Indian J Pure Appl Math (2026) 57:658–669
https://doi.org/10.1007/s13226-024-00540-4

ORIGINAL RESEARCH

Cubic residues and their new distributive properties

Xiaoge Liu · Tianping Zhang

Received: 9 December 2023 / Accepted: 10 January 2024 / Published online: 3 February 2024

Abstract Let $M_k(p)$ denote the number of all integers $1 \le a \le p-1$ such that $a+a^k$ and $a-a^k$ are cubic residues modulo p. We obtain some identities or asymptotic formulae for $M_2(p)$ and $M_3(p)$ by using the properties of Gauss sums and third-order Dirichlet character. Through these results, the distributive properties of cubic residues have been fully characterized.

Keywords Cubic residue · Third-order Dirichlet character · Gauss sums · Dirichlet character sums

Mathematics Subject Classification 11A15 · 11L40

1 Introduction

In 2002, Z. H. Sun [5] considered an interesting question of whether there exist three consecutive integers which are all quadratic residues of modulo an odd prime p. He obtained that for any $n \in \mathbb{Z}$ with $n \not\equiv 0, \pm 1 \bmod p$, then

$$\left(\frac{n-1}{p}\right) = \left(\frac{n}{p}\right) = \left(\frac{n+1}{p}\right) \Leftrightarrow n \equiv \frac{(x^2+1)^2}{4x^3-4x} \bmod p, \text{ for some } x \in \mathbb{Z}.$$

Later, Z. W. Sun proposed the following conjectures in a speech, which are

1. For any prime $p \ge 101$, there is at least one integer a, such that a, $a+\overline{a}$ and $a-\overline{a}$ are quadratic residues modulo p.
2. For any prime $p \ge 18$, there is at least one quadratic non-residue a modulo p, such that $a+\overline{a}$ and $a-\overline{a}$ are quadratic residues modulo p.

Let $N(p, 1)$ denote the number of all integers $1 \le a \le p-1$ such that a, $a+\overline{a}$ and $a-\overline{a}$ are quadratic residues modulo p. Let $N(p, -1)$ denote the number of all integers $1 \le a \le p-1$ such that a is a quadratic non-residue modulo p, $a+\overline{a}$ and $a-\overline{a}$ are quadratic residues modulo p. T. T. Wang and X. X. Lv [8] got some

Communicated by B. Sury.

Xiaoge Liu and Tianping Zhang contributed equally to this work.

X. Liu · T. Zhang
School of Mathematics and Statistics, Shaanxi Normal University, Xi'an 710119, Shaanxi, China
E-mail: xiaogeliu@snnu.edu.cn

T. Zhang (✉)
Research Center for Number Theory and Its Applications, Northwest University, Xi'an 710127, Shaanxi, China
E-mail: tpzhang@snnu.edu.cn

identities or asymptotic formulae for $N(p,1)$ and $N(p,-1)$. Very recently, the above results have been improved by X. G. Liu and Y. Y. Meng [4]. In addition, it can be seen from the conclusions that the above conjecture is highly likely to be correct.

Furthermore, let $N_k(p)$ denote the number of all integers $1 \le a \le p-1$ such that $a+\overline{a}$ and $a-\overline{a}$ are k-th residues modulo p. Some scholars obtained several identities or asymptotic formulae for $N_3(p)$, $N_4(p)$ and $N_6(p)$, which can be found in [3], [6], [7] and [10].

Inspired by the above articles, we consider the following issues. Is there an integer a with $1 \le a \le p-1$ such that $a+a^k$ and $a-a^k$ are both the cubic residues of modulo p? For example, when $p=31$, $12+12^2$ and $12-12^2$, $13+13^2$ and $13-13^2$, $18+18^2$ and $18-18^2$, $19+19^2$ and $19-19^2$ are the cubic residues modulo 31. Besides, $14+14^3$ and $14-14^3$, $17+17^3$ and $17-17^3$ are also the cubic residues modulo 31.

Now let $M_k(p)$ denote the number of all integers $1 \le a \le p-1$ such that $a+a^k$ and $a-a^k$ are both cubic residues modulo p. It seems that the characterization of $M_k(p)$ had never been addressed in any previous literature. By using the properties of Gauss sums and third-order Dirichlet character, we will get some identities or asymptotic formulae for $M_2(p)$ and $M_3(p)$. Especially, note that if $p \equiv 2 \bmod 3$, for any integer a with $(a,p)=1$, a is a cubic residue of modulo p. So this situation is trivial. Thus, we will only discuss the case of $p \equiv 1 \bmod 3$ below.

Theorem 1.1 *Let p be an odd prime with $p \equiv 1 \bmod 3$. Then we have*

$$M_2(p) = \begin{cases} \frac{1}{9}(p+4d-11), & \text{if 2 is a cubic residue modulo } p, \\ \frac{1}{9}(p-5)+E_1(p), & \text{if 2 is a cubic non-residue modulo } p, \end{cases}$$

where d is uniquely determined by $4p=d^2+27b^2$, d, b are integers with $b>0$ and $d \equiv 1 \bmod 3$. Besides, $|E_1(p)| \le \frac{2}{3}\sqrt{p}$.

