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ORIGINAL RESEARCH

Ground state solutions to critical Schrödinger–Possion system with steep potential well

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Abstract We study the following critical Schrödinger-Possion system with steep potential well

$$\begin{cases} -\Delta u + (1 + \lambda V(x))u + \phi u = f(u) + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a positive parameter, $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function and f is a continuous subcritical nonlinearity. Under some certain assumptions on V and f, for any $\lambda \geq \lambda_0 > 0$, we prove the existence of a ground state solution via variational methods. Moreover, the concentration behavior of the ground state solution is also described as $\lambda \to \infty$. Our results extends that in Jiang[11](J. Differ. Equ. 2011) and Zhao[21](J. Differ. Equ. 2013) to the critical growth case.

Keywords Schrödinger-Possion system · Critical growth · Ground states

Mathematics Subject Classification 35J20 · 35J65 · 35J60

1 Introduction and main results

In this article, we study the following Schrödinger-Possion system

$$\begin{cases} -\Delta u + (1 + \lambda V(x))u + \phi u = f(u) + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{SP}$$

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where $\lambda > 0$ is a positive parameter and V, f satisfy some suitable assumptions. As we all know, if $\lambda = 0$, system (SP) is a variant type of the following system

$$\begin{cases} -\Delta u + u + \phi u = f(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

which was first proposed by Benci and Fortunato [4] in 1998 on a bounded domain, and was related to the Hartree equation. It was obtained when looking for the existence of standing waves for the nonlinear Schrödinger equation interacting with an unknown electrostatic field

$$\begin{cases} i\hbar \dfrac{\partial \Psi(x,t)}{\partial t} = -\dfrac{\hbar^2}{2m}\Delta\Psi + \tilde{V}(x)\Psi + \phi\Psi - f(|\Psi|), & \text{in } \mathbb{R}^3 \times \mathbb{R}^3, \\ -\Delta\phi = |\Psi|^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where i is the imaginary unit, $\hbar$ is the Planck constant, m is the mass of the field Ψ, $\tilde{V}(x)$ is a given external potential and $f(\exp(i\theta)\xi) = \exp(i\theta)f(\xi)$ for $\theta, \xi \in \mathbb{R}$ is a nonlinear function which describes the interaction among many particles.

In recent years, a great deal of work has been devoted to the study of standing waves, i.e.,

$$\Psi(x,t) = e^{-iEt}u(x),$$

which leads to the stationary Schrödinger-Possion system (1.1) with $\frac{\hbar^2}{2m} = 1$, $\tilde{V}(x) = 1 - E$. After Benci and Fortunato [5,6] studied the model

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

for $p \in (4,6)$, D'Aprile and Mugnai [7] proved the existence of radially symmetric solitary waves with $p \in [4,6)$ and obtained solutions as Mountain-Pass critical points for the associated energy functional. Inspired by [5–7], Azzollini et al. [2] proved the existence of nontrivial solutions by supposing that the nonlinearity satisfies Berestycki-Lions type assumptions. And in [1], Azzollini and Pomponio proved the existence of a ground state solution of the following system

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-2}u + u^5, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

for $p \in (2,6)$, they obtained the results by Nehari manifold and extended the results of [5,6] to the critical growth. Whereafter, Yin et al. [18] proved the existence of ground state solutions of the system (1.1) by using the method introduced in [10] to construct a special Palais-Smale sequence that combines with Pohožaev identity where f is asymptotically 2-linear.

Considering the potential $V(x)$, there are also some works about Schrödinger-Possion system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

Since $V(x)$ might not be radial, one cannot work in the radial functions space directly. On the one hand, Wang and Zhou [16] assumed that the potential V satisfies the *global condition*

$$0 < \inf_{x\in\mathbb{R}^3} V(x) < \lim_{|x|\to\infty} \inf V(x) = V_\infty,$$

which was introduced by Rabinowitz [14], they obtained the existence and non-existence of positive solutions for the problem (1.3) with asymptotically nonlinearities. On the other hand, Jiang and Zhou [11] first applied the steep potential well conditions to Schrödinger-Poisson system, and proved the existence of solutions when the steep potential is $(1 + \lambda V(x))$ and the nonlinearity is $|u|^{p-2}u$ with $p \in (2,3) \cup [4,6)$. Moreover, they also studied the asymptotic behavior of solutions by combining domains approximation with priori estimates. Then, Zhao et al. [21] considered a case allowing the potential $V(x)$ change sign. Using variational setting of [8], they obtained the existence and asymptotic behavior of nontrivial solutions when the steep potential is $\lambda V(x)$ and the nonlinearity is $|u|^{p-2}u$ with $p \in (3, 6)$. For the critical case, Yin and Wu [17] studied the following Schrödinger equation

$$-\Delta u + (1 + \lambda V(x))u = f(u) + |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N, \ N \geq 3, \lambda > 0,$$

and established the existence and concentration of the ground state solution. Whereafter, in [19], they studied the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + (1 + \lambda V(x))u + \phi u = \mu |u|^{p-2}u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $p \in (3, 6)$ and $\lambda, \mu > 0$, they obtained the existence and concentration of ground state solutions of system (1.4).

Motivated by the above facts, we focus our attention on the existence and concentration of ground state solutions of the critical Schrödinger-Poisson with steep potential and the more general subcritical nonlinearity $f(u)$.

Now, we assume that V and f satisfy the following conditions:

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$, $V(x) \geq 0$ *for all* $x \in \mathbb{R}^3$, *there exists* $V_0 > 0$ *such that*

$$meas(\{x \in \mathbb{R}^3 : V(x) \leq V_0\}) < +\infty;$$

(V_2) *the set* $\Omega := int V^{-1}(0)$ *is non-empty;*

(V_3) $V(x)$ *is weakly differentiable such that* $\nabla V(x) \cdot x \in L^\sigma(\mathbb{R}^3)$ *for some* $\sigma \in \left[\frac{3}{2}, \infty\right]$,

$$2V(x) + \nabla V(x) \cdot x \geq 0 \ \ a.e \ \ x \in \mathbb{R}^3.$$

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$, $\lim\limits_{t \to 0} \frac{f(t)}{t} = 0$ *and* $\lim\limits_{|t| \to \infty} \frac{f(t)}{t^5} = 0$;

(f_2) $\lim\limits_{t \to +\infty} \frac{F(t)}{t^4} = +\infty$;

(f_3) $f(t)t \geq 3F(t) \geq 0, \forall t \in \mathbb{R}$.

There exist some functions satisfying $(V_1) - (V_3)$. Here, a function provided by

$$V(x) = \begin{cases} a\left(\dfrac{1}{5} - \dfrac{1}{1 + |x|^2}\right), & |x| \geq 2, \\ 0, & |x| \leq 2, \end{cases}$$

where $a > 5$ is a positive constant.

Indeed, there are some functions which satisfy conditions $(f_1) - (f_3)$. For instance, a function is taken by

$$f(u) = b|u|^{p-2}u \ \text{ where } 2 < p < 6 \ \text{ and } \ b > 0.$$

Our main theorems read as follows:

Theorem 1.1 *If* $(V_1) - (V_3)$ *and* $(f_1) - (f_3)$ *hold, there exists* $\lambda_0 > 0$ *such that* $\lambda \geq \lambda_0$, *system* (SP) *admits at least a ground state solution.*

Theorem 1.2 *Assume that* u_{λ_n} *are the solutions obtained in Theorem 1.1 and* Ω *is defined by* (V_2), *then* $u_{\lambda_n} \to \tilde{u}$ *in* E, *as* $\lambda_n \to +\infty$, *where* $\tilde{u} \in H_0^1(\Omega)$ *is a nontrivial solution of*

$$\begin{cases} -\Delta u + u + \phi u = f(u) + |u|^4 u, & in\ \Omega, \\ -\Delta \phi = u^2, & on\ \partial\Omega. \end{cases}$$

Remark 1.1 In [20], Zhang and Zou obtained a ground state solution by the penalization skills. In Theorem 1.1, we do not use the penalization approach. Motivated by L. Jeanjean [10], we construct Nehari-Pohožaev-Palais-Smale sequence to obtain the boundedness of sequence. Therefore, the new results will be established.

