$, a contradiction. Hence, $n \leq 2$.

Suppose $n = 2$ and $|\mathcal{R}_i| \geq 3$ for atleast one i, we assume $|\mathcal{R}_1| \geq 3$, then $(e_1, v) \sim (e, v) \sim (e_2, v) \sim (e_1, v)$ and $(e_1, v) \sim (f, v) \sim (v, v) \sim (e_1, v)$, where $v = e_1 + e_2$, $e = (a, 0)$, $f = (a, 1)$ and $1 \neq a \in U(\mathcal{R}_1)$, are two different cycles in $Cr_2(\mathcal{R})$, a contradiction. Hence, $|\mathcal{R}| = 2$ for $i = 1, 2$. This shows that $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Now, suppose $n = 1$. If $|\mathcal{R}| \geq 5$, then $(1, 1) \sim (u_1, 1) \sim (u_2, 1) \sim (1, 1)$ and $(1, 1) \sim (u_1, 1) \sim (u_3, 1) \sim (1, 1)$, where $1 \neq u_i \in U(\mathcal{R})$, are two different cycles in $Cr_2(\mathcal{R})$, a contradiction. Hence, $|\mathcal{R}| \leq 4$. If $\mathcal{R} \cong \mathbb{Z}_2$, then $Cr_2(\mathcal{R}) \cong K_1$ is not unicyclic. If $\mathcal{R} \cong \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then $Cr_2(\mathcal{R}) \cong K_2 \oplus K_2$ is not unicyclic. If $\mathcal{R} \cong \mathbb{F}_4$, then $Cr_2(\mathcal{R}) \cong K_3 \oplus K_{3,3}$ is not unicyclic.
Conversely, if $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $Cr_2(\mathcal{R}) \cong K_3$ is a unicyclic graph. □

Theorem 4.4 *Let $\mathcal{R}$ be a finite commutative ring with unity. Then $Cr_2(\mathcal{R})$ is a tree if and only if $\mathcal{R} \cong \mathbb{Z}_2$.*

Proof Assume that $Cr_2(\mathcal{R})$ is a tree. Since $\mathcal{R}$ is finite, $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where $\mathcal{R}_i$ is a local ring for each i and $n \geq 1$. If $n \geq 2$, then $(e_1, u) \sim (e_2, u) \sim (u, u) \sim (e_1, u)$, where $u = e_1 + e_2 + \cdots + e_n \in U(\mathcal{R})$, is a cycle in $Cr_2(\mathcal{R})$, a contradiction. Hence, $n = 1$, which implies that $\mathcal{R}$ is a local ring. If $|\mathcal{R}| \geq 5$, then $|U(\mathcal{R})| \geq 4$, this implies $|U_{reg}(\mathcal{R})| \geq 4$. Let $1 \neq u_1, u_2 \in U(\mathcal{R})$, then $(u_1, 1) \sim (u_2, 1) \sim (1, 1) \sim (u_1, 1)$ is a cycle in $Cr_2(\mathcal{R})$, a contradiction. Hence, $|\mathcal{R}| \leq 4$. If $\mathcal{R} \cong \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then $Cr_2(\mathcal{R}) \cong K_2 \oplus K_2$ is disconnected, a contradiction. If $\mathcal{R} \cong \mathbb{F}_4$, then $Cr_2(\mathcal{R}) \cong K_3 \oplus K_{3,3}$ contains a cycle, a contradiction. Hence, $\mathcal{R} \cong \mathbb{Z}_2$.
Conversely, if $\mathcal{R} \cong \mathbb{Z}_2$, then $Cr_2(\mathcal{R}) \cong K_1$ is a tree. □

Theorem 4.5 [13] *A graph G is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.*