Theorem 1.2 *Let p be an odd prime with $p \equiv 7 \bmod 12$. Then we have*

$$M_3(p) = \begin{cases} \frac{1}{9}(p+d-17), & \text{if 2 is a cubic residue modulo } p, \\ \frac{1}{9}(p-11)+E_2(p), & \text{if 2 is a cubic non-residue modulo } p, \end{cases}$$

where $|E_2(p)| \le \frac{2}{9}\sqrt{p}$.

Theorem 1.3 *Let p be an odd prime with $p \equiv 1 \bmod 12$. Then we have*

$$M_3(p) = \begin{cases} \frac{1}{9}(p-23)+E_3(p), & \text{if 2 is a cubic residue modulo } p, \\ \frac{1}{9}(p-11)+E_4(p), & \text{if 2 is a cubic non-residue modulo } p, \end{cases}$$

where $|E_3(p)| \le 2\sqrt{p}$, $|E_4(p)| \le 2\sqrt{p}$.

From Theorem 1.1, we immediately have

Corollary 1.4 *Let $p \ge 97$ be an odd prime with $p \equiv 1 \bmod 3$. Then there exists an integer a with $1 \le a \le p-1$ such that $a+a^2$ and $a-a^2$ are the cubic residues of modulo p.*

Corollary 1.5 *Let $p \ge 373$ be an odd prime with $p \equiv 1 \bmod 3$. Then there exists an integer b with $1 \le b \le p-1$ such that $b+b^3$ and $b-b^3$ are the cubic residues of modulo p.*

Corollary 1.6 *Let p be an odd prime with $p \equiv 1 \bmod 3$. If 2 is a cubic residue modulo p, then*

$$p+4d-11 \equiv 0 \bmod 9.$$

Example 1.7 Let $p=31$. We can know 2 is a cubic residue modulo 31 from $4^3 \equiv 2 \bmod 31$. In addition, we can obtain $d=4$ from $4 \times 31 = 4^2 + 27 \times 2^2$ and $4 \equiv 1 \bmod 3$. Thus,

$$31 + 4 \times 4 - 11 = 36 \equiv 0 \bmod 9.$$

Let $p = 43$. We can know 2 is a cubic residue modulo 43 from $20^3 \equiv 2 \mod 43$. In addition, we can obtain $d = -8$ from $4 \times 43 = (-8)^2 + 27 \times 2^2$ and $-8 \equiv 1 \mod 3$. Thus,

$$43 - 4 \times 8 - 11 = 0 \equiv 0 \mod 9.$$

Let $p = 109$. We can know 2 is a cubic residue modulo 109 from $57^3 \equiv 2 \mod 109$. In addition, we can obtain $d = -2$ from $4 \times 109 = (-2)^2 + 27 \times 4^2$ and $-2 \equiv 1 \mod 3$. Thus,

$$109 - 4 \times 2 - 11 = 90 \equiv 0 \mod 9.$$

Let $p = 223$. We can know 2 is a cubic residue modulo 223 from $68^3 \equiv 2 \mod 223$. In addition, we can obtain $d = 28$ from $4 \times 223 = 28^2 + 27 \times 2^2$ and $28 \equiv 1 \mod 3$. Thus,

$$223 + 4 \times 28 - 11 = 324 \equiv 0 \mod 9.$$

From Theorem 1.2, we immediately have

Corollary 1.8 *Let p be an odd prime with $p \equiv 7 \bmod 12$. If 2 is a cubic residue modulo p, then*

$$p + d - 17 \equiv 0 \bmod 9.$$

Example 1.9 Let $p = 31$. We can know 2 is a cubic residue modulo 31 from $4^3 \equiv 2 \mod 31$. In addition, we can obtain $d = 4$ from $4 \times 31 = 4^2 + 27 \times 2^2$ and $4 \equiv 1 \mod 3$. Thus,

$$31 + 4 - 17 = 18 \equiv 0 \mod 9.$$

Let $p = 223$. We can know 2 is a cubic residue modulo 223 from $68^3 \equiv 2 \mod 223$. In addition, we can obtain $d = 28$ from $4 \times 223 = 28^2 + 27 \times 2^2$ and $28 \equiv 1 \mod 3$. Thus,

$$223 + 28 - 17 = 234 \equiv 0 \mod 9.$$

Let $p = 283$. We can know 2 is a cubic residue modulo 283 from $120^3 \equiv 2 \mod 283$. In addition, we can obtain $d = -32$ from $4 \times 283 = (-32)^2 + 27 \times 2^2$ and $-32 \equiv 1 \mod 3$. Thus,

$$283 - 32 - 17 = 234 \equiv 0 \mod 9.$$

Let $p = 307$. We can know 2 is a cubic residue modulo 307 from $52^3 \equiv 2 \mod 307$. In addition, we can obtain $d = 16$ from $4 \times 307 = 16^2 + 27 \times 6^2$ and $16 \equiv 1 \mod 3$. Thus,

$$307 + 16 - 17 = 306 \equiv 0 \mod 9.$$

2 Some lemmas

In the following, we will use the properties of Gauss sums, which can be found in reference [1]. Here, we only introduce the classical Gauss sums $\tau(\chi)$, which is defined as

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right).$$

Especially, if χ is any primitive Dirichlet character mod q, then we have

$$|\tau(\chi)| = \sqrt{q}.$$

To prove our Theorems, we need the several following Lemmas.