Remark 1.2 Compared with [1,2,5,6], we consider the potential function $1+\lambda V(x)>1$ in our equation which is more complicated than their equations and we get richer results. And when λ is taken as the appropriate value, the steep potential in this artical can obtain the potential $V(x)$ which satisfies the global condition in [16]. So, our potential condition is weaker than that of [16]. Then, we extend the results of [11,21] to the critical growth case and extend the results of [17] to the Schrödinger-Poisson system. Furthermore, the subcritical nonlinearities in [1,5,6,11,19,21] also satisfies the conditions $(f_1)-(f_3)$ which f satisfies in this artical.

This artical is organized as follows. In Section 2, we give some notations and important lemmas. In Section 3, we present the proof of Theorem 1.1. In Section 4, we establish the behavior of ground state solutions and give the proof of Theorem 1.2.

2 Preliminaries

Henceforth we use the following notations.

- $\|u\| := \left(\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,\mathrm{d}\,x\right)^{\frac{1}{2}}$, for $u\in H^1(\mathbb{R}^3)$.
- $\|u\|_{D^{1,2}(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}\,x\right)^{\frac{1}{2}}$.
- $|u|_p := \left(\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}\,x\right)^{\frac{1}{p}}$, $\forall\, p\in[1,+\infty)$.
- $S := \inf\limits_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}\,x}{\left(\int_{\mathbb{R}^3}|u|^6\,\mathrm{d}\,x\right)^{\frac{1}{3}}}$.
- $\langle\cdot,\cdot\rangle$ denotes action of dual.
- $o_n(1)$ denotes a quantity which goes to zero as $n\to\infty$.
- c, c_i, C, C_i $(i=0,1,2,\ldots)$ denote various positive constants.

Firstly, we recall that by the Lax-Milgram theorem, for every $u\in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u\in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi_u=u^2$ with $\phi_u(x)=\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}\,\mathrm{d}\,y$ (see [4]). Moreover, $\phi_u>0$ when $u\neq 0$, and by using Hardy-Littlewood-Sobolev inequality (see [12]), we have

$$\int_{\mathbb{R}^3}\phi_u(x)u^2\,\mathrm{d}\,x=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\,\mathrm{d}\,x\,\mathrm{d}\,y\le S^{-2}\bar{S}^{-4}\|u\|^4,$$

where $\bar{S}=\inf\limits_{u\in H^1(\mathbb{R}^3)\setminus\{0\}}\frac{\left(\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,\mathrm{d}\,x\right)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}^3}|u|^6\,\mathrm{d}\,x\right)^{\frac{1}{3}}}$.

Then, for any $\lambda>0$, we define

$$E=\left\{u\in D^{1,2}(\mathbb{R}^3):\int_{\mathbb{R}^3}\lambda V(x)u^2\,\mathrm{d}\,x<+\infty\right\}.$$

Assume that (V_3) holds, for every $x\in\mathbb{R}^3$, we obtain

$$V(x)=\int_0^1\frac{\mathrm{d}}{\mathrm{d}\,t}(t^2V(tx))\,\mathrm{d}\,t=\int_0^1 t(2V(tx)+\nabla V(tx)\cdot tx)\,\mathrm{d}\,t\ge 0.$$

Therefore, E is a Hilbert space with inner product as follows

$$\langle u,v\rangle_\lambda=\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v+(1+\lambda V(x))uv)\,\mathrm{d}\,x,$$

and the norm

$$\|u\|_\lambda=\left(\int_{\mathbb{R}^3}\left(|\nabla u|^2+(1+\lambda V(x))u^2\right)\,\mathrm{d}\,x\right)^{\frac{1}{2}}.$$

From $V(x)\ge 0$, for any $\lambda>0$, we get

$$\int_{\mathbb{R}^3}u^2\,\mathrm{d}\,x\le\|u\|^2\le\|u\|_\lambda^2. \tag{2.1}$$

By (2.1), Hölder and Sobolev inequalities, for any $2 \leq \alpha \leq 6$, we have

$$\int_{\mathbb{R}^3} |u|^\alpha \, \mathrm{d}\,\mathrm{x} \leq \left(\int_{\mathbb{R}^3} u^2 \, \mathrm{d}\,\mathrm{x}\right)^{\frac{6-\alpha}{4}} \left(\int_{\mathbb{R}^3} u^6 \, \mathrm{d}\,\mathrm{x}\right)^{\frac{\alpha-2}{4}} \leq S^{\frac{6-3\alpha}{4}} \|u\|_\lambda^\alpha, \ \text{ as } \lambda \geq 0 \tag{2.2}$$

Thus, it implies that the embedding $E \hookrightarrow H^1(\mathbb{R}^3)$ is continuous. The functional $I_\lambda : E \to \mathbb{R}$ is defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (1+\lambda V(x))u^2\right) \, \mathrm{d}\,\mathrm{x} + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \, \mathrm{d}\,\mathrm{x} - \int_{\mathbb{R}^3} F(u) \, \mathrm{d}\,\mathrm{x} - \frac{1}{6}\int_{\mathbb{R}^3} u^6 \, \mathrm{d}\,\mathrm{x}.$$

It simply obtains that I_λ is well defined C^1-functional with derivative given by

$$\begin{aligned} \langle I_\lambda'(u), v\rangle = &\int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + (1+\lambda V(x))uv) \, \mathrm{d}\,\mathrm{x} + \int_{\mathbb{R}^3} \phi_u uv \, \mathrm{d}\,\mathrm{x} - \int_{\mathbb{R}^3} f(u)v \, \mathrm{d}\,\mathrm{x} \\ &- \int_{\mathbb{R}^3} u^4 uv \, \mathrm{d}\,\mathrm{x}, \ \ \forall\, v \in E. \end{aligned}$$

Thus, u is a critical point of I_λ if and only if (u, ϕ_u) is a solution of (SP).

Lemma 2.1 *Suppose that (f_1), (f_3) and (V_1) hold, then*
(i) there exists $w \in E\backslash\{0\}$ such that $I_\lambda(w) \leq 0$;
(ii) $c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0$, where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \ I(\gamma(1)) < 0\}$.

Proof (i) For fixed $u \in E\backslash\{0\}$, by (f_3), we have, as $t \to +\infty$,

$$\begin{aligned} I_\lambda(tu) &= \frac{t^2}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (1+\lambda V(x))u^2\right) \, \mathrm{d}\,\mathrm{x} + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_u u^2 \, \mathrm{d}\,\mathrm{x} - \int_{\mathbb{R}^3} F(tu) \, \mathrm{d}\,\mathrm{x} - \frac{t^6}{6}\int_{\mathbb{R}^3} u^6 \, \mathrm{d}\,\mathrm{x} \\ &\leq \frac{t^2}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (1+\lambda V(x))u^2\right) \, \mathrm{d}\,\mathrm{x} + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_u u^2 \, \mathrm{d}\,\mathrm{x} - \frac{t^6}{6}\int_{\mathbb{R}^3} u^6 \, \mathrm{d}\,\mathrm{x} \\ &\to -\infty. \end{aligned}$$

Thus, taking $w = tu$ for t large, we obtain $I(w) \leq 0$.
(ii) By (f_1), for $\varepsilon = \frac{1}{2} > 0$, there exists $C_{\frac{1}{2}} > 0$ such that

$$|f(t)| \leq \frac{1}{2}|t| + C_{\frac{1}{2}}|t|^5, \ \ \forall\, t \in \mathbb{R}. \tag{2.3}$$

