Indian J Pure Appl Math (2026) 57:670–678
https://doi.org/10.1007/s13226-024-00565-9

ORIGINAL RESEARCH

Characterization of $*$-(strongly) regular rings in terms of $\mathcal{G}$-projections

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Received: 14 December 2023 / Accepted: 9 February 2024 / Published online: 9 March 2024

Abstract A unit-picker is a map $\mathcal{G}$ that associates to every ring R a well-defined set $\mathcal{G}(R)$ of central units in R which contains 1_R and is invariant under isomorphisms of rings and closed under taking inverses, and which satisfies certain set containment conditions for quotient rings, corner rings and matrix rings. Let $\mathcal{G}$ be a unit-picker. An element q of a ring R is $\mathcal{G}$-idempotent, a special kind of the strongly regular element, if $q^2 = uq$ for some unit picker u of R, or equivalently, $q = ue$, where u is a unit picker and e is an idempotent of R. In a ring R with involution $*$, projections are self-adjoint idempotents. As a natural generalization of projections, an element q of a ring R is called a $\mathcal{G}$-projection if $q^2 = uq = uq^*$ for some self-adjoint unit-picker u of a $*$-ring R, or equivalently, $q = up$, where p is a projection. We characterize $*$-(strongly) regular rings in terms of the $\mathcal{G}$-projection element.

Keywords ($\mathcal{G}$)-idempotent element · ($\mathcal{G}$)-projection element · $*$-(strongly) regular ring

Mathematics Subject Classification 15A09 · 15A27 · 16U80 · 16U99 · 16W10

1 Introduction

Let R be an associative ring with identity. A ring R is *regular* (in the sense of von Neumann) if for every $a \in R$ the principal right ideal aR is generated by an idempotent, or equivalently there exists $b \in R$ such that $a = aba$; and an element a in a ring R is called *strongly regular* if $a \in a^2R \cap Ra^2$. It is shown that $a \in R$ is strongly regular if and only if $a = ue = eu$ where $e^2 = e \in R$ and u is a unit in R if and only if $a^2 = ua = au$ for a unit u of R in [14]. The authors of [16,19] and [15] introduced a generalization of the idempotent element which is also a special kind of the strongly regular element: an element q of a ring R is a $\mathcal{G}$-*idempotent* (or a *quasi idempotent*), if $q^2 = uq$ for some unit-picker (or central unit) u of R, or equivalently, $q = ue$, where u is a unit-picker (or central unit) and e is an idempotent of R. For a ring R, the set of $\mathcal{G}$-idempotent elements is denoted by $\mathcal{G}$-$E(R)$. Notice that a $\mathcal{G}$-idempotent is a strongly regular element, but the converse is not true. Also, UU-rings, UJ-rings and UNJ-rings in the general setting of $\mathcal{G}$ were investigated by Zhou in [19]. (See more information UU-rings, UJ-rings and UNJ-rings in [2,10–12])

An *involution* of a ring R is an operation $* : R \to R$ such that

$$(x + y)^* = x^* + y^*,\ (xy)^* = y^*x^*, \text{ and } (x^*)^* = x,$$

Communicated by B. Sury.

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for all $x, y \in R$. A ring R with an involution $*$ is called a $*$-*ring*. An element p in a $*$-ring R is called a *projection* if p is a self-adjoint idempotent, i.e., $p^* = p$ and $p = p^2$.

A $*$-ring R is called $*$-*regular* if for every element a in R there exists a projection p such that $aR = pR$, equivalently, R is regular and involution is proper (i.e., $aa^* = 0$ implies $a^* = 0$ for all $a \in R$) in [1]. Let R be a $*$-ring. In [7] were studied basic properties and examples of $*$-regular rings. Also, the authors investigated some characterisations of $*$-regular rings in [3].

An element $a \in R$ is called $*$-*strongly regular* if there exist $p \in PE(R)$ and $u \in U(R)$ such that $a = pu = up$. A ring R is called $*$-*strongly regular* if every element of R is $*$-strongly regular. Clearly, $*$-strongly regular elements are strongly regular, and so $*$-strongly regular rings are strongly regular. However, the converse is not true in general. Recently, in [8] and [5] were studied the $*$-π- regularity and strongly $*$-π-regularity, respectively. In [4] was introduced the structure of $*$-periodic rings as well.

