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ORIGINAL RESEARCH

Theta function identities and vanishing coefficient results for continued fractions of order sixty-four

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Abstract Eight continued fractions of order sixty-four are derived from a general continued fraction identity of Ramanujan. A general theta-function identity is proved for the eight continued fractions and partition-theoretic results are obtained therefrom. Some vanishing coefficient arising for the continued fractions and their reciprocals are also offered.

Keywords Continued fraction · Theta-function identities · Partition of integer · Colour partition · Vanishing coefficients

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1 Introduction

Throughout the paper, for any complex numbers δ and q, define the q-product as

$$(\delta; q)_0 = 1, \qquad (\delta; q)_t := \prod_{k=1}^{t-1}(1-\delta q^k) \quad \text{for} \quad t \geq 1. \quad (\delta; q)_\infty := \prod_{t=0}^{\infty}\left(1-\delta q^t\right); \qquad |q| < 1. \tag{1.1}$$

For brevity, we will write

$$(\delta_1; q)_\infty(\delta_2; q)_\infty(\delta_3; q)_\infty \ldots (\delta_m; q)_\infty = (\delta_1, \delta_2, \delta_3, \ldots, \delta_m; q)_\infty .$$

Ramanujan's general theta-function $\mathfrak{f}(w, z)$ [2, p. 34] is defined as

$$\mathfrak{f}(w, z) = \sum_{t=-\infty}^{\infty} w^{t(t+1)/2} z^{t(t-1)/2}, \qquad |wz| < 1. \tag{1.2}$$

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Also, Jacobi's triple product identity [2, p. 35, Entry 19] is given as

$$\mathfrak{f}(w,z)=(-w;wz)_\infty(-z;wz)_\infty(wz;wz)_\infty=(-w,-z,wz;wz)_\infty. \tag{1.3}$$

Three important special cases of $\mathfrak{f}(w,z)$ [2, p. 36, Entry 22 (i)-(iii)] are given by

$$\phi(q):=\mathfrak{f}(q,q)=\sum_{t=-\infty}^{\infty}q^{t^2}, \tag{1.4}$$

$$\psi(q):=\mathfrak{f}(q,q^3)=\sum_{t=0}^{\infty}q^{t(t+1)/2}, \tag{1.5}$$

and

$$f(-q):=\mathfrak{f}(-q,-q^2)=\sum_{t=-\infty}^{\infty}(-1)^t q^{t(3t-1)/2}. \tag{1.6}$$

Ramanujan [2, p. 24, Entry 12] in his notebook recorded the following general continued fraction identity: Let a, b and q are complex numbers such that $|ab|<1$ and $|q|<1$, or for some integer m, $a=b^{2m+1}$. Then

$$\frac{(a^2q^3;q^4)_\infty(b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty(b^2q;q^4)_\infty}=\cfrac{1}{1-ab+\cfrac{(a-bq)(b-aq)}{(1-ab)(q^2+1)+\cfrac{(a-bq^3)(b-aq^3)}{(1-ab)(q^4+1)+\cdots}}}. \tag{1.7}$$

As special cases of (1.7), we derived the following eight continued fractions of order sixty-four:

$$E_1(q)=q^{15/2}\frac{\mathfrak{f}(-q,-q^{63})}{\mathfrak{f}(-q^{31},-q^{33})}=\cfrac{q^{15/2}(1-q)}{(1-q^{16})+\cfrac{q^{16}(1-q^{15})(1-q^{17})}{(1-q^{16})(1+q^{32})+\cfrac{q^{16}(1-q^{47})(1-q^{49})}{(1-q^{16})(1+q^{64})+\cdots}}}, \tag{1.8}$$

$$E_2(q)=q^{13/2}\frac{\mathfrak{f}(-q^3,-q^{61})}{\mathfrak{f}(-q^{29},-q^{35})}=\cfrac{q^{13/2}(1-q^3)}{(1-q^{16})+\cfrac{q^{16}(1-q^{13})(1-q^{19})}{(1-q^{16})(1+q^{32})+\cfrac{q^{16}(1-q^{45})(1-q^{51})}{(1-q^{16})(1+q^{64})+\cdots}}}, \tag{1.9}$$

$$E_3(q)=q^{11/2}\frac{\mathfrak{f}(-q^5,-q^{59})}{\mathfrak{f}(-q^{27},-q^{37})}=\cfrac{q^{11/2}(1-q^5)}{(1-q^{16})+\cfrac{q^{16}(1-q^{11})(1-q^{21})}{(1-q^{16})(1+q^{32})+\cfrac{q^{16}(1-q^{43})(1-q^{53})}{(1-q^{16})(1+q^{64})+\cdots}}}, \tag{1.10}$$

$$E_4(q)=q^{9/2}\frac{\mathfrak{f}(-q^7,-q^{57})}{\mathfrak{f}(-q^{25},-q^{39})}=\cfrac{q^{9/2}(1-q^7)}{(1-q^{16})+\cfrac{q^{16}(1-q^{9})(1-q^{23})}{(1-q^{16})(1+q^{32})+\cfrac{q^{16}(1-q^{41})(1-q^{55})}{(1-q^{16})(1+q^{64})+\cdots}}}, \tag{1.11}$$

$$E_5(q)=q^{7/2}\frac{\mathfrak{f}(-q^9,-q^{55})}{\mathfrak{f}(-q^{23},-q^{41})}=\cfrac{q^{7/2}(1-q^9)}{(1-q^{16})+\cfrac{q^{16}(1-q^{7})(1-q^{25})}{(1-q^{16})(1+q^{32})+\cfrac{q^{16}(1-q^{39})(1-q^{57})}{(1-q^{16})(1+q^{64})+\cdots}}},$$

$$E_6(q)=q^{5/2}\frac{\mathfrak{f}(-q^{11},-q^{53})}{\mathfrak{f}(-q^{21},-q^{43})}=\cfrac{q^{5/2}(1-q^{11})}{(1-q^{16})+\cfrac{q^{16}(1-q^{5})(1-q^{27})}{(1-q^{16})(1+q^{32})+\cfrac{q^{16}(1-q^{37})(1-q^{59})}{(1-q^{16})(1+q^{64})+\cdots}}}, \tag{1.12}$$

$$E_7(q)=q^{3/2}\frac{\mathfrak{f}(-q^{13},-q^{51})}{\mathfrak{f}(-q^{19},-q^{45})}=\cfrac{q^{3/2}(1-q^{13})}{(1-q^{16})+\cfrac{q^{16}(1-q^{3})(1-q^{29})}{(1-q^{16})(1+q^{32})+\cfrac{q^{16}(1-q^{35})(1-q^{61})}{(1-q^{16})(1+q^{64})+\cdots}}}, \tag{1.13}$$

$$E_8(q)=q^{1/2}\frac{\mathfrak{f}(-q^{15},-q^{49})}{\mathfrak{f}(-q^{17},-q^{47})}=\cfrac{q^{1/2}(1-q^{15})}{(1-q^{16})+\cfrac{q^{16}(1-q)(1-q^{31})}{(1-q^{16})(1+q^{32})+\cfrac{q^{16}(1-q^{33})(1-q^{63})}{(1-q^{16})(1+q^{64})+\cdots}}}. \tag{1.14}$$

