Indian J Pure Appl Math (2026) 57:597–627
https://doi.org/10.1007/s13226-024-00543-1

ORIGINAL RESEARCH

Multi-step inertial algorithms for equilibrium, fixed point, general systems of variational inequalities and split feasibility problems

Haiying Li · Jiaoying He · Fenghui Wang

Received: 14 October 2023 / Accepted: 12 January 2024 / Published online: 27 January 2024

Abstract In this paper, we present two novel multi-step inertial iterative methods to approximate a common element that combines equilibrium problems with other problems in real Hilbert spaces. Firstly, by combining equilibrium problems, fixed point problems and a general system of variational inequalities, we adopt a conjugate gradient method to solve them. Secondly, by integrating equilibrium problems, fixed point problems and split feasibility problems, we employ a S-iteration process to address them. Under appropriate assumptions and mild conditions, we obtain strong convergence theorems of our proposed algorithms. Finally, some numerical experiments are provided to test and verify the efficiency of the proposed algorithms.

Keywords Multi-step inertial effect · Equilibrium problem · Fixed point problem · Variational inequality · Split feasibility problem

Mathematics Subject Classification 47H09 · 47J25 · 47H10 · 49J40 · 90C99

1 Introduction

Throughout this article, H is a real Hilbert space and C is a nonempty closed convex subset of H, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$ on H. Let $f : H \times H \to \mathbb{R}$ be a bifuntion with $f(x, x) = 0$ for all $x \in C$. The Equilibrium Problem (shortly, EP [2,23]) for the bifunction f on C is stated as follows:

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of EP (1.1) is denoted by $EP(f)$.

The EP is a general model in the sense that it unifies many important mathematical problems, such as variational inequality problems, fixed point problems, complementarity problems, Nash equilibrium problems (see, e.g., [2,3,8,10,15,22,23,33,35,45]). In addition, the EP has direct applications in many fields, such as network problems, electricity markets. It is why the EP has been attracted the interest and considered by

Communicated by K Sandeep.

H. Li (✉)· J. He
College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China
E-mail: lihaiying@htu.edu.cn

J. He
E-mail: hjy8423@163.com

F. Wang
Faculty of Mathematical Sciences, Luoyang Normal University, Luoyang 471934, China
E-mail: wfenghui@163.com

researchers in recent years. At present, strategies for solving the EP mainly include the proximal point methods [12,21], the projection algorithms [9,37], the gap function methods [20,25] and extragradient methods [36,38].

An extremely significant research field in nonlinear analysis is fixed point theory. Recall that the Fixed Point Problem (shortly, FPP) for a mapping $T : H \to H$ is said as follows:

$$\text{find } x^* \in H \text{ such that } Tx^* = x^*. \tag{1.2}$$

The set of solutions of FPP (1.2) is denoted by $F(T)$.

There exist several methods for solving the FPP. One of the most popular methods is Mann's method [19] which was first presented in 1953. The form is as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.3}$$

where $\{\alpha_n\} \subset [0, 1]$. Reich proved in 1979 [26] that if the sequence $\{\alpha_n\}$ satisfies $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ then the sequence $\{x_n\}$ generated by (1.3) converges weakly to a fixed point of a nonexpansive mapping T. Moreover, conjugate gradient methods [24] are popular acceleration methods of the steepest descent method. The conjugate gradient direction of f at x_n is that

$$d_{n+1} = -\nabla f(x_n) + \beta_n d_n, \tag{1.4}$$

where $\{\beta_n\} \subset (0, +\infty)$. In 2018, Dong et al. [11] introduced the following algorithm:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ d_{n+1} = \dfrac{1}{\lambda}(Tw_n - w_n) + \beta_n d_n, \\ y_n = w_n + \lambda d_{n+1}, \\ x_{n+1} = \mu\gamma_n w_n + (1 - \mu\gamma_n)y_n, \end{cases} \tag{1.5}$$

where $\lambda, \mu > 0, \theta_n, \beta_n$ and γ_n are real sequences. They proved the weak convergence of the algorithm (1.5) for a nonexpansive mapping T under mild conditions.

We consider the following problem of finding $(x^*, y^*) \in C \times C$ satisfying

$$\begin{cases} \langle \mu Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \\ \langle \nu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C \end{cases} \tag{1.6}$$

which is called a General System of Variational Inequalities, where $A, B : C \to H$ are two nonlinear mappings, $\mu, \nu > 0$ are two fixed constants. The set of solutions of (1.6) is denoted by $GV(A, B)$.

In [6], Ceng et al. introduced the following iterative algorithm for solving (1.6):

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \nu Bx_n), \\ x_{n+1} = \alpha_n x_0 + \beta_n x_n + \gamma_n TP_C(y_n - \mu Ax_n), \end{cases} \tag{1.7}$$

where α_n, β_n and γ_n are real sequences, T is a nonexpansive mapping, P_C is a metric projection. A strong convergence theorem was obtained under some suitable conditions.

In addition, we focus on another mathematical problem: the Split Feasibility Problem (shortly, SFP). It was introduced by Censor and Elfving in 1994 [7]. The SFP can be described as finding a point $x^* \in C$ such that

$$Jx^* \in Q, \tag{1.8}$$

where $J : H_1 \to H_2$ is a bounded linear operator, and C and Q are nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively. The solution set of the SFP (1.8) is denoted by Γ.

A CQ algorithm was first proposed by Byrne [4] to solve the SFP, which generates the sequence $\{x_n\}$ through the following iteration:

$$x_{n+1} = P_C(I - \gamma J^*(I - P_Q)J)x_n,$$

where $\gamma \in (0, \frac{2}{\|J\|^2})$, J^* denotes the adjoint operator of J, I is the identity operator on H_1 and P_C and P_Q are metric projections onto C and Q. Generally, the calculation of projections is difficult when sets C and Q are general convex sets. To overcome this, Yang [40] introduced a relaxed CQ algorithm by two half-spaces C_n and Q_n, where

$$C_n = \{x \in H_1 : c(x_n) + \langle \phi_n, x - x_n \rangle \leq 0\}, \quad \phi_n \in \partial c(x_n)$$
$$Q_n = \{y \in H_2 : q(Jx_n) + \langle \psi_n, y - Jx_n \rangle \leq 0\}, \quad \psi_n \in \partial q(Jx_n),$$

where c and q are two convex and lower semicontinuous functions. Afterwards, relaxed CQ methods were widely applied (see, e.g., [27,30]). In recent years, some authors introduced ball-relaxed CQ algorithms, where C_n and Q_n are replaced by $\overline{C}_n$ and $\overline{Q}_n$ (see, e.g., [16,17,31]).

As for various iterative schemes, accelerating the convergence rate is the main purpose. It is the reason why inertial-type approaches, based on the heavy ball methods, have attracted a large number of scholars to investigate. Adding an extrapolation term $\alpha_n(x_n - x_{n-1})$ indicates that the proposed algorithm has an inertial effect. The strategy of adding inertial technique is that the next iteration will be composed of the previous two iterations. Recently, many authors have studied different inertial types to accelerate the convergence process, such as double inertial extrapolation steps, alternated inertial terms and multi-step inertial terms (see, e.g., [1,13,27,28,30–32,34,41,44]). Among them, we follow the interest in multi-step inertial effects, which takes the form of

$$w_n = x_n + \sum_{i \in N} \alpha_{i,n}(x_{n-i} - x_{n-i-1}).$$

A large amount of numerical experiments have shown that this technique can improve the convergence speed of proposed algorithms. Therefore, many authors apply it in various fields (see, e.g., [1,3,10,11,13,28,30–34,41,44,45]).

In this study, we first pay close attention to searching for common solutions of monotone equilibrium problems, fixed point problems of nonexpansive mappings and general systems of variational inequalities in real Hilbert spaces. Yazdi et al. [43] proposed a hybrid viscosity approximation method in 2023. Their algorithm is as follows:

$$\begin{cases} u_n = s_n x_n + (1 - s_n) y_n, \\ f(v_n, y) + \dfrac{1}{r_n} \langle y - v_n, v_n - u_n \rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(v_n - \nu B v_n), \\ y_n = P_C(z_n - \mu A z_n), \\ x_{n+1} = \alpha_n \varphi(x_n) + (1 - \alpha_n) T_n y_n, \end{cases} \tag{1.9}$$

where α_n, r_n and s_n are real sequences, T_n is an infinite family of nonexpansive mappings. They proved a strong convergence theorem under some appropriate conditions and assumptions.

Moreover, we also focus on finding common elements of monotone equilibrium problems, split feasibility problems and fixed point problems of nonexpansive mappings. Recently, Li et al. [16] introduced a strategy with alternated inertial and obtained a strong convergence theorem under some suitable conditions. This algorithm is given as follows:

$$w_n = \begin{cases} x_n, & if\ n = even, \\ x_n + \theta_n(x_n - x_{n-1}), & if\ n = odd. \end{cases}$$
$$\begin{cases} f(z_n, y) + \dfrac{1}{r_n} \langle y - z_n, z_n - w_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = (1 - \alpha_n) w_n + \alpha_n z_n, \\ x_{n+1} = \beta_n x_0 + \beta_n P_{\overline{C}_n}(y_n - \gamma_n J^*(I - P_{\overline{Q}_n}) J y_n), \end{cases} \tag{1.10}$$

where α_n, β_n, θ_n and r_n are real sequences, and

$$\gamma_n = \frac{\rho_n \|(I - P_{\overline{Q}_n}) J y_n\|^2}{2(\|J^*(I - P_{\overline{Q}_n}) J y_n\|^2)},$$

Motivated and inspired by the foregoing works, in this paper, we first propose a new multi-step inertial algorithm for approximating a common element of a monotone equilibrium problem, a fixed point problem of a quasinonexpansive mapping and a general system of variational inequalities in real Hilbert spaces via the conjugate gradient method. Later, we establish a strong convergence theorem under some suitable conditions on parameters. In addition, we blend a multi-step inertial term with a S-iteration process for finding a common element of a monotone equilibrium problem, a fixed point problem of a quasinonexpansive mapping and a split feasibility problem. A strong convergence result of the algorithm is obtained. In the end, numerical experimental results show the performance of proposed algorithms by comparing them with the existing algorithms.

This paper is organized as follows. In Sect. 2, we present some fundamental definitions, lemmas and preliminary results needed in the sequel. Two multi-step inertial algorithms and analyses are investigated in Sects. 3 and 4. Section 5 comprises numerical experiments to illustrate the numerical behavior of proposed algorithms and compare them with relevant algorithms. In Sect. 6, we make a brief summary of this paper.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The strong (weak) convergence of a sequence $\{x_n\}$ to x is denoted by $x_n \to x$ ($x_n \rightharpoonup x$), respectively. We have the following facts:

(1) For any $x, y \in H$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle; \tag{2.1}$$

(2) For any $x, y, z \in H$, $a, b, c \in [0,1]$ and $a+b+c=1$,

$$\|ax+by+cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - ac\|x-z\|^2 - bc\|y-z\|^2. \tag{2.2}$$

Definition 2.1 [9] The metric projection from H onto C at $x \in H$ is defined by

$$P_C x = \arg\min\left\{\|x-y\| : y \in C\right\}.$$

Further, for $x \in H$, we have

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall y \in C. \tag{2.3}$$

We suppose that the bifunction $f : C \times C \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(A1) $f(x,x) = 0, \ \forall x \in C$;
(A2) f is monotone, that is, $f(x,y) + f(y,x) \leq 0, \ \forall x, y \in C$;
(A3) For each $x, y, z \in C$, $\limsup_{t\to 0} f(tz + (1-t)x, y) \leq f(x,y)$;
(A4) For each $x \in C$, $y \mapsto f(x,y)$ is convex and lower semicontinuous.

If a bifunction $f : C \times C \to \mathbb{R}$ satisfies assumptions (A1)-(A4) and $r > 0$, $x \in C$, there exists a unique $z \in C$ [43] such that

$$f(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \ \ \forall y \in C.$$

Definition 2.2 [39] A mapping $T : H \to H$ with $F(T) \neq \emptyset$.

(1) T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x-y\|, \quad \forall x, y \in H;$$

(2) T is called quasinonexpansive if

$$\|Tx - y\| \leq \|x-y\|, \quad \forall x \in H, y \in F(T);$$

(3) T is called firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x-y\rangle, \quad \forall x, y \in H;$$

(3) $I - T$ is called demiclosed at zero if $\{x_n\} \subset H$,

$$x_n \rightharpoonup x \text{ and } \|Tx_n - x_n\| \to 0 \Rightarrow x \in F(T).$$

It is known that the mapping P_C is firmly nonexpansive and nonexpansive.

Definition 2.3 [43] A nonlinear operator A whose domain $D(A) \subseteq H$ and the range $R(A) \subseteq H$ is said to be η-inverse strongly monotone (η-ism) if there exists $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|Ax - Ay\|^2, \quad \forall x, y \in D(A).$$

Definition 2.4 [39] A function $g : H \to (-\infty, +\infty)$ is proper.

(1) A vector $u \in H$ is subgradient of g at a point x if

$$\langle y - x, u\rangle + g(x) \leq g(y), \quad \forall y \in H.$$

(2) The subdifferential of g is the set-valued operator $\partial g : H \to 2^H$.

$$\partial g(x) = \{u \in H : \langle y - x, u\rangle + g(x) \leq g(y), \ \forall y \in H\}.$$

(3) g is called lower semicontinuous if $x_n \to x$ implies

$$g(x) \leq \liminf_{n\to\infty} g(x_n).$$

(4) g is called weakly lower semicontinuous if $x_n \rightharpoonup x$ implies

$$g(x) \leq \liminf_{n\to\infty} g(x_n).$$

Lemma 2.5 *[14] Let $T : H \to H$ be a nonexpansive mapping. Let $\{x_n\} \subset H$ and $x \in H$. Suppose that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to \infty$. Then $x \in F(T)$.*

Lemma 2.6 *[16] Let $g : H \to (-\infty, +\infty]$ be an α-strongly convex function. Then, for all $x, y \in H$ and $\alpha > 0$,*

$$g(x) + \langle \phi, y - x\rangle + \frac{\alpha}{2}\|y - x\|^2 \leq g(y), \quad \forall \phi \in \partial g(x).$$

Lemma 2.7 *[43] Let $C \subseteq H$, $A : C \to H$ be η-ism and $\mu \in (0, 2\eta)$. Then, $I - \mu A$ is nonexpansive.*

Lemma 2.8 *[8] Assume that $f : C \times C \longrightarrow \mathbb{R}$ satisfies assumptions (A1)-(A4). For any $r > 0$ and $x \in H$, define a mapping $T_r^f : H \to C$ as follows:*

$$T_r^f(x) = \left\{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\right\}.$$

Then, the following hold:

(i) T_r^f is a single-valued mapping.
(ii) T_r^f is firmly nonexpansive, that is, $\|T_r^f x - T_r^f y\|^2 \leq \langle T_r^f x - T_r^f y, x - y\rangle$.
(iii) $F(T_r^f) = EP(f)$.
(iv) $EP(f)$ is closed and convex.

Lemma 2.9 *[43] Assume that $f : C \times C \longrightarrow \mathbb{R}$ satisfies assumptions (A1)-(A4). For any $r, s > 0$ and $x \in H$, the mapping T_r^f satisfies the following inequality:*

$$\|T_r^f x - T_s^f y\| \leq \|x - y\| + |\frac{r - s}{r}| \cdot \|T_r^f x - x\|, \quad \forall x, y \in C.$$

Lemma 2.10 *[39] Let $\{a_n\}$ be a sequence of nonnegative real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n + c_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ be a sequence of real numbers in (0,1), $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers, such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\limsup_{n\to\infty} b_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n b_n| < \infty$.
(iii) $\sum_{n=1}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11 *[6] For given* $x^*, y^* \in C$, (x^*, y^*) *is a solution of problem (1.6) if and only if* x^* *is a fixed point of the mapping* $G : C \to C$*, which is defined by*

$$G(x) = P_C(I - \mu A)P_C(I - \nu B)x, \quad \forall x \in C,$$

where $y^* = P_C(x^* - \nu Bx^*)$.