So, a natural generalization of projection is $\mathcal{G}$-projection. An element q of a $*$-ring R is called $\mathcal{G}$-*projection* if $q^2 = uq = uq^*$ for some self-adjoint unit-picker u of a $*$-ring R. The element q is also called a u-*projection*. Denote by $\mathcal{G}$-$PE(R)$ the set of all $\mathcal{G}$-projection elements of R. Clearly, a $\mathcal{G}$-projection is also a special kind of the strongly regular element of a $*$-ring R.

In this paper, we define and give some characterization of $\mathcal{G}$-projection elements. Also, in section 3, we describe structure of $*$-strongly regular rings and $*$-regular rings in terms of a $\mathcal{G}$-idempotent and a $\mathcal{G}$-projection.

Throughout this paper, R will be an associative ring with identity. We write $E(R)$, $CE(R)$, $U(R)$, $UC(R)$ and $PE(R)$ to denote the set of all idempotents, the set of central elements, the set of units, the set of central units and the set of all projections elements of R, respectively. Here, $\mathbb{Z}$ is the ring of integers.

2 $\mathcal{G}$-projection elements

Definition 2.1 [15,16,19] Recall that a unit-picker is a map $\mathcal{G}$ that associates to every ring R a well-defined set $\mathcal{G}(R)$ of central units in R such that $\mathcal{G}(R)$ is invariant under isomorphism of rings and the following are satisfied:

(1) $1_R \in \mathcal{G}(R)$ and $g^{-1} \in \mathcal{G}(R)$ whenever $g \in \mathcal{G}(R)$.
(2) $(\mathcal{G}(R) + I)/I \subseteq \mathcal{G}(R/I)$ for any ideal $I \lhd R$.
(3) $e\mathcal{G}(R) \subseteq \mathcal{G}(eRe)$ for any $e^2 = e \in R$.
(4) $\mathcal{G}(\mathbb{M}_n(R)) \subseteq \mathcal{G}(\mathbb{M}_n(C(R)))$ for any $n > 1$.

Example 2.2 ([16], Example 2.2) The following $\mathcal{G}_1, \mathcal{G}_{\pm 1}, \mathcal{G}_{\sqrt[n]{1}}, \mathcal{G}_{\sqrt[\infty]{1}}, \mathcal{G}_{\sqrt[n^\infty]{1}}$ and $\mathcal{G}_f$ are examples of unit-pickers:

(1) $\mathcal{G}_1(R) = \{1\}$.
(2) $\mathcal{G}_{\pm 1}(R) = \{1, -1\}$.
(3) $\mathcal{G}_{\sqrt[n]{1}}(R) = \{g \in UC(R) : g^n = 1\}$, where n is a fixed positive integer.
(4) $\mathcal{G}_{\sqrt[\infty]{1}}(R) = \{g \in UC(R) : g^k = 1 \text{ for some } k \geq 1\}$.
(5) $\mathcal{G}_{\sqrt[n^\infty]{1}} = \{g \in UC(R) : g^{n^k} = 1 \text{ for some } k \geq 1\}$, where n is a fixed positive integer.
(6) $\mathcal{G}_f(R) = UC(R)$, here the symbol f comes from the word full.

Definition 2.3 An element q of a $*$-ring R is $\mathcal{G}$-*projection* if $q^2 = uq = uq^*$ for some self-adjoint unit-picker u of a $*$-ring R . The element q is also called a u-*projection*. Denote by $\mathcal{G}$-$PE(R)$ the set of all $\mathcal{G}$-projection elements of R.