Together with (2.1) and (2.2), one derives

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (1+\lambda V(x))u^2\right) \, \mathrm{d}\,\mathrm{x} + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \, \mathrm{d}\,\mathrm{x} - \frac{1}{4}\int_{\mathbb{R}^3} |u|^2 \, \mathrm{d}\,\mathrm{x} - \frac{C_{\frac{1}{2}}}{6}\int_{\mathbb{R}^3} u^6 \, \mathrm{d}\,\mathrm{x} \\ &\geq \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (1+\lambda V(x))u^2\right) \, \mathrm{d}\,\mathrm{x} - \frac{1}{4}\int_{\mathbb{R}^3} |u|^2 \, \mathrm{d}\,\mathrm{x} - \frac{C_{\frac{1}{2}}}{6}\int_{\mathbb{R}^3} u^6 \, \mathrm{d}\,\mathrm{x} \\ &\geq \frac{1}{4}\|u\|_\lambda^2 - \frac{C_{\frac{1}{2}}}{6} S^{-3}\|u\|_\lambda^6. \end{aligned}$$

According to the above inequality, there exists $\rho > 0$ such that $I_\lambda(u) > 0$ for $\|u\|_\lambda = \rho$.
By

$$S := \inf_{u\in D^{1,2}(\mathbb{R}^3)\backslash\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}\,\mathrm{x}}{\left(\int_{\mathbb{R}^3} |u|^6 \, \mathrm{d}\,\mathrm{x}\right)^{\frac{1}{3}}},$$

it is well known that S is achieved by the function

$$U_\varepsilon(x) = \frac{C\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x-y|^2)^{\frac{1}{2}}},$$

where $C > 0$ and $\varepsilon > 0$. We can easily obtain that $\int_{\mathbb{R}^3} |\nabla U_\varepsilon|^2 \, \mathrm{d}\,\mathrm{x} = \int_{\mathbb{R}^3} |U_\varepsilon|^6 \, \mathrm{d}\,\mathrm{x} = S^{\frac{3}{2}}$, see [22].

Lemma 2.2 *Under the conditions* (V_1), (V_2) *and* $(f_1)-(f_3)$, *there exists* $v \in E\backslash\{0\}$ *such that*

$$0 < c_\lambda \le \max_{t\ge 0} I_\lambda(tv) < \frac{1}{3}S^{\frac{3}{2}},$$

where $\lambda > 0$.

Proof The same demonstrations probed Lemma 3.5 in [9], we give a sketch of the proof for the readers convenience.

Consider $v_\varepsilon(x) = \frac{\varphi U_\varepsilon}{\|\varphi U_\varepsilon\|_6}$ where $\varphi \in C_0^\infty(\Omega, [0, 1])$ and

$$\varphi(x) := \begin{cases} 1, & \text{if } x \in B_R(y), \\ 0, & \text{if } x \notin B_{2R}(y), \end{cases}$$

where $B_R(y) \subset B_{2R}(y) \subset \Omega$. By the proof of Lemma 2.1, there exists $t_\varepsilon > 0$ such that $I_\lambda(t_\varepsilon v_\varepsilon) = \max_{t\ge 0} I_\lambda(tv_\varepsilon)$. From $\frac{\mathrm{d}\, \mathrm{I}_\lambda(tv_\varepsilon(x))}{\mathrm{d}\, \mathrm{t}}|_{t=t_\varepsilon} = 0$, (V_2) and (f_3), we have

$$\begin{aligned} 0 &= t_\varepsilon \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \, \mathrm{d\,x} + t_\varepsilon \int_{\mathbb{R}^3} (1+\lambda V(x))v_\varepsilon^2 \, \mathrm{d\,x} + t_\varepsilon^3 \int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 \, \mathrm{d\,x} - \int_{\mathbb{R}^3} f(t_\varepsilon v_\varepsilon)v_\varepsilon \, \mathrm{d\,x} - t_\varepsilon^5 \\ &= t_\varepsilon \int_\Omega |\nabla v_\varepsilon|^2 \, \mathrm{d\,x} + t_\varepsilon \int_\Omega v_\varepsilon^2 \, \mathrm{d\,x} + t_\varepsilon^3 \int_\Omega \phi_{v_\varepsilon} v_\varepsilon^2 \, \mathrm{d\,x} - \int_\Omega f(t_\varepsilon v_\varepsilon)v_\varepsilon \, \mathrm{d\,x} - t_\varepsilon^5 \\ &\le t_\varepsilon \int_\Omega |\nabla v_\varepsilon|^2 \, \mathrm{d\,x} + t_\varepsilon \int_\Omega v_\varepsilon^2 \, \mathrm{d\,x} + t_\varepsilon^3 \int_\Omega \phi_{v_\varepsilon} v_\varepsilon^2 \, \mathrm{d\,x} - t_\varepsilon^5. \end{aligned} \tag{2.4}$$

Combining with (2.4), we have

$$t_\varepsilon^4 \le \int_\Omega |\nabla v_\varepsilon|^2 \, \mathrm{d}x + \int_\Omega v_\varepsilon^2 \, \mathrm{d}x + t_\varepsilon^2 \int_\Omega \phi_{v_\varepsilon} v_\varepsilon^2 \, \mathrm{d}x. \tag{2.5}$$

From [22], we can also obtain

$$\int_\Omega |\nabla v_\varepsilon|^2 \, \mathrm{d}x = S + O(\varepsilon^{\frac{1}{2}}), \tag{2.6}$$

and

$$\int_\Omega |v_\varepsilon|^p \, \mathrm{d}x = \begin{cases} O(\varepsilon^{\frac{p}{4}}), & p \in [2, 3), \\ O(\varepsilon^{\frac{p}{4}}|\ln \varepsilon|), & p = 3\,, \\ O(\varepsilon^{\frac{6-p}{4}}), & p \in (3, 6), \end{cases} \tag{2.7}$$

where $O(\varepsilon^{\frac{p}{4}})$, $O(\varepsilon^{\frac{p}{4}}|\ln\varepsilon|)$ and $O(\varepsilon^{\frac{6-p}{4}})$ are nonnegative functions. It follows from (2.5)-(2.7) and $\int_{\mathbb{R}^3} \phi_{v_\varepsilon} v_\varepsilon^2 \, \mathrm{d}x \le c|v_\varepsilon|_{\frac{12}{5}}^4$ that $t_\varepsilon \le C$, where C is independent of $\varepsilon > 0$ small. We claim that there exists a constant $C > 0$ such that $t_\varepsilon \ge C > 0$ for $\varepsilon > 0$ small. Otherwise, there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ such that $t_{\varepsilon_n} \to 0$ as $n \to \infty$. So, up to subsequence, one gets $t_{\varepsilon_n} v_{\varepsilon_n} \to 0$ in E as $n \to \infty$. Thus,

$$0 < c_\lambda \le \max_{t\ge 0} I_\lambda(tv_{\varepsilon_n}) = I_\lambda(t_{\varepsilon_n} v_{\varepsilon_n}) \to 0.$$

This is a contradiction. As in the proof of Lemma 2.2 in [17], by (f_2), for any $M > 0$, there exists $T_M > 0$ such that $t \in [T_M, +\infty)$, we obtain

$$F(t) \ge Mt^4.$$

Hence for ε small enough, it deduces

$$\int_{|x-y|\le\varepsilon^{\frac{1}{2}}} F(v_\varepsilon) \, \mathrm{d}x \ge CM\varepsilon^{\frac{1}{2}}.$$

Together with the arbitrariness of M and (2.7), one has

$$\lim_{\varepsilon\to 0^+}\frac{\int_{|x-y|\le\varepsilon^{\frac{1}{2}}}F(v_\varepsilon)\,\mathrm{d}x}{\int_\Omega |v_\varepsilon|^2\,\mathrm{d}x}=+\infty. \tag{2.8}$$