Lemma 2.12 *[16] Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that there exists a subsequence* $\{a_{n_j}\}$ *of* $\{a_n\}$ *such that* $a_{n_j} < a_{n_j+1}$ *for all* $j \in \mathbb{N}$*. Then, there exists a nondecreasing sequence* $\{m_k\}$ *of* $\mathbb{N}$ *such that* $m_k \to \infty$ *as* $k \to \infty$ *and the following properties are satisfied by all number* $k \in \mathbb{N}$*:*

$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1}.$$

In fact, m_k *is the largest number* n *in the set* $\{1, 2, 3, ..., k\}$ *such that* $a_n < a_{n+1}$.

3 The first algorithm and convergence analysis

In this section, we propose a new multi-step inertial algorithm for finding a common solution of monotone equilibrium problems, fixed point problems of quasinonexpansive mappings and general systems of variational inequalities in real Hilbert spaces and provide convergence analysis.

Let C be a closed convex subset of H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (A1)-(A4), $A, B : C \to H$ be η_1-ism, η_2-ism, respectively. $T : H \to H$ is a quasinonexpansive mapping such that $(I - T)$ is demiclosed at zero and $\varphi : H \to H$ is a strict contraction function with coefficient $\delta \in (0, 1)$. We denote the solution set by $\Omega := EP(f) \cap F(T) \cap F(G) \neq \emptyset$. We establish the convergence of the algorithm under the following conditions on control parameters:

Condition 1

(C1) $\sum_{n=1}^{\infty} \max_{i\in N} |\alpha_{i,n}| \sum_{i\in N} \|x_{n-i} - x_{n-i-1}\| < \infty$;
(C2) $\{\gamma_n\} \subset (0, 1)$, $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$, and $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$;
(C3) $\{\beta_n\}$ is a positive sequence satisfying $\beta_n + \beta_{n-1} = \circ(\gamma_n)$;
(C4) $\{r_n\} \subset (a, \infty)$ for some $a > 0$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$;
(C5) $\{Tz_n - z_n\}$ is bounded.

Now, we are in a position to give the description of our algorithm.

Algorithm 3.1

Initialization. Let $n \in \mathbb{N}_+$, $N := \{0, 1, \cdots, n-1\}$ and $\{\alpha_{i,n}\}_{i\in N} \subset (-1, 1)^n$. Choose $\mu \in (0, 2\eta_1)$, $\nu \in (0, 2\eta_2)$, $\lambda > 0$, $x_0 \in H$, $x_{i-1} = x_0$, $i \in N \setminus \{0\}$ and $d_0 = \dfrac{1}{\lambda}(Tx_0 - x_0)$.

Iterative Step. Given the iterates x_{n-i} and x_{n-i-1} for each $n > 1$.

Step 1. Compute

$$w_n = x_n + \sum_{i\in N} \alpha_{i,n}(x_{n-i} - x_{n-i-1}).$$

Step 2. Find $u_n \in C$ such that

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0, \quad \forall y \in C.$$

Step 3. Compute

$$y_n = P_C(I - \nu B)u_n$$

and

$$z_n = P_C(I - \mu A)y_n.$$

Step 4. Compute

$$v_n = z_n + \lambda d_{n+1},$$

where $d_{n+1} = \frac{1}{\lambda}(Tz_n - z_n) + \beta_n d_n$.
Step 5. Compute

$$x_{n+1} = \gamma_n \varphi(x_n) + (1 - \gamma_n)v_n.$$

Set $n := n + 1$ and return to **Step 1**.

Lemma 3.1 *Assume that conditions (C1)-(C5) hold. Let $\{x_n\}$, $\{z_n\}$ and $\{d_n\}$ be sequences generated by Algorithm 3.1. Then, for each $p \in \Omega$, the following results hold:*

(a) the sequences $\{d_n\}$ and $\{x_n\}$ are bounded.
(b) $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$.
(c) $\|Tz_n - z_n\| \to 0$, as $n \to \infty$.

Proof (a) First, $\{d_n\}$ is a bounded sequence by induction. This is true trivially for $n = 0$. Let $M_0 = \max\{\|d_0\|, \frac{2}{\lambda}\sup_{n\geq 1}\|Tz_n - z_n\|\}$. According to the condition (C5), we know that $M_0 < \infty$. Assume that $\|d_n\| \leq M_0$ is true for some n. We will show that it continues to hold for $n + 1$. From (C3), we get

$$\begin{aligned}
\|d_{n+1}\| &= \|\frac{1}{\lambda}(Tz_n - z_n) + \beta_n d_n\| \\
&\leq \frac{1}{\lambda}\|Tz_n - z_n\| + \beta_n\|d_n\| \\
&\leq \frac{1}{\lambda}\cdot\frac{\lambda}{2}M_0 + \frac{1}{2}M_0 \\
&= M_0.
\end{aligned}$$

It implies that $\|d_n\| \leq M_0$ for all $n \geq 0$, that is, $\{d_n\}$ is bounded.

Next, we show that $\{x_n\}$ is bounded. Indeed, from Lemma 2.8, we know that $u_n = T_{r_n}^f w_n$.

$$\|u_n - p\| = \|T_{r_n}^f w_n - p\| \leq \|w_n - p\|. \tag{3.1}$$

From the definition of w_n, we have

$$\begin{aligned}
\|w_n - p\| &= \|x_n + \sum_{i\in N}\alpha_{i,n}(x_{n-i} - x_{n-i-1}) - p\| \\
&\leq \|x_n - p\| + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\|.
\end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$\|u_n - p\| \leq \|x_n - p\| + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\|. \tag{3.3}$$

Using the triangle inequality, we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\gamma_n\varphi(x_n) + (1 - \gamma_n)v_n - p\| \\
&\leq \gamma_n\|\varphi(x_n) - p\| + (1 - \gamma_n)\|v_n - p\| \\
&\leq \gamma_n\delta\|x_n - p\| + \gamma_n\|\varphi(p) - p\| + (1 - \gamma_n)\|v_n - p\|.
\end{aligned} \tag{3.4}$$

According to the definition of v_n, we have

$$\begin{aligned}\|v_n - p\| &= \|z_n + \lambda d_{n+1} - p\| \\ &= \|z_n + \lambda(\frac{1}{\lambda}(Tz_n - z_n) + \beta_n d_n) - p\| \\ &\le \|Tz_n - p\| + \lambda\beta_n\|d_n\| \\ &\le \|z_n - p\| + \lambda\beta_n M_0. \end{aligned} \tag{3.5}$$

From Lemma 2.7,

$$\begin{aligned}\|z_n - p\| &= \|P_C(I - \mu A)y_n - p\| \\ &= \|P_C(I - \mu A)P_C(I - \nu B)u_n - p\| \\ &\le \|u_n - p\|. \end{aligned} \tag{3.6}$$

In addition, from the condition (C3), there exists $M_1 > 0$ such that

$$\frac{\beta_n}{\gamma_n} \le M_1, \quad \forall n \ge n_0. \tag{3.7}$$

Substituting (3.3), (3.5), (3.6), (3.7) into (3.4), we obtain

$$\begin{aligned}\|x_{n+1} - p\| &\le \gamma_n\delta\|x_n - p\| + \gamma_n\|\varphi(p) - p\| + (1-\gamma_n)\|v_n - p\| \\ &\le \gamma_n\delta\|x_n - p\| + \gamma_n\|\varphi(p) - p\| + (1-\gamma_n)(\|z_n - p\| + \lambda\beta_n M_0) \\ &\le \gamma_n\delta\|x_n - p\| + \gamma_n\|\varphi(p) - p\| + (1-\gamma_n)(\|u_n - p\| + \lambda\beta_n M_0) \\ &\le \gamma_n\delta\|x_n - p\| + \gamma_n\|\varphi(p) - p\| + (1-\gamma_n)(\|x_n - p\| \\ &\quad + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\| + \lambda\beta_n M_0) \\ &\le \gamma_n\delta\|x_n - p\| + \gamma_n\|\varphi(p) - p\| + (1-\gamma_n)\|x_n - p\| \\ &\quad + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\| + \lambda\beta_n M_0 \\ &\le [1 - \gamma_n(1-\delta)]\|x_n - p\| + \gamma_n(1-\delta)\Big[\frac{\|\varphi(p) - p\|}{1-\delta} + \frac{\lambda\beta_n M_0}{\gamma_n(1-\delta)}\Big] \\ &\quad + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\| \\ &\le [1 - \gamma_n(1-\delta)]\|x_n - p\| + \gamma_n(1-\delta)\Big[\frac{\|\varphi(p) - p\|}{1-\delta} + \frac{\lambda M_0 M_1}{1-\delta}\Big] \\ &\quad + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\| \\ &\le \max\Big\{\|x_n - p\|, \frac{\|\varphi(p) - p\| + \lambda M_0 M_1}{1-\delta}\Big\} + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\| \\ &\le \max\Big\{\|x_{n-1} - p\|, \frac{\|\varphi(p) - p\| + \lambda M_0 M_1}{1-\delta}\Big\} + \sum_{i\in N}|\alpha_{i,n-1}|\cdot\|x_{n-i-1} - x_{n-i-2}\| \\ &\quad + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\| \\ &\le \cdots \\ &\le \max\Big\{\|x_{n_0} - p\|, \frac{\|\varphi(p) - p\| + \lambda M_0 M_1}{1-\delta}\Big\} + M_2, \end{aligned}$$

where $M_2 := \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\| + \cdots + \sum_{i\in N}|\alpha_{i,n_0}|\cdot\|x_{n_0-i} - x_{n_0-i-1}\| < \infty$.

Thus, we infer that the sequence $\{x_n\}$ is bounded, so are $\{w_n\}$, $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{v_n\}$ and $\{\varphi(x_n)\}$.

(b) We show that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$.

Indeed, from the definition of x_{n+1}, we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\gamma_n\varphi(x_n)+(1-\gamma_n)v_n-(\gamma_{n-1}\varphi(x_{n-1})+(1-\gamma_{n-1})v_{n-1})\|\\ &= \|\gamma_n(\varphi(x_n)-\varphi(x_{n-1}))+(1-\gamma_n)(v_n-v_{n-1})+(\gamma_n-\gamma_{n-1})\varphi(x_{n-1})\\ &\quad+(\gamma_{n-1}-\gamma_n)v_{n-1}\|\\ &\le \gamma_n\delta\|x_n-x_{n-1}\|+(1-\gamma_n)\|v_n-v_{n-1}\|+|\gamma_n-\gamma_{n-1}|(\|\varphi(x_{n-1})\|+\|v_{n-1}\|). \end{aligned}\tag{3.8}$$

Now the estimate

$$\begin{aligned}\|v_n-v_{n-1}\| &= \|z_n+\lambda d_{n+1}-(z_{n-1}+\lambda d_n)\|\\ &= \|z_n+Tz_n-z_n+\lambda\beta_n d_n-(z_{n-1}+Tz_{n-1}-z_{n-1}+\lambda\beta_{n-1}d_{n-1})\|\\ &\le \|z_n-z_{n-1}\|+\|Tz_n-z_n\|+\|Tz_{n-1}-z_{n-1}\|+\lambda\|\beta_n d_n-\beta_{n-1}d_{n-1}\|\\ &\le \|z_n-z_{n-1}\|+\|Tz_n-z_n\|+\|Tz_{n-1}-z_{n-1}\|+\lambda(\beta_n+\beta_{n-1})M_0. \end{aligned}\tag{3.9}$$

From Lemma 2.7 and Lemma 2.9, we get

$$\begin{aligned}\|z_n-z_{n-1}\| &= \|P_C(I-\mu A)P_C(I-\nu B)u_n-P_C(I-\mu A)P_C(I-\nu B)u_{n-1}\|\\ &\le \|u_n-u_{n-1}\|\\ &= \|T^f_{r_n}w_n-T^f_{r_{n-1}}w_{n-1}\|\\ &\le \|w_n-w_{n-1}\|+\left|\frac{r_n-r_{n-1}}{r_n}\right|\|T^f_{r_n}w_n-w_n\|. \end{aligned}\tag{3.10}$$

From the definition of w_n, we have

$$\begin{aligned}\|w_n-w_{n-1}\| &= \|x_n+\sum_{i\in N}\alpha_{i,n}(x_{n-i}-x_{n-i-1})-(x_{n-1}+\sum_{i\in N}\alpha_{i,n-1}(x_{n-i-1}-x_{n-i-2}))\|\\ &\le \|x_n-x_{n-1}\|+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|+\sum_{i\in N}|\alpha_{i,n-1}|\cdot\|x_{n-i-1}-x_{n-i-2}\|. \end{aligned}\tag{3.11}$$

Substituting (3.11) into (3.10),

$$\begin{aligned}\|z_n-z_{n-1}\| &\le \|x_n-x_{n-1}\|+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|+\sum_{i\in N}|\alpha_{i,n-1}|\cdot\|x_{n-i-1}-x_{n-i-2}\|\\ &\quad+\left|\frac{r_n-r_{n-1}}{r_n}\right|\|T^f_{r_n}w_n-w_n\|\\ &\le \|x_n-x_{n-1}\|+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|+\sum_{i\in N}|\alpha_{i,n-1}|\cdot\|x_{n-i-1}-x_{n-i-2}\|\\ &\quad+|r_n-r_{n-1}|\frac{M_3}{a}, \end{aligned}\tag{3.12}$$

where $M_3:=\sup_{n\ge1}\{\|T^f_{r_n}w_n-w_n\|\}>0$. Moreover,

$$\|Tz_n-z_n\|+\|Tz_{n-1}-z_{n-1}\|\le\frac{\lambda}{2}M_0+\frac{\lambda}{2}M_0=\lambda M_0.\tag{3.13}$$

Putting (3.9), (3.12), (3.13) into (3.8), we obtain

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \gamma_n\delta\|x_n-x_{n-1}\|+(1-\gamma_n)\|v_n-v_{n-1}\|+|\gamma_n-\gamma_{n-1}|(\|\varphi(x_{n-1})\|+\|v_{n-1}\|)\\ &\le \gamma_n\delta\|x_n-x_{n-1}\|+(1-\gamma_n)(\|z_n-z_{n-1}\|+\|Tz_n-z_n\|+\|Tz_{n-1}-z_{n-1}\|\\ &\quad+\lambda(\beta_n+\beta_{n-1})M_0)+|\gamma_n-\gamma_{n-1}|(\|\varphi(x_{n-1})\|+\|v_{n-1}\|)\\ &\le \gamma_n\delta\|x_n-x_{n-1}\|+(1-\gamma_n)(\|x_n-x_{n-1}\|+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|\\ &\quad+\sum_{i\in N}|\alpha_{i,n-1}|\cdot\|x_{n-i-1}-x_{n-i-2}\|+|r_n-r_{n-1}|\frac{M_3}{a}+\lambda M_0+\lambda(\beta_n+\beta_{n-1})M_0) \end{aligned}$$

$$
\begin{aligned}
&+|\gamma_n-\gamma_{n-1}|(\|\varphi(x_{n-1})\|+\|v_{n-1}\|)\\
&\le[1-\gamma_n(1-\delta)]\|x_n-x_{n-1}\|+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|\\
&+\sum_{i\in N}|\alpha_{i,n-1}|\cdot\|x_{n-i-1}-x_{n-i-2}\|+|r_n-r_{n-1}|\frac{M_3}{a}+\lambda M_0+\lambda(\beta_n+\beta_{n-1})M_0\\
&+|\gamma_n-\gamma_{n-1}|(\|\varphi(x_{n-1})\|+\|v_{n-1}\|)\\
&\le[1-\gamma_n(1-\delta)]\|x_n-x_{n-1}\|+\gamma_n(1-\delta)\Big[\frac{(\beta_n+\beta_{n-1})\lambda M_0}{\gamma_n(1-\delta)}+\frac{\beta_n\lambda M_0}{\gamma_n(1-\delta)\beta_n}\Big]\\
&+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|+\sum_{i\in N}|\alpha_{i,n-1}|\cdot\|x_{n-i-1}-x_{n-i-2}\|+|r_n-r_{n-1}|\frac{M_3}{a}\\
&+|\gamma_n-\gamma_{n-1}|(\|\varphi(x_{n-1})\|+\|v_{n-1}\|).
\end{aligned}\tag{3.14}
$$