Theorem 4.6 *Let $\mathcal{R}$ be a finite commutative ring with unity. Then $Cr_2(\mathcal{R})$ is outerplanar if and only if $\mathcal{R}$ is isomorphic to one of the following rings: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof Assume $Cr_2(\mathcal{R})$ is outerplanar. Since $\mathcal{R}$ is finite, $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where $\mathcal{R}_i$ is a local ring for each i and $n \geq 1$. Suppose $n \geq 3$, then the set $\{(e_1, u), (e_2, u), (e_3, u), (u, u)\}$, $u = e_1 + e_2 + \cdots + e_n \in U(\mathcal{R})$, induces a subgraph which is isomorphic to K_4, leading to a contradiction by Theorem 4.5. Hence, $n \leq 2$.
Suppose $n = 2$ and $|\mathcal{R}_i| \geq 3$ for atleast one i, then $U(\mathcal{R}_i) \geq 2$. Let us take $|\mathcal{R}_1| \geq 3$. Then the subgraph generated by the set $\{(e_1, v), (e_2, v), (e, v), (f, v)\}$, where $v = e_1 + e_2$, $e = (a, 0)$, $f = (a, 1)$ and $1 \neq a \in U(\mathcal{R}_1)$, is K_4 in $Cr_2(\mathcal{R})$, a contradiction by Theorem 4.5. Hence, $|\mathcal{R}_i| = 2$ for $i = 1, 2$. This shows that $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
Now, suppose $n = 1$ i.e., $\mathcal{R}$ is a local ring. If $|\mathcal{R}| \geq 5$ then $|U(\mathcal{R})| \geq 4$. Let $1 \neq u_1, u_2, u_3 \in U(\mathcal{R})$, then the subgraph induced by the set $\{(1, 1), (u_1, 1), (u_2, 1), (u_3, 1)\}$ is K_4 in $Cr_2(\mathcal{R})$, leading to a contradiction by Theorem 4.5. Hence, $|\mathcal{R}| \leq 4$. If $\mathcal{R}$ is a field and $|\mathcal{R}| = 4$, then $Cr_2(\mathcal{R}) \cong K_3 \oplus K_{3,3}$ contains $K_{2,3}$ as a subgraph, a contradiction. Hence, $\mathcal{R} \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.
Converse follows from Figures 6 and 7 and the proof is complete. □

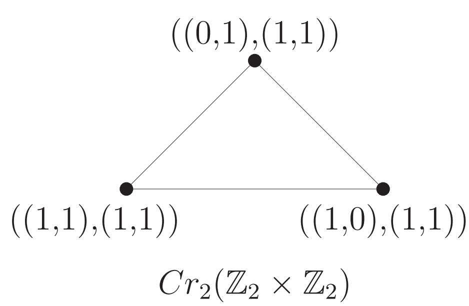


Fig. 6 .

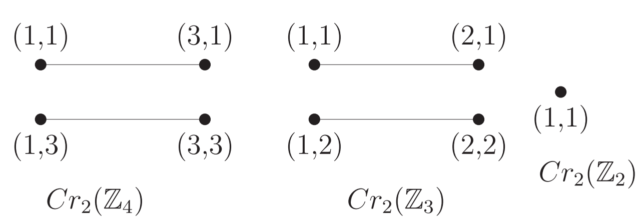


Fig. 7 .

Theorem 4.7 [12] *Let G be a connected graph, then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2$, C_4 or C_5.*

Theorem 4.8 *Let $\mathcal{R}$ be a finite commutative ring with unity. Then $Cr_2(\mathcal{R})$ is a split graph if and only if $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.*

Proof Assume that $Cr_2(\mathcal{R})$ is a split graph. Since $\mathcal{R}$ is finite, $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where $\mathcal{R}_i$ is a local ring for each i and $n \geq 1$. Suppose $n \geq 3$. If $|\mathcal{R}_i| \geq 3$, for atleast one i. Without loss of generality, we assume $|\mathcal{R}_1| \geq 3$. Then the induced subgraph generated by the set $\{(e_1 + e_3, u), (e_1 + e_2, u), (e_2, v), (e_3, v)\}$, where $u = e_1 + e_2 + \cdots + e_n$, $v = (a, 1, 1, \ldots, 1)$ and $1 \neq a \in U(\mathcal{R}_1)$, is C_4 in $Cr_2(\mathcal{R})$, a contradiction by Theorem 4.7. Hence, $|\mathcal{R}_i| \leq 2$ for each i. This implies that $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.
Suppose $n = 2$ and $|\mathcal{R}_i| \geq 3$ for atleast one i, say $|\mathcal{R}_1| \geq 3$. Then the induced subgraph generated by the set $\{(e_1, u), (e_2, u), (e, u), (u, u)\}$, where $u = e_1 + e_2$, $e = (a, 1)$ and $1 \neq a \in U(\mathcal{R}_1)$, is C_4 in $Cr_2(\mathcal{R})$, a contradiction by Theorem 4.7. Hence, $|\mathcal{R}_i| = 2$ for $i = 1, 2$.
Finally, suppose $n = 1$ i.e., $\mathcal{R}$ is a local ring. If $|\mathcal{R}| \geq 3$, then $|U(\mathcal{R})| \geq 2$. Let $1 \neq a \in U(\mathcal{R})$, then the induced subgraph generated by the set $\{(1, 1), (a, 1), (1, a), (a, a^{-1})\}$ is isomprphic to $2K_2$, which is again a contradiction by Theorem 4.7. Hence, $\mathcal{R} \cong \mathbb{Z}_2$.
Conversely, if $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, then $Cr_2(\mathcal{R})$ is a complete graph and hence, a split graph. □

5 Genus of clear graphs

In this section, we will examine the characterization of finite commutative rings $\mathcal{R}$ for which $Cr_2(\mathcal{R})$ displays a genus of one.