(1.15)

To obtain $E_1(q)$, $E_2(q)$, $E_3(q)$, $E_4(q)$, $E_5(q)$, $E_6(q)$, $E_7(q)$ and $E_8(q)$, we first replace q by q^{16} in (1.7), set $\{a=q^{15/2}, b=q^{17/2}\}$, $\{a=q^{13/2}, b=q^{19/2}\}$, $\{a=q^{11/2}, b=q^{21/2}\}$, $\{a=q^{9/2}, b=q^{23/2}\}$, $\{a=q^{7/2}, b=q^{25/2}\}$, $\{a=q^{5/2}, b=q^{27/2}\}$, $\{a=q^{3/2}, b=q^{29/2}\}$ and $\{a=q^{1/2}, b=q^{31/2}\}$ and then simplify using (1.3) and the results that $\{(q^{65};q^{64})_\infty=(q;q^{64})_\infty/(1-q)\}$, $\{(q^{67};q^{64})_\infty=(q^3;q^{64})_\infty/(1-q^3)\}$, $\{(q^{69};q^{64})_\infty=(q^5;q^{64})_\infty/(1-q^5)\}$, $\{(q^{71};q^{64})_\infty=(q^7;q^{64})_\infty/(1-q^7)\}$, $\{(q^{73};q^{64})_\infty=(q^9;q^{64})_\infty/(1-q^9)\}$, $\{(q^{75};q^{64})_\infty=(q^{11};q^{64})_\infty/(1-q^{11})\}$,$\{(q^{77};q^{64})_\infty=(q^{13};q^{64})_\infty/(1-q^{13})\}$ and $\{(q^{79};q^{64})_\infty=(q^{15};q^{64})_\infty/(1-q^{15})\}$, respectively.

The main objective of this paper is to establish some theta-function identities for the continued fractions $E_1(q)$, $E_2(q)$, $E_3(q)$, $E_4(q)$, $E_5(q)$, $E_6(q)$, $E_7(q)$ and $E_8(q)$ and to give some applications. In Sect. 2, we establish some theta-function identities for the continued fractions. In Sect. 3, we show that colour partition identities can be obtained from the theta-function identities of continued fractions with suitable examples. We also offer some vanishing coefficient results for the eight continued fractions and their reciprocals in Section 4.

2 Theta-function identities

Theorem 2.1 *For $m=1,2,3,4,5,6,7$ and 8, we have*

$$\frac{1}{E_m(q)}\pm E_m(q)=\frac{\mathfrak{f}\big(\pm q^{16-(2m-1)},\pm q^{16+(2m-1)}\big)\phi(\mp q^{16})}{q^{(16-(2m-1))/2}\mathfrak{f}\big(-q^{2m-1},-q^{32-(2m-1)}\big)\psi(q^{32})}.$$

Proof We give the proof of the case $m=1$ only and omit other cases as they follow similarly.

From (1.8), we obtain

$$\frac{1}{\sqrt{E_1(q)}}-\sqrt{E_1(q)}=\frac{\mathfrak{f}(-q^{31},-q^{33})-q^{15/2}\mathfrak{f}(-q,-q^{63})}{\sqrt{q^{15/2}\mathfrak{f}(-q,-q^{63})\mathfrak{f}(-q^{31},-q^{33})}}. \tag{2.1}$$

From [2, p. 46, Entry 30 (ii) and (iii)], we note that

$$\mathfrak{f}(w,z)=\mathfrak{f}(w^3z,wz^3)+w\mathfrak{f}(z/w,w^5z^3). \tag{2.2}$$

Setting $\{w=-q^{15/2}, z=q^{17/2}\}$ and $\{w=q^{15/2}, z=-q^{17/2}\}$ in (2.2), we obtain

$$\mathfrak{f}(-q^{15/2},q^{17/2})=\mathfrak{f}(-q^{31},-q^{33})-q^{15/2}\mathfrak{f}(-q,-q^{63}) \tag{2.3}$$

and

$$\mathfrak{f}(q^{15/2}, -q^{17/2}) = \mathfrak{f}(-q^{31}, -q^{33}) + q^{15/2}\mathfrak{f}(-q, -q^{63}), \tag{2.4}$$

respectively. Again, from [2, p. 46, Entry 30 (i)], we note that

$$\mathfrak{f}(w, wz^2)\mathfrak{f}(z, w^2z) = \mathfrak{f}(w, z)\psi(wz). \tag{2.5}$$

Setting $w = -q$ and $z = -q^{31}$ in (2.5), we obtain

$$\mathfrak{f}(-q, -q^{63})\mathfrak{f}(-q^{31}, -q^{33}) = \mathfrak{f}(-q, -q^{31})\psi(q^{32}). \tag{2.6}$$

Employing (2.3) and (2.6) in (2.1), we find that

$$\frac{1}{\sqrt{E_1(q)}} - \sqrt{E_1(q)} = \frac{\mathfrak{f}(-q^{15/2}, q^{17/2})}{\sqrt{q^{15/2}\mathfrak{f}(-q, -q^{31})\psi(q^{32})}}. \tag{2.7}$$

Similarly, from (1.8) and applying (2.4), we obtain

$$\frac{1}{\sqrt{E_1(q)}} + \sqrt{E_1(q)} = \frac{\mathfrak{f}(q^{15/2}, -q^{17/2})}{\sqrt{q^{15/2}\mathfrak{f}(-q, -q^{31})\psi(q^{32})}}. \tag{2.8}$$