According to the conditions (C1)-(C4), we apply Lemma 2.10 to (3.14) to obtain that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

(c) We show that $\|Tz_n-z_n\|\to 0$, as $n\to\infty$. Indeed, since $p\in\Omega$, we know that $p=P_C(P_C(p-\nu Bp)-\mu AP_C(p-\nu Bp))=P_C(I-\mu A)P_C(I-\nu B)p$. From Lemma 2.11, we have $q=P_C(I-\nu B)p$. Observe that

$$
\begin{aligned}
\|y_n-q\|^2&=\|P_C(I-\nu B)u_n-q\|^2\\
&\le\|P_C(I-\nu B)u_n-P_C(I-\nu B)p\|^2\\
&\le\|(I-\nu B)u_n-(I-\nu B)p\|^2\\
&=\|u_n-p\|^2-2\nu\langle u_n-p,Bu_n-Bp\rangle+\nu^2\|Bu_n-Bp\|^2\\
&\le\|u_n-p\|^2-2\nu\eta_2\|Bu_n-Bp\|^2+\nu^2\|Bu_n-Bp\|^2\\
&=\|u_n-p\|^2-\nu(2\eta_2-\nu)\|Bu_n-Bp\|^2.
\end{aligned}\tag{3.15}
$$

Similarly,

$$\|z_n-p\|^2\le\|y_n-q\|^2-\mu(2\eta_1-\mu)\|Ay_n-Aq\|^2.\tag{3.16}$$

Combining (3.15) and (3.16),

$$\|z_n-p\|^2\le\|u_n-p\|^2-\mu(2\eta_1-\mu)\|Ay_n-Aq\|^2-\nu(2\eta_2-\nu)\|Bu_n-Bp\|^2.\tag{3.17}$$

Because $\mu\in(0,2\eta_1)$, $\nu\in(0,2\eta_2)$, we get

$$\|z_n-p\|^2\le\|u_n-p\|^2.\tag{3.18}$$

From (2.2),

$$
\begin{aligned}
\|x_{n+1}-p\|^2&=\|\gamma_n\varphi(x_n)+(1-\gamma_n)v_n-p\|^2\\
&=\|\gamma_n(\varphi(x_n)-p)+(1-\gamma_n)(v_n-p)\|^2\\
&\le\gamma_n\|\varphi(x_n)-p\|^2+(1-\gamma_n)\|v_n-p\|^2.
\end{aligned}\tag{3.19}
$$

By (3.5),

$$
\begin{aligned}
\|v_n-p\|^2&\le(\|z_n-p\|+\lambda\beta_nM_0)^2\\
&=\|z_n-p\|^2+2\lambda\beta_nM_0\|z_n-p\|+(\lambda\beta_nM_0)^2\\
&=\|z_n-p\|^2+\beta_n(2\lambda\beta_nM_0\|z_n-p\|+\beta_n(\lambda M_0)^2)\\
&\le\|z_n-p\|^2+\beta_nM_4,
\end{aligned}\tag{3.20}
$$

where $M_4:=\sup_{n\ge1}\{2\lambda\beta_nM_0\|z_n-p\|+\beta_n(\lambda M_0)^2\}>0$.

In addition, by definition of w_n,

$$\begin{aligned}\|w_n - p\|^2 &= \|x_n + \sum_{i\in N}\alpha_{i,n}(x_{n-i} - x_{n-i-1}) - p\|^2\\ &= \|x_n - p\|^2 + \|\sum_{i\in N}\alpha_{i,n}(x_{n-i} - x_{n-i-1})\|^2 + 2\sum_{i\in N}\alpha_{i,n}\langle x_n - p, x_{n-i} - x_{n-i-1}\rangle\\ &\le \|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}|\sum_{i\in N}|\alpha_{i,n}|\|x_{n-i} - x_{n-i-1}\|^2 + 2\sum_{i\in N}|\alpha_{i,n}|\|x_n - p\|\|x_{n-i} - x_{n-i-1}\|\\ &\le \|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}|\|x_{n-i} - x_{n-i-1}\|(\sum_{i\in N}|\alpha_{i,n}|\|x_{n-i} - x_{n-i-1}\| + 2\|x_n - p\|)\\ &\le \|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\|M_5, \qquad (3.21)\end{aligned}$$

where $M_5 := \sup_{n\ge1}\{\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\| + 2\|x_n - p\|\} < \infty$. Combining (3.1), (3.17) and (3.21), we get

$$\begin{aligned}\|z_n - p\|^2 &\le \|u_n - p\|^2 - \mu(2\eta_1 - \mu)\|Ay_n - Aq\|^2 - \nu(2\eta_2 - \nu)\|Bu_n - Bp\|^2\\ &\le \|w_n - p\|^2 - \mu(2\eta_1 - \mu)\|Ay_n - Aq\|^2 - \nu(2\eta_2 - \nu)\|Bu_n - Bp\|^2\\ &\le \|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\|M_5 - \mu(2\eta_1 - \mu)\|Ay_n - Aq\|^2\\ &\quad - \nu(2\eta_2 - \nu)\|Bu_n - Bp\|^2. \qquad (3.22)\end{aligned}$$

Putting (3.20), (3.22) into (3.19), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \gamma_n\|\varphi(x_n) - p\|^2 + (1-\gamma_n)\|v_n - p\|^2\\ &\le \gamma_n\|\varphi(x_n) - p\|^2 + (1-\gamma_n)(\|z_n - p\|^2 + \beta_n M_4)\\ &\le \gamma_n\|\varphi(x_n) - p\|^2 + (1-\gamma_n)(\|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\|M_5\\ &\quad - \mu(2\eta_1 - \mu)\|Ay_n - Aq\|^2 - \nu(2\eta_2 - \nu)\|Bu_n - Bp\|^2 + \beta_n M_4)\\ &\le \gamma_n\|\varphi(x_n) - p\|^2 + \|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\|M_5\\ &\quad - \mu(2\eta_1 - \mu)\|Ay_n - Aq\|^2 - \nu(2\eta_2 - \nu)\|Bu_n - Bp\|^2 + \beta_n M_4,\end{aligned}$$

that is,

$$\begin{aligned}&\mu(2\eta_1 - \mu)\|Ay_n - Aq\|^2 + \nu(2\eta_2 - \nu)\|Bu_n - Bp\|^2\\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\|M_5 + \beta_n M_4\\ &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i} - x_{n-i-1}\|M_5 + \beta_n M_4. \qquad (3.23)\end{aligned}$$

From $\mu \in (0, 2\eta_1)$, $\nu \in (0, 2\eta_2)$, conditions (C1), (C3) and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we can see

$$\lim_{n\to\infty}\|Ay_n - Aq\| = 0 \text{ and } \lim_{n\to\infty}\|Bu_n - Bp\| = 0. \qquad (3.24)$$

Furthermore,

$$\begin{aligned}\|z_n - p\|^2 &= \|P_C(I - \mu A)y_n - p\|^2\\ &= \|P_C(I - \mu A)y_n - P_C(I - \mu A)q\|^2\\ &\le \langle (I - \mu A)y_n - (I - \mu A)q, z_n - p\rangle\\ &= \frac{1}{2}(\|(I - \mu A)y_n - (I - \mu A)q\|^2 + \|z_n - p\|^2\end{aligned}$$

$$- \|y_n - z_n + p - q - \mu(Ay_n - Aq)\|^2),$$

which implies

$$\begin{aligned}\|z_n - p\|^2 &\le \|(I-\mu A)y_n - (I-\mu A)q\|^2 - \|y_n - z_n + p - q - \mu(Ay_n - Aq)\|^2\\ &\le \|y_n - p\|^2 - \|y_n - z_n + p - q - \mu(Ay_n - Aq)\|^2\\ &\le \|y_n - p\|^2 - (\|y_n - z_n + p - q\|^2 + \mu^2\|Ay_n - Aq\|^2\\ &\quad - 2\mu\langle y_n - z_n + p - q, Ay_n - Aq\rangle\\ &\le \|y_n - p\|^2 - \|y_n - z_n + p - q\|^2 + 2\mu\|y_n - z_n + p - q\| \cdot \|Ay_n - Aq\|. \qquad (3.25)\end{aligned}$$

In the same way,

$$\begin{aligned}\|y_n - q\|^2 \le \frac{1}{2}(&\|(I-\nu B)u_n - (I-\nu B)p\|^2 + \|y_n - p\|^2\\ &- \|u_n - y_n + q - p - \nu(Bu_n - Bp)\|^2),\end{aligned}$$

that is,

$$\begin{aligned}\|y_n - q\|^2 &\le \|(I-\nu B)u_n - (I-\nu B)p\|^2 - \|u_n - y_n + q - p - \nu(Bu_n - Bp)\|^2\\ &\le \|u_n - p\|^2 - (\|u_n - y_n + q - p\|^2 + \nu^2\|Bu_n - Bp\|^2\\ &\quad - 2\nu\langle u_n - y_n + q - p, Bu_n - Bp\rangle)\\ &\le \|u_n - p\|^2 - \|u_n - y_n + q - p\|^2 + 2\nu\|u_n - y_n + q - p\| \cdot \|Bu_n - Bp\|. \qquad (3.26)\end{aligned}$$

Substituting (3.1), (3.21) into (3.26), we obtain

$$\begin{aligned}\|y_n - q\|^2 &\le \|u_n - p\|^2 - \|u_n - y_n + q - p\|^2 + 2\nu\|u_n - y_n + q - p\| \cdot \|Bu_n - Bp\|\\ &\le \|w_n - p\|^2 - \|u_n - y_n + q - p\|^2 + 2\nu\|u_n - y_n + q - p\| \cdot \|Bu_n - Bp\|\\ &\le \|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\|M_5 - \|u_n - y_n + q - p\|^2\\ &\quad + 2\nu\|u_n - y_n + q - p\| \cdot \|Bu_n - Bp\|. \qquad (3.27)\end{aligned}$$

Combining (3.25) and (3.27),

$$\begin{aligned}&\|z_n - p\|^2\\ &\le \|x_n - p\|^2 - \|u_n - y_n + q - p\|^2 - \|y_n - z_n + p - q\|^2 + \sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\|M_5\\ &\quad + 2\nu\|u_n - y_n + q - p\| \cdot \|Bu_n - Bp\| + 2\mu\|y_n - z_n + p - q\| \cdot \|Ay_n - Aq\|. \qquad (3.28)\end{aligned}$$

It follows from (2.2) and (3.20) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\gamma_n\varphi(x_n) + (1-\gamma_n)v_n - p\|^2\\ &\le \gamma_n\|\varphi(x_n) - p\|^2 + (1-\gamma_n)\|v_n - p)\|^2\\ &\le \gamma_n\|\varphi(x_n) - \varphi(p)\|^2 + \gamma_n\|\varphi(p) - p\|^2 + 2\gamma_n\langle\varphi(x_n) - \varphi(p), \varphi(p) - p\rangle\\ &\quad + (1-\gamma_n)\|v_n - p\|^2\\ &\le \gamma_n\delta^2\|x_n - p\|^2 + \gamma_n\|\varphi(p) - p\|^2 + 2\gamma_n\delta\|x_n - p\| \cdot \|\varphi(p) - p\|\\ &\quad + (1-\gamma_n)(\|z_n - p\|^2 + \beta_n M_4)\\ &\le \gamma_n\delta\|x_n - p\|^2 + \gamma_n(\|\varphi(p) - p\|^2 + M_6) + (1-\gamma_n)(\|z_n - p\|^2 + \beta_n M_4), \qquad (3.29)\end{aligned}$$

where $M_6 := \sup_{n\ge 1}\{2\delta\|x_n - p\| \cdot \|\varphi(p) - p\|\} > 0$.
Substituting (3.28) into (3.29), we have

$$\|x_{n+1} - p\|^2 \le \gamma_n\delta\|x_n - p\|^2 + \gamma_n(\|\varphi(p) - p\|^2 + M_6) + (1-\gamma_n)(\|z_n - p\|^2 + \beta_n M_4)$$

$$
\begin{aligned}
&\leq \gamma_n\delta\|x_n-p\|^2+\gamma_n(\|\varphi(p)-p\|^2+M_6)+(1-\gamma_n)\Big(\|x_n-p\|^2\\
&\quad-\|u_n-y_n+q-p\|^2-\|y_n-z_n+p-q\|^2+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|M_5\\
&\quad+2\nu\|u_n-y_n+q-p\|\cdot\|Bu_n-Bp\|+2\mu\|y_n-z_n+p-q\|\cdot\|Ay_n-Aq\|+\beta_nM_4\Big)\\
&\leq[1-\gamma_n(1-\delta)]\|x_n-p\|^2+\gamma_n(\|\varphi(p)-p\|^2+M_6)-(1-\gamma_n)\|u_n-y_n+q-p\|^2\\
&\quad-(1-\gamma_n)\|y_n-z_n+p-q\|^2+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|M_5\\
&\quad+2\nu\|u_n-y_n+q-p\|\cdot\|Bu_n-Bp\|+2\mu\|y_n-z_n+p-q\|\cdot\|Ay_n-Aq\|+\beta_nM_4\\
&\leq\|x_n-p\|^2+\gamma_n(\|\varphi(p)-p\|^2+M_6)-(1-\gamma_n)\|u_n-y_n+q-p\|^2\\
&\quad-(1-\gamma_n)\|y_n-z_n+p-q\|^2+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|M_5\\
&\quad+2\nu\|u_n-y_n+q-p\|\cdot\|Bu_n-Bp\|+2\mu\|y_n-z_n+p-q\|\cdot\|Ay_n-Aq\|+\beta_nM_4,
\end{aligned}
$$

which impiles

$$
\begin{aligned}
&(1-\gamma_n)\|u_n-y_n+q-p\|^2+(1-\gamma_n)\|y_n-z_n+p-q\|^2\\
&\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+\gamma_n(\|\varphi(p)-p\|^2+M_6)+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|M_5\\
&\quad+2\nu\|u_n-y_n+q-p\|\cdot\|Bu_n-Bp\|+2\mu\|y_n-z_n+p-q\|\cdot\|Ay_n-Aq\|+\beta_nM_4\\
&\leq(\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+\gamma_n(\|\varphi(p)-p\|^2+M_6)+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|M_5\\
&\quad+2\nu\|u_n-y_n+q-p\|\cdot\|Bu_n-Bp\|+2\mu\|y_n-z_n+p-q\|\cdot\|Ay_n-Aq\|+\beta_nM_4.
\end{aligned}
\tag{3.30}
$$

From (3.24), conditions (C1), (C2), (C3) and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, we can see

$$
\lim_{n\to\infty}\|u_n-y_n+q-p\|=0 \text{ and } \lim_{n\to\infty}\|y_n-z_n+p-q\|=0. \tag{3.31}
$$

We obtain from (4.26) that

$$
\|u_n-z_n\|\leq\|u_n-y_n+q-p\|+\|y_n-z_n+p-q\|\to 0, \text{ as } n\to\infty. \tag{3.32}
$$