The term genus of a graph refers to the minimum integer value, denoted as $\gamma(G)$, that allows the graph to be represented on a sphere with a certain number of handles or loops without any crossings. Put simply, it indicates how many handles need to be added to a sphere so that the graph can be drawn on it without intersecting itself. For instance, planar graphs have a genus of 0 as they can be accommodated on a sphere without crossings. In the context of determining the genus for complete and complete bipartite graphs, we consider the subsequent findings.

Lemma 5.1 [13] $\gamma(K_m) = \lceil \frac{(m-3)(m-4)}{12} \rceil$ *if $m \geq 3$. In particular, $\gamma(K_m) = 1$ if $m = 5, 6, 7$.*

Lemma 5.2 [13] $\gamma(K_{n,m}) = \lceil \frac{(n-2)(m-2)}{4} \rceil$ *if $n, m \geq 2$. In particular, $\gamma(K_{4,4}) = \gamma(K_{3,m}) = 1$ if $m = 3, 4, 5, 6$. Also, $\gamma(K_{5,4}) = \gamma(K_{6,4}) = \gamma(K_{m,3}) = 2$ if $m = 7, 8, 9, 10$.*

Theorem 5.1 [15] *The genus of a graph is the sum of the genera of its blocks.*

Now, we can outline the distinctive features of finite commutative rings $\mathcal{R}$ in situations where $Cr_2(\mathcal{R})$ displays a genus of one.

Theorem 5.2 *Let $\mathcal{R}$ be a finite commutative ring with unity. Then $\gamma(Cr_2(\mathcal{R})) = 1$ if and only if $\mathcal{R}$ is one of the following rings: $\mathbb{F}_4$, $\mathbb{Z}_5$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof Assume that $\gamma(Cr_2(\mathcal{R})) = 1$. Since $\mathcal{R}$ is finite, $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$, where $\mathcal{R}_i$ is a local ring for each i and $n \geq 1$. Suppose $n \geq 4$. Then the subgraph induced by the set $S = \{(e_1, u), (e_2, u), (e_3, u), (e_4, u), (e_1 + e_2, u), (e_1 + e_3, u), (e_1 + e_4, u), (e_2 + e_3, u)\}$, where $u = e_1 + e_2 + \cdots + e_n \in U(\mathcal{R})$, is K_8 in $Cr_2(\mathcal{R})$. Thus, by Lemma 5.1, $\gamma(Cr_2(\mathcal{R})) > 1$, a contradiction. Hence, $n \leq 3$.
Suppose $n = 3$. If $|\mathcal{R}_i| \geq 3$ for atleast one i, then $|U(\mathcal{R}_i)| \geq 2$. Let us assume $|\mathcal{R}_1| \geq 3$. Since $u_1 = (1, 1, 1) \in U(\mathcal{R})$ implies $u_1 = (1, 1, 1) \in U_{reg}(\mathcal{R})$. Also, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, $e_4 = (1, 1, 0)$, $e_5 = (0, 1, 1)$, $e_6 = (1, 0, 1)$, $e_7 = (1, 1, u) \in U_{reg}(\mathcal{R})$. Then the subgraph in $Cr_2(\mathcal{R})$ induced by the set $S = \{(e_1, v), (e_2, v), (e_3, v), (e_1 + e_2, v), (e_2 + e_3, v), (e_1 + e_3, v), (e, v), (v, v)\}$, where $v = e_1 + e_2 + e_3$, $e = (a, 1, 1)$ and $1 \neq a \in U(\mathcal{R}_1)$, is K_8 in $Cr_2(\mathcal{R})$. Thus, by Lemma 5.1, $\gamma(Cr_2(\mathcal{R})) > 1$, a contradiction. Hence, $|\mathcal{R}_1| = 2$. Similarly, we can show that $|\mathcal{R}_2| = |\mathcal{R}_3| = 2$. Hence, $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
Suppose $n = 2$ and $|\mathcal{R}_i| \geq 5$ for atleast one i. Without loss of generality, we can assume that $|\mathcal{R}_1| \geq 5$. Then the subgraph induced by the set $\{(e_1, w), (e_2, w), (f_1, w), (f_2, w), (f_3, w), (f_4, w), (f_5, w), (f_6, w)\}$, where $w = e_1 + e_2$, $f_1 = (a_1, 1)$, $f_2 = (a_2, 1)$, $f_3 = (a_3, 1)$, $f_4 = (a_1, 0)$, $f_5 = (a_2, 0)$, $f_6 = (a_3, 0)$ and $1 \neq a_i \in U(\mathcal{R}_1)$, is K_8 in $Cr_2(\mathcal{R})$. Thus, by Lemma 5.1, $\gamma(Cr_2(\mathcal{R})) > 1$, a contradiction. Hence, $|\mathcal{R}_i| \leq 4$ for $i = 1, 2$.