Lemma 2.1 *Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order Dirichlet character λ mod p, we have the identity*

$$\tau^3(\lambda) + \tau^3(\overline{\lambda}) = dp,$$

where d is the same as the definition in Theorem 1.1.

Proof See Theorem 1 in [11]. □

Lemma 2.2 *Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order Dirichlet character λ mod p, we have the identities*

$$\sum_{a=1}^{p-1} \lambda(a-a^2) + \sum_{a=1}^{p-1} \overline{\lambda}(a-a^2) = d, \tag{1}$$

$$\sum_{a=1}^{p-1} \lambda(a+a^2) + \sum_{a=1}^{p-1} \overline{\lambda}(a+a^2) = d. \tag{2}$$

Proof Note that $\lambda(-1) = 1$ and $\lambda^2 = \overline{\lambda}$. From the properties of the reduced residue system mod p and Gauss sums, we can obtain

$$\begin{aligned}
&\sum_{a=1}^{p-1} \lambda(a-a^2) \\
&= \sum_{a=1}^{p-1} \lambda(a)\lambda(a-1) \\
&= \frac{1}{\tau(\overline{\lambda})} \sum_{a=1}^{p-1} \lambda(a)\lambda(a-1) \sum_{b=1}^{p-1} \overline{\lambda}(b(a-1)) e\left(\frac{b(a-1)}{p}\right) \\
&= \frac{1}{\tau(\overline{\lambda})} \sum_{b=1}^{p-1} \overline{\lambda}(b) e\left(\frac{-b}{p}\right) \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ab}{p}\right) \\
&= \frac{\tau(\lambda)}{\tau(\overline{\lambda})} \sum_{b=1}^{p-1} \overline{\lambda}^2(b) e\left(\frac{-b}{p}\right) \\
&= \frac{\tau^2(\lambda)}{\tau(\overline{\lambda})} = \frac{\tau^3(\lambda)}{p}.
\end{aligned} \tag{3}$$

Similarly, we have

$$\sum_{a=1}^{p-1} \overline{\lambda}(a-a^2) = \frac{\tau^3(\overline{\lambda})}{p}. \tag{4}$$

Combining (3), (4) and Lemma 2.1, we can obtain

$$\begin{aligned}
&\sum_{a=1}^{p-1} \lambda(a-a^2) + \sum_{a=1}^{p-1} \overline{\lambda}(a-a^2) \\
&= \frac{\tau^3(\lambda)}{p} + \frac{\tau^3(\overline{\lambda})}{p} \\
&= d,
\end{aligned}$$

which proves that formula (1) holds. Using the same method, we can prove the formula (2).

This proves Lemma 2.2. □

Lemma 2.3 *Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order Dirichlet character λ mod p, we have the identity*

$$\begin{aligned}
&\sum_{a=1}^{p-1} \lambda(a+a^2)\lambda(a-a^2) + \sum_{a=1}^{p-1} \overline{\lambda}(a+a^2)\overline{\lambda}(a-a^2) \\
&= d + \frac{1}{p} \cdot \left(\lambda(2)\tau^3(\lambda) + \overline{\lambda}(2)\tau^3(\overline{\lambda})\right).
\end{aligned}$$

Proof For any integer a with $(a, p) = 1$. Since λ is a third-order Dirichlet character mod p, we have $\lambda^3(a) = 1$. Thus we have

$$\sum_{a=1}^{p-1} \lambda(a - a^2)\lambda(a + a^2) = \sum_{a=1}^{p-1} \lambda(a^2 - a^4) = \sum_{a=1}^{p-1} \lambda(a - \overline{a}),$$
$$\sum_{a=1}^{p-1} \overline{\lambda}(a - a^2)\overline{\lambda}(a + a^2) = \sum_{a=1}^{p-1} \overline{\lambda}(a^2 - a^4) = \sum_{a=1}^{p-1} \overline{\lambda}(a - \overline{a}).$$

Applying Lemma 2.3 in [3], we can obtain

$$\begin{aligned}
&\sum_{a=1}^{p-1} \lambda(a + a^2)\lambda(a - a^2) + \sum_{a=1}^{p-1} \overline{\lambda}(a + a^2)\overline{\lambda}(a - a^2) \\
&= \sum_{a=1}^{p-1} \lambda(a - \overline{a}) + \sum_{a=1}^{p-1} \overline{\lambda}(a - \overline{a}) \\
&= d + \frac{1}{p} \cdot (\lambda(2)\tau^3(\lambda) + \overline{\lambda}(2)\tau^3(\overline{\lambda})).
\end{aligned}$$

This proves Lemma 2.3. □

Lemma 2.4 *Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order Dirichlet character λ mod p, we have the identity*

$$\sum_{a=1}^{p-1} \lambda(a + a^2)\overline{\lambda}(a - a^2) = \sum_{a=1}^{p-1} \overline{\lambda}(a + a^2)\lambda(a - a^2) = -2.$$