Using Lemma 2.8 (ii), we have

$$
\begin{aligned}
\|u_n-p\|^2&=\|T_{r_n}^f w_n-p\|^2\\
&\leq\langle w_n-p,u_n-p\rangle\\
&=\frac{1}{2}(\|w_n-p\|^2+\|u_n-p\|^2-\|u_n-w_n\|^2).
\end{aligned}
$$

Hence, we obtain

$$
\|u_n-p\|^2\leq\|w_n-p\|^2-\|u_n-w_n\|^2. \tag{3.33}
$$

Combining (3.18), (3.20), (3.21) and (3.33), we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2&=\|\gamma_n\varphi(x_n)+(1-\gamma_n)v_n-p\|^2\\
&\leq\gamma_n\|\varphi(x_n)-p\|^2+(1-\gamma_n)\|v_n-p\|^2\\
&\leq\gamma_n\|\varphi(x_n)-p\|^2+(1-\gamma_n)(\|z_n-p\|^2+\beta_nM_4)\\
&\leq\gamma_n\|\varphi(x_n)-p\|^2+(1-\gamma_n)(\|u_n-p\|^2+\beta_nM_4)\\
&\leq\gamma_n\|\varphi(x_n)-p\|^2+(1-\gamma_n)(\|w_n-p\|^2-\|u_n-w_n\|^2+\beta_nM_4)\\
&\leq\gamma_n\|\varphi(x_n)-p\|^2+\|w_n-p\|^2-(1-\gamma_n)\|u_n-w_n\|^2+\beta_nM_4
\end{aligned}
$$

$$
\begin{aligned}
&\leq \gamma_n \|\varphi(x_n) - p\|^2 + \|x_n - p\|^2 - (1-\gamma_n)\|u_n - w_n\|^2 \\
&\quad + \beta_n M_4 + \sum_{i\in N} |\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\| M_5.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
(1-\gamma_n)\|u_n - w_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n \|\varphi(x_n) - p\|^2 \\
&\quad + \beta_n M_4 + \sum_{i\in N} |\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\| M_5 \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \gamma_n \|\varphi(x_n) - p\|^2 \\
&\quad + \beta_n M_4 + \sum_{i\in N} |\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\| M_5. \qquad (3.34)
\end{aligned}
$$

Using conditions (C1), (C2), (C3) and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we can see

$$
\|u_n - w_n\| \to 0, \quad \text{as} \quad n \to \infty. \qquad (3.35)
$$

Moreover,

$$
\begin{aligned}
\|w_n - x_n\| &= \|x_n + \sum_{i\in N} \alpha_{i,n}(x_{n-i} - x_{n-i-1}) - x_n\| \\
&\leq \sum_{i\in N} |\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\| \to 0, \quad \text{as} \quad n \to \infty. \qquad (3.36)
\end{aligned}
$$

From (3.32) and (3.35), we obtain

$$
\|z_n - w_n\| \leq \|z_n - u_n\| + \|u_n - w_n\| \to 0, \quad \text{as} \quad n \to \infty. \qquad (3.37)
$$

From (3.36) and (3.37), we have

$$
\|z_n - x_n\| \leq \|z_n - w_n\| + \|w_n - x_n\| \to 0, \quad \text{as} \quad n \to \infty. \qquad (3.38)
$$

By the definition of x_{n+1}, we have

$$
\begin{aligned}
\|x_{n+1} - v_n\| &= \|\gamma_n \varphi(x_n) + (1-\gamma_n)v_n - v_n\| \\
&= \gamma_n \|\varphi(x_n) - v_n\| \to 0, \quad \text{as} \quad n \to \infty. \qquad (3.39)
\end{aligned}
$$

From Lemma 3.1 (b) and (3.39), we obtain

$$
\|x_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \|x_n - z_n\| \to 0, \quad \text{as} \quad n \to \infty. \qquad (3.40)
$$

Combining (3.39) and (3.40), we have

$$
\|v_n - z_n\| \leq \|v_n - x_{n+1}\| + \|x_{n+1} - z_n\| \to 0, \quad \text{as} \quad n \to \infty. \qquad (3.41)
$$

Note that

$$
v_n - z_n = z_n + \lambda d_{n+1} - z_n = T z_n - z_n + \lambda \beta_n d_n,
$$

that is,

$$
T z_n - z_n = v_n - z_n - \lambda \beta_n d_n.
$$

Therefore,

$$
\begin{aligned}
\|T z_n - z_n\| &\leq \|v_n - z_n\| + \lambda \beta_n \|d_n\| \\
&\leq \|v_n - z_n\| + \lambda \beta_n M_0 \to 0, \quad \text{as} \quad n \to \infty. \qquad (3.42)
\end{aligned}
$$

□

Theorem 3.2 *Assume that conditions (C1)-(C5) hold. The sequence $\{x_n\}$ generated by Algorithm 3.1 converges to $p \in \Omega$ in norm, where $p = P_\Omega(\varphi(p))$ is a unique solution of the variational inequality*

$$\langle \varphi(p) - p, y - p\rangle \le 0, \quad \forall y \in \Omega.$$

Proof Since $P_\Omega\varphi$ is a contraction on Ω and the Banach Contraction Principle, there exists a unique element $p \in \Omega$ such that $p = P_\Omega(\varphi(p))$. Applying (2.1),

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\gamma_n\varphi(x_n) + (1-\gamma_n)v_n - p\|^2 \\
&\le (1-\gamma_n)^2\|v_n - p\|^2 + 2\gamma_n\langle\varphi(x_n) - p, x_{n+1} - p\rangle \\
&\le (1-\gamma_n)^2\|v_n - p\|^2 + 2\gamma_n\delta\|x_n - p\| \cdot \|x_{n+1} - p\| + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\le (1-\gamma_n)^2\|v_n - p\|^2 + \gamma_n\delta\|x_n - p\|^2 + \gamma_n\delta\|x_{n+1} - p\|^2 + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle.
\end{aligned} \tag{3.43}$$

Putting (3.1), (3.18), (3.20), (3.21) into (3.43), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\gamma_n)^2(\|z_n - p\|^2 + \beta_n M_4) + \gamma_n\delta\|x_n - p\|^2 + \gamma_n\delta\|x_{n+1} - p\|^2 \\
&\quad + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\le (1-\gamma_n)^2(\|u_n - p\|^2 + \beta_n M_4) + \gamma_n\delta\|x_n - p\|^2 + \gamma_n\delta\|x_{n+1} - p\|^2 \\
&\quad + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\le (1-\gamma_n)^2(\|w_n - p\|^2 + \beta_n M_4) + \gamma_n\delta\|x_n - p\|^2 + \gamma_n\delta\|x_{n+1} - p\|^2 \\
&\quad + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\le (1-\gamma_n)^2(\|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\|M_5 + \beta_n M_4) \\
&\quad + \gamma_n\delta\|x_n - p\|^2 + \gamma_n\delta\|x_{n+1} - p\|^2 + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\le (1sps2\gamma_n + \gamma_n\delta + \gamma_n^2)\|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\|M_5 + \beta_n M_4 \\
&\quad + \gamma_n\delta\|x_{n+1} - p\|^2 + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \frac{1sps2\gamma_n + \gamma_n\delta + \gamma_n^2}{1-\gamma_n\delta}\|x_n - p\|^2 + \frac{1}{1-\gamma_n\delta}\Big[\sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\|M_5 \\
&\quad + \beta_n M_4 + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle\Big] \\
&\le \Big[1 - \frac{2\gamma_n(1-\delta)}{1-\gamma_n\delta}\Big]\|x_n - p\|^2 + \frac{2\gamma_n(1-\delta)}{1-\gamma_n\delta}\Big[\frac{\gamma_n\|x_n - p\|^2}{2(1-\delta)} + \frac{\langle\varphi(p) - p, x_{n+1} - p\rangle}{1-\delta}\Big] \\
&\quad + \frac{\sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\|M_5 + \beta_n M_4}{1-\gamma_n\delta}.
\end{aligned} \tag{3.44}$$

According to Lemma 3.1 (a), we have that $\{x_n\}$ is bounded. Therefore, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$, which converges weakly to some $\tilde{x} \in H$, such that

$$\limsup_{n\to\infty}\langle\varphi(p) - p, x_n - p\rangle = \lim_{j\to\infty}\langle\varphi(p) - p, x_{n_j} - p\rangle = \langle\varphi(p) - p, \tilde{x} - p\rangle. \tag{3.45}$$

We show $\tilde{x} \in \Omega$. Indeed, from (3.38) and Lemma 3.1 (c), we have $z_{n_j} \rightharpoonup \tilde{x}$ and $\lim_{j\to\infty}\|Tz_{n_j} - z_{n_j}\| = 0$. It follows from the demiclosedness of $(I - T)$ that $\tilde{x} \in F(T)$. In addition, since $u_n = T^f_{r_n}w_n$, we obtain

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \ge 0, \quad \forall y \in C.$$

From the monotonicity of f, we get

$$\frac{1}{r_n}\langle y-u_n, u_n-w_n\rangle \geq f(y,u_n), \quad \forall y\in C.$$

Replacing n by n_j,

$$\frac{1}{r_{n_j}}\langle y-u_{n_j}, u_{n_j}-w_{n_j}\rangle \geq f(y,u_{n_j}), \quad \forall y\in C.$$

From (3.35) and (3.36), we have $w_{n_j}\rightharpoonup \tilde{x}$ and $u_{n_j}\rightharpoonup \tilde{x}$. It follows from (A4) and (C4) that $f(y,\tilde{x})\leq 0$ for all $y\in C$. Let $y_t = ty+(1-t)\tilde{x}$, $\forall t\in(0,1]$ and $y\in C$. This implies that $y_t\in C$ and $f(y_t,\tilde{x})\leq 0$. By applying assumptions (A1) and (A2), we get

$$0 = f(y_t,y_t)\leq tf(y_t,y)+(1-t)f(y_t,\tilde{x})\leq tf(y_t,y).$$

Letting $t\to 0$, we obtain $f(\tilde{x},y)\geq 0$ for all $y\in C$. This implies $\tilde{x}\in EP(f)$.
On the other hand, from (3.32),

$$\|z_n-u_n\| = \|P_C(I-\mu A)P_C(I-\nu B)u_n-u_n\| = \|Gu_n-u_n\|\to 0, \quad \text{as} \quad n\to\infty. \tag{3.46}$$

Applying Lemma 2.5 that $\tilde{x}\in F(G)$. Consequently, $\tilde{x}\in\Omega$. From (2.3) and (3.45), we obtain

$$\langle\varphi(p)-p,\tilde{x}-p\rangle\leq 0.$$

According to Lemma 3.1 (b), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle\varphi(p)-p,x_{n+1}-p\rangle\\
&\leq \limsup_{n\to\infty}\langle\varphi(p)-p,x_{n+1}-x_n\rangle+\limsup_{n\to\infty}\langle\varphi(p)-p,x_n-p\rangle\\
&= \langle\varphi(p)-p,\tilde{x}-p\rangle\\
&\leq 0.
\end{aligned}\tag{3.47}$$

Applying Lemma 2.10 to (4.3) and using (3.47) together with conditions (C1), (C2) and (C4), we deduce that $\lim_{n\to\infty}\|x_n-p\|=0$ as desired. □

Remark 3.3 If the inertial parameters $\{\alpha_{i,n}\}\subset[0,1]$, the condition (C1) can be written as

$$\sum_{n=1}^{\infty}\max_{i\in N}\alpha_{i,n}\sum_{i\in N}\|x_{n-i}-x_{n-i-1}\|<\infty. \tag{3.48}$$

The condition (3.48) can be enforced by a simple online updating rule, that is, for each $i\in N$ and $\alpha_i\in[0,1]$,

$$\alpha_{i,n}=\min\{\alpha_i,c_{i,n}\},$$

where $c_{i,n}$ and $\max\{c_{i,n}\}\sum_{i\in N}\|x_{n-i}-x_{n-i-1}\|$ is summable. In this case, we can choose

$$c_{i,n}=\frac{c_i}{n^{1+\epsilon}\sum_{i\in N}\|x_{n-i}-x_{n-i-1}\|}, \quad c_i>0, \epsilon>0.$$

4 The second algorithm and convergence analysis

In this section, we combine a multi-step inertial term with the S-iteration process for finding a common element of monotone equilibrium problems, split feasibility problems and fixed point problems of quasinonexpansive mappings in real Hilbert spaces and provide convergence analysis.

Suppose that $\varphi : H_1 \to H_1$ is a strict contraction function with constant $\delta \in (0, 1)$, $J : H_1 \to H_2$ is a bounded linear operator and $S : H_1 \to H_1$ is a quasinonexpansive mapping such that $(I - S)$ is demiclosed at zero. Let C and Q be nonempty, closed and convex sets of real Hilbert spaces H_1 and H_2, respectively, which are defined as follows:

$$C = \{x \in H_1 : c(x) \leq 0\}, \quad Q = \{y \in H_2 : q(y) \leq 0\},$$

where $c : H_1 \to \mathbb{R}$ is an α-strongly convex function and $q : H_2 \to \mathbb{R}$ is a β-strongly convex function. It is easy to check that c and q are lower semicontinuous. We assume that ∂c and ∂q are bounded operators. We introduce two specific half spaces

$$\overline{C}_n = \{x \in H_1 : c(t_n) + \langle \phi_n, x - t_n\rangle + \frac{\alpha}{2}\|x - t_n\|^2 \leq 0\},$$
$$\overline{Q}_n = \{y \in H_2 : q(Jt_n) + \langle \psi_n, y - Jt_n\rangle + \frac{\beta}{2}\|y - Jt_n\|^2 \leq 0\},$$

where $\phi_n \in \partial c(t_n)$ and $\psi_n \in \partial q(Jt_n)$. Moreover, using Lemma 2.6, we know $C \subset \overline{C}_n \subset C_n$ and $Q \subset \overline{Q}_n \subset Q_n$. Conditions on the control parameters are as follows:

Condition 2

(C1) $\sum_{n=1}^{\infty} \max_{i\in N} |\alpha_{i,n}| \sum_{i\in N} \|x_{n-i} - x_{n-i-1}\| < \infty$;
(C6) $\{\gamma_n\} \subset (0, 1)$, $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(C7) $\{\beta_n\} \subset (a, b)$, where $a, b \in (0, 1)$;
(C8) $\{\zeta_n\}, \{\xi_n\} \subset (0, 1)$ and $\gamma_n + \zeta_n + \xi_n = 1$;
(C9) $\{\varepsilon_n\} \subset (0, 1)$, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$;
(C10) $\{\rho_n\} \subset (c, d)$, where $c, d \in (0, 2)$.

Algorithm 4.1

Initialization. Let $n \in \mathbb{N}_+$, $N := \{0, 1, \cdots, n-1\}$ and $\{\alpha_{i,n}\}_{i\in N} \subset (-1, 1)^n$. Choose $\theta > 0$, $x_0 \in H$, $x_{i-1} = x_0$, $i \in N \setminus \{0\}$.

Iterative Step. Given the iterates x_{n-i} and x_{n-i-1} for each $n > 1$.

Step 1. Compute

$$w_n = x_n + \sum_{i\in N} \alpha_{i,n}(x_{n-i} - x_{n-i-1}).$$

Step 2. Find $u_n \in C$ such that

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0, \quad \forall y \in C.$$

Step 3. Compute

$$t_n = (1 - \theta_n)w_n + \theta_n u_n,$$

where $0 < \theta_n < \overline{\theta}_n$ and

$$\overline{\theta}_n = \begin{cases} \min\left\{\dfrac{\theta}{2}, \dfrac{\varepsilon_n}{\|u_n - x_n\|}\right\}, & if\ u_n \neq x_n, \\ \dfrac{\theta}{2}, & otherwise. \end{cases} \tag{4.1}$$

Step 4. Compute

$$v_n = t_n - \tau_n[(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n],$$

where

$$\tau_n = \begin{cases} \dfrac{\rho_n \|(I - P_{\overline{C}_n})t_n\|^2 + \rho_n \|(I - P_{\overline{Q}_n})Jt_n\|^2}{\|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\|^2}, \\ \quad if\ \|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\| \neq 0, \\ 0, \quad otherwise\,. \end{cases}$$

Step 5. Compute

$$z_n = \beta_n S v_n + (1 - \beta_n)t_n.$$

Step 6. Compute

$$x_{n+1} = \gamma_n \varphi(x_n) + \zeta_n S v_n + \xi_n S z_n.$$

Set $n := n + 1$ and return to **Step 1**.