From (f_3), one has

$$\int_{|x-y|\ge\varepsilon^{\frac{1}{2}}}F(v_\varepsilon)\,\mathrm{d}x\ge -C\int_{|x-y|\ge\varepsilon^{\frac{1}{2}}}|v_\varepsilon|^2\,\mathrm{d}x\ge -C\int_\Omega |v_\varepsilon|^2\,\mathrm{d}x. \tag{2.9}$$

Thus, by (V_2), (2.6)–(2.9), $t_\varepsilon\le C$, Hölder inequality and the property of ϕ_u, we get

$$\begin{aligned}
I_\lambda(t_\varepsilon v_\varepsilon)=&\frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\,\mathrm{d}x+\frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}(1+\lambda V(x))v_\varepsilon^2\,\mathrm{d}x+\frac{t_\varepsilon^4}{4}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2\,\mathrm{d}x\\
&-\int_{\mathbb{R}^3}F(t_\varepsilon v_\varepsilon)\,\mathrm{d}x-\frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3}v_\varepsilon^6\,\mathrm{d}x\\
=&\frac{t_\varepsilon^2}{2}\int_\Omega|\nabla v_\varepsilon|^2\,\mathrm{d}x+\frac{t_\varepsilon^2}{2}\int_\Omega v_\varepsilon^2\,\mathrm{d}x+\frac{t_\varepsilon^4}{4}\int_\Omega\phi_{v_\varepsilon}v_\varepsilon^2\,\mathrm{d}x\\
&-\int_\Omega F(t_\varepsilon v_\varepsilon)\,\mathrm{d}x-\frac{t_\varepsilon^6}{6}\int_\Omega v_\varepsilon^6\,\mathrm{d}x\\
\le&\frac{t_\varepsilon^2}{2}\int_\Omega|\nabla v_\varepsilon|^2\,\mathrm{d}x+\frac{t_\varepsilon^2}{2}\int_\Omega v_\varepsilon^2\,\mathrm{d}x+\frac{t_\varepsilon^4}{4S}\left(\int_\Omega|v_\varepsilon|^{\frac{12}{5}}\,\mathrm{d}x\right)^{\frac{5}{3}}\\
&-\int_\Omega F(t_\varepsilon v_\varepsilon)\,\mathrm{d}x-\frac{t_\varepsilon^6}{6}\int_\Omega v_\varepsilon^6\,\mathrm{d}x\\
\le&\max_{t\ge0}\left(\frac{t^2}{2}\int_\Omega|\nabla v_\varepsilon|^2\,\mathrm{d}x-\frac{t^6}{6}\int_\Omega v_\varepsilon^6\,\mathrm{d}x\right)+\frac{t_\varepsilon^2}{2}\int_\Omega v_\varepsilon^2\,\mathrm{d}x\\
&+\frac{t_\varepsilon^4}{4S}\left(\int_\Omega|v_\varepsilon|^{\frac{12}{5}}\,\mathrm{d}x\right)^{\frac{5}{3}}-\int_\Omega F(t_\varepsilon v_\varepsilon)\,\mathrm{d}x\\
\le&\frac{1}{3}S^{\frac{3}{2}}+O(\varepsilon^{\frac{1}{2}})+CS^{-1}O(\varepsilon)-\int_{|x-y|\le\varepsilon^{\frac{1}{2}}}F(t_\varepsilon v_\varepsilon)\,\mathrm{d}x+C\int_\Omega v_\varepsilon^2\,\mathrm{d}x\\
<&\frac{1}{3}S^{\frac{3}{2}},
\end{aligned}$$

which implies that $c_\lambda<\frac{1}{3}S^{\frac{3}{2}}$.

Denote the Nehari-Pohožaev functional $J_\lambda: H^1(\mathbb{R}^3)\to\mathbb{R}$ by

$$\begin{aligned}
J_\lambda(u):=&\frac{3}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}u^2\,\mathrm{d}x+\frac{\lambda}{2}\int_{\mathbb{R}^3}(V(x)-\nabla V(x)\cdot x)u^2\,\mathrm{d}x+\frac{3}{4}\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x\\
&-\int_{\mathbb{R}^3}(2f(u)u-3F(u))\,\mathrm{d}x-\frac{3}{2}\int_{\mathbb{R}^3}u^6\,\mathrm{d}x,
\end{aligned}$$

for $u\in H^1(\mathbb{R}^3)$. Hence, we will establish the Nehari-Pohožaev-Palais-Smale sequence and obtain the boundedness of the sequence as the following Lemma.

Lemma 2.3 *If (f_1), (f_3) and $(V_1)-(V_3)$ hold, there exists a bounded sequence $\{u_n\}\subset E$ satisfying*

$$\begin{aligned}
&I_\lambda(u_n)\to c_\lambda>0;\\
&I_\lambda'(u_n)\to 0;\\
&J_\lambda(u_n)\to 0.
\end{aligned}$$

Proof Borrowing the idea of L. Jeanjean [10], we define the continuous map

$$h : \mathbb{R} \times H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3), \quad h(s, v)(x) = e^{2s} v(e^s(x-y)) \text{ for } s \in \mathbb{R},\ v \in H^1(\mathbb{R}^3) \text{ and } x \in \mathbb{R}^3.$$

We consider the following auxiliary functional

$$\begin{aligned}\tilde{I}_\lambda(s, v) = I_\lambda(h(s, v)) =& \frac{e^{3s}}{2}\int_{\mathbb{R}^3} |\nabla v|^2 \,\mathrm{d}x + \frac{e^s}{2}\int_{\mathbb{R}^3}\left(1 + \lambda V\left(\frac{x}{e^s} + y\right)\right) v^2 \,\mathrm{d}x \\ &+ \frac{e^{3s}}{4}\int_{\mathbb{R}^3} \phi_v v^2 \,\mathrm{d}x - \frac{1}{e^{3s}}\int_{\mathbb{R}^3} F(e^{2s}v)\,\mathrm{d}x - \frac{e^{9s}}{6}\int_{\mathbb{R}^3} v^6 \,\mathrm{d}x.\end{aligned} \tag{2.10}$$

It is easy to check that $\tilde{I}_\lambda \in C^1(\mathbb{R} \times H^1(\mathbb{R}^3), \mathbb{R})$. Let $y \in \Omega$ and $y \in B_r(y) \subset \Omega$ for some $r > 0$. Choose $v \in C_0^\infty(\mathbb{R}^3)$ such that $\mathrm{supp} v \subset B_r(0)$. Set $u_s(x) = e^{2s}v(e^s(x-y))$, then $\mathrm{supp} u_s(x) \subset B_r(y)$ for $s > 0$. Together with (2.10), we have

$$\begin{aligned}\tilde{I}_\lambda(s, v) = I_\lambda(h(s, v)) =& \frac{e^{3s}}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \frac{e^s}{2}\int_\Omega v^2 \,\mathrm{d}x + \frac{e^{3s}}{4}\int_\Omega \phi_v v^2 \,\mathrm{d}x \\ &- \frac{1}{e^{3s}}\int_\Omega F(e^{2s}v)\,\mathrm{d}x - \frac{e^{9s}}{6}\int_\Omega v^6 \,\mathrm{d}x \to -\infty \text{ as } s \to \infty.\end{aligned}$$

So, there exists $s_0 \in \mathbb{R}$ such that $\tilde{I}_\lambda(s_0, v) < 0$. Thus, we denote the family of paths is given by

$$\tilde{\Gamma} = \left\{\tilde{\gamma} \in C([0,1], \mathbb{R} \times H^1(\mathbb{R}^3)) : \tilde{\gamma}(0) = (0,0),\ \tilde{I}_\lambda(\tilde{\gamma}(1)) < 0\right\}.$$