Combining (2.7) and (2.8), we obtain

$$\frac{1}{E_1(q)} - E_1(q) = \frac{\mathfrak{f}(-q^{15/2}, q^{17/2})\mathfrak{f}(q^{15/2}, -q^{17/2})}{q^{15/2}\mathfrak{f}(-q, -q^{31})\psi(q^{32})}. \tag{2.9}$$

Again, from [2, p. 46, Entry 30 (iv)], we note that

$$\mathfrak{f}(w, z)\mathfrak{f}(-w, -z) = \mathfrak{f}(-w^2, -z^2)\phi(-wz). \tag{2.10}$$

Setting $w = -q^{15/2}$ and $z = q^{17/2}$ in (2.10), we obtain

$$\mathfrak{f}(-q^{15/2}, q^{17/2})\mathfrak{f}(q^{15/2}, -q^{17/2}) = \mathfrak{f}(-q^{15}, -q^{17})\phi(q^{16}). \tag{2.11}$$

Employing (2.11) in (2.9), we obtain

$$\frac{1}{E_1(q)} - E_1(q) = \frac{\mathfrak{f}(-q^{15}, -q^{17})\phi(q^{16})}{q^{15/2}\mathfrak{f}(-q, -q^{31})\psi(q^{32})}. \tag{2.12}$$

Squaring (2.7), we obtain

$$\frac{1}{E_1(q)} + E_1(q) = \frac{\mathfrak{f}^2(-q^{15/2}, q^{17/2})}{q^{15/2}\mathfrak{f}(-q, -q^{31})\psi(q^{32})} + 2. \tag{2.13}$$

From [2, p. 46, Entry 30 (v),(vi)], we note that

$$\mathfrak{f}^2(w, z) = \mathfrak{f}(w^2, z^2)\phi(wz) + 2w\mathfrak{f}(z/w, w^3z)\psi(w^2z^2). \tag{2.14}$$

Setting $w = -q^{15/2}$ and $z = q^{17/2}$ in (2.14), we obtain

$$\mathfrak{f}^2(-q^{15/2}, q^{17/2}) = \mathfrak{f}(q^{15}, q^{17})\phi(-q^{16}) - 2q^{15/2}\mathfrak{f}(-q, -q^{31})\psi(q^{32}). \tag{2.15}$$

Employing (2.15) in (2.13), we obtain

$$\frac{1}{E_1(q)} + E_1(q) = \frac{\mathfrak{f}(q^{15}, q^{17})\phi(-q^{16})}{q^{15/2}\mathfrak{f}(-q, -q^{31})\psi(q^{32})}. \tag{2.16}$$

Thus, proof of the theorem (that is, identities (2.12) and (2.16)) is complete for the case $m = 1$. □

Corollary 2.2 *For* $(u, v) = \{(1, 8), (2, 7), (3, 6), (4, 5)\}$, *we have*

$$\left(\frac{1}{E_u(q)} - E_u(q)\right)\left(\frac{1}{E_v(q)} - E_v(q)\right) = \frac{\phi^2(q^{16})}{q^8\psi^2(q^{32})}. \tag{2.17}$$

Proof We give the proof of the case $(u, v) = (1, 8)$ and omit other cases as the proofs are analogous. Setting $m = 8$ in Theorem 2.1, we obtain

$$\frac{1}{E_8(q)} - E_8(q) = \frac{\mathfrak{f}(-q, -q^{31})\phi(q^{16})}{q^{1/2}\mathfrak{f}(-q^{15}, -q^{17})\psi(q^{32})}. \tag{2.18}$$

Multiplying (2.12) and (2.18), we obtain

$$\left(\frac{1}{E_1(q)} - E_1(q)\right)\left(\frac{1}{E_8(q)} - E_8(q)\right) = \frac{\phi^2(q^{16})}{q^8\psi^2(q^{32})}. \tag{2.19}$$

□

Theorem 2.3 *For a positive integer k and* $i = 1, 2, 3, 4, 5, 6, 7$ *and* 8, *we have*

$$E_i^k(q)E_i^k(-q) = \begin{cases} E_i^k(q^2), & if \quad k \equiv 0 \pmod 4 \\ -E_i^k(q^2), & if \quad k \equiv 2 \pmod 4). \end{cases}$$

Proof Take $i = 1$, then from (1.8), we obtain

$$E_1^k(q)E_1^k(-q) = (-1)^{15k/2}q^{15k}\frac{\mathfrak{f}^k(-q, -q^{63})}{\mathfrak{f}^k(-q^{31}, -q^{33})} \times \frac{\mathfrak{f}^k(q, q^{63})}{\mathfrak{f}^k(q^{31}, q^{33})}. \tag{2.20}$$

Setting $\{w = q, z = q^{63}\}$ and $\{w = q^{31}, z = q^{33}\}$ in (2.10), we find that

$$\mathfrak{f}(q, q^{63})\mathfrak{f}(-q, -q^{63}) = \mathfrak{f}(-q^2, -q^{126})\phi(-q^{64}), \tag{2.21}$$

and

$$\mathfrak{f}(q^{31}, q^{33})\mathfrak{f}(-q^{31}, -q^{33}) = \mathfrak{f}(-q^{62}, -q^{66})\phi(-q^{64}), \tag{2.22}$$

respectively. Employing (2.21) and (2.22) in (2.20), we obtain

$$E_1^k(q)E_1^k(-q) = (-1)^{15k/2}q^{15k}\frac{\mathfrak{f}^k(-q^2, -q^{126})\phi^k(-q^{64})}{\mathfrak{f}^k(-q^{62}, -q^{66})\phi^k(-q^{64})} = (-1)^{15k/2}E_1^k(q^2). \tag{2.23}$$

Using fact that $15k/2$ is even if $k \equiv 0 \pmod 4$ and odd if $k \equiv 2 \pmod 4$ in (2.23), we complete the proof of the case $i = 1$. The cases $i = 2, 3, 4, 5, 6, 7$ and 8 of the theorem follow similarly, so omitted. □

3 Partition theoretic results

In this section, we show that colour partition identities can be obtained from the theta-function identities established in Theorem 2.1. As example, we deduce two partition-theoretic identities from the theta-function identities of the continued fraction $E_1(q)$ using colour partition of integer. One can also obtain colour partition identities from the other theta-function identities of Theorem 2.1. First, we will give the definition of colour partition of integer and its generating functions.