Lemma 4.1 *Let $\{t_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 4.1. Then, for each $p \in EP(f) \cap \Gamma \cap F(S)$, the following inequality holds:*

$$\|v_n - p\|^2 \leq \|t_n - p\|^2 - \rho_n(2 - \rho_n)\frac{(\|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\|^2}.$$

Proof Let $p \in EP(f) \cap \Gamma \cap F(S)$, $p \in EP(f) \Rightarrow p = T^f_{r_n} p$, $p = P_C p = P_{\overline{C}_n} p$ and $Jp = P_Q Jp = P_{\overline{Q}_n} Jp$. From the definition of v_n, we have

$$\begin{aligned} \|v_n - p\|^2 &= \|t_n - \tau_n[(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n] - p\|^2 \\ &= \|t_n - p\|^2 + \tau_n^2 \|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\|^2 \\ &\quad - 2\tau_n \langle (I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n, t_n - p\rangle. \end{aligned} \tag{4.2}$$

Since $P_{\overline{C}_n}$ is a firmly nonexpansive mapping, we deduced that $I - P_{\overline{C}_n}$ is firmly nonexpansive. Similarly, $I - P_{\overline{Q}_n}$ is also firmly nonexpansive. Hence, we get

$$\begin{aligned} &\langle (I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n, t_n - p\rangle \\ &= \langle (I - P_{\overline{C}_n})t_n - (I - P_{\overline{C}_n})p, t_n - p\rangle + \langle J^*(I - P_{\overline{Q}_n})Jt_n - J^*(I - P_{\overline{Q}_n})Jp, t_n - p\rangle \\ &= (I - P_{\overline{C}_n})t_n - (I - P_{\overline{C}_n})p, t_n - p\rangle + \langle (I - P_{\overline{Q}_n})Jt_n - (I - P_{\overline{Q}_n})Jp, Jt_n - Jp\rangle \\ &\geq \|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2. \end{aligned} \tag{4.3}$$

From (4.2), (4.3) and the definition of τ_n, we obtain

$$\begin{aligned} \|v_n - p\|^2 &\leq \|t_n - p\|^2 + \tau_n^2 \|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\|^2 \\ &\qquad - 2\tau_n(\|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2) \\ &\leq \|t_n - p\|^2 \\ &\quad + \rho_n^2 \frac{(\|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\|^4} \|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\|^2 \\ &\quad - 2\rho_n \frac{\|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2}{\|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\|^2}(\|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2) \end{aligned}$$

$$\leq \|t_n - p\|^2 - \rho_n(2-\rho_n)\frac{(\|(I-P_{\overline{C}_n})t_n\|^2 + \|(I-P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I-P_{\overline{C}_n})t_n + J^*(I-P_{\overline{Q}_n})Jt_n\|^2}.$$

This completes the proof of the lemma. □

Lemma 4.2 *Assume that (C1), (C6)-(C10) hold. Let $\{x_n\}$, $\{w_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{z_n\}$ be sequences generated by Algorithm 4.1. Then, for each $p \in EP(f) \cap \Gamma \cap F(S)$,*

(a) the sequence $\{x_n\}$ is bounded.
(b) the following inequality holds:

$$\begin{aligned}\|x_{n+1}-p\|^2 \leq{}& \|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|M_5 + \gamma_n\|\varphi(x_n)-p\|^2\\ &- \beta_n\xi_n(1-\beta_n)\|Sv_n - t_n\|^2 - \zeta_n\xi_n\|Sv_n - Sz_n\|^2\\ &- \rho_n\beta_n\xi_n(2-\rho_n)\frac{(\|(I-P_{\overline{C}_n})t_n\|^2 + \|(I-P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I-P_{\overline{C}_n})t_n + J^*(I-P_{\overline{Q}_n})Jt_n\|^2}.\end{aligned}$$

Proof (a) First, we show that the sequence $\{x_n\}$ is bounded. Indeed, From Lemma 4.1 and the condition of ρ_n, we get

$$\|v_n - p\| \leq \|t_n - p\|. \tag{4.4}$$

From the definition of v_n and (4.4),

$$\begin{aligned}\|z_n - p\| &= \|\beta_n Sv_n + (1-\beta_n)t_n - p\|\\ &\leq \beta_n\|Sv_n - p\| + (1-\beta_n)\|t_n - p\|\\ &\leq \beta_n\|v_n - p\| + (1-\beta_n)\|t_n - p\|\\ &\leq \beta_n\|t_n - p\| + (1-\beta_n)\|t_n - p\|\\ &= \|t_n - p\|.\end{aligned} \tag{4.5}$$

Also, from (3.1),

$$\begin{aligned}\|t_n - p\| &= \|(1-\theta_n)w_n + \theta_n u_n - p\|\\ &\leq (1-\theta_n)\|w_n - p\| + \theta_n\|u_n - p\|\\ &\leq \|w_n - p\|.\end{aligned} \tag{4.6}$$

From (3.2), (4.5) and (4.6), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\gamma_n\varphi(x_n) + \zeta_n Sv_n + \xi_n Sz_n - p\|\\ &\leq \gamma_n\|\varphi(x_n)-p\| + \zeta_n\|Sv_n - p\| + \xi_n\|Sz_n - p\|\\ &\leq \gamma_n\delta\|x_n - p\| + \gamma_n\|\varphi(p)-p\| + \zeta_n\|v_n - p\| + \xi_n\|z_n - p\|\\ &\leq \gamma_n\delta\|x_n - p\| + \gamma_n\|\varphi(p)-p\| + (1-\gamma_n)\|w_n - p\|\\ &\leq \gamma_n\delta\|x_n - p\| + \gamma_n\|\varphi(p)-p\| + (1-\gamma_n)\|x_n - p\| + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|\\ &\leq [1-\gamma_n(1-\delta)]\|x_n - p\| + \gamma_n(1-\delta)\frac{\|\varphi(p)-p\|}{1-\delta} + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|\\ &\leq \max\left\{\|x_n - p\|, \frac{\|\varphi(p)-p\|}{1-\delta}\right\} + \sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|\\ &\leq \cdots\\ &\leq \max\left\{\|x_{n_0} - p\|, \frac{\|\varphi(p)-p\|}{1-\delta}\right\} + M_2.\end{aligned}$$

Thus, we infer that the sequence $\{x_n\}$ is bounded, so are $\{w_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{\varphi(x_n)\}$.

(b) From (2.2), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\gamma_n\varphi(x_n)+\zeta_n Sv_n+\xi_n Sz_n-p\|^2\\ &\le \gamma_n\|\varphi(x_n)-p\|^2+\zeta_n\|Sv_n-p\|^2+\xi_n\|Sz_n-p\|^2-\zeta_n\xi_n\|Sv_n-Sz_n\|^2\\ &\le \gamma_n\|\varphi(x_n)-p\|^2+\zeta_n\|v_n-p\|^2+\xi_n\|z_n-p\|^2-\zeta_n\xi_n\|Sv_n-Sz_n\|^2. \qquad (4.7)\end{aligned}$$

From Lemma 4.1 and the definition of z_n, we obtain

$$\begin{aligned}\|z_n-p\|^2 &= \|\beta_n Sv_n+(1-\beta_n)t_n-p\|^2\\ &= \beta_n\|Sv_n-p\|^2+(1-\beta_n)\|t_n-p\|^2-\beta_n(1-\beta_n)\|Sv_n-t_n\|^2\\ &\le \beta_n\|v_n-p\|^2+(1-\beta_n)\|t_n-p\|^2-\beta_n(1-\beta_n)\|Sv_n-t_n\|^2\\ &\le \beta_n\Big[\|t_n-p\|^2-\rho_n(2-\rho_n)\frac{(\|(I-P_{\overline{C}_n})t_n\|^2+\|(I-P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I-P_{\overline{C}_n})t_n+J^*(I-P_{\overline{Q}_n})Jt_n\|^2}\Big]\\ &\quad +(1-\beta_n)\|t_n-p\|^2-\beta_n(1-\beta_n)\|Sv_n-t_n\|^2\\ &\le \|t_n-p\|^2-\rho_n\beta_n(2-\rho_n)\frac{(\|(I-P_{\overline{C}_n})t_n\|^2+\|(I-P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I-P_{\overline{C}_n})t_n+J^*(I-P_{\overline{Q}_n})Jt_n\|^2}\\ &\quad -\beta_n(1-\beta_n)\|Sv_n-t_n\|^2. \qquad (4.8)\end{aligned}$$

Putting (3.21), (4.4), (4.6), (4.8) into (4.7), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \gamma_n\|\varphi(x_n)-p\|^2+\zeta_n\|v_n-p\|^2+\xi_n\|z_n-p\|^2-\zeta_n\xi_n\|Sv_n-Sz_n\|^2\\ &\le \gamma_n\|\varphi(x_n)-p\|^2+\zeta_n\|t_n-p\|^2+\xi_n\Big[\|t_n-p\|^2\\ &\quad -\rho_n\beta_n(2-\rho_n)\frac{(\|(I-P_{\overline{C}_n})t_n\|^2+\|(I-P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I-P_{\overline{C}_n})t_n+J^*(I-P_{\overline{Q}_n})Jt_n\|^2}-\beta_n(1-\beta_n)\|Sv_n-t_n\|^2\Big]\\ &\quad -\zeta_n\xi_n\|Sv_n-Sz_n\|^2\\ &\le \|w_n-p\|^2+\gamma_n\|\varphi(x_n)-p\|^2-\beta_n\xi_n(1-\beta_n)\|Sv_n-t_n\|^2-\zeta_n\xi_n\|Sv_n-Sz_n\|^2\\ &\quad -\rho_n\beta_n\xi_n(2-\rho_n)\frac{(\|(I-P_{\overline{C}_n})t_n\|^2+\|(I-P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I-P_{\overline{C}_n})t_n+J^*(I-P_{\overline{Q}_n})Jt_n\|^2}\\ &\le \|x_n-p\|^2+\sum_{i\in N}|\alpha_{i,n}|\cdot\|x_{n-i}-x_{n-i-1}\|M_5+\gamma_n\|\varphi(x_n)-p\|^2\\ &\quad -\beta_n\xi_n(1-\beta_n)\|Sv_n-t_n\|^2-\zeta_n\xi_n\|Sv_n-Sz_n\|^2\\ &\quad -\rho_n\beta_n\xi_n(2-\rho_n)\frac{(\|(I-P_{\overline{C}_n})t_n\|^2+\|(I-P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I-P_{\overline{C}_n})t_n+J^*(I-P_{\overline{Q}_n})Jt_n\|^2}.\end{aligned}$$

□

Theorem 4.3 *Assume that conditions (C1), (C6)-(C10) hold. In addition, the solution set $EP(f)\cap\Gamma\cap F(S)$ is nonempty. Then, the sequence $\{x_n\}$ generated by Algorithm 4.1 converges strongly to an element $p=P_{EP(f)\cap\Gamma\cap F(S)}(\varphi(p))$, where p is a unique solution of the variational inequality*

$$\langle\varphi(p)-p,y-p\rangle\le 0,\quad \forall y\in EP(f)\cap\Gamma\cap F(S).$$

Proof Combining (2.1), (3.21), (4.4), (4.5) and (4.6), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\gamma_n\varphi(x_n)+\zeta_n Sv_n+\xi_n Sz_n-p\|^2\\ &\le \|\zeta_n(Sv_n-p)+\xi_n(Sz_n-p)\|^2+2\gamma_n\langle\varphi(x_n)-p,x_{n+1}-p\rangle\\ &= \zeta_n^2\|Sv_n-p\|^2+\xi_n^2\|Sz_n-p\|^2+2\zeta_n\xi_n\langle Sv_n-p,Sz_n-p\rangle\\ &\quad +2\gamma_n\langle\varphi(x_n)-\varphi(p),x_{n+1}-p\rangle+2\gamma_n\langle\varphi(p)-p,x_{n+1}-p\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq \zeta_n^2\|Sv_n - p\|^2 + \xi_n^2\|Sz_n - p\|^2 + 2\zeta_n\xi_n\|Sv_n - p\| \cdot \|Sz_n - p\| \\
&\quad + 2\gamma_n\delta\|x_n - p\| \cdot \|x_{n+1} - p\| + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\leq \zeta_n^2\|Sv_n - p\|^2 + \xi_n^2\|Sz_n - p\|^2 + \zeta_n\xi_n\|Sv_n - p\|^2 + \zeta_n\xi_n\|Sz_n - p\|^2 \\
&\quad + \gamma_n\delta\|x_n - p\|^2 + \gamma_n\delta\|x_{n+1} - p\|^2 + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&= \zeta_n(\zeta_n + \xi_n)\|Sv_n - p\|^2 + \xi_n(\zeta_n + \xi_n)\|Sz_n - p\|^2 + \gamma_n\delta(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\
&\quad + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\leq \zeta_n(1 - \gamma_n)\|v_n - p\|^2 + \xi_n(1 - \gamma_n)\|z_n - p\|^2 + \gamma_n\delta(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\
&\quad + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\leq (1 - \gamma_n)^2\|w_n - p\|^2 + \gamma_n\delta(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle \\
&\leq (1 - \gamma_n)^2\|x_n - p\|^2 + \sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\|M_5 + \gamma_n\delta\|x_{n+1} - p\|^2 \\
&\quad + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \frac{1sps2\gamma_n + \gamma_n\delta + \gamma_n^2}{1 - \gamma_n\delta}\|x_n - p\|^2 + \frac{1}{1 - \gamma_n\delta}\Big[\sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\|M_5 \\
&\quad + 2\gamma_n\langle\varphi(p) - p, x_{n+1} - p\rangle\Big] \\
&\leq \Big[1 - \frac{2\gamma_n(1 - \delta)}{1 - \gamma_n\delta}\Big]\|x_n - p\|^2 + \frac{2\gamma_n(1 - \delta)}{1 - \gamma_n\delta}\Big[\frac{\gamma_n\|x_n - p\|^2}{2(1 - \delta)} \\
&\quad + \frac{\langle\varphi(p) - p, x_{n+1} - p\rangle}{1 - \delta}\Big] + \frac{\sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\|M_5}{1 - \gamma_n\delta}. \qquad (4.9)
\end{aligned}
$$

The sequence $\{\|x_n - p\|\}$ converges by considering two possible cases.