The minimax level of $\tilde{I}_\lambda$:

$$\tilde{c}_\lambda = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{I}_\lambda(\tilde{\gamma}(t)). \tag{2.11}$$

According to $\Gamma = \left\{h \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma}\right\}$, we have $c_\lambda \le \tilde{c}_\lambda$. Distinctly, one attains that $\{0\} \times \Gamma \subset \tilde{\Gamma}$ and thus $\tilde{c}_\lambda \le c_\lambda$. So, $\tilde{c}_\lambda = c_\lambda$. By the definition of the infimum, for every $n \in \mathbb{N}$, there exists $\gamma_n \subset \Gamma$ such that

$$\max_{t \in [0,1]} I_\lambda(\gamma_n(t)) \le c_\lambda + \frac{1}{n}.$$

Let $\tilde{\gamma}_n(t) = (0, \gamma_n(t))$, one has $\tilde{\gamma}_n(t) \subset \tilde{\Gamma}$. Combining with $\tilde{c}_\lambda = c_\lambda$, we conclude

$$\max_{t \in [0,1]} \tilde{I}(\tilde{\gamma}_n(t)) \le c_\lambda + \frac{1}{n}.$$

By (2.11) and Theorem 2.8 in [15], there exists a sequence $\{(s_n, v_n)\}_{n \in \mathbb{N}}$ in $\mathbb{R} \times H^1(\mathbb{R}^3)$ such that, as $n \to \infty$

$$\begin{aligned}&\tilde{I}_\lambda(s_n, v_n) \to c_\lambda, \\ &\tilde{I}'_\lambda(s_n, v_n) \to 0 \text{ in } (\mathbb{R} \times H^1(\mathbb{R}^3))^{-1}, \\ &\min_{t \in [0,1]} \|(s_n, v_n) - \tilde{\gamma}_n(t)\|_{\mathbb{R} \times H^1(\mathbb{R}^3)} \to 0,\end{aligned}$$

where the norm of $(s, v) \in \mathbb{R} \times H^1(\mathbb{R}^3)$ is denoted by $\|(s, v)\| = (s^2 + \|v\|^2)^{\frac{1}{2}}$. Let $u_n := h(s_n, v_n)$, we have $I_\lambda(u_n) \to c_\lambda$, and, for every $(\tau, \omega) \in \mathbb{R} \times H^1(\mathbb{R}^3)$,

$$\langle \tilde{I}_\lambda(s_n, v_n), (\tau, \omega)\rangle = \langle I'_\lambda(h(s_n, v_n)), h(s_n, \omega)\rangle + J_\lambda(h(s_n, v_n))\tau \to 0. \tag{2.12}$$

Taking $(\tau, \omega) = (1, 0)$, we have $J_\lambda(u_n) \to 0$ as $n \to \infty$. For every $\psi \in H^1(\mathbb{R}^3)$, set $\tau = 0$ and $\omega_n = e^{-2s_n}\psi(e^{-s_n}(x-y))$, we imply that

$$\langle \tilde{I}'_\lambda(s_n, \omega_n), (0, \omega)\rangle = \langle I'_\lambda(h(s_n, v_n)), h(s_n, \omega_n)\rangle = \langle I'_\lambda(h(s_n, v_n)), \psi\rangle \to 0, \quad \text{as } n \to \infty.$$

Thereby, $I_\lambda'(u_n) \to 0$. By (f_3) and (V_3), we have

$$\begin{aligned}
c_\lambda + o_n(1)\|u_n\|_\lambda \geq & I_\lambda(u_n) - \frac{1}{3}J_\lambda(u_n)\\
=& \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_{\mathbb{R}^3} V(x)u_n^2 \,\mathrm{d}x\\
&+ \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(u_n)\,\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x\\
&- \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 \,\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x - \frac{\lambda}{6}\int_{\mathbb{R}^3} V(x)u_n^2 \,\mathrm{d}x\\
&+ \frac{\lambda}{6}\int_{\mathbb{R}^3} (\nabla V(x)\cdot x)u_n^2 \,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x\\
&+ \frac{2}{3}\int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x - \int_{\mathbb{R}^3} F(u_n)\,\mathrm{d}x\\
=& \frac{1}{3}\int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x + \frac{\lambda}{6}\int_{\mathbb{R}^3} (2V(x) + \nabla V(x)\cdot x)u_n^2 \,\mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x\\
&+ \frac{2}{3}\int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x - 2\int_{\mathbb{R}^3} F(u_n)\,\mathrm{d}x\\
\geq & \frac{1}{3}\int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x.
\end{aligned}$$

Therefore combining (2.1), (2.2) and (2.3), one gets

$$\begin{aligned}
\|u\|_\lambda^2 &= \langle I_\lambda'(u_n), u_n\rangle - \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x + \int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x\\
&\leq \langle I_\lambda'(u_n), u_n\rangle + \int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x + \int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x\\
&\leq o_n(1)\|u_n\|_\lambda + \frac{1}{2}\|u_n\|_\lambda^2 + 3(C_{\frac{1}{2}} + 1)(c_\lambda + o_n(1)\|u_n\|_\lambda).
\end{aligned}$$

Hence, $\{u_n\}$ is bounded in E.

Lemma 2.4 *If (V_1), (V_2) and $(f_1)-(f_3)$ hold. For some $\lambda_0 > 0$, $\lambda \geq \lambda_0$ and $0 < c_\lambda < \frac{1}{3}S^{\frac{3}{2}}$, let $\{u_n\} \subset E$ be a bounded $(PS)_{c_\lambda}$ sequence of I_λ, uniformly in n and λ, then there exists $u \neq 0$ such that $u_n \rightharpoonup u$ and $I_\lambda'(u) = 0$.*

Proof By Lemma 2.2 and 2.3, there exists a bounded sequence $\{u_n\} \subset E$, along a subsequence denoted as $\{u_n\}$. And there exists u such that $u_n \rightharpoonup u$ in E, $I_\lambda(u_n) \to c_\lambda < \frac{1}{3}S^{\frac{3}{2}}$ and $I_\lambda'(u_n) \to 0$.

Firstly, we need prove $I_\lambda'(u) = 0$. In fact, we only need to prove that $\langle I_\lambda'(u_n), v\rangle = 0$ for all $v \in C_c^\infty(\mathbb{R}^3)$. It is easy to see that

$$\int_{\mathbb{R}^3} (\nabla u_n \nabla v + (1+\lambda V(x))u_n v + \phi_{u_n} u_n v)\,\mathrm{d}x \to \int_{\mathbb{R}^3} (\nabla u \nabla v + (1+\lambda V(x))uv + \phi_u uv)\,\mathrm{d}x, \quad (2.13)$$

$$\int_{\mathbb{R}^3} (f(u_n) + |u_n|^4 u_n)v\,\mathrm{d}x \to \int_{\mathbb{R}^3} (f(u) + |u|^4 u)v\,\mathrm{d}x. \quad (2.14)$$

Combining (2.13) and (2.14), we obtain that $I_\lambda'(u) = 0$.