A partition of a natural number n can be defined as finite sequence of positive integers $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that

$$\sum_{j=1}^{k}\alpha_j = n; \qquad \alpha_j \geq \alpha_{j+1},$$

where α_j are called parts or summands of the partition.

A summand in a partition of n is said to have r different colours if each part of the partition is assumed to have r distinct copies or colours. For any natural numbers n and r, let $\xi_r(n)$ count the total number of partitions of n with each part having r different colours. For example, if each part of partition of 3 has 2 colours, say green (signified by the suffix g) and black (signified by the suffix b), then the number of 2 colour partition of 3 is 10, namely 3_g, 3_b, 2_g+1_g, 2_g+1_b, 2_b+1_g, 2_b+1_b, $1_g+1_g+1_g$, $1_g+1_g+1_b$, $1_g+1_b+1_b$, $1_b+1_b+1_b$. The generating function of $\xi_r(n)$ is given by

$$\sum_{n=0}^{\infty}\xi_r(n)q^n=\frac{1}{(q;q)_\infty^r}. \tag{3.1}$$

For positive integers s, m and r, the generating function of the total number of partitions of n with parts congruent to s modulo m and each part has r colours is given by

$$\frac{1}{(q^s;q^m)_\infty^r}. \tag{3.2}$$

For convenience, we write

$$(q^{r\pm};q^t)_\infty := (q^r,q^{t-r};q^t)_\infty, \tag{3.3}$$

where r and t are natural numbers with $r<t$.

Theorem 3.1 *For any integer $n\geq 15$, let $\mathcal{E}_1(n)$:= number of partitions of n into parts $\equiv \pm1$, ±15, ±17 or $\pm32 \pmod{64}$ such that the parts $\equiv \pm1$ and $\pm32 \pmod{64}$ have 2 colours, $\mathcal{E}_2(n)$:= number of partitions of n into parts $\equiv \pm15, \pm17, \pm31$ or $\pm32 \pmod{64}$ such that parts $\equiv \pm31$ and $\pm32 \pmod{64}$ have 2 colours and $\mathcal{E}_3(n)$:= number of partitions of n into parts $\equiv \pm1, \pm16$ and $\pm31 \pmod{64}$ with 2 colours. Then*

$$\mathcal{E}_1(n)-\mathcal{E}_2(n-15)-\mathcal{E}_3(n)=0.$$

Proof Employing (1.3), (1.4), (1.5) and (1.8) in the theta-function identity (2.12) in Theorem 2.1, we obtain

$$\frac{(q^{31\pm};q^{64})_\infty}{q^{15/2}(q^{1\pm};q^{64})_\infty}-q^{15/2}\frac{(q^{1\pm};q^{64})_\infty}{(q^{31\pm};q^{64})_\infty}-\frac{(q^{15\pm};q^{32})_\infty(q^{32\pm};q^{32})_\infty^6}{q^{15/2}(q^{1\pm};q^{32})_\infty(q^{16};q^{16})_\infty^2(q^{64};q^{64})^4}=0. \tag{3.4}$$

Changing the products in the third term of (3.4) to base q^{64} and simplifying, we obtain

$$\frac{(q^{31\pm};q^{64})_\infty}{(q^{1\pm};q^{64})_\infty}-q^{15}\frac{(q^{1\pm};q^{64})_\infty}{(q^{31\pm};q^{64})_\infty}-\frac{(q^{15\pm},q^{17\pm};q^{64})_\infty(q^{32\pm};q^{64})_\infty^2}{(q^{1\pm},q^{31\pm};q^{64})_\infty(q^{16\pm};q^{64})_\infty^2}=0. \tag{3.5}$$

Dividing (3.5) by $(q^{1\pm,15\pm,17\pm,31\pm};q^{64})_\infty(q^{32\pm},q^{64})_\infty^2$, we obtain

$$\frac{1}{(q^{15\pm,17\pm};q^{64})_\infty(q^{1\pm,32\pm};q^{64})_\infty^2}-\frac{q^{15}}{(q^{15\pm,17\pm};q^{64})_\infty(q^{31\pm,32\pm};q^{64})_\infty^2}-\frac{1}{(q^{1\pm,16\pm,31\pm};q^{64})_\infty^2}=0. \tag{3.6}$$

Equation (3.6) is equivalent to

$$\sum_{n=0}^{\infty}\mathcal{E}_1(n)q^n-q^{15}\sum_{n=0}^{\infty}\mathcal{E}_2(n)q^n-\sum_{n=0}^{\infty}\mathcal{E}_3(n)q^n=0, \tag{3.7}$$

where we take $\mathcal{E}_1(0)=\mathcal{E}_2(0)=\mathcal{E}_3(0)=1$. Comparing the coefficients of q^n on both sides of (3.7), we complete the proof. □

Example: By enumerating the relevant partitions of $n=15$, one can verify that $\mathcal{E}_1(15)=17$, $\mathcal{E}_2(0)=1$ and $\mathcal{E}_3(15)=16$ satisfy Theorem 3.1.

Theorem 3.2 *For any integer $n \geq 15$, let $\mathcal{W}_1(n) :=$ number of partitions of n into parts $\equiv \pm 1, \pm 16, \pm 30$ or $\pm 32 \pmod{64}$ such that the parts $\equiv \pm 1$ and $\pm 16 \pmod{64}$ have 2 colours, $\mathcal{W}_2(n) :=$ number of partitions of n into parts $\equiv \pm 16, \pm 30, \pm 31$ or $\pm 32 \pmod{64}$ such that parts $\equiv \pm 16$ and $\pm 31 \pmod{64}$ have 2 colours and $\mathcal{W}_3(n) :=$ number of partitions of n into parts $\equiv \pm 1, \pm 15, \pm 17$ and $\pm 31 \pmod{64}$ such that the parts $\equiv \pm 1$ and $\pm 31 \pmod{64}$ have 2 colours. Then*

$$\mathcal{W}_1(n) + \mathcal{W}_2(n-15) - \mathcal{W}_3(n) = 0.$$

Proof Proceeding as in the proof of Theorem 3.1, identity (2.16) can be expressed as

$$\frac{1}{(q^{30\pm,32\pm};q^{64})_\infty (q^{1\pm,16\pm};q^{64})_\infty^2} + \frac{q^{15}}{(q^{30\pm,32\pm};q^{64})_\infty (q^{16\pm,31\pm};q^{64})_\infty^2}$$

$$- \frac{1}{(q^{15\pm,17\pm};q^{64})_\infty (q^{1\pm,31\pm};q^{64})_\infty^2} = 0. \tag{3.8}$$