Suppose $|\mathcal{R}_i| \geq 3$ for all i, then $Cr_2(\mathcal{R})$ contains K_8 generated by the set $\{(e_1, w), (e_2, w), (g_1, w), (g_2, w), (g_3, w), (g_4, w), (g_5, w), (w, w)\}$, where $w = e_1 + e_2$, $g_1 = (b_1, 1)$, $g_2 = (1, b_2)$, $g_3 = (b_1, 0)$, $g_4 = (0, b_2)$, $g_5 = (b_1, b_2)$ and $1 \neq b_i \in U(\mathcal{R}_i)$ for $i = 1, 2$. Thus, by Lemma 5.1, $\gamma(Cr_2(\mathcal{R})) > 1$, a contradiction. Moreover, if $|\mathcal{R}_1| = |\mathcal{R}_2| = 2$ then by Theorem 4.2, $\gamma(Cr_2(\mathcal{R})) = 0$, a contradiction. Thus $\mathcal{R}$ may be $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_2 \times \mathbb{F}_4$. If $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then by Theorem 5.1, a contradiction. Moreover, if $\mathcal{R} \cong \mathbb{Z}_2 \times \mathbb{F}_4$, then the subgraph generated by the set $\{((0, 1), (1, w)), ((0, w), (1, w)), ((0, w^{-1}), (1, w)), ((1, w), (1, w)), ((1, 1), (1, w)), ((0, 1), (1, w^{-1})), ((0, w), (1, w^{-1})), ((0, w^{-1}), (1, w^{-1})), ((1, 1), (1, w^{-1}))\}$ is $k_{5,4}$, a contradiction.

Finally, suppose $n = 1$ i.e., $\mathcal{R}$ is a local ring. Suppose $\mathcal{R}$ is a field and $|\mathcal{R}| \geq 7$. Then $\{1, u_1, u_1^{-1}, u_2, u_2^{-1}, u_3\} \in U(\mathcal{R})$ such that $Cr_2(\mathcal{R})$ has a subgraph induced by the set $\{(1, u_1), (u_1, u_1), (u_2, u_1), (u_1^{-1}, u_1), (u_3, u_1), (1, u_1^{-1}), (u_1, u_1^{-1}), (u_2, u_1^{-1}), (u_1^{-1}, u_1^{-1})\}$ which is isomorphic to $K_{5,4}$, a contradiction by Lemma 5.2. If $|\mathcal{R}| = 2, 3$ then by Theorem 4.2, $\gamma(Cr_2(\mathcal{R})) = 0$, a contradiction. Hence, $\mathcal{R} \cong \mathbb{F}_4$ or $\mathbb{Z}_5$.

Now, suppose $\mathcal{R}$ is a local ring which is not a field. If $|\mathcal{R}| \geq 9$ then $|U(\mathcal{R})| \geq 6$. Then $Cr_2(\mathcal{R})$ has $K_{5,4}$ induced by the set $\{(1, u_1), (u_1, u_1), (u_2, u_1), (u_1^{-1}, u_1), (u_3, u_1), (1, u_1^{-1}), (u_1, u_1^{-1}), (u_2, u_1^{-1}), (u_1^{-1}, u_1^{-1})\}$, a contradiction by Lemma 5.2. Hence, $|\mathcal{R}| \leq 8$. If $|\mathcal{R}| = 4$ *or* 8, then by Theorem 4.2, $\gamma(Cr_2(\mathcal{R})) = 0$, a contradiction.