Proof Using the same method as proving Lemma 2.2, we have

$$\begin{aligned}
&\sum_{a=1}^{p-1} \lambda(a + a^2)\overline{\lambda}(a - a^2) \\
&= \sum_{a=1}^{p-1} \lambda(a + 1)\overline{\lambda}(a - 1) \\
&= \sum_{a=0}^{p-1} \lambda(a + 1)\overline{\lambda}(a - 1) - 1 \\
&= \sum_{a=1}^{p-1} \lambda(a + 2)\overline{\lambda}(a) - 1 \\
&= \frac{1}{\tau(\overline{\lambda})} \sum_{a=1}^{p-1} \lambda(a + 2)\overline{\lambda}(a) \sum_{b=1}^{p-1} \overline{\lambda}(b(a + 2)) e\left(\frac{b(a + 2)}{p}\right) - 1 \\
&= \frac{1}{\tau(\overline{\lambda})} \sum_{b=1}^{p-1} \overline{\lambda}(b) e\left(\frac{2b}{p}\right) \sum_{a=1}^{p-1} \overline{\lambda}(a) e\left(\frac{ab}{p}\right) - 1 \\
&= \frac{\tau(\overline{\lambda})}{\tau(\overline{\lambda})} \sum_{b=1}^{p-1} \overline{\lambda}(b)\lambda(b) e\left(\frac{2b}{p}\right) - 1 \\
&= \frac{\tau(\overline{\lambda})}{\tau(\overline{\lambda})} \sum_{b=1}^{p-1} e\left(\frac{2b}{p}\right) - 1 = -2.
\end{aligned}$$

Similarly, we also have

$$\sum_{a=1}^{p-1} \overline{\lambda}(a+a^2)\lambda(a-a^2) = -2.$$

This proves Lemma 2.4. □

Lemma 2.5 *Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order Dirichlet character $\lambda \bmod p$, we have the identities*

$$\sum_{a=1}^{p-1} \lambda(a+a^3) + \sum_{a=1}^{p-1} \overline{\lambda}(a+a^3) = -2 + \chi_2(-1)\frac{\overline{\lambda}(2)\tau^3(\lambda) + \lambda(2)\tau^3(\overline{\lambda})}{p}, \tag{5}$$

$$\sum_{a=1}^{p-1} \lambda(a-a^3) + \sum_{a=1}^{p-1} \overline{\lambda}(a-a^3) = -2 + \frac{\overline{\lambda}(2)\tau^3(\lambda) + \lambda(2)\tau^3(\overline{\lambda})}{p}. \tag{6}$$

Proof Note that $\lambda = \overline{\lambda}^2$. From the properties of Gauss sums and Legendre's symbol, we can obtain

$$\begin{aligned}
&\sum_{a=1}^{p-1} \lambda(a+a^3)\\
&= \sum_{a=1}^{p-1} \lambda(a)\lambda(a^2+1)\\
&= \frac{1}{\tau(\overline{\lambda})} \sum_{a=1}^{p-1} \lambda(a)\lambda(a^2+1) \sum_{b=1}^{p-1} \overline{\lambda}\left(b(a^2+1)\right) e\left(\frac{b(a^2+1)}{p}\right)\\
&= \frac{1}{\tau(\overline{\lambda})} \sum_{b=1}^{p-1} \overline{\lambda}(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ba^2}{p}\right)\\
&= \frac{1}{\tau(\overline{\lambda})} \sum_{b=1}^{p-1} \overline{\lambda}(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} (1+\chi_2(a))\, \overline{\lambda}(a) e\left(\frac{ba}{p}\right)\\
&= \frac{1}{\tau(\overline{\lambda})} \sum_{b=1}^{p-1} \overline{\lambda}(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \overline{\lambda}(a) e\left(\frac{ba}{p}\right) + \frac{1}{\tau(\overline{\lambda})} \sum_{b=1}^{p-1} \overline{\lambda}(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \chi_2\overline{\lambda}(a) e\left(\frac{ba}{p}\right)\\
&= -1 + \frac{\tau(\chi_2\overline{\lambda})\tau(\chi_2)}{\tau(\overline{\lambda})}.
\end{aligned} \tag{7}$$

Using Lemma 2.2 in [2], we can obtain

$$\tau(\chi_2\overline{\lambda}) = \frac{\tau^2(\lambda)}{\chi_2(-1)\lambda(2)\tau(\chi_2)}. \tag{8}$$

Combining (7) and (8), we can get

$$\sum_{a=1}^{p-1} \lambda(a+a^3) = -1 + \chi_2(-1)\frac{\overline{\lambda}(2)\tau^3(\lambda)}{p}. \tag{9}$$

Similarly, we have

$$\sum_{a=1}^{p-1} \overline{\lambda}(a+a^3) = -1 + \chi_2(-1)\frac{\lambda(2)\tau^3(\overline{\lambda})}{p}. \tag{10}$$

Combining (9) and (10), we can get (5). Using the same method of proving (5), we can prove (6).
This proves Lemma 2.5. □

Lemma 2.6 *Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order Dirichlet character λ mod p, we have the identity*

$$\sum_{a=1}^{p-1} \lambda(a+a^3)\overline{\lambda}(a-a^3) + \sum_{a=1}^{p-1} \overline{\lambda}(a+a^3)\lambda(a-a^3)$$
$$= \begin{cases} -4, & \text{if } p \equiv 7 \bmod 12, \\ -4 + A_1(p), & \text{if } p \equiv 1 \bmod 12, \end{cases}$$

where $|A_1(p)| \le 6\sqrt{p}$.