Noting the generating functions, we write (3.8) as

$$\sum_{n=0}^{\infty} \mathcal{W}_1(n) q^n + q^{15} \sum_{n=0}^{\infty} \mathcal{W}_2(n) q^n - \sum_{n=0}^{\infty} \mathcal{W}_3(n) q^n = 0, \tag{3.9}$$

where we take $\mathcal{W}_1(0) = \mathcal{W}_2(0) = \mathcal{W}_3(0) = 1$. The required result follows easily from (3.9). □

Example: By enumerating the relevant partitions of $n = 15$, one can see that $\mathcal{W}_1(16) = 19$, $\mathcal{W}_2(1) = 0$ and $\mathcal{W}_3(16) = 19$ satisfy Theorem 3.2.

4 Vanishing coefficients

Recently, many authors have studied vanishing coefficients in the arithmetic progressions of several q-series expansions. One can see [1], Hirschhorn [3] and references there in for details. In this section, we obtain vanishing coefficient results arising from the q-product expansion of the continued fractions $E_i(q) (i = 1, 2, 3, 4, 5, 6, 7, 8)$ and their reciprocals. From [2, p. 34, Entry 18(iv)], we note here that, for integers $t > s > 0$,

$$\mathfrak{f}(-q^t, -q^{-s}) = -q^{-s} \mathfrak{f}(-q^s, -q^{t-2s}). \tag{4.1}$$

Theorem 4.1 *If*

$$E_1^*(q) := q^{-15/2} E_1(q) = \frac{\mathfrak{f}(-q, -q^{63})}{\mathfrak{f}(-q^{31}, -q^{33})} = \sum_{n=0}^{\infty} \zeta_n q^n \quad \textit{and} \quad \frac{1}{E_1^*(q)} = \sum_{n=0}^{\infty} \zeta_n' q^n ,$$

then

$$(i) \quad \zeta_{32n+16} = 0 \quad \textit{and} \quad (ii) \quad \zeta'_{32n+31} = 0.$$

Proof Write

$$\sum_{n=0}^{\infty} \zeta_n q^n = \frac{\mathfrak{f}(-q, -q^{63})}{\mathfrak{f}(-q^{31}, -q^{33})} = \frac{\mathfrak{f}\left(-q, -q^{63}\right) \mathfrak{f}\left(q^{31}, q^{33}\right)}{\mathfrak{f}\left(-q^{31}, -q^{33}\right) \mathfrak{f}\left(q^{31}, q^{33}\right)}. \tag{4.2}$$

From [2, p. 45, Entry 29], we note that, if a, b, c and d are complex numbers satisfying $ab = cd$, then

$$\mathfrak{f}(a, b)\mathfrak{f}(c, d) = \mathfrak{f}(ac, bd)\mathfrak{f}(ad, bc) + a\mathfrak{f}(b/c, ac^2 d)\mathfrak{f}(b/d, acd^2). \tag{4.3}$$

Setting $a=-q, b=-q^{63}, c=q^{31}, d=q^{33}$ in (4.3), we obtain

$$\mathfrak{f}(-q,-q^{63})\mathfrak{f}(q^{31},q^{33})=\mathfrak{f}(-q^{32},-q^{96})\mathfrak{f}(-q^{34},-q^{94})-q\mathfrak{f}(-q^{32},-q^{96})\mathfrak{f}(-q^{30},-q^{98}). \tag{4.4}$$

Setting $w=q^{31}$ and $z=q^{33}$ in (2.10), we obtain

$$\mathfrak{f}(q^{31},q^{33})\mathfrak{f}(-q^{31},-q^{33})=\mathfrak{f}(-q^{62},-q^{66})\phi(-q^{64}). \tag{4.5}$$

Employing (4.5) and (4.11) in (4.2), we obtain

$$\sum_{n=0}^{\infty}\zeta_n q^n=\frac{\mathfrak{f}(-q^{32},-q^{96})\mathfrak{f}(-q^{34},-q^{94})-q\mathfrak{f}(-q^{32},-q^{96})\mathfrak{f}(-q^{30},-q^{98})}{\mathfrak{f}(-q^{62},-q^{66})\phi(-q^{64})}. \tag{4.6}$$

Extracting the terms containing q^{2n} in (4.6) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\zeta_{2n} q^n=\frac{\mathfrak{f}(-q^{16},-q^{48})\mathfrak{f}(-q^{17},-q^{47})}{\mathfrak{f}(-q^{31},-q^{33})\phi(-q^{32})}=\frac{\mathfrak{f}(-q^{16},-q^{48})\mathfrak{f}(-q^{17},-q^{47})\mathfrak{f}(q^{31},q^{33})}{\mathfrak{f}(-q^{31},-q^{33})\mathfrak{f}(q^{31},q^{33})\phi(-q^{32})}. \tag{4.7}$$

Setting $a=-q^{17}, b=-q^{47}, c=q^{31}$ and $d=q^{33}$ in (4.3), we obtain

$$\mathfrak{f}(-q^{17},-q^{47})\mathfrak{f}(q^{31},q^{33})=\mathfrak{f}(-q^{48},-q^{80})\mathfrak{f}(-q^{50},-q^{78})-q^{17}\mathfrak{f}(-q^{16},-q^{112})\mathfrak{f}(-q^{14},-q^{114}). \tag{4.8}$$

Applying (4.5) and (4.8) in (4.7), we obtain

$$\sum_{n=0}^{\infty}\zeta_{2n} q^n=\frac{\mathfrak{f}(-q^{16},-q^{48})\left(\mathfrak{f}(-q^{48},-q^{80})\mathfrak{f}(-q^{50},-q^{78})-q^{17}\mathfrak{f}(-q^{16},-q^{112})\mathfrak{f}(-q^{14},-q^{114})\right)}{\mathfrak{f}(-q^{62},-q^{66})\phi(-q^{32})\phi(-q^{64})}. \tag{4.9}$$