Secondly, we claim $u \neq 0$. If $u_n \rightharpoonup 0$, one of two following cases holds:

$$\text{(i)} \ \lim_{n\to\infty} \sup_{z\in\mathbb{R}^3} \int_{B_r(z)} |u_n|^2 \,\mathrm{d}x = 0;$$

$$\text{(ii)} \ \lim_{n\to\infty} \sup_{z\in\mathbb{R}^3} \int_{B_r(z)} |u_n|^2 \,\mathrm{d}x > 0.$$

If the case (i) holds, by Lemma 1.21 in [15], $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for all $q \in (2,6)$. So, we have $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x \le c|u_n|^4_{\frac{12}{5}} \to 0$. Thus

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x \to 0. \tag{2.15}$$

Since $u_n \to 0$ in $L^q_{loc}(\mathbb{R}^3)$ with $q \in (2,6)$, we derive

$$\int_{|x|\le R} |u_n|^q \,\mathrm{d}x \to 0.$$

By (f_1), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(u_n)| \le \varepsilon(u_n^2 + |u_n|^6) + C_\varepsilon |u_n|^q, \quad q \in (2,6).$$

Thus, by the Lebesgue dominated convergence theorem, it derives

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n)\,\mathrm{d}x = 0. \tag{2.16}$$

Similarly, $\lim\limits_{n\to\infty} \int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x = 0$. Thus,

$$\begin{aligned} I_\lambda(u_n) &= \frac{1}{2}\|u_n\|_\lambda^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(u_n)\,\mathrm{d}x - \frac{1}{6}|u_n|_6^6 \\ &= \frac{1}{2}\|u_n\|_\lambda^2 - \frac{1}{6}|u_n|_6^6 + o_n(1), \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} o_n(1) = \langle I'_\lambda(u_n), u_n\rangle &= \|u_n\|_\lambda^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x - |u_n|_6^6 \\ &= \|u_n\|_\lambda^2 - |u_n|_6^6. \end{aligned} \tag{2.18}$$

Taking a subsequence denoted as $\{u_n\}$, by (2.18), there exists $l \ge 0$ such that

$$\lim_{n\to\infty} \|u_n\|_\lambda^2 = \lim_{n\to\infty} |u_n|_6^6 = l.$$

If $l = 0$, one has $\|u_n\|_\lambda \to 0$ in E. Thus, $I_\lambda(u_n) \to 0$. This contradicts with the fact that $I_\lambda(u_n) \to c_\lambda > 0$. Otherwise, the case of $l > 0$ holds. By $I_\lambda(u_n) \to c_\lambda > 0$, (2.17) and (2.18), one gets

$$3c_\lambda \ge l + o_n(1). \tag{2.19}$$

Together with the definition of S, we have

$$\|u_n\|_\lambda^2 \ge \int_{\mathbb{R}^3} |\nabla u_n|^2 \,\mathrm{d}x \ge S\left(\int_{\mathbb{R}^3} |u_n|^6 \,\mathrm{d}x\right)^{\frac{1}{3}}.$$

Passing the limit in the above inequality, we obtain $l \ge S^{\frac{3}{2}}$ as $n \to \infty$. Then, combining with (2.19), we conclude $c_\lambda \ge \frac{1}{3}S^{\frac{3}{2}}$. It contradicts Lemma 2.2. Thereby, the case (ii) holds, there exists a positive constant $\theta > 0$ such that

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^3} \int_{B_r(z)} |u_n|^2 \,\mathrm{d}x = \theta > 0.$$

For $R > 0$, we define

$$D_R := \{x \in \mathbb{R}^3 \backslash B_R : V(x) \ge V_0\},$$

and

$$G_R := \{x \in \mathbb{R}^3 \backslash B_R : V(x) < V_0\}.$$

Similar to the method of [3], as $n \to \infty$, one has

$$\begin{aligned}\int_{D_R} u_n^2 \,\mathrm{d}x &\leq \frac{1}{1+\lambda V_0} \int_{\mathbb{R}^3} (1+\lambda V(x)) u_n^2 \,\mathrm{d}x \\ &\leq \frac{1}{1+\lambda V_0} \|u_n\|_\lambda^2 \\ &\leq \frac{C}{1+\lambda V_0},\end{aligned}$$

where $\|u_n\|_\lambda^2 \leq C$ independent of n and λ. Taking $\lambda \geq \frac{4C}{\theta V_0}$, one gets

$$\int_{D_R} u_n^2 \,\mathrm{d}x \leq \frac{C}{1+\lambda V_0} \leq \frac{C}{\lambda V_0} \leq \frac{\theta}{4}, \tag{2.20}$$

uniformly in n. Combining Hölder and Sobolev inequalities, one has

$$\begin{aligned}\int_{G_R} u_n^2 \,\mathrm{d}x &= \left(\int_{G_R} |u_n|^p \,\mathrm{d}x\right)^{\frac{2}{p}} \left(\int_{G_R} 1 \,\mathrm{d}x\right)^{\frac{p-2}{p}} \\ &\leq \|u_n\|_\lambda^2 (\mathrm{meas}(G_R))^{\frac{p-2}{p}} \\ &\leq C (\mathrm{meas}(G_R))^{\frac{p-2}{p}},\end{aligned} \tag{2.21}$$

where $p \in (2, 6]$. Because $\mathrm{meas}(G_R) \to 0$ as $R \to 0$ by (V_1), for any $\delta > 0$ and taking $\delta < \frac{\theta}{4}$, there exists R' such that $R > R'$, and we obtain that

$$\int_{G_R} u_n^2 \,\mathrm{d}x \leq \frac{\theta}{4}, \tag{2.22}$$

uniformly in n. From $u_n \to 0$ in $L^q_{loc}(\mathbb{R}^3)$ with $q \in (2, 6)$, (2.20) and (2.22), we have

$$\begin{aligned}\theta = \lim_{n\to\infty} \sup_{z\in\mathbb{R}^3} \int_{B_r(z)} |u_n|^2 \,\mathrm{d}x &\leq \lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^2 \,\mathrm{d}x \\ &= \lim_{n\to\infty} \left(\int_{B_R} |u_n|^2 \,\mathrm{d}x + \int_{B_R^c} |u_n|^2 \,\mathrm{d}x\right) \\ &= \lim_{n\to\infty} \left(\int_{D_R} u_n^2 \,\mathrm{d}x + \int_{G_R} u_n^2 \,\mathrm{d}x\right) \\ &\leq \frac{\theta}{2}.\end{aligned}$$

This is a contradiction. Hence, there exists $u \neq 0$ such that $u_n \rightharpoonup u$ in E and $I'_\lambda(u) = 0$. The proof is completed.

3 The Proof of the main result

In this section, the nontrivial critical points of functional I_λ on E are established by Nehari-Pohožaev-Palais-Smale sequence. Moreover, we will obtain ground state solutions for (SP).

Now, we start to prove Theorem 1.1.

Proof of Theorem 1.1. From Lemmas 2.1-2.3, there exists a bounded sequence $\{u_n\} \subset E$ such that $I_\lambda(u_n) \to c_\lambda < \frac{1}{3}S^{\frac{3}{2}}$, $I'_\lambda(u_n) \to 0$ and $J_\lambda(u_n) \to 0$. Taking a subsequence, still denoted by $\{u_n\}$ satisfying that $u_n \rightharpoonup u$ in E, $I_\lambda(u_n) \to c_\lambda < \frac{1}{3}S^{\frac{3}{2}}$, $I'_\lambda(u_n) \to 0$, $J_\lambda(u_n) \to 0$ and $\|u_n\|_\lambda^2 \le C$ where C independent of n and λ. Obviously, there exists $u \neq 0$ such that $I'_\lambda(u) = 0$ by Lemma 2.4. Combining $I'_\lambda(u) = 0$ and Pohožaev identity Lemma 2.2 in [13], one gets

$$\begin{aligned} P_\lambda(u) :=& \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \frac{3}{2}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_{\mathbb{R}^3} (3V(x) + \nabla V(x)\cdot x)u^2 \,\mathrm{d}x \\ &+ \frac{5}{4}\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x - 3\int_{\mathbb{R}^3} F(u)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^6 \,\mathrm{d}x \\ =& 0. \end{aligned}$$