Proof Using the same method as proving Lemma 2.4, we have

$$\begin{aligned}
&\sum_{a=1}^{p-1} \lambda(a+a^3)\overline{\lambda}(a-a^3) \\
&= \sum_{a=1}^{p-1} \lambda(a^2+1)\overline{\lambda}(a^2-1) \\
&= \sum_{a=1}^{p-1} (1+\chi_2(a))\lambda(a+1)\overline{\lambda}(a-1) \\
&= -2 + \sum_{a=1}^{p-1} \chi_2(a)\lambda(a+1)\overline{\lambda}(a-1).
\end{aligned}$$

Similarly, we have

$$\sum_{a=1}^{p-1} \overline{\lambda}(a+a^3)\lambda(a-a^3) = -2 + \sum_{a=1}^{p-1} \chi_2(a)\overline{\lambda}(a+1)\lambda(a-1).$$

Thus,

$$\begin{aligned}
&\sum_{a=1}^{p-1} \lambda(a+a^3)\overline{\lambda}(a-a^3) + \sum_{a=1}^{p-1} \overline{\lambda}(a+a^3)\lambda(a-a^3) \\
&= -4 + \sum_{a=1}^{p-1} \chi_2(a)\lambda(a+1)\overline{\lambda}(a-1) + \sum_{a=1}^{p-1} \chi_2(a)\overline{\lambda}(a+1)\lambda(a-1).
\end{aligned} \tag{11}$$

If $p \equiv 7 \bmod 12$, using $\chi_2(-1) = -1$ and $\lambda(-1) = 1$, we can obtain

$$\begin{aligned}
&\sum_{a=1}^{p-1} \chi_2(a)\lambda(a+1)\overline{\lambda}(a-1) = \sum_{a=1}^{p-1} \chi_2(a)\lambda(a+1)\lambda^2(a-1) \\
&= \sum_{a=1}^{p-1} \chi_2(a)\lambda(a^2-1)\lambda(a-1) = -\sum_{a=1}^{p-1} \chi_2(a)\lambda(a^2-1)\lambda(a+1) \\
&= -\sum_{a=1}^{p-1} \chi_2(a)\lambda^2(a+1)\lambda(a-1) = -\sum_{a=1}^{p-1} \chi_2(a)\overline{\lambda}(a+1)\lambda(a-1).
\end{aligned}$$

So we have

$$\sum_{a=1}^{p-1} \chi_2(a)\lambda(a+1)\overline{\lambda}(a-1) + \sum_{a=1}^{p-1} \chi_2(a)\overline{\lambda}(a+1)\lambda(a-1) = 0. \tag{12}$$

Thus, combining (11) and (12), we can get

$$\sum_{a=1}^{p-1} \lambda(a+a^3)\overline{\lambda}(a-a^3) + \sum_{a=1}^{p-1} \overline{\lambda}(a+a^3)\lambda(a-a^3) = -4,$$

where $p \equiv 7 \bmod 12$.

If $p \equiv 1 \bmod 12$, from the deep estimation by Weil [9], we have

$$\left|\sum_{a=1}^{p-1} \chi_2(a)\lambda(a+1)\overline{\lambda}(a-1)\right| \le 3\sqrt{p}, \qquad \left|\sum_{a=1}^{p-1} \chi_2(a)\overline{\lambda}(a+1)\lambda(a-1)\right| \le 3\sqrt{p}. \tag{13}$$

Thus, combining (11) and (13), we can get

$$\sum_{a=1}^{p-1} \lambda(a+a^3)\overline{\lambda}(a-a^3) = -4 + A_1(p),$$

where

$$A_1(p) = \sum_{a=1}^{p-1} \chi_2(a)\lambda(a+1)\overline{\lambda}(a-1) + \sum_{a=1}^{p-1} \chi_2(a)\overline{\lambda}(a+1)\lambda(a-1).$$

This proves Lemma 2.6. □

Lemma 2.7 *Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order Dirichlet character λ mod p, we have the identity*

$$\sum_{a=1}^{p-1} \lambda(a+a^3)\lambda(a-a^3) + \sum_{a=1}^{p-1} \overline{\lambda}(a+a^3)\overline{\lambda}(a-a^3)$$
$$= \begin{cases} -2 + \dfrac{\overline{\lambda}(2)\tau^3(\lambda) + \lambda(2)\tau^3(\overline{\lambda})}{p}, & \text{if } p \equiv 7 \bmod 12, \\ -2 + \dfrac{\overline{\lambda}(2)\tau^3(\lambda) + \lambda(2)\tau^3(\overline{\lambda})}{p} + A_2(p), & \text{if } p \equiv 1 \bmod 12, \end{cases}$$

where $|A_2(p)| \le 6\sqrt{p}$.