Again, extracting the terms containing q^{2n} in (4.9) and replacing q^2 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\zeta_{4n} q^n&=\frac{\mathfrak{f}(-q^{8},-q^{24})\mathfrak{f}(-q^{24},-q^{40})\mathfrak{f}(-q^{25},-q^{39})}{\mathfrak{f}(-q^{31},-q^{33})\phi(-q^{16})\phi(-q^{32})}\\&=\frac{\mathfrak{f}(-q^{8},-q^{24})\mathfrak{f}(-q^{24},-q^{40})\mathfrak{f}(-q^{25},-q^{39})\mathfrak{f}(q^{31},q^{33})}{\mathfrak{f}(-q^{31},-q^{33})\phi(-q^{16})\phi(-q^{32})\mathfrak{f}(q^{31},q^{33})}.\end{aligned} \tag{4.10}$$

Setting $a=-q^{25}, b=-q^{39}, c=q^{31}$ and $d=q^{33}$ in (4.3), we obtain

$$\mathfrak{f}(-q^{25},-q^{39})\mathfrak{f}(q^{31},q^{33})=\mathfrak{f}(-q^{56},-q^{72})\mathfrak{f}(-q^{58},-q^{70})-q^{25}\mathfrak{f}(-q^{8},-q^{120})\mathfrak{f}(-q^{6},-q^{122}). \tag{4.11}$$

Applying (4.5) and (4.11) in (4.10), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\zeta_{4n} q^n=&\frac{\mathfrak{f}(-q^{8},-q^{24})\mathfrak{f}(-q^{24},-q^{40})}{\mathfrak{f}(-q^{62},-q^{66})\phi(-q^{16})\phi(-q^{32})\phi(-q^{64})}\Big[\mathfrak{f}(-q^{56},-q^{72})\mathfrak{f}(-q^{58},-q^{70})\\&-q^{25}\mathfrak{f}(-q^{8},-q^{120})\mathfrak{f}(-q^{6},-q^{122})\Big].\end{aligned} \tag{4.12}$$

Extracting the terms containing q^{2n} and replacing q^2 by q in (4.12), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\zeta_{8n} q^n&=\frac{\mathfrak{f}(-q^{4},-q^{12})\mathfrak{f}(-q^{12},-q^{20})\mathfrak{f}(-q^{28},-q^{36})\mathfrak{f}(-q^{29},-q^{35})}{\phi(-q^{8})\phi(-q^{16})\phi(-q^{32})\mathfrak{f}(-q^{31},-q^{33})}\\&=\frac{\mathfrak{f}(-q^{4},-q^{12})\mathfrak{f}(-q^{12},-q^{20})\mathfrak{f}(-q^{28},-q^{36})\mathfrak{f}(-q^{29},-q^{35})\mathfrak{f}(q^{31},q^{33})}{\phi(-q^{8})\phi(-q^{16})\phi(-q^{32})\mathfrak{f}(-q^{31},-q^{33})\mathfrak{f}(q^{31},q^{33})}.\end{aligned} \tag{4.13}$$

Setting $a=-q^{29}$, $b=-q^{35}$, $c=q^{31}$ and $d=q^{33}$ in (4.3), we obtain

$$\mathfrak{f}(-q^{29},-q^{35})\mathfrak{f}(q^{31},q^{33})=\mathfrak{f}(-q^{60},-q^{68})\mathfrak{f}(-q^{62},-q^{66})-q^{29}\mathfrak{f}(-q^{4},-q^{124})\mathfrak{f}(-q^{2},-q^{126}). \quad (4.14)$$

Applying (4.5) and (4.14) in (4.13), we obtain

$$\sum_{n=0}^{\infty}\zeta_{8n}q^n=\frac{\mathfrak{f}(-q^4,-q^{12})\mathfrak{f}(-q^{12},-q^{20})\mathfrak{f}(-q^{28},-q^{36})}{\mathfrak{f}(-q^{62},-q^{66})\phi(-q^8)\phi(-q^{16})\phi(-q^{32})\phi(-q^{64})}\Big[\mathfrak{f}(-q^{60},-q^{68})\mathfrak{f}(-q^{62},-q^{66})$$

$$-q^{29}\mathfrak{f}(-q^4,-q^{124})\mathfrak{f}(-q^2,-q^{126})\Big]. \quad (4.15)$$

Again, extracting the terms containing q^{2n} and replacing q^2 by q in (4.15), we obtain

$$\sum_{n=0}^{\infty}\zeta_{16n}q^n=\frac{\mathfrak{f}(-q^2,-q^6)\mathfrak{f}(-q^6,-q^{10})\mathfrak{f}(-q^{14},-q^{18})\mathfrak{f}(-q^{30},-q^{34})}{\phi(-q^4)\phi(-q^8)\phi(-q^{16})\phi(-q^{32})}. \quad (4.16)$$

The right hand side of (4.16) contains no term involving q^{2n+1}, so extracting terms containing q^{2n+1}, we arrive at (i).

To prove (ii), we note that

$$\sum_{n=0}^{\infty}\zeta'_n q^n=\frac{\mathfrak{f}(-q^{31},-q^{33})}{\mathfrak{f}(-q,-q^{63})}=\frac{\mathfrak{f}\left(-q^{31},-q^{33}\right)\mathfrak{f}\left(q,q^{63}\right)}{\mathfrak{f}\left(-q,-q^{63}\right)\mathfrak{f}\left(q,q^{63}\right)}. \quad (4.17)$$

Setting $a=-q^{31}$, $b=-q^{33}$, $c=q$, $d=q^{63}$ in (4.3), we obtain

$$\mathfrak{f}(-q^{31},-q^{33})\mathfrak{f}(q,q^{63})=\mathfrak{f}(-q^{32},-q^{96})\mathfrak{f}(-q^{94},-q^{34})-q^{31}\mathfrak{f}(-q^{32},-q^{96})\mathfrak{f}(-q^{-30},-q^{158}). \quad (4.18)$$

Setting $s=30$ and $t=158$ in (4.1), we obtain

$$\mathfrak{f}(-q^{158},-q^{-30})=-q^{-30}\mathfrak{f}(-q^{30},-q^{98}). \quad (4.19)$$

Employing (4.19) in (4.18), we obtain

$$\mathfrak{f}(-q^{31},-q^{33})\mathfrak{f}(q,q^{63})=\mathfrak{f}(-q^{32},-q^{96})\mathfrak{f}(-q^{94},-q^{34})+q\mathfrak{f}(-q^{32},-q^{96})\mathfrak{f}(-q^{30},-q^{98}). \quad (4.20)$$