Case I Assume that there exists $n_1 \in \mathbb{N}$ such that $\{x_n\}$ is nonincreasing for all $n \geq n_1$. It implies that $\lim_{n\to\infty}\|x_n - p\|$ exists. According to Lemma 4.2 (b), we get,

$$
\begin{aligned}
&\beta_n\xi_n(1 - \beta_n)\|Sv_n - t_n\|^2 + \zeta_n\xi_n\|Sv_n - Sz_n\|^2 \\
&+ \rho_n\beta_n\xi_n(2 - \rho_n)\frac{(\|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\|^2} \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\|M_5 + \gamma_n\|\varphi(x_n) - p\|^2.
\end{aligned}
$$

Using Condition 2, we obtain

$$
\|Sv_n - t_n\| \to 0, \quad \text{as } n \to \infty. \qquad (4.10)
$$

$$
\|Sv_n - Sz_n\| \to 0, \quad \text{as } n \to \infty. \qquad (4.11)
$$

And

$$
\frac{(\|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\|^2} \to 0, \quad \text{as } n \to \infty. \qquad (4.12)
$$

In addition,

$$
\begin{aligned}
\frac{(\|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2)^2}{\|(I - P_{\overline{C}_n})t_n + J^*(I - P_{\overline{Q}_n})Jt_n\|^2} &\geq \frac{(\|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2)^2}{2(\|(I - P_{\overline{C}_n})t_n\|^2 + \|J^*(I - P_{\overline{Q}_n})Jt_n\|^2)} \\
&\geq \frac{(\|(I - P_{\overline{C}_n})t_n\|^2 + \|(I - P_{\overline{Q}_n})Jt_n\|^2)^2}{2(\|(I - P_{\overline{C}_n})t_n\|^2 + \|J\|^2\|(I - P_{\overline{Q}_n})Jt_n\|^2)}
\end{aligned}
$$

$$\geq \frac{(\|(I-P_{\overline{C}_n})t_n\|^2 + \|(I-P_{\overline{Q}_n})Jt_n\|^2)^2}{2\max\{1, \|J\|^2\}(\|(I-P_{\overline{C}_n})t_n\|^2 + \|(I-P_{\overline{Q}_n})Jt_n\|^2)}$$
$$\geq \frac{\|(I-P_{\overline{C}_n})t_n\|^2 + \|(I-P_{\overline{Q}_n})Jt_n\|^2}{2\max\{1, \|J\|^2\}}.$$

Taking the limit $n \to \infty$ on both sides, we get

$$\|(I-P_{\overline{C}_n})t_n\| \to 0, \quad \text{and} \quad \|(I-P_{\overline{Q}_n})Jt_n\| \to 0. \tag{4.13}$$

Using (4.10) and (4.11),

$$\begin{aligned}\|z_n - Sz_n\| &= \|\beta_n Sv_n + (1-\beta_n)t_n - Sz_n\| \\ &\leq \beta_n\|Sv_n - Sz_n\| + (1-\beta_n)\|t_n - Sz_n\| \\ &\leq \beta_n\|Sv_n - Sz_n\| + (1-\beta_n)\|t_n - Sv_n\| + (1-\beta_n)\|Sv_n - Sz_n\| \\ &\leq \|Sv_n - Sz_n\| + \|t_n - Sv_n\| \to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{4.14}$$

Utilizing (4.10) again,

$$\|z_n - Sv_n\| = \|\beta_n Sv_n + (1-\beta_n)t_n - Sv_n\|$$
$$\leq (1-\beta_n)\|t_n - Sv_n\| \to 0, \quad \text{as } n \to \infty. \tag{4.15}$$
$$\|z_n - t_n\| \leq \|z_n - Sv_n\| + \|Sv_n - t_n\| \to 0, \quad \text{as } n \to \infty. \tag{4.16}$$

Using (4.13), we have

$$\begin{aligned}\|v_n - t_n\| &= \|t_n - \tau_n[(I-P_{\overline{C}_n})t_n + J^*(I-P_{\overline{Q}_n})Jt_n] - t_n\| \\ &\leq \tau_n\|(I-P_{\overline{C}_n})t_n\| + \tau_n\|J\| \cdot \|(I-P_{\overline{Q}_n})Jt_n\| \to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{4.17}$$

From (4.1), we get $\theta_n\|u_n - x_n\| \leq \varepsilon_n$. Combining $\sum_{n=1}^{\infty}\varepsilon_n < \infty$,

$$\theta_n\|u_n - x_n\| \to 0, \quad \text{as } n \to \infty. \tag{4.18}$$

Using (3.36) and (4.18), we have

$$\begin{aligned}\|t_n - w_n\| &= \|(1-\theta_n)w_n + \theta_n u_n - w_n\| \\ &\leq \theta_n\|u_n - w_n\| \\ &\leq \theta_n\|u_n - x_n\| + \theta_n\|x_n - w_n\| \to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{4.19}$$

Therefore,

$$\|t_n - x_n\| \leq \|t_n - w_n\| + \|w_n - x_n\| \to 0, \quad \text{as } n \to \infty. \tag{4.20}$$

In addition,

$$\begin{aligned}\|w_n - u_n\| &= \|x_n + \sum_{i\in N}\alpha_{i,n}(x_{n-i} - x_{n-i-1}) - u_n\| \\ &\leq \frac{1}{\theta_n}\theta_n\|u_n - x_n\| + \sum_{i\in N}|\alpha_{i,n}| \cdot \|x_{n-i} - x_{n-i-1}\| \to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{4.21}$$

Combining (4.10) and (4.20),

$$\|Sv_n - x_n\| \leq \|Sv_n - t_n\| + \|t_n - x_n\| \to 0, \quad \text{as } n \to \infty. \tag{4.22}$$

Combining (4.11) and (4.22),

$$\|Sz_n - x_n\| \leq \|Sz_n - Sv_n\| + \|Sv_n - x_n\| \to 0, \quad \text{as } n \to \infty. \tag{4.23}$$

Therefore, we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\gamma_n \varphi(x_n) + \zeta_n S v_n + \xi_n S z_n - x_n\| \\ &\le \gamma_n \|\varphi(x_n) - x_n\| + \zeta_n \|S v_n - x_n\| + \xi_n \|S z_n - x_n\| \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{4.24}$$

From Lemma 4.2, we know that the sequence $\{x_n\}$ is bounded. Thus, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x}$ as $i \to \infty$ and

$$\limsup_{n\to\infty} \langle \varphi(p) - p, x_n - p\rangle = \lim_{i\to\infty} \langle \varphi(p) - p, x_{n_i} - p\rangle = \langle \varphi(p) - p, \tilde{x} - p\rangle. \tag{4.25}$$

Next, we show that $\tilde{x} \in EP(f) \cap \Gamma \cap F(S)$. Indeed, from (3.36), (4.16), (4.17) and (4.20), we have $w_{n_i} \rightharpoonup \tilde{x}$, $t_{n_i} \rightharpoonup \tilde{x}$, $v_{n_i} \rightharpoonup \tilde{x}$ and $z_{n_i} \rightharpoonup \tilde{x}$ as $i \to \infty$. And by (4.14), $\lim_{n\to\infty} \|z_n - S z_n\| = 0$. From the demiclosedness of $(I - S)$, we get $\tilde{x} \in F(S)$.

From Algorithm 4.1, we know $u_n = T_{r_n}^f w_n$. Similarly arguments as the proof of Theorem 3.2, we conclude that $\tilde{x} \in EP(f)$.

Furthermore, $P_{\overline{C}_{n_i}} t_{n_i} \in \overline{C}_{n_i}$, by the definition of $\overline{C}_{n_i}$,

$$c(t_{n_i}) + \langle \phi_{n_i}, P_{\overline{C}_{n_i}} t_{n_i} - t_{n_i}\rangle + \frac{\alpha}{2}\|P_{\overline{C}_{n_i}} t_{n_i} - t_{n_i}\|^2 \le 0,$$

where $\phi_{n_i} \in \partial c(t_{n_i})$. According to (4.13) the boundedness of ϕ_{n_i}, we have

$$\begin{aligned} c(t_{n_i}) &\le \langle \phi_{n_i}, t_{n_i} - P_{\overline{C}_{n_i}} t_{n_i}\rangle - \frac{\alpha}{2}\|P_{\overline{C}_{n_i}} t_{n_i} - t_{n_i}\|^2 \\ &\le \|\phi_{n_i}\| \cdot \|(I - P_{\overline{C}_{n_i}}) t_{n_i}\| - \frac{\alpha}{2}\|(I - P_{\overline{C}_{n_i}}) t_{n_i}\|^2 \\ &\le M_7 \|(I - P_{\overline{C}_{n_i}}) t_{n_i}\| - \frac{\alpha}{2}\|(I - P_{\overline{C}_{n_i}}) t_{n_i}\|^2 \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

where $\|\phi_{n_i}\| \le M_7 > 0$.

By the fact that $t_{n_i} \rightharpoonup \tilde{x}$ and c is weakly lower semi-continuous,

$$c(\tilde{x}) \le \liminf_{i\to\infty} c(t_{n_i}) \le 0.$$

This implies $\tilde{x} \in C$. Since J is a bounded linear operator, then $J t_{n_i} \rightharpoonup J\tilde{x}$. Similarly, from $P_{\overline{Q}_{n_i}} J t_{n_i} \in \overline{Q}_{n_i}$ and (4.13), we also obtain

$$q(J\tilde{x}) \le \liminf_{i\to\infty} q(J t_{n_i}) \le 0.$$

It implies $J\tilde{x} \in Q$. Therefore, $\tilde{x} \in EP(f) \cap \Gamma \cap F(S)$. In addition, according to (2.3) and (4.25), we have

$$\limsup_{n\to\infty} \langle \varphi(p) - p, x_n - p\rangle = \lim_{i\to\infty} \langle \varphi(p) - p, x_{n_i} - p\rangle = \langle \varphi(p) - p, \tilde{x} - p\rangle \le 0. \tag{4.26}$$

By the use of (4.24),

$$\begin{aligned} &\limsup_{n\to\infty} \langle \varphi(p) - p, x_{n+1} - p\rangle \\ &\le \limsup_{n\to\infty} \langle \varphi(p) - p, x_{n+1} - x_n\rangle + \limsup_{n\to\infty} \langle \varphi(p) - p, x_n - p\rangle \\ &\le 0. \end{aligned} \tag{4.27}$$

Combining (4.9), (4.27) and Condition 2, we conclude from Lemma 2.10 that $\lim_{n\to\infty} \|x_n - p\| = 0$.

Case II There exists a subsequence $\{\|x_{n_j} - p\|\}$ of $\{\|x_n - p\|\}$ such that $\|x_{n_j} - p\| < \|x_{n_j+1} - p\|$ for $\forall j \in \mathbb{N}$. From Lemma 2.12, there exists a nondecreasing sequence $m_k \subset \mathbb{N}$ such that $m_k \to \infty$, as $k \to \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - p\| \le \|x_{m_k+1} - p\| \quad \text{and} \quad \|x_k - p\| \le \|x_{m_k+1} - p\|. \tag{4.28}$$

According to Lemma 4.2 (b), we get,

$$\begin{aligned}
&\beta_{m_k}\xi_{m_k}(1-\beta_{m_k})\|Sv_{m_k}-t_{m_k}\|^2+\zeta_{m_k}\xi_{m_k}\|Sv_{m_k}-Sz_{m_k}\|^2\\
&+\rho_{m_k}\beta_{m_k}\xi_{m_k}(2-\rho_{m_k})\frac{(\|(I-P_{\overline{C}_{m_k}})t_{m_k}\|^2+\|(I-P_{\overline{Q}_{m_k}})Jt_{m_k}\|^2)^2}{\|(I-P_{\overline{C}_{m_k}})t_{m_k}+J^*(I-P_{\overline{Q}_{m_k}})Jt_{m_k}\|^2}\\
&\le \|x_{m_k}-p\|^2-\|x_{m_k+1}-p\|^2+\sum_{i\in N}|\alpha_{i,m_k}|\cdot\|x_{m_k-i}-x_{m_k-i-1}\|M_5+\gamma_{m_k}\|\varphi(x_{m_k})-p\|^2\\
&\le \sum_{i\in N}|\alpha_{i,m_k}|\cdot\|x_{m_k-i}-x_{m_k-i-1}\|M_5+\gamma_{m_k}\|\varphi(x_{m_k})-p\|^2.
\end{aligned}$$

Taking the limit, we obtain

$$\begin{aligned}
&\|Sv_{m_k}-t_{m_k}\|\to 0,\quad \text{as } k\to\infty.\\
&\|Sv_{m_k}-Sz_{m_k}\|\to 0,\quad \text{as } k\to\infty.
\end{aligned}$$

And

$$\frac{(\|(I-P_{\overline{C}_{m_k}})t_{m_k}\|^2+\|(I-P_{\overline{Q}_{m_k}})Jt_{m_k}\|^2)^2}{\|(I-P_{\overline{C}_{m_k}})t_{m_k}+J^*(I-P_{\overline{Q}_{m_k}})Jt_{m_k}\|^2}\to 0,\quad \text{as } k\to\infty.$$

By the similar arguments as in Case I, it is easy to see that

$$\limsup_{k\to\infty}\langle\varphi(p)-p,x_{m_k+1}-p\rangle\le 0. \tag{4.29}$$

Combining (4.9) and (4.28)

$$\begin{aligned}
\|x_k-x^*\|\le\|x_{m_k+1}-p\|^2&\le\Big[1-\frac{2\gamma_{m_k}(1-\delta)}{1-\gamma_{m_k}\delta}\Big]\|x_{m_k}-p\|^2+\frac{2\gamma_{m_k}(1-\delta)}{1-\gamma_{m_k}\delta}\Big[\frac{\gamma_{m_k}\|x_{m_k}-p\|^2}{2(1-\delta)}\\
&\quad+\frac{\langle\varphi(p)-p,x_{m_k+1}-p\rangle}{1-\delta}\Big]+\frac{\sum_{i\in N}|\alpha_{i,m_k}|\cdot\|x_{m_k-i}-x_{m_k-i-1}\|M_5}{1-\gamma_{m_k}\delta}\\
&\le\Big[1-\frac{2\gamma_{m_k}(1-\delta)}{1-\gamma_{m_k}\delta}\Big]\|x_{m_k+1}-p\|^2+\frac{2\gamma_{m_k}(1-\delta)}{1-\gamma_{m_k}\delta}\Big[\frac{\gamma_{m_k}\|x_{m_k}-p\|^2}{2(1-\delta)}\\
&\quad+\frac{\langle\varphi(p)-p,x_{m_k+1}-p\rangle}{1-\delta}\Big]+\frac{\sum_{i\in N}|\alpha_{i,m_k}|\cdot\|x_{m_k-i}-x_{m_k-i-1}\|M_5}{1-\gamma_{m_k}\delta}\\
&\le\frac{\gamma_{m_k}\|x_{m_k}-p\|^2}{2(1-\delta)}+\frac{\langle\varphi(p)-p,x_{m_k+1}-p\rangle}{1-\delta}\\
&\quad+\frac{1-\gamma_{m_k}\delta}{2\gamma_{m_k}(1-\delta)}\frac{\sum_{i\in N}|\alpha_{i,m_k}|\cdot\|x_{m_k-i}-x_{m_k-i-1}\|M_5}{1-\gamma_{m_k}\delta}.
\end{aligned}$$

So it follows from (4.29) and Condition 2 that $\lim_{k\to\infty}\|x_k-x^*\|=0$. This finishes the proof. □

Remark 4.4 (a) If $\alpha=\beta=0$, then $\overline{C}_n$ and $\overline{Q}_n$ are converted to half spaces C_n and Q_n.

(b) If $\alpha>0,\beta>0$, it is easy to observe that $\overline{C}_n$ is a nonempty closed ball, which has radius $\frac{1}{\alpha}\sqrt{\|\phi_n\|^2-2\alpha c(t_n)}$ centred at $t_n-\frac{1}{\alpha}\phi_n$ and $\overline{Q}_n$ is a nonempty closed ball, which has radius $\frac{1}{\beta}\sqrt{\|\psi_n\|^2-2\beta q(Jt_n)}$ centred at $Jt_n-\frac{1}{\beta}\psi_n$.