Proposition 2.4 *Let R be a $*$-ring.*

(1) An element q is a u-projection where u is a self-adjoint unit-picker if and only if $q = up$, where p is a projection and u is a self-adjoint unit-picker.
(2) If q is a u-projection where u is a self-adjoint unit-picker then q is a self-adjoint.
(3) Every $\mathcal{G}$-projection is $\mathcal{G}$-idempotent.
(4) Every $\mathcal{G}$-projection is $*$-strongly regular.
(5) An element q is a $\mathcal{G}$-idempotent if and only if q^* is a $\mathcal{G}$-idempotent.
(6) If q is a u-idempotent where u is a unit-picker then and $u^* - q^*$ is a u^*-idempotent.
(7) If q is a u-projection where u is a self-adjoint unit-picker, then $u - q$ and $u^{-1}q$ are $\mathcal{G}$-projections.

Proof (1) Let q be a u-projection. Then $q^2 = uq = uq^*$ where u is a self-adjoint unit-picker. Since

$$(u^{-1}q)^2 = (u^{-1}q)(u^{-1}q) = u^{-1}u^{-1}q^2 = u^{-1}u^{-1}uq = u^{-1}q$$

and

$$(u^{-1}q)^* = (u^{-1})^*q^* = (u^*)^{-1}q^* = u^{-1}q^* = u^{-1}q,$$

we obtain that $u^{-1}q$ is a projection. Hence $u^{-1}q = p$ implies $q = up$, where p is a projection.

For the converse, suppose that $q = up$ where $p^2 = p = p^*$ and u is a self-adjoint unit-picker. Since $q^* = (up)^* = u^*p^* = up$, we get $q^2 = (up)(up) = uup^2 = u(up) = uq = uq^*$. Hence q is a u-projection.

(2) Let q be a u-projection. By (1), there exist a projection p and a self-adjoint unit-picker u such that $q = up = up^*$. So, $q^* = (up)^* = u^*p^* = up = q$.

(3) let q be a $\mathcal{G}$-projection. By (1), there exist a projection p and a (self-adjoint) unit-picker u such that $q = up$. Since every projections is an idempotent, we get q is a $\mathcal{G}$-idempotent.

(4) This follows from [18, Corollary 3.9] and (1).

(5) Let $q = ue$ where u is a unit-picker of R and $e^2 = e \in R$. Since $UC(R) = UC(R)^*$, we obtain that

$$(q^*)^2 = (ue)^*(ue)^* = e^*u^*e^*u^* = u^*u^*(ee)^* = u^*u^*e^* = u^*q^*,$$

as desired. The proof of the reverse implication is similar.

(6) Since q is a u-idempotent, we get $q^2 = uq$. Then

$$\begin{aligned}(u^* - q^*)^2 &= (u^* - q^*)(u^* - q^*)\\ &= u^*u^* - u^*q^* - q^*u^* + q^*q^*\\ &= u^*u^* - u^*q^* - u^*q^* + (qq)^* \ \ (= (qu)^*)\\ &= u^*u^* - u^*q^* - u^*q^* + u^*q^*\\ &= u^*(u^* - q^*).\end{aligned}$$

(7) Let q be a u-projection, i.e., $q^2 = uq = uq^*$ where u is a self-adjoint unit-picker. Since $(u - q)^2 = u^2 - uq - qu + q^2 = u^2 - uq - uq + uq = u(u - q) = u(u - q)^*$, we have $(u - q)$ is a $\mathcal{G}$-projection. On the other hand, as $u^{-1}q \in PE(R)$, we get $u^{-1}q$ is $\mathcal{G}$-projection. □

Example 2.5 Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and consider the involution map $* : R \to R$ defined by $(a, b)^* = (b, a)$. Then R is a $*$-ring. The element $q := (1, 0)$ is a $\mathcal{G}$-idempotent of R which is not a $\mathcal{G}$-projection. Since $q^2 = uq$, where $u = (1, 1)$ is a central self-adjoint, we have q is a $\mathcal{G}$-idempotent but $q^* \neq q$ so $q = (1, 0)$ is not a $\mathcal{G}$-projection.

Example 2.6 (1) Let $R = \mathbb{T}_2(\mathbb{Z}_2)$ be the 2×2 upper triangular matrix ring and consider the involution map $* : R \to R$ defined by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \longmapsto \begin{pmatrix} c & b \\ 0 & a \end{pmatrix}$. It is easy to check that $q := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a $\mathcal{G}$-projection (since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$) but it is not a projection.