Setting $w=q$ and $z=q^{63}$ in (2.10), we obtain

$$\mathfrak{f}(q,q^{63})\mathfrak{f}(-q,-q^{63})=\mathfrak{f}(-q^2,-q^{126})\phi(-q^{64}). \quad (4.21)$$

Employing (4.20) and (4.21) in (4.17), we obtain

$$\sum_{n=0}^{\infty}\zeta'_n q^n=\frac{\mathfrak{f}(-q^{32},-q^{96})\mathfrak{f}(-q^{94},-q^{34})+q\mathfrak{f}(-q^{32},-q^{96})\mathfrak{f}(-q^{30},-q^{98})}{\mathfrak{f}(-q^2,-q^{126})\phi(-q^{64})}. \quad (4.22)$$

Extracting the terms containing q^{2n+1} in (4.22), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\zeta'_{2n+1}q^n=\frac{\mathfrak{f}(-q^{16},-q^{48})\mathfrak{f}(-q^{15},-q^{49})}{\mathfrak{f}(-q,-q^{63})\phi(-q^{32})}=\frac{\mathfrak{f}(-q^{16},-q^{48})\mathfrak{f}(-q^{15},-q^{49})\mathfrak{f}(q,q^{63})}{\mathfrak{f}(-q,-q^{63})\mathfrak{f}(q,q^{63})\phi(-q^{32})}. \quad (4.23)$$

Setting $a=-q^{15}$, $b=-q^{49}$, $c=q$ and $d=q^{63}$ in (4.3), we obtain

$$\mathfrak{f}(-q^{15},-q^{49})\mathfrak{f}(q,q^{63})=\mathfrak{f}(-q^{16},-q^{112})\mathfrak{f}(-q^{78},-q^{50})-q^{15}\mathfrak{f}(-q^{48},-q^{80})\mathfrak{f}(-q^{-14},-q^{142}). \quad (4.24)$$

Setting $s=14$ and $t=142$ in (4.1), we obtain

$$\mathfrak{f}(-q^{142},-q^{-14})=-q^{-14}\mathfrak{f}(-q^{14},-q^{114}). \tag{4.25}$$

Employing (4.25) in (4.24), we obtain

$$\mathfrak{f}(-q^{15},-q^{49})\mathfrak{f}(q,q^{63})=\mathfrak{f}(-q^{16},-q^{112})\mathfrak{f}(-q^{78},-q^{50})+q\mathfrak{f}(-q^{48},-q^{80})\mathfrak{f}(-q^{14},-q^{114}). \tag{4.26}$$

Applying (4.21) and (4.26) in (4.23), we obtain

$$\sum_{n=0}^{\infty}\zeta'_{2n+1}q^n=\frac{\mathfrak{f}(-q^{16},-q^{48})\Big[\mathfrak{f}(-q^{16},-q^{112})\mathfrak{f}(-q^{78},-q^{50})+q\mathfrak{f}(-q^{48},-q^{80})\mathfrak{f}(-q^{14},-q^{114})\Big]}{\mathfrak{f}(-q^2,-q^{126})\phi(-q^{32})\phi(-q^{64})}. \tag{4.27}$$

Extracting the terms containing q^{2n+1} in (4.27), dividing by q and replacing q^2 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\zeta'_{4n+3}q^n&=\frac{\mathfrak{f}(-q^8,-q^{24})\mathfrak{f}(-q^{24},-q^{40})\mathfrak{f}(-q^7,-q^{57})}{\mathfrak{f}(-q,-q^{63})\phi(-q^{16})\phi(-q^{32})}\\&=\frac{\mathfrak{f}(-q^8,-q^{24})\mathfrak{f}(-q^{24},-q^{40})\mathfrak{f}(-q^7,-q^{57})\mathfrak{f}(q,q^{63})}{\mathfrak{f}(-q,-q^{63})\mathfrak{f}(q,q^{63})\phi(-q^{16})\phi(-q^{32})}.\end{aligned} \tag{4.28}$$

Setting $a=-q^7$, $b=-q^{57}$, $c=q$ and $d=q^{63}$ in (4.3), we obtain

$$\mathfrak{f}(-q^7,-q^{57})\mathfrak{f}(q,q^{63})=\mathfrak{f}(-q^8,-q^{120})\mathfrak{f}(-q^{70},-q^{58})-q^7\mathfrak{f}(-q^{56},-q^{72})\mathfrak{f}(-q^{-6},-q^{134}). \tag{4.29}$$

Setting $s=6$ and $t=134$ in (4.1), we obtain

$$\mathfrak{f}(-q^{134},-q^{-6})=-q^{-6}\mathfrak{f}(-q^6,-q^{122}). \tag{4.30}$$

Employing (4.30) in (4.29), we obtain

$$\mathfrak{f}(-q^7,-q^{57})\mathfrak{f}(q,q^{63})=\mathfrak{f}(-q^8,-q^{120})\mathfrak{f}(-q^{70},-q^{58})+q\mathfrak{f}(-q^{56},-q^{72})\mathfrak{f}(-q^6,-q^{122}). \tag{4.31}$$

Applying (4.21) and (4.31) in (4.28), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\zeta'_{4n+3}q^n=\frac{\mathfrak{f}(-q^8,-q^{24})\mathfrak{f}(-q^{24},-q^{40})}{\mathfrak{f}(-q^2,-q^{126})\phi(-q^{16})\phi(-q^{32})\phi(-q^{64})}\Big[&\mathfrak{f}(-q^8,-q^{120})\mathfrak{f}(-q^{70},-q^{58})\\&+q\mathfrak{f}(-q^{56},-q^{72})\mathfrak{f}(-q^6,-q^{122})\Big].\end{aligned} \tag{4.32}$$

Extracting the terms containing q^{2n+1} in (4.32), dividing by q and replacing q^2 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\zeta'_{8n+7}q^n&=\frac{\mathfrak{f}(-q^4,-q^{12})\mathfrak{f}(-q^{12},-q^{20})\mathfrak{f}(-q^{28},-q^{36})\mathfrak{f}(-q^3,-q^{61})}{\mathfrak{f}(-q,-q^{63})\phi(-q^8)\phi(-q^{16})\phi(-q^{32})}\\&=\frac{\mathfrak{f}(-q^4,-q^{12})\mathfrak{f}(-q^{12},-q^{20})\mathfrak{f}(-q^{28},-q^{36})\mathfrak{f}(-q^3,-q^{61})\mathfrak{f}(q,q^{63})}{\mathfrak{f}(-q,-q^{63})\mathfrak{f}(q,q^{63})\phi(-q^8)\phi(-q^{16})\phi(-q^{32})}.\end{aligned} \tag{4.33}$$