According to $2 \times \langle I'_\lambda(u), u\rangle - P_\lambda(u)$, one deduces

$$\begin{aligned} J_\lambda(u) =& \frac{3}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_{\mathbb{R}^3} (V(x) - \nabla V(x)\cdot x)u^2 \,\mathrm{d}x \\ &+ \frac{3}{4}\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} (2f(u)u - 3F(u))\,\mathrm{d}x - \frac{3}{2}\int_{\mathbb{R}^3} u^6 \,\mathrm{d}x \\ =& 0. \end{aligned} \tag{3.1}$$

Combining $J_\lambda(u) = 0$, (V_3) and (f_3), one gets

$$\begin{aligned} I_\lambda(u) =& I_\lambda(u) - \frac{1}{3}J_\lambda(u) \\ =& \frac{1}{3}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{\lambda}{6}\int_{\mathbb{R}^3} (2V(x) + \nabla V(x)\cdot x)u^2 \,\mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}^3} u^6 \,\mathrm{d}x \\ &+ \frac{2}{3}\int_{\mathbb{R}^3} f(u)u \,\mathrm{d}x - 2\int_{\mathbb{R}^3} F(u)\,\mathrm{d}x \\ \ge& 0. \end{aligned}$$

Together with $u \neq 0$, one has $I_\lambda(u) > 0$. It follows from $I_\lambda(u_n) \to c_\lambda$, $J_\lambda(u_n) \to 0$, Fatou's Lemma, (V_3), (f_3) and (3.1) that

$$\begin{aligned} c_\lambda =& \lim_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{3}J_\lambda(u_n)\right) \\ =& \lim_{n\to\infty}\left(\frac{1}{3}\int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x + \frac{\lambda}{6}\int_{\mathbb{R}^3} (2V(x) + \nabla V(x)\cdot x)u_n^2 \,\mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x \right. \\ &\left. + \frac{2}{3}\int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x - 2\int_{\mathbb{R}^3} F(u_n)\,\mathrm{d}x\right) \\ \ge& \frac{1}{3}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{\lambda}{6}\int_{\mathbb{R}^3} (2V(x) + \nabla V(x)\cdot x)u^2 \,\mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}^3} u^6 \,\mathrm{d}x \\ &+ \frac{2}{3}\int_{\mathbb{R}^3} f(u)u \,\mathrm{d}x - 2\int_{\mathbb{R}^3} F(u)\,\mathrm{d}x \\ =& I_\lambda(u) - \frac{1}{3}J_\lambda(u) \\ =& I_\lambda(u). \end{aligned}$$

So, $0 < I_\lambda(u) \le c_\lambda < \frac{1}{3}S^{\frac{3}{2}}$.

To find a least energy solution for problem (SP), we define

$$m_\lambda := \inf_{u\in M} I_\lambda(u), \quad M := \{u \in E\backslash\{0\} \mid I'_\lambda(u) = 0\}.$$

By the argument of the above, M is not empty. To complete our proof, we claim m_λ can be achieved in M. Suppose that $u_n \neq 0$ satisfying $0 < I_\lambda(u_n) \to m_\lambda \le c_\lambda < \frac{1}{3}S^{\frac{3}{2}}$ and $I'_\lambda(u_n) = 0$. Along a subsequence, still denoted as $\{u_n\}$, there holds that $0 < I_\lambda(u_n) \to m_\lambda < \frac{1}{3}S^{\frac{3}{2}}$ and $I'_\lambda(u_n) = 0$ for all n. Thus, by (V_3) and (f_3), we have

$$\begin{aligned} m_\lambda + o_n(1) \geq & I_\lambda(u_n) - \frac{1}{3}J_\lambda(u_n) \\ = & \frac{1}{3}\int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x + \frac{\lambda}{6}\int_{\mathbb{R}^3}(2V(x) + \nabla V(x)\cdot x)u_n^2 \,\mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x \\ & + \frac{2}{3}\int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x - 2\int_{\mathbb{R}^3} F(u_n)\,\mathrm{d}x \\ \geq & \frac{1}{3}\int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x. \end{aligned}$$

Therefore, by (2.1), (2.2) and (2.3), one gets

$$\begin{aligned} \|u_n\|_\lambda^2 &= \langle I'_\lambda(u_n), u_n\rangle - \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x + \int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x \\ &\leq \langle I'_\lambda(u_n), u_n\rangle + \int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x + \int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x \\ &\leq o_n(1)\|u_n\|_\lambda + \frac{1}{2}\int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x + C_{\frac{1}{2}}\int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x + \int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x \\ &\leq o_n(1)\|u_n\|_\lambda + \frac{1}{2}\|u_n\|_\lambda^2 + 3(C_{\frac{1}{2}} + 1)(m_\lambda + o_n(1)\|u\|_\lambda). \end{aligned}$$

Hence, $\{u_n\}$ is bounded in E. Together with Lemma 2.4, there exists $u \neq 0$ such that $u_n \rightharpoonup u$ in E. Thus, we obtain $I'_\lambda(u) = 0$. By (f_3), (3.1) and Fatou's Lemma, we conclude

$$\begin{aligned} m_\lambda + o_n(1) \leq & I_\lambda(u) - \frac{1}{3}J_\lambda(u) \\ = & \frac{1}{3}\int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{\lambda}{6}\int_{\mathbb{R}^3}(2V(x) + \nabla V(x)\cdot x)u^2 \,\mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}^3} u^6 \,\mathrm{d}x \\ & + \frac{2}{3}\int_{\mathbb{R}^3} f(u)u \,\mathrm{d}x - 2\int_{\mathbb{R}^3} F(u)\,\mathrm{d}x \\ \leq & \liminf_{n\to\infty}\left(\frac{1}{3}\int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x + \frac{\lambda}{6}\int_{\mathbb{R}^3}(2V(x) + \nabla V(x)\cdot x)u_n^2 \,\mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}^3} u_n^6 \,\mathrm{d}x \right. \\ & \left. + \frac{2}{3}\int_{\mathbb{R}^3} f(u_n)u_n \,\mathrm{d}x - 2\int_{\mathbb{R}^3} F(u_n)\,\mathrm{d}x\right) \\ = & \liminf_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{3}J_\lambda(u_n)\right) \\ = & m_\lambda. \end{aligned}$$

The proof is completed.

4 Concentration of ground state solutions

In this section, we will discuss the concentration behavior of ground state solutions for (SP) as $\lambda_n \to \infty$. Now, we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Due to the conclusion of Theorem 1.1, we have

$$I_\lambda(u_n) = m_\lambda \leq c_\lambda < \frac{1}{3}S^{\frac{3}{2}} \quad \text{and} \quad I'_\lambda(u_\lambda) = 0.$$

Define $u_n := u_{\lambda_n}$, there exists a sequence $\{u_n\}$ such that $I'_{\lambda_n}(u_n) = 0$ and

$$I_{\lambda_n}(u_n) = m_{\lambda_n} < \frac{1}{3}S^{\frac{3}{2}}.$$

From the Theorem 1.1, by $2 \times \langle I'_\lambda(u_n), u_n\rangle - P_\lambda(u_n)$, we have $J_{\lambda_n}(u_n) = 0$. As in the proof of Lemma 2.3, it follows that

$$\|u_n\|^2_{\lambda_n} \le Cm_{\lambda_n} < CS^{\frac{3}{2}}. \tag{4.1}$$

Hence, $\{u_n\}$ is bounded in E, i.e., along a subsequence, still denoted by $\{u_n\}$, there exists $u \in E$ such that $u_n \rightharpoonup u$ in E. We claim that $u_n \to u \neq 0$ in E. Indeed, as $\lambda_n \to \infty$, we have

$$\begin{aligned}\int_{D_R} u_n^2\,\mathrm{d}x &\le \frac{1}{1+\lambda_n V_0}\int_{\mathbb{R}^3}(1+\lambda V(x))u_n^2\,\mathrm{d}x\\ &\le \frac{C}{1+\lambda_n V_0} \to 0,\end{aligned} \tag{4.2}$$