(2) Let $R = \mathbb{M}_n(F_2)$ and consider the involution map $* : R \to R$ defined by, if $A = (a_{ij}) \in \mathbb{M}_n(F_2)$, A^* is the transpose of (a^*_{ij}). Note that $u = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is a self-adjoint unit-picker, $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a projection and $q = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is not a projection. Since $q = up$, the element q is a $\mathcal{G}$-projection.

Example 2.7 Let $\mathbb{R}$ be the set of real numbers and $R = \mathbb{R} \times \mathbb{R}$. Consider the involution map $* : R \to R$ defined by $(a, b)^* = (b, a)$. The element $a := (2, 3)$ is a $*$-strongly regular element but it is not a $\mathcal{G}$-projection by Proposition 2.4(2).

We have the following diagram:

$$\begin{array}{ccccc} Idempotent & \Longrightarrow & \mathcal{G}\text{-}Idempotent & \Longrightarrow & Strongly\ Regular \\ \Uparrow & & \Uparrow & & \Uparrow \\ Projection & \Longrightarrow & \mathcal{G}\text{-}Projection & \Longrightarrow & *\text{-}Strongly\ Regular \end{array} \tag{1}$$

Lemma 2.8 *Let R be a $*$-ring.*

(1) Let $a \in R$ such that $a = up$ for some self-adjoint unit-picker u and a projection p. Then a is central if and only if p is central.

(2) If every $\mathcal{G}$-idempotent is a $\mathcal{G}$-projection then $\mathcal{G}$-idempotents of R are central.

Proof (1) Assume $a \in R$ which is the form $a = up$ is central. Then $p = u^{-1}a$. Since u is a self-adjoint unit-picker, p is central.

The converse is clear.

(2) Let a be a $\mathcal{G}$-idempotent in R. Then $a^2 = ua$ where $u \in \mathcal{G}(R)$. For each $r \in R$, consider the element $q = a + (u - a)ra$ of R. Now,

$$\begin{aligned} q^2 &= [a + (u - a)ra][a + (u - a)ra] \\ &= a^2 + a(u - a)ra + (u - a)ra^2 + (u - a)ra(u - a)ra \\ &= ua + u(u - a)ra \\ &= uq. \end{aligned}$$

implies that q is a $\mathcal{G}$-idempotent. By hypothesis, q is a $\mathcal{G}$-projection. By Propostion 2.4(2), $a = a^*$ and $q = q^*$. Therefore

$$a + (u - a)ra = q = q^* = (a + (u - a)ra)^* = a^* + a^*r^*(u - a)^* = a + ar^*(u - a).$$

Notice that $(u - a)^* = u - a$ by Propostion 2.4(7). Now $a + (u - a)ra = a + ar^*(u - a)$, which implies that

$$(u - a)ra = ar^*(u - a).$$

Multiplying (1) by $(u - a)$ from the left and right, respectively, we get

$$ar^*u = ar^*a \text{ and } ura = ara \text{ (i.e., } (u - a)Ra = 0).$$

Hence, $ur^*a = ar^*a = ar^*u$ implies that $r^*au = ar^*u$ which gives $r^*a = ar^*$. □

Lemma 2.9 *Assume that units of a ∗-ring R are self-adjoint. Then* $\mathcal{G}\text{-}E(R) = \mathcal{G}\text{-}PE(R)$ *if and only if, for each* $a, b \in \mathcal{G}\text{-}E(R)$, $a^*a = aa^*$ *and* $b^*b = 0$ *implies* $b^2 = 0$ *(or* $b^* = b = 0$*).*

Proof Assume that $\mathcal{G}\text{-}E(R) = \mathcal{G}\text{-}PE(R)$ and $a, b \in \mathcal{G}\text{-}E(R)$. By Proposition 2.4(2), $a = a^*$ and $b^* = b$. Clearly, $a^*a = aa^*$. Since $b^2 = ub$ for some (self-adjoint) unit-picker u of R, $b^* = b$ and $b^*b = 0$, we obtain that $b^*bu = bbu = 0$. Hence $b^2 = bu = 0$ which implies $b = 0$.