Proof Using the same methods as proving Lemma 2.5, the properties of Gauss sums and Legendre's symbol, we can obtain

$$\begin{aligned} &\sum_{a=1}^{p-1} \lambda(a+a^3)\lambda(a-a^3) \\ &= \sum_{a=1}^{p-1} \lambda(a^2)\lambda(a^4-1) \\ &= \sum_{a=1}^{p-1} (1+\chi_2(a))\,\lambda(a)\lambda(a^2-1) \\ &= \sum_{a=1}^{p-1} \lambda(a)\lambda(a^2-1) + \sum_{a=1}^{p-1} \chi_2(a)\lambda(a)\lambda(a^2-1) \\ &= -1 + \frac{\overline{\lambda}(2)\tau^3(\lambda)}{p} + \sum_{a=1}^{p-1} \chi_2(a)\lambda(a)\lambda(a^2-1). \end{aligned}$$

Similarly, we have

$$\sum_{a=1}^{p-1}\overline{\lambda}(a+a^3)\overline{\lambda}(a-a^3)=-1+\frac{\lambda(2)\tau^3(\overline{\lambda})}{p}+\sum_{a=1}^{p-1}\chi_2(a)\overline{\lambda}(a)\overline{\lambda}(a^2-1).$$

Thus,

$$\begin{aligned}&\sum_{a=1}^{p-1}\lambda(a+a^3)\lambda(a-a^3)+\sum_{a=1}^{p-1}\overline{\lambda}(a+a^3)\overline{\lambda}(a-a^3)\\&=-2+\frac{\overline{\lambda}(2)\tau^3(\lambda)+\lambda(2)\tau^3(\overline{\lambda})}{p}+\sum_{a=1}^{p-1}\chi_2(a)\lambda(a)\lambda(a^2-1)+\sum_{a=1}^{p-1}\chi_2(a)\overline{\lambda}(a)\overline{\lambda}(a^2-1).\end{aligned}\tag{14}$$

If $p\equiv 7 \bmod 12$, then $\chi_2(-1)=-1$. Using the properties of the reduced residue system mod p, then we can get

$$\sum_{a=1}^{p-1}\chi_2(a)\lambda(a)\lambda(a^2-1)=\sum_{a=1}^{p-1}\chi_2(-a)\lambda(-a)\lambda((-a)^2-1)=-\sum_{a=1}^{p-1}\chi_2(a)\lambda(a)\lambda(a^2-1),$$

which implies that when $p\equiv 7 \bmod 12$,

$$\sum_{a=1}^{p-1}\chi_2(a)\lambda(a)\lambda(a^2-1)=\sum_{a=1}^{p-1}\chi_2(a)\overline{\lambda}(a)\overline{\lambda}(a^2-1)=0.\tag{15}$$

Thus, combining (14) and (15), we can get when $p\equiv 7 \bmod 12$,

$$\sum_{a=1}^{p-1}\lambda(a+a^3)\lambda(a-a^3)+\sum_{a=1}^{p-1}\overline{\lambda}(a+a^3)\overline{\lambda}(a-a^3)=-2+\frac{\overline{\lambda}(2)\tau^3(\lambda)+\lambda(2)\tau^3(\overline{\lambda})}{p}.$$

If $p\equiv 1 \bmod 12$, from the deep estimation by Weil [9] we have

$$\left|\sum_{a=1}^{p-1}\chi_2(a)\lambda(a)\lambda(a^2-1)\right|\le 3\sqrt{p},\qquad \left|\sum_{a=1}^{p-1}\chi_2(a)\overline{\lambda}(a)\overline{\lambda}(a^2-1)\right|\le 3\sqrt{p}.\tag{16}$$

Thus, combining (14) and (16), we can get when $p\equiv 1 \bmod 12$,

$$\begin{aligned}&\sum_{a=1}^{p-1}\lambda(a+a^3)\lambda(a-a^3)+\sum_{a=1}^{p-1}\overline{\lambda}(a+a^3)\overline{\lambda}(a-a^3)\\&=-2+\frac{\overline{\lambda}(2)\tau^3(\lambda)+\lambda(2)\tau^3(\overline{\lambda})}{p}+A_2(p),\end{aligned}\tag{17}$$

where

$$A_2(p)=\sum_{a=1}^{p-1}\chi_2(a)\lambda(a)\lambda(a^2-1)+\sum_{a=1}^{p-1}\chi_2(a)\overline{\lambda}(a)\overline{\lambda}(a^2-1).$$

This proves Lemma 2.7. □

3 Proofs of the theorems

In this section, we shall prove our Theorems. From the properties of Dirichlet character, if λ is a third-order Dirichlet character mod p, for any integer a with $(a, p) = 1$, we have

$$1 + \lambda(a) + \overline{\lambda}(a) = \begin{cases} 3, & \text{if } a \text{ is a cubic residue modulo} p, \\ 0, & \text{if } a \text{ is a cubic non-residue modulo} p. \end{cases}$$