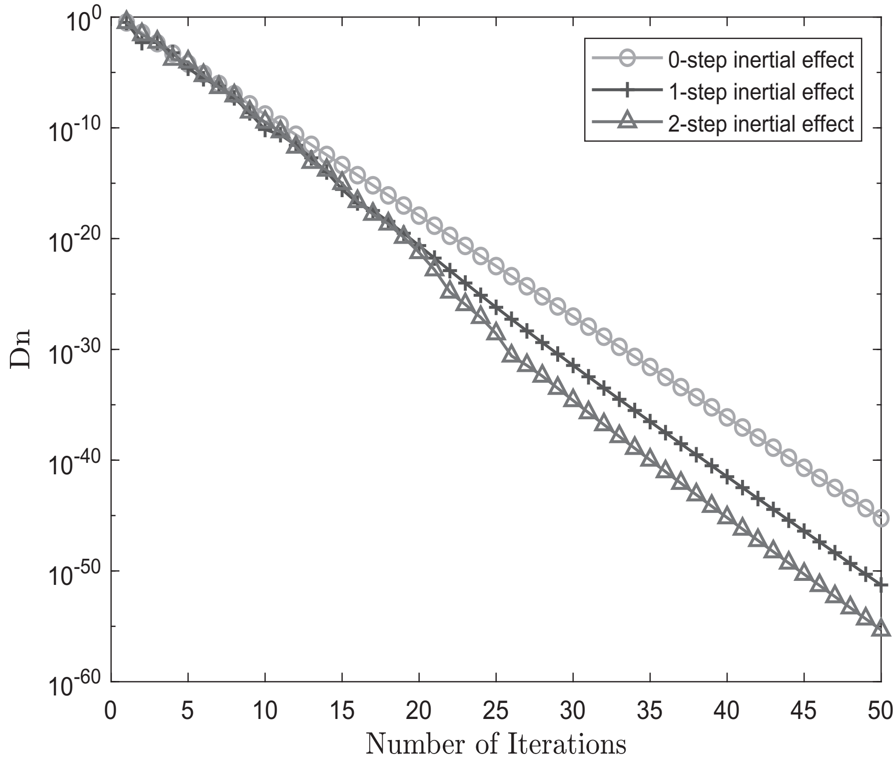


Fig. 1 Algorithm 3.1 with different inertial steps

5 Numerical experiments

In this section, we give two numerical examples to show the numerical results of the proposed algorithms. We demonstrate the advantages of the proposed algorithms by comparing them with previous iterative schemes in the literature. All the programs are implemented in MATLAB 2021a on Intel(R) Core(TM) i5-5200U CPU @ 2.20GHz, RAM 8.00 GB.

Example 5.1 [43] Let $H = l^2$ be a real Hilbert space where $x = (x_1, x_2, x_3, \ldots)$ is a real sequence. Let $C := \{x : \|x\| \leq 10\}$. Define

$$f(x, y) = -4\|x\|^2 + 3\langle x, y\rangle + \|y\|^2.$$

By proving, the bifunction f satisfies assumptions (A1)-(A4). It is easy to check that the mapping

$$T_{r_n}^f(x) = \frac{x}{5r_n + 1}.$$

Test 1. First, we perform an experiment to show the numerical behavior of different inertial steps for Algorithm 3.1. Suppose $Tx = x$, $Ax = x/6$ is 3-ism and $Bx = x/3$ is 2-ism. Therefore, $\Omega := EP(f) \cap F(T) \cap F(G) = \{0\} \neq \emptyset$. The initial $x_0 = x_1 = x_2 = x_3 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, \ldots)$.

The data parameters are as follows:

- $\mu = 2$, $\nu = 1$, $\lambda = 1/2$, $\delta = 1/10$, $\alpha_{0,n} = 1/(10n + 10)$, $\alpha_{1,n} = -5/(2n + 10)^2$, $\beta_n = 1/(n + 20)^2$, $r_n = n/(4n - 1)$ and $\gamma_n = 1/(10n + 10)$.

The maximum number of iterations 50 is used as a stopping criterion. Figure 1 shows the numerical illustration $D_n = \|x_n - 0\|$, where D_n is used to measure the computational error of algorithms at n-th step.

Test 2. This test compares our Algorithm 3.1 with the Algorithm 3.1 proposed by Yazdi et al. [43] and the Algorithm 4.4 suggested by Cai et al. [5]. The data of the algorithms are as follows:

- For Alg 3.1 with 1-step inertial effect, we take $\mu = 2$, $\nu = 1$, $\lambda = 1$, $\delta = 1/10$, $\alpha_{0,n} = 3/(2.4n + 10)$, $\beta_n = 1/(n + 1)^2$, $r_n = n/(4n - 1)$ and $\gamma_n = 1/(10n + 10)$.

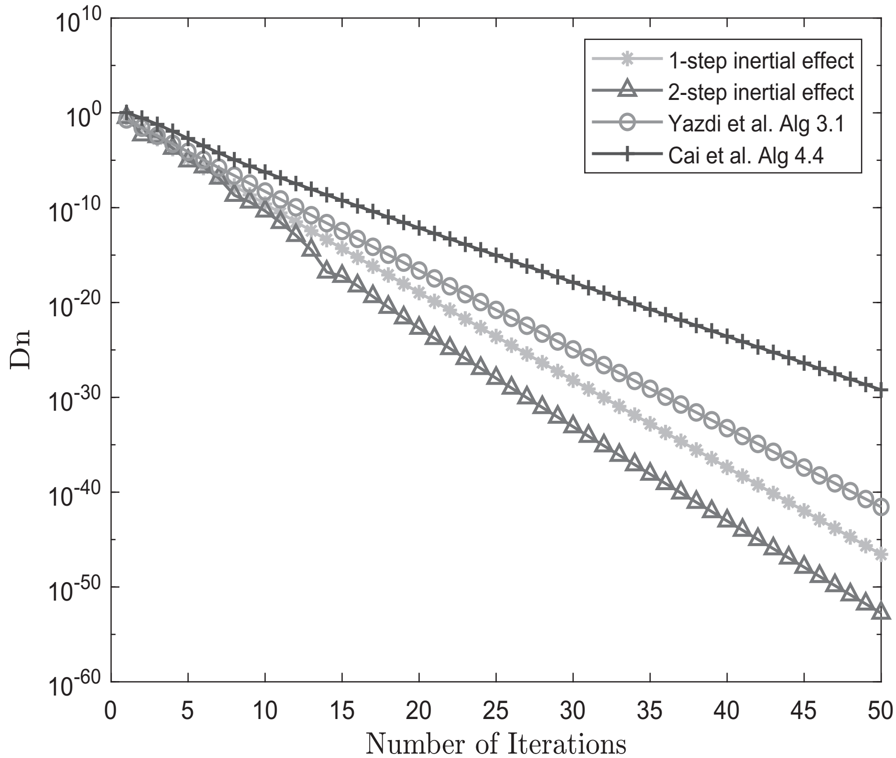


Fig. 2 Comparison of different algorithms

- For Alg 3.1 with 2-step inertial effect, we take $\mu = 2$, $\nu = 1$, $\lambda = 1$, $\delta = 1/10$, $\alpha_{0,n} = 3/(2.4n + 10)$, $\alpha_{1,n} = -5/(2.5n + 10)^2$, $\beta_n = 1/(n + 1)^2$, $r_n = n/(4n - 1)$ and $\gamma_n = 1/(10n + 10)$.
- For the Alg 3.1 proposed by Yazdi et al., we set $\mu = 2$, $\nu = 1$, $t_n = 1/(n+1)$, $\kappa = 1/10$, $\alpha_n = 1/(10n+10)$ and $r_n = n/(4n - 1)$.
- For the Alg 4.4 suggested by Cai et al., we set $\mu = 2$, $\nu = 1$, $s_n = 1/(n + 1)$, $\kappa = 1/10$, $\rho = 1/2$, $\alpha_n = 1/(10n + 10)$, $\beta_n = 1/(n + 2)$ and $r_n = n/(4n - 1)$.

Suppose $T_n x = Tx = 1/2x$, $Ax = x/6$ and $Bx = x/3$. Choose the same initial values x_0, x_1, x_2, x_3 as **Test 1**. The maximum number of iterations 50 is also used as a stopping criterion. Figure 2 shows the the numerical behavior $D_n = \|x_n - 0\|$ of different algorithms.

Example 5.2 [18,42] Consider the LASSO problem:

$$\begin{aligned} &\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Ax - y\|_2^2, \\ &s.t. \quad \|x\|_1 \le t, \end{aligned} \tag{5.1}$$

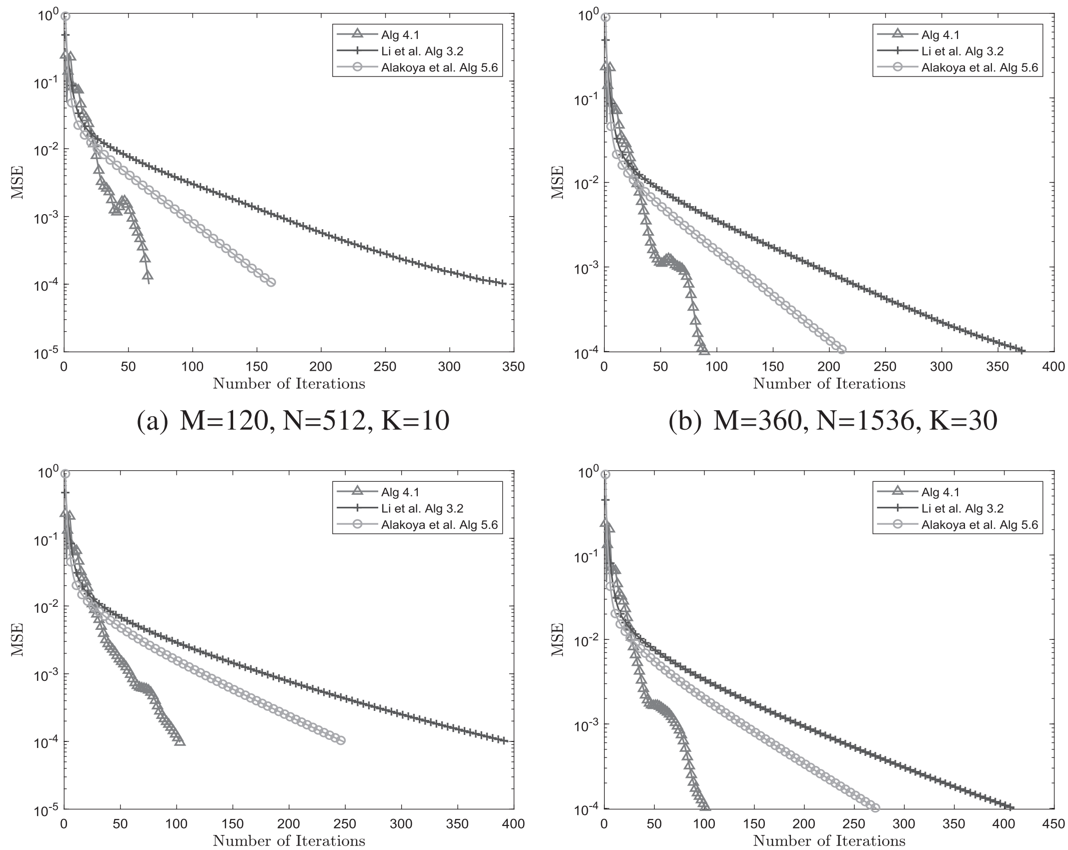
where $A \in \mathbb{R}^{M\times N}$ $(M < N)$, $y \in \mathbb{R}^M$ and $t > 0$ is a given constant. The system matrix A is generated from a normal distribution with mean zero and unit variance. The true sparse signal $x^\dagger \in \mathbb{R}^N$ is generated from a uniform distribution in [-2,2] with random K position nonzero while the rest is kept zero. The sample data is $y = Ax^\dagger$. The observation y is generated by white Gaussian noise with signal-to-noise ratio SNR=40. The process is started with $t = K$ and initial points are picked randomly.

The solution of minimization problem (5.1) is equivalent to the ℓ_0-norm solution of the underdetermined linear system with a certain matrix A. Let $C = \{x \in \mathbb{R}^N : \|x\|_1 \le t\}$ and $Q = \{y\}$. Then the minimization problem (5.1) can be seen as the problem SFP and, since the projection onto the closed convex C does not have a closed form solution, we utilize the subgradient projections [27,29,34]. Define a convex function $c(x) = \|x\|_1 - t$ and denote the level set C_n by:

$$C_n = \{x : c(x_n) + \langle \phi_n, x - x_n\rangle \le 0\rangle\},$$

Table 1 Numerical results for solving the LASSO problem

(M, N, K)	Iter. Alg 4.1	Li et al. Alg 3.2	Alakoya et al. Alg 5.6	CPU Alg 4.1	Li et al. Alg 3.2	Alakoya et al. Alg 5.6
(120,512,10)	66	344	164	0.078	0.182	0.063
(360, 1536,30)	89	374	214	0.877	2.834	0.955
(720,3072,60)	102	395	249	4.983	10.612	5.084
(1200,5120,100)	104	410	272	15.768	21.191	15.102



(a) M=120, N=512, K=10 (b) M=360, N=1536, K=30

(c) M=720, N=3072, K=60 (d) M=1200, N=5120, K=100

Fig. 3 Compare MSE of algorithms under different M, N, K

where $\phi_n \in \partial c(x_n)$. The subdifferential ∂c at x_n is

$$\partial c(x_n) = \begin{cases} 1, & x_n > 0, \\ [-1, 1], & x_n = 0, \\ -1, & x_n < 0. \end{cases}$$

And the orthogonal projection on C_n can be calculated by

$$P_{C_n}(x) = \begin{cases} x, & if\ c(x_n) + \langle \phi_n, x - x_n \rangle \le 0, \\ x - \dfrac{c(x_n) + \langle \phi_n, x - x_n \rangle}{\|\phi_n\|^2}\phi_n, & otherwise. \end{cases}$$

The bifunction $f(x, y)$ is still used as given in Example 5.1,

$$f(x, y) = -4\|x\|^2 + 3\langle x, y\rangle + \|y\|^2.$$

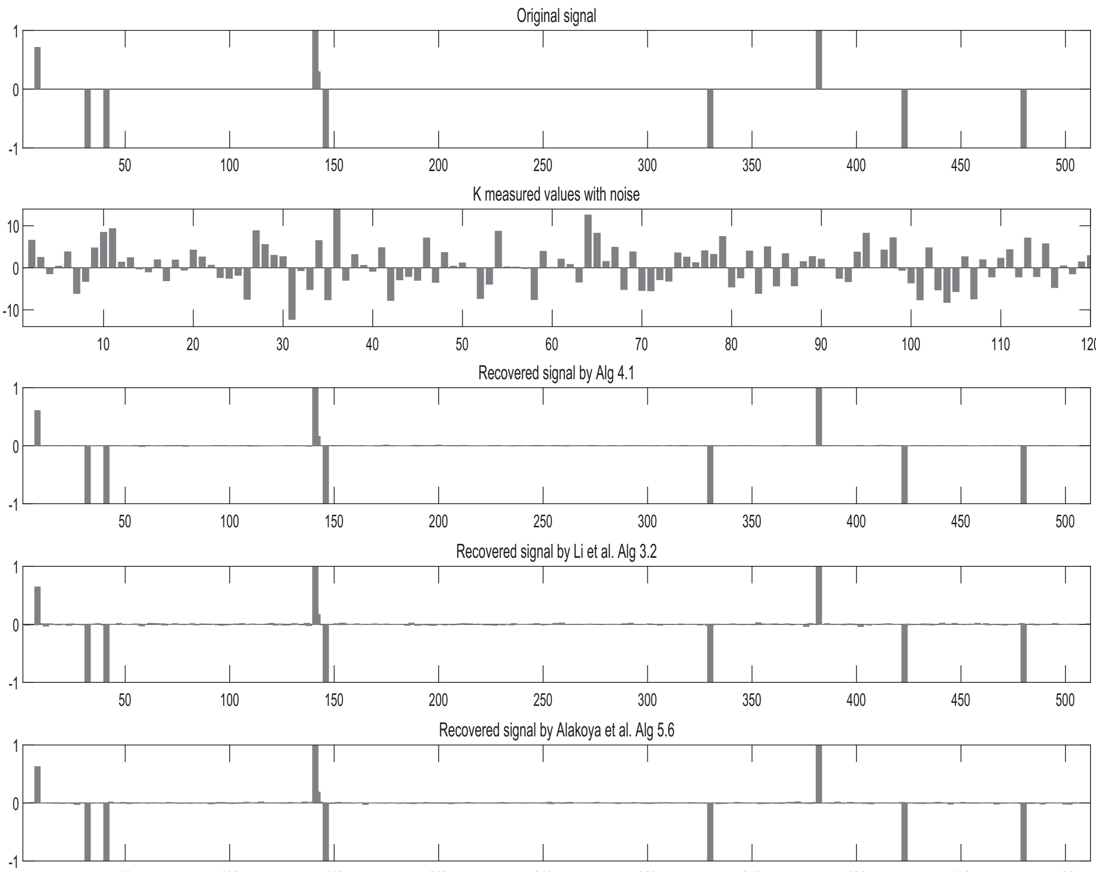


Fig. 4 From top to bottom: original signal, observed signal, restored signal by algorithms with M=120, N=512, K=10