For the converse, by Proposition 2.4(3), $\mathcal{G}\text{-}PE(R) \subseteq \mathcal{G}\text{-}E(R)$, let $a \in \mathcal{G}\text{-}E(R)$, where $a^2 = ua$ for some unit-picker u of R. Take $b := ua - aa^*$. Then

$$b^* = (ua - aa^*)^* = (ua)^* - (aa^*)^* = u^*a^* - a^*a = ua^* - a^*a.$$

By hypothesis, we have $ua = aa^*$ and $ua^* = a^*a$. Since $(a^2)^* = (aa)^* = a^*a^*$ and $(a^2)^* = (au)^* = ua^*$, we obtain that $b^*b = 0$. Hence $a^2 = ua = ua^*$, as desired. □

Lemma 2.8 (or 2.9) yields the following.

Corollary 2.10 *Let R be a ∗-ring and* $\mathcal{G}\text{-}E(R) = \mathcal{G}\text{-}PE(R)$*. Then each* $\mathcal{G}$*-idempotents of R is central.*

3 ∗-(strongly) regular rings with $\mathcal{G}$-projections

According to Koliha and Patricio ([9]), the *commutant* and the *double commutant* of an element $a \in R$ are defined by

$$comm(a) = \{x \in R | xa = ax\}$$

and

$$comm^2(a) = \{x \in R | xy = yx \text{ for all } y \in comm(a)\}.$$

It is known that a ring R is strongly regular if and only if for each $a \in R$, there exists an idempotent $e \in comm^2(a)$ such that $a + e \in U(R)$ and $ae = 0$.

Let R be a ∗-ring. An element a of R is called ∗-strongly regular if there exists a projection $p \in comm^2(a)$ such that $a + p \in U(R)$ and $ap = 0$. A ring R is ∗-strongly regular if every element of R is ∗-strongly regular.

Theorem 3.1 *A $*$-ring R is $*$-strongly regular if and only if R is strongly regular and every $\mathcal{G}$-idempotent of R is $\mathcal{G}$-projection.*

Proof Let R be a $*$-strongly regular. Then it is a strongly regular ring. Now we show that an arbitrary $\mathcal{G}$-idempotent q of R is a $\mathcal{G}$-projection. Since q is $*$-strongly regular, then there exists a projection $p \in comm^2(q)$ such that $q + p \in U(R)$ and $qp = 0$. Notice that p is a $\mathcal{G}$-projection. So there exists a self-adjoint unit-picker, say v, in R such that $p^2 = vp$. It follows that $q = v - p$ is a $\mathcal{G}$-projection by Proposition 2.4(7).

Assume R is a strongly regular ring and every $\mathcal{G}$-idempotent of R is $\mathcal{G}$-projection. Then, for every $a \in R$, there exists a unique $p^2 = p \in comm^2(a)$ such that $a + p = u \in U(R)$ and $ap = 0$. We show that p is a projection. Since $p^2 = p \in E(R)$, we obtain that $p \in \mathcal{G}\text{-}E(R)$. By hypothesis, $p \in \mathcal{G}\text{-}PE(R)$, i.e., p is a u-projection. By Proposition 2.4(2), $p^2 = p = p^*$ as desired. □

An element a of a ring R is said to *group invertible*, $R^\sharp$ denotes the set of all group invertibles, if there is a $a^\sharp$ such that

$$aa^\sharp a = a, \quad a^\sharp aa^\sharp = a^\sharp, \quad a^\sharp a = aa^\sharp.$$

Clearly, R is a strongly regular ring if and only if $R^\sharp = R$. (see [18])

Corollary 3.2 *A $*$-ring R is $*$-strongly regular if and only if $R^\sharp = R$ and every $\mathcal{G}$-idempotent of R is $\mathcal{G}$-projection.*

Theorem 3.3 *The following are equivalent for a $*$-ring R:*

(1) R is $*$-strongly regular,
(2) For each $a \in R$, there exists $b \in R$ such that $a = aba$, $b = bab$ and $ab = ba \in PE(R)$,
(3) For each $a \in R$, there exists $b \in R$ such that $a = aba$, $b = bab$ and $ab = ba \in \mathcal{G}\text{-}PE(R)$.