Firstly, we prove Theorem 1.1. Combining Lemmas 2.1–2.4, we obtain

$$\begin{aligned} M_2(p) &= \frac{1}{9} \sum_{\substack{a=1 \\ (a+a^2,p)=1 \\ (a-a^2,p)=1}}^{p-1} \left(1 + \lambda(a + a^2) + \overline{\lambda}(a + a^2)\right) \left(1 + \lambda(a - a^2) + \overline{\lambda}(a - a^2)\right) \\ &= \frac{1}{9} \sum_{a=1}^{p-1} \left(1 + \lambda(a + a^2) + \overline{\lambda}(a + a^2)\right) \left(1 + \lambda(a - a^2) + \overline{\lambda}(a - a^2)\right) \\ &\quad - \frac{2}{9} \left(1 + \lambda(2) + \overline{\lambda}(2)\right) \\ &= \frac{p-1}{9} + \frac{1}{9} \sum_{a=1}^{p-1} \left(\lambda(a - a^2) + \overline{\lambda}(a - a^2)\right) + \frac{1}{9} \sum_{a=1}^{p-1} \left(\lambda(a + a^2) + \overline{\lambda}(a + a^2)\right) \\ &\quad + \frac{1}{9} \sum_{a=1}^{p-1} \left(\lambda(a + a^2)\overline{\lambda}(a - a^2) + \overline{\lambda}(a + a^2)\lambda(a - a^2)\right) \\ &\quad + \frac{1}{9} \sum_{a=1}^{p-1} \left(\lambda(a + a^2)\lambda(a - a^2) + \overline{\lambda}(a + a^2)\overline{\lambda}(a - a^2)\right) - \frac{2}{9} \left(1 + \lambda(2) + \overline{\lambda}(2)\right) \\ &= \frac{1}{9} \left(p + 3d - 7 - 2\lambda(2) - 2\overline{\lambda}(2)\right) + \frac{1}{9p} \left(\lambda(2)\tau^3(\lambda) + \overline{\lambda}(2)\tau^3(\overline{\lambda})\right). \end{aligned} \tag{18}$$

If 2 is a cubic residue modulo p, then $\lambda(2) = 1$. Combining (18) and Lemma 2.1, we have

$$M_2(p) = \frac{p + 4d - 11}{9}.$$

If 2 is a cubic non-residue modulo p, then $1 + \lambda(2) + \overline{\lambda}(2) = 0$. Combining (18) and Lemma 2.1, we have

$$M_2(p) = \frac{p - 5}{9} + E_1(p),$$

where

$$E_1(p) = \frac{\left(2 - \overline{\lambda}(2)\right) \tau^3(\lambda) + (2 - \lambda(2))\, \tau^3(\overline{\lambda})}{9p}.$$

Utilizing $|\tau(\lambda)| = \sqrt{p}$, we have $|E_1(p)| \leq \frac{2}{3}\sqrt{p}$.

This completes the proof of Theorem 1.1.

Then we prove Theorems 1.2 and 1.3. When $p \equiv 7 \bmod 12$, combining Lemmas 2.5–2.7, we can obtain

$$\begin{aligned} M_3(p) &= \frac{1}{9} \sum_{\substack{a=1 \\ (a+a^3,p)=1 \\ (a-a^3,p)=1}}^{p-1} \left(1 + \lambda(a + a^3) + \overline{\lambda}(a + a^3)\right) \left(1 + \lambda(a - a^3) + \overline{\lambda}(a - a^3)\right) \\ &= \frac{1}{9} \sum_{a=1}^{p-1} \left(1 + \lambda(a + a^3) + \overline{\lambda}(a + a^3)\right) \left(1 + \lambda(a - a^3) + \overline{\lambda}(a - a^3)\right) \end{aligned}$$

$$
\begin{aligned}
&-\frac{2}{9}\left(1+\lambda(2)+\overline{\lambda}(2)\right)\\
&=\frac{p-1}{9}+\frac{1}{9}\sum_{a=1}^{p-1}\left(\lambda(a-a^3)+\overline{\lambda}(a-a^3)\right)+\frac{1}{9}\sum_{a=1}^{p-1}\left(\lambda(a+a^3)+\overline{\lambda}(a+a^3)\right)\\
&\quad+\frac{1}{9}\sum_{a=1}^{p-1}\left(\lambda(a+a^3)\overline{\lambda}(a-a^3)+\overline{\lambda}(a+a^3)\lambda(a-a^3)\right)\\
&\quad+\frac{1}{9}\sum_{a=1}^{p-1}\left(\lambda(a+a^3)\lambda(a-a^3)+\overline{\lambda}(a+a^3)\overline{\lambda}(a-a^3)\right)-\frac{2}{9}\left(1+\lambda(2)+\overline{\lambda}(2)\right)\\
&=\frac{1}{9}\left(p-13-2\lambda(2)-2\overline{\lambda}(2)\right)+\frac{1}{9p}\left(\overline{\lambda}(2)\tau^3(\lambda)+\lambda(2)\tau^3(\overline{\lambda})\right). \qquad (19)
\end{aligned}
$$

If 2 is a cubic residue modulo p, then $\lambda(2)=1$. Combining (19) and Lemma 2.1, we have

$$M_3(p)=\frac{p+d-17}{9}.$$

If 2 is a cubic non-residue modulo p, then $1+\lambda(2)+\overline{\lambda}(2)=0$. Combining (19) and Lemma 2.1, we have

$$M_3(p)=\frac{p-11}{9}+E_2(p),$$

where

$$|E_2(p)|=\left|\frac{1}{9p}\left(\overline{\lambda}(2)\tau^3(\lambda)+\lambda(2)\tau^3(\overline{\lambda})\right)\right|\leq\frac{2}{9}\sqrt{p}.$$