Some numerical results are given by using our Algorithm 4.1 with multi-step inertial, Li et al. Algorithm 3.2 [16] and Alakoya et al. Algorithm 5.6 [1]. The data of the algorithms are as follows:

- For Alg 4.1, we take $\theta = 1/2$, $\delta = 1/2$, $\varepsilon_n = 1/(n+10)$, $\beta_n = 1/(2n)$, $\zeta_n = 1/(2n)$, $\rho_n = n/(1.5n+3)$, $\gamma_n = 1/(n+100)^4$ and $S = I$.
- For the Alg 3.2 proposed by Li et al., we set $\alpha = 1$, $\theta = 1/2$, $r_n = n/(4n-1)$, $\rho_n = n/(n+3)$, $\beta_n = 1/(n+50)$, $\eta_n = 1/(n+10)$ and $\varepsilon_n = 1/(n+10)$.
- For the Alg 5.6 suggested by Alakoya et al., we set $\kappa = 1/2$, $\nu = 1$, $\delta_n = 1/10$, $\sigma_n = n/(n+2)$, $\rho_n = n/(n+3)$, $\mu_n = 1/(10n+100)$, $\beta_n = n/(n+5)$, $\alpha_n = 1/(n+100)$, $r_n = n/(4n-1)$ and $S = T = I$.

In our experiments, we consider four cases as follows:
Case 1: $M = 120$, $N = 512$ and $K = 10$;
Case 2: $M = 360$, $N = 1536$ and $K = 30$;
Case 3: $M = 720$, $N = 3072$ and $K = 60$;
Case 4: $M = 1200$, $N = 5120$ and $K = 100$.
The stopping criterion is defined by the following Mean Squared Error (MSE):

$$\text{MSE} = \frac{\|x_n - x^\dagger\|^2}{N} < 10^{-4}.$$

Table 1 and Figure 3 illustrate the number of iterations that the algorithms satisfy MSE in different dimensions.
Figure 4 and 5 show the illustrations of the original signal and the recovered signal in case $M = 120$, $N = 512$, $K = 10$ and $M = 1200$, $N = 5120$, $K = 100$.

Remark 5.3 From Example 5.1 and Example 5.2, we have the following remarks.

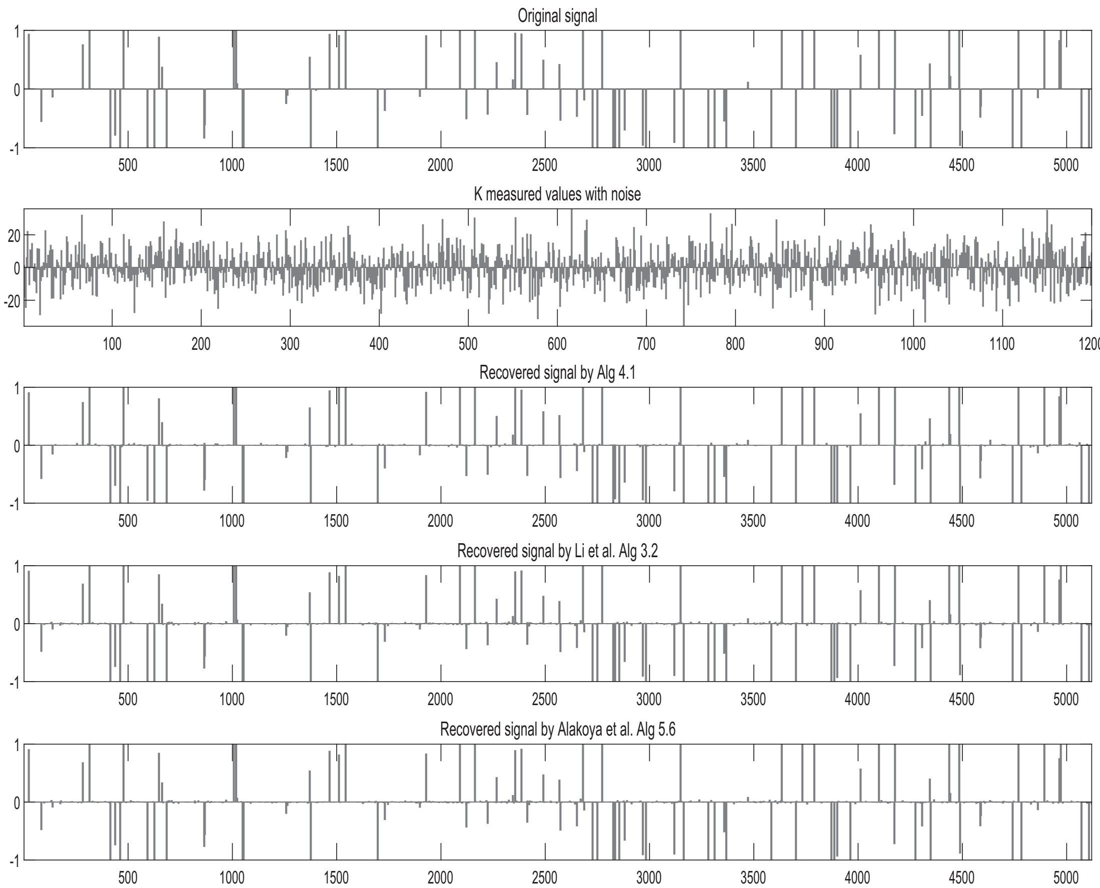


Fig. 5 From top to bottom: original signal, observed signal, restored signal by algorithms with M=1200, N=5120, K=100

(i) In **Test 1**, it can be observed from Figure 1 that the 2-step inertial algorithm converges faster than the 1-step inertial algorithm which in turn is faster than the 0-step inertial algorithm. It also indicates that the multi-step inertial method is meaningful. From Figure 2 in **Test 2**, our algorithms with inertial effects are rate of convergence faster than existing algorithms [5,43]. In addition, it can be intuitively seen from the Figure 2 that the iteration error D_n is minimal compared to the other two algorithms.

(ii) Based on Example 5.2, it can be observed that Algorithm 4.1 with multi-step inertial has competitive power as compared to other inertial-type algorithms. Comparing Algorithm 4.1 with other known algorithms in the literature [1,16], it can be observed from Table 1 and Figures 3 that under MSE$< 10^{-4}$, Algorithm 4.1 has significant advantages in terms of iteration times. In addition, it can be seen in Figure 4 and 5 that we plot more accurate signal recovery images than other algorithms [1,16].

6 Conclusions

In this paper, we propose two novel multi-step inertial iterative methods to approximate a common element that combines a monotone equilibrium problem with other problems in real Hilbert spaces. Firstly, we adopt the conjugate gradient method to solve by combining equilibrium problems, fixed point problems and a general system of variational inequalities. Secondly, we employ the S-iteration process to by integrating equilibrium problems, fixed point problems and split feasibility problems. Two strong convergence theorems are established under appropriate assumptions and mild conditions. We present some numerical experiments to illustrate its performance and compared with some known ones in the literature. We found that our proposed algorithms have advantages through preliminary numerical results.

Funding This work was supported by the National Natural Science Foundation of China(Nos.11771126; 11971216).

References

1. T.O. Alakoya, O.T. Mewomo, Viscosity S-iteration method with inertial technique and self-adaptive step size for split variational inclusion, equilibrium and fixed point problems. Comput. Appl. Math. 41 (2022), Paper No. 39, 31 pp.
2. E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems. Math. Student 63 (1994), 123-145.
3. R.I. Bot, E.R. Csetnek, N. Nimana, Gradient-type penalty method with inertial effects for solving constrained convex optimization problems with smooth data. Optim. Lett. 12 (2018), 17-33.
4. C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Problems 18 (2002), 441–453.
5. G. Cai, Y, Shehu, O.S. IyiolaL, The modified viscosity implicit rules for variational inequality problems and fixed point problems of nonexpansive mappings in Hilbert spaces. RACSAM. 113 (2019), 3545-3562.
6. L.C. Ceng, C. Wang, J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities. Math. Methods Oper. Res. 67 (2008), 375-390.
7. Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space. Numer. Algorithms 8 (1994), 221–239.
8. P.I. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces J. Nonlinear Convex Anal. 6 (2005), 117-136.
9. B.V. Dinh, D.S. Kim, Projection algorithms for solving nonmonotone equilibrium problems in Hilbert space. J. Comput. Appl. Math. 302 (2016), 106-117.
10. Q.L. Dong, Y.Y. Lu, J. Yang, The extragradient algorithm with inertial effects for solving the variational inequality. Optimization 65 (2016), 2217-2226.
11. Q.L. Dong, H.B. Yuan, Y.J. Cho, Th.M. Rassias, Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings. Optim. Lett. 12 (2018), 87-102.
12. A.N. Iusem, W. Sosa, On the proximal point method for equilibrium problems in Hilbert spaces. Optimization 59 (2010), 1259-1274.
13. O.S. Iyiola, Y. Shehu, Alternated inertial method for nonexpansive mappings with applications. J. Nonlinear Convex Anal. 21 (2020), 1175-1189.
14. J.S. Jung, Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. 302 (2005), 509-520.
15. G.M. Korpelevich, The extragradient method for finding saddle points and other problems. Ekon. Mat. Metody 12 (1976), 747-756.
16. H. Li, Y. Wu, F. Wang, Convergence analysis for solving equilibrium problems and split feasibility problems in Hilbert spaces. Optimization 72 (2023), 1863-1898.
17. X. Li, X. Liu, F. Gao, The ball-relaxed gradient-projection algorithm for split feasibility problem. J. Funct. Spaces 2022, Art.ID 9991466, 6 pp.
18. X. Ma, H. Liu, X. Li, The iterative method for solving the proximal split feasibility problem with an application to LASSO problem. Comput. Appl. Math. 41 (2022), Paper No. 5, 18 pp.
19. W.R. Mann, Mean value methods in iteration. Proc. Amer. Math. Soc. 4 (1953), 506-510.
20. G. Mastroeni, Gap functions for equilibrium problems. J. Global Optim. 27 (2003), 411–426.
21. A. Moudafi, Proximal point algorithm extended to equilibrium problems. J. Nat. Geom. 15 (1999), 91-100.
22. L.D. Muu, T.D. Quoc, Regularization algorithms for solving monotone Ky Fan inequalities with application to a Nash-Cournot equilibrium model. J. Optim. Theory Appl. 142 (2009), 185-204.
23. H. Nikaido, K. Isoda, Note on non-cooperative convex games. Pacific J. Math. 5 (1955), 807-815.
24. J. Nocedal, S.J. Wright, Numerical Optimization. Springer Series in Operations Research and Financial Engineering, 2nd ed., Springer, Berlin, 2006.
25. T.D. Quoc, L.D. Muu, Iterative methods for solving monotone equilibrium problems via dual gap functions. Comput. Optim. Appl. 51 (2012), 709-728.
26. S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. 67 (1979), 274-276.
27. D.R. Sahu, Y.J. Cho, Q.L. Dong, M.R. Kashyap, X.H. Li, Inertial relaxed CQ algorithms for solving a split feasibility problem in Hilbert spaces. Numer. Algorithms 87 (2021), 1075-1095.
28. Y. Shehu, Q.L. Dong, L. Liu, J.C. Yao, Alternated inertial subgradient extragradient method for equilibrium problems. TOP 31 (2023), 1–30.
29. Y. Shehu, A. Gibali, New inertial relaxed method for solving split feasibilities. Optim. Lett. 15 (2021), 2109-2126.
30. S. Suantai, B. Panyanak, S. Kesornprom, P. Cholamjiak, Inertial projection and contraction methods for split feasibility problem applied to compressed sensing and image restoration. Optim. Lett. 16 (2022), 1725-1744.
31. G.H. Taddele, P. Kumam, V. Berinde, An extended inertial Halpern-type ball-relaxed CQ algorithm for multiple-sets split feasibility problem. Ann. Funct. Anal. 13 (2022), Paper No. 48, 38 pp.
32. B. Tan, S.Y. Cho, J.C. Yao, Accelerated inertial subgradient extragradient algorithms with non-monotonic step sizes for equilibrium problems and fixed point problems. J. Nonlinear Var. Anal. 6 (2022), 89-122.
33. B. Tan, S. Li, Adaptive inertial subgradient extragradient methods for finding minimum-norm solutions of pseudomonotone variational inequalities. J. Ind. Manag. Optim. 19 (2023), 7640-7659.
34. B. Tan, X. Qin, X. Wang, Alternated inertial algorithms for split feasibility problems. Numer. Algorithms (2023) https://doi.org/10.1007/s11075-023-01589-8.
35. D.V. Thong, Q.L. Dong, X.H. Li, H.V. Thang, P.V. Nghia, N.T.C. Van, Two extragradient methods for solving variational inequalities in real Hilbert spaces. Linear Nonlinear Anal. 7 (2021), 387-412.
36. D.Q. Tran, M.L. Dung, V.H. Nguyen, Extragradient algorithms extended to equilibrium problems. Optimization 57 (2008), 749-776.
37. N.T. Vinh, A new projection algorithm for solving constrained equilibrium problems in Hilbert spaces. Optimization 68 (2019), 1447-1470.

38. P.T. Vuong, J.J. Strodiot, V.H. Nguyen, On extragradient-viscosity methods for solving equilibrium and fixed point problems in a Hilbert space. Optimization 64 (2015), 429-451.
39. H.K. Xu, Iterative algorithms for nonlinear operators. J. London Math. Soc.(2) 66 (2002), 240-256.
40. Q. Yang, The relaxed CQ algorithm solving the split feasibility problem.Inverse Problems 20 (2004), 1261-1266.
41. Y. Yao, O.S. Iyiola, Y. Shehu, Subgradient extragradient method with double inertial steps for variational inequalities. J. Sci. Comput. 90 (2022), Paper No. 71, 29 pp.
42. Z. Yao, S. Vong, Two inertial-type algorithms for solving the split feasibility problem. Optimization (2022) https://doi.org/10.1080/02331934.2022.2070066.
43. M. Yazdi, S. Hashemi Sababe, A hybrid viscosity approximation method for a common solution of a general system of variational inequalities, an equilibrium problem, and fixed point problems. J. Comput. Math. 41 (2023), 153-172.
44. C. Zhang, Q.L. Dong, J. Chen, Multi-step inertial proximal contraction algorithms for monotone variational inclusion problems. Carpathian J. Math. 36 (2020), 159-177.
45. Z. Zhou, B. Tan, S.X. Li, Adaptive hybrid steepest descent algorithms involving an inertial extrapolation term for split monotone variational inclusion problems. Math. Methods Appl. Sci. 45 (2022), 8835-8853.