Setting $a=-q^3$, $b=-q^{61}$, $c=q$ and $d=q^{63}$ in (4.3), we obtain

$$\mathfrak{f}(-q^3,-q^{61})\mathfrak{f}(q,q^{63})=\mathfrak{f}(-q^4,-q^{124})\mathfrak{f}(-q^{66},-q^{62})-q^3\mathfrak{f}(-q^{60},-q^{68})\mathfrak{f}(-q^{-2},-q^{130}). \tag{4.34}$$

Setting $s = 2$ and $t = 130$ in (4.1), we obtain

$$\mathfrak{f}(-q^{130}, -q^{-2}) = -q^{-2}\mathfrak{f}(-q^2, -q^{126}). \tag{4.35}$$

Employing (4.35) in (4.34), we obtain

$$\mathfrak{f}(-q^3, -q^{61})\mathfrak{f}(q, q^{63}) = \mathfrak{f}(-q^4, -q^{124})\mathfrak{f}(-q^{66}, -q^{62}) + q\mathfrak{f}(-q^{60}, -q^{68})\mathfrak{f}(-q^2, -q^{126}). \tag{4.36}$$

Applying (4.21) and (4.36) in (4.33), we obtain

$$\sum_{n=0}^{\infty} \zeta'_{8n+7}q^n = \frac{\mathfrak{f}(-q^4, -q^{12})\mathfrak{f}(-q^{12}, -q^{20})\mathfrak{f}(-q^{28}, -q^{36})}{\mathfrak{f}(-q^2, -q^{126})\phi(-q^8)\phi(-q^{16})\phi(-q^{32})\phi(-q^{64})}\Big[\mathfrak{f}(-q^4, -q^{124})\mathfrak{f}(-q^{66}, -q^{62}) + q\mathfrak{f}(-q^{60}, -q^{68})\mathfrak{f}(-q^2, -q^{126})\Big]. \tag{4.37}$$

Extracting the terms containing q^{2n+1} in (4.37), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \zeta'_{16n+15}q^n = \frac{\mathfrak{f}(-q^2, -q^6)\mathfrak{f}(-q^6, -q^{10})\mathfrak{f}(-q^{14}, -q^{18})\mathfrak{f}(-q^{30}, -q^{34})}{\phi(-q^4)\phi(-q^8)\phi(-q^{16})\phi(-q^{32})}. \tag{4.38}$$

The right hand side of (4.38) contains no term containing q^{2n+1}, so extracting terms containing q^{2n+1} from (4.38), we complete the proof. □

In the following table, we present vanishing coefficients of the q-series expresssions connected with the continued fractions $E_2(q)$, $E_3(q)$, $E_4(q)$, $E_5(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$ and their reciprocals. Proofs are identical to the proof of Theorem 4.1.

q-series/Continued fractions	Vanishing coefficients
$E_2^*(q) := q^{-13/2}E_2(q) = \dfrac{\mathfrak{f}(-q^3, -q^{61})}{\mathfrak{f}(-q^{29}, -q^{35})} = \sum_{n=0}^{\infty} \lambda_n q^n$	$\lambda_{32n+13} = 0$
$\dfrac{1}{E_2^*(q)} = \sum_{n=0}^{\infty} \lambda'_n q^n$	$\lambda'_{32n+26} = 0$
$E_3^*(q) := q^{-11/2}E_3(q) = \dfrac{\mathfrak{f}(-q^5, -q^{59})}{\mathfrak{f}(-q^{27}, -q^{37})} = \sum_{n=0}^{\infty} \alpha_n q^n$	$\alpha_{32n+6} = 0$
$\dfrac{1}{E_3^*(q)} = \sum_{n=0}^{\infty} \alpha'_n q^n$	$\alpha'_{32n+17} = 0$
$E_4^*(q) := q^{-9/2}E_4(q) = \dfrac{\mathfrak{f}(-q^7, -q^{57})}{\mathfrak{f}(-q^{25}, -q^{39})} = \sum_{n=0}^{\infty} \gamma_n q^n$	$\gamma_{32n+27} = 0$
$\dfrac{1}{E_4^*(q)} = \sum_{n=0}^{\infty} \gamma'_n q^n$	$\gamma'_{32n+4} = 0$
$E_5^*(q) := q^{-7/2}E_5(q) = \dfrac{\mathfrak{f}(-q^9, -q^{55})}{\mathfrak{f}(-q^{23}, -q^{41})} = \sum_{n=0}^{\infty} \beta_n q^n$	$\beta_{32n+12} = 0$
$\dfrac{1}{E_5^*(q)} = \sum_{n=0}^{\infty} \beta'_n q^n$	$\beta'_{32n+19} = 0$
$E_6^*(q) := q^{-5/2}E_6(q) = \dfrac{\mathfrak{f}(-q^{11}, -q^{53})}{\mathfrak{f}(-q^{21}, -q^{43})} = \sum_{n=0}^{\infty} \delta_n q^n$	$\delta_{32n+25} = 0$
$\dfrac{1}{E_6^*(q)} = \sum_{n=0}^{\infty} \delta'_n q^n$	$\delta'_{32n+30} = 0$
$E_7^*(q) := q^{-3/2}E_7(q) = \dfrac{\mathfrak{f}(-q^{13}, -q^{51})}{\mathfrak{f}(-q^{19}, -q^{45})} = \sum_{n=0}^{\infty} \epsilon_n q^n$	$\epsilon_{32n+2} = 0$
$\dfrac{1}{E_7^*(q)} = \sum_{n=0}^{\infty} \epsilon'_n q^n$	$\epsilon'_{32n+5} = 0$
$E_8^*(q) := q^{-7/2}E_8(q) = \dfrac{\mathfrak{f}(-q^9, -q^{55})}{\mathfrak{f}(-q^{23}, -q^{41})} = \sum_{n=0}^{\infty} \mu_n q^n$	$\mu_{32n+7} = 0$
$\dfrac{1}{E_8^*(q)} = \sum_{n=0}^{\infty} \mu'_n q^n$	$\mu'_{32n+8} = 0$

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interests.

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