When $p\equiv 1 \bmod 12$. Since $\chi_2(-1)=1$, there must exist an integer r with $1<r<p-1$ such that $r^2\equiv -1 \bmod p$. Besides, $\lambda(r)=\lambda(-r)=1$. Combining Lemmas 2.5–2.7, we can obtain

$$
\begin{aligned}
M_3(p)&=\frac{1}{9}\sum_{\substack{a=1\\(a+a^3,p)=1\\(a-a^3,p)=1}}^{p-1}\left(1+\lambda(a+a^3)+\overline{\lambda}(a+a^3)\right)\left(1+\lambda(a-a^3)+\overline{\lambda}(a-a^3)\right)\\
&=\frac{1}{9}\sum_{a=1}^{p-1}\left(1+\lambda(a+a^3)+\overline{\lambda}(a+a^3)\right)\left(1+\lambda(a-a^3)+\overline{\lambda}(a-a^3)\right)\\
&\quad-\frac{4}{9}\left(1+\lambda(2)+\overline{\lambda}(2)\right)\\
&=\frac{p-1}{9}+\frac{1}{9}\sum_{a=1}^{p-1}\left(\lambda(a-a^3)+\overline{\lambda}(a-a^3)\right)+\frac{1}{9}\sum_{a=1}^{p-1}\left(\lambda(a+a^3)+\overline{\lambda}(a+a^3)\right)\\
&\quad+\frac{1}{9}\sum_{a=1}^{p-1}\left(\lambda(a+a^3)\overline{\lambda}(a-a^3)+\overline{\lambda}(a+a^3)\lambda(a-a^3)\right)\\
&\quad+\frac{1}{9}\sum_{a=1}^{p-1}\left(\lambda(a+a^3)\lambda(a-a^3)+\overline{\lambda}(a+a^3)\overline{\lambda}(a-a^3)\right)-\frac{4}{9}\left(1+\lambda(2)+\overline{\lambda}(2)\right)\\
&=\frac{1}{9}\left(p-15-4\lambda(2)-4\overline{\lambda}(2)\right)+\frac{1}{3p}\left(\overline{\lambda}(2)\tau^3(\lambda)+\lambda(2)\tau^3(\overline{\lambda})\right)+\frac{A_1(p)+A_2(p)}{9}. \qquad (20)
\end{aligned}
$$

If 2 is a cubic residue modulo p, then $\lambda(2)=1$. Combining (20) and Lemma 2.1, we have

$$M_3(p)=\frac{p-23}{9}+E_3(p),$$

where

$$|E_3(p)| = \left|\frac{1}{3p}(\tau^3(\lambda) + \tau^3(\overline{\lambda})) + \frac{A_1(p) + A_2(p)}{9}\right| \le 2\sqrt{p}.$$

If 2 is a cubic non-residue modulo p, then $1 + \lambda(2) + \overline{\lambda}(2) = 0$. Combining (20) and Lemma 2.1, we have

$$M_3(p) = \frac{p-11}{9} + E_4(p),$$

where

$$|E_4(p)| = \left|\frac{1}{3p}\left(\overline{\lambda}(2)\tau^3(\lambda) + \lambda(2)\tau^3(\overline{\lambda})\right) + \frac{A_1(p) + A_2(p)}{9}\right| \le 2\sqrt{p}.$$

This completes the proofs of Theorems 1.2 and 1.3.

Acknowledgements The authors would like to show their great gratitude to the anonymous referees for their very helpful and detailed comments.

Author contributions Both the authors have equally contributed to this work. Both the authors read and approved the final manuscript.

Funding This work is supported by the National Natural Science Foundation of China (Nos. 12271320, 11871317), and the Natural Science Basic Research Plan for Distinguished Young Scholars in Shaanxi Province of China (No. 2021JC-29).

Declarations

Competing interests The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. T. M. Apostol (1976). *Introduction to analytic number theory*, Springer-Verlag, New York.
2. L. Chen (2018). On the classical Gauss sums and their some properties. *Symmetry* **10**, 625. https://doi.org/10.3390/sym10110625
3. J. Y. Hu and Z. Y. Chen (2020). On distribution properties of cubic residues. *AIMS Math.* **5**(6), 6051–6060. https://doi.org/10.3934/math.2020388
4. X. G. Liu and Y. Y. Meng (2023). On the quadratic residues and their distribution properties. *Open Math.* **21**(1), 20230123. https://doi.org/10.1515/math-2023-0123
5. Z. H. Sun (2002). Consecutive numbers with the same Legendre symbol. *Proc. Amer. Math. Soc* **130**(9), 2503–2507.
6. J. L. Su and J. F. Zhang (2020). On the quartic residues and their new distribution properties. *Mathematics* **8**(8), 1337. https://doi.org/10.3390/math8081337
7. J. L. Su and J. Zhang (2021). On the sixth residues and some new properties of their distribution. *J. Math.* **2021** Art. ID 6652045 (8pp). https://doi.org/10.1155/2021/6652045
8. T. T. Wang and X. X. Lv (2020). The quadratic residues and some of their new distribution properties. *Symmetry* **12**(3), 421. https://doi.org/10.3390/sym12030421
9. A. Weil (1974). Basic Number Theory. Springer-Verlag, New York.
10. X. D. Yuan and W. P. Zhang (2023). On cubic residues and related problems. *Indian J. Pure Ap. Mat.* **54**(3), 806–815. https://doi.org/10.1007/s13226-022-00299-6
11. W. P. Zhang and J. Y. Hu (2018). The number of solutions of the diagonal cubic congruence equation mod p. *Math. Rep.* **20**, 73–80.