uniformly in n. From Hölder and Sobolev inequalities, (2.2), (3.2) and meas$(G_R) \to 0$, we obtain,

$$\begin{aligned}\int_{B_R^c}|u_n|^q\,\mathrm{d}x &= \left(\int_{B_R^c} u_n^6\,\mathrm{d}x\right)^{\frac{q-2}{4}}\left(\int_{B_R^c}|u_n|^2\,\mathrm{d}x\right)^{\frac{6-q}{4}}\\ &\le S^{\frac{3(2-q)}{4}}\left(\int_{B_R^c}|\nabla u_n|^2\,\mathrm{d}x\right)^{\frac{3(q-2)}{4}}\left(\int_{D_R}|u_n|^2\,\mathrm{d}x + \int_{G_R}|u_n|^2\,\mathrm{d}x\right)^{\frac{6-q}{4}}\\ &\le C\|u_n\|_\lambda^{\frac{3(q-2)}{2}}\left(\frac{C}{1+\lambda_n V_0} + C(\mathrm{meas}(G_R))^{\frac{p-2}{p}}\right)^{\frac{6-q}{4}} \to 0,\end{aligned}$$

where $B_R^c = \{x \in \mathbb{R}^3, |x| \ge R\}$, $p, q \in (2,6)$ and uniformly in n. According, as $\lambda_n \to \infty$,

$$\int_{B_R^c}\left||u_n|^q - |u|^q\right|\,\mathrm{d}x \le \int_{B_R^c}|u_n|^q\,\mathrm{d}x + \int_{B_R^c}|u|^q\,\mathrm{d}x \to 0, \quad \text{as} \quad R \to \infty.$$

Since $u_n \to u$ in $L^q_{loc}(\mathbb{R}^3)$ with $q \in (2,6)$, we derive

$$\int_{|x|\le R}|u_n|^q\,\mathrm{d}x \to \int_{|x|\le R}|u|^q\,\mathrm{d}x.$$

Therefore, $u_n \to u$ in $L^q(\mathbb{R}^3)$ in E as $\lambda_n \to \infty$. Set $v_n := u_n - u$. Since $\{u_n\}$ is bounded in E, we have

$$\|u_n\|^2_\lambda = \|u\|^2_\lambda + \|v_n\|^2_\lambda, \tag{4.3}$$

$$|u_n|_6^6 = |u|_6^6 + |v_n|_6^6 + o_n(1). \tag{4.4}$$

Similar to the proof of (2.15) and (2.16), we derive

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,\mathrm{d}x &= \int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x + o_n(1),\\ \int_{\mathbb{R}^3}F(u_n)\,\mathrm{d}x &= \int_{\mathbb{R}^3}F(u)\,\mathrm{d}x + o_n(1),\\ \int_{\mathbb{R}^3}f(u_n)u_n\,\mathrm{d}x &= \int_{\mathbb{R}^3}f(u)u\,\mathrm{d}x + o_n(1).\end{aligned} \tag{4.5}$$

According to $I_\lambda(u_n) \to c_\lambda$, $I_\lambda(u) \ge 0$, (2.15) and (4.3)-(4.5), we have

$$\begin{aligned}I_\lambda(u_n) &= \frac{1}{2}\|u_n\|^2_\lambda + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,\mathrm{d}x - \int_{\mathbb{R}^3}F(u_n)\,\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}|u_n|^6\,\mathrm{d}x\\ &= \frac{1}{2}(\|u\|^2_\lambda + \|v_n\|^2_\lambda) + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x - \int_{\mathbb{R}^3}F(u)\,\mathrm{d}x - \frac{1}{6}(|u|_6^6 + |v_n|_6^6) + o_n(1)\\ &= I_\lambda(u) + \frac{1}{2}\|v_n\|^2_\lambda - \frac{1}{6}|v_n|_6^6\\ &\ge \frac{1}{2}\|v_n\|^2_\lambda - \frac{1}{6}|v_n|_6^6.\end{aligned} \tag{4.6}$$

Similarly, from $I'_\lambda(u)=0$ and (4.3)-(4.5), we obtain

$$\begin{aligned} o_n(1)=\langle I'_\lambda(u_n),u_n\rangle &= \|u_n\|_\lambda^2+\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,\mathrm{d}x-\int_{\mathbb{R}^3}f(u_n)u_n\,\mathrm{d}x-|u_n|_6^6 \\ &= \|v_n\|_\lambda^2-|v_n|_6^6. \end{aligned} \tag{4.7}$$

Extracting a sequence denoted as $\{v_n\}$, by (4.7), there exists $l_1\geq 0$ such that $\lim\limits_{n\to\infty}\|v_n\|_\lambda^2=\lim\limits_{n\to\infty}|v_n|_6^6=l_1$. If $l_1=0$, it deduces that $\|v_n\|_\lambda\to 0$, the proof is completed. Otherwise, $l_1>0$. From $I_\lambda(u_n)\to c_\lambda>0$, (4.5) and (4.7), one deduces

$$3c_\lambda=l_1+o_n(1). \tag{4.8}$$

Combining with the definition of S, we have

$$\|v_n\|_\lambda^2\geq\int_{\mathbb{R}^3}|\nabla v_n|^2\,\mathrm{d}x\geq S\left(\int_{\mathbb{R}^3}|v_n|^6\,\mathrm{d}x\right)^{\frac{1}{3}}.$$

Passing the limit in the above inequality, we obtain $l_1\geq S^{\frac{3}{2}}$ as $n\to\infty$. Then, together with (4.8), there holds $c_\lambda\geq\frac{1}{3}S^{\frac{3}{2}}$, which contradicts Lemma 2.2. Thus, we obtain that $u_n\to u\neq 0$ in E and then $I'_\lambda(u)=0$. Together with (4.1), we obtain

$$\lambda_n\int_{\mathbb{R}^3}V(x)u_n^2\,\mathrm{d}x\leq\|u_n\|_{\lambda_n}^2<CS^{\frac{3}{2}}. \tag{4.9}$$

Combining (4.9) and Fatou's Lemma, we have

$$\begin{aligned} 0&\leq\int_{\mathbb{R}^3}V(x)u^2\,\mathrm{d}x \\ &\leq\lim_{n\to\infty}\inf\int_{\mathbb{R}^3}V(x)u_n^2\,\mathrm{d}x \\ &\leq\lim_{n\to\infty}\inf\frac{1}{\lambda_n}\|u_n\|_\lambda^2 \\ &\leq\lim_{n\to\infty}\inf\frac{C}{\lambda_n}\to 0. \end{aligned}$$

So, $\int_{\mathbb{R}^3}V(x)u^2\,\mathrm{d}x=0$. Thus, by (V_2), one deduces that $u=0$ a.e, $x\in\mathbb{R}^3\backslash\Omega$ and $u\in H_0^1(\Omega)$. Define the functional I_0 on $H_0^1(\Omega)$ by

$$I_0(u)=\frac{1}{2}\int_\Omega(|\nabla u|^2+u^2)\,\mathrm{d}x+\frac{1}{4}\int_\Omega\phi_u u^2\,\mathrm{d}x-\int_\Omega F(u)\,\mathrm{d}x-\frac{1}{6}\int_\Omega u^6\,\mathrm{d}x.$$

Being $u\in H_0^1(\Omega)$, we obtain

$$\int_\Omega(\nabla u\cdot\nabla v+uv)\,\mathrm{d}x+\int_\Omega\phi_u v\,\mathrm{d}x=\int_\Omega f(u)v\,\mathrm{d}x+\int_\Omega|u|^4uv\,\mathrm{d}x,\;\; v\in H_0^1(\Omega)$$

i.e., $I'_0(u)=0$.
The proof is completed.

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