Converse follows from Figure 8, 9, and 10. □

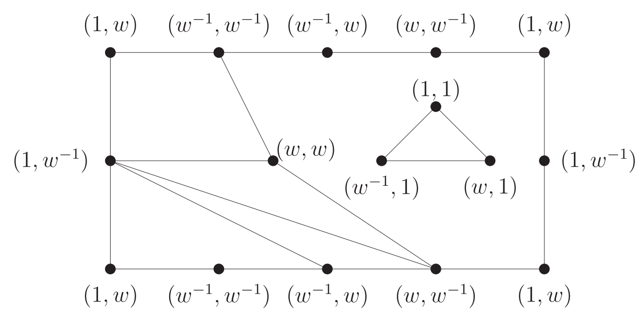


Fig. 8 Toroidal embedding of $Cr_2(\mathbb{F}_4)$

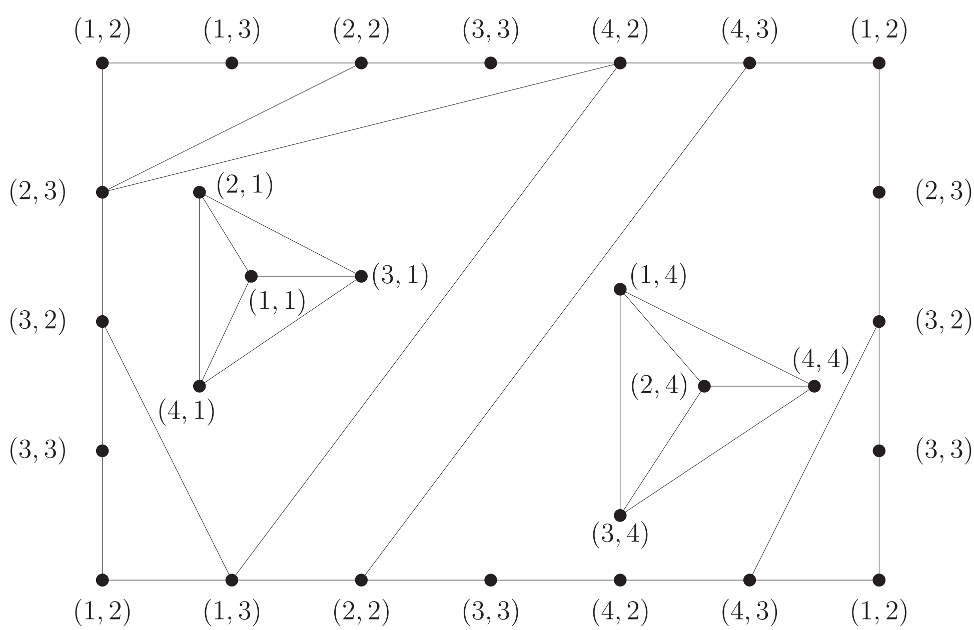


Fig. 9 Toroidal embedding of $Cr_2(\mathbb{Z}_5)$

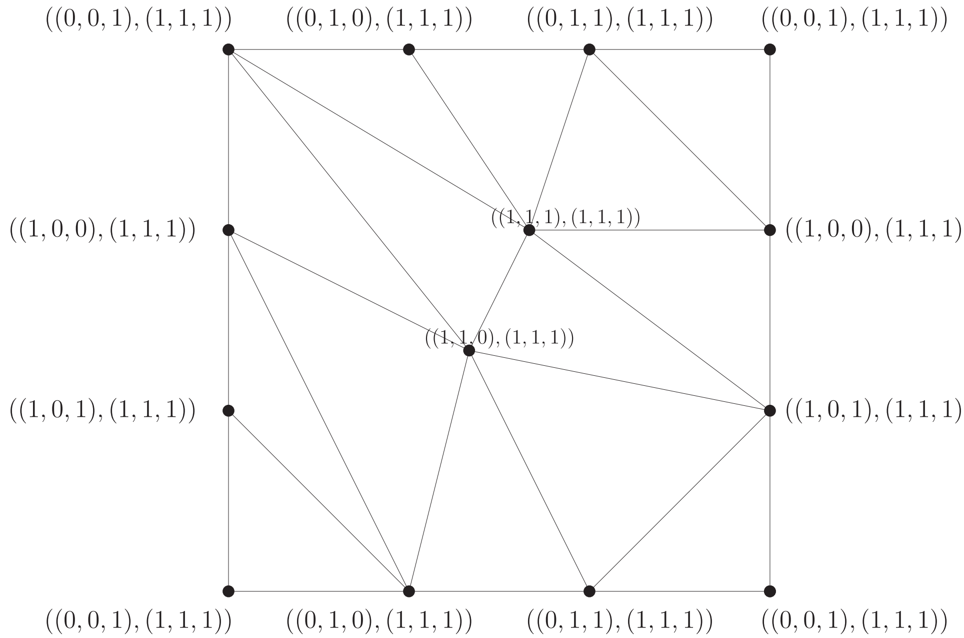


Fig. 10 Toroidal embedding of $Cr_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

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