Proof (1) $\Rightarrow$ (2) This implication follows from [18, Theorem 3.8].
(2) $\Rightarrow$ (3) This implication follows from by diagram (1).
(3) $\Rightarrow$ (1) Suppose, for each $a \in R$, there exists $b \in R$ such that $a = aba$, $b = bab$ and $ab = ba \in \mathcal{G}\text{-}PE(R)$. By Proposition 2.4(2), $(ab)^* = ab$. Let $p := 1 - ab$. Then,

$$p^2 = (1 - ab)(1 - ab) = 1 - ab - ab + abab = 1 - ab$$
$$p^* = (1 - ab)^* = 1 - (ab)^* = 1 - ab$$

which imply that $p^2 = p = p^*$, i.e., $p \in \mathcal{G}\text{-}PE(R)$. Since $(a + p)(b + p) = 1$ and $p \in comm^2(a)$, we obtain that the element $a \in R$ is $*$-strongly regular. □

Corollary 3.4 *A $*$-ring R is $*$-strongly regular if and only if, for each $a \in R$, there exist $u \in U(R)$ and $b \in \mathcal{G}\text{-}PE(R)$ such that $ab = a = ub = bu$.*

Proof Let R is a $*$-strongly regular $*$-ring and $a \in R$. Then there exists a $p \in PE(R)$ such that $p \in comm^2(a)$, $a + p = u \in U(R)$ and $ap = 0$. Let $b := 1 - p$. By Proposition 2.4(7) and by diagram (1), b is a $\mathcal{G}$-projection. Since $a + p = u$, $ap = 0$ and $b = 1 - p$, we obtain that

$ap = up - p^2$ implies $p^2 = p = up$, $a = u - up$ implies $a = u(1 - p) = ub$, $ab = a - ap$ implies $ab = a$, $ba = a - pa$ implies $ba = a$.

For the converse, assume that there exists a unit-picker, say v, such that $b^2 = vb$. Let $c := v^{-1}u^{-1}au^{-1}v^{-1}$. Then

$$aca = a \qquad cac = b$$
$$ac = v^{-1}b \qquad ca = v^{-1}b$$

which imply that $ac = ca = v^{-1}b$. By Proposition 2.4 (7), we get that $ac = ca = v^{-1}b \in \mathcal{G}\text{-}PE(R)$. Hence R is a $*$-strongly regular ring by Theorem 3.3. □

Lemma 3.5 *Let a be a $\mathcal{G}$-projection element a $*$-ring R such that $a = up$ where u is a self-adjoint unit-picker and p is a projection. Then $aR = pR$.*

Proof Let $x \in aR$. Then there exists an element y in R such that $x = ay$. So

$$x = ay = (up)y = p(uy)$$

which implies that $x \in pR$. On the other hand, let $x \in pR$. Then $x = py$ for some $y \in R$. Since $p = u^{-1}a$ is a $\mathcal{G}$-projection by Proposition 2.4(7), we get

$$x = (u^{-1}a)y = a(u^{-1}y) \in aR.$$

They imply $aR = pR$. □

Lemma 3.6 *The following are equivalent for a $*$-ring R:*

(1) R is $*$-regular,
(2) For every $a \in R$, there exists a $\mathcal{G}$-projection $b := up$, where u is a self-adjoint unit-picker and p is a projection, such that $aR = bR = pR$,
(3) Every finitely generated right ideal of R is generated by a $\mathcal{G}$-projection of R.
(4) For every $a \in R$, $Ra = Ra^*a$ and $Ra = Rb^*a = Rba$ where b is a $\mathcal{G}$-projection.

Proof (1) $\Rightarrow$ (2) Let $a \in R$. By (1), there exists a projection b such that $aR = bR$. By diagram (1), $b = up$ where u is a self-adjoint unit-picker and p is a projection. Now, $aR = bR = pR$ by the previous paragraph.

(2) $\Rightarrow$ (3) Recall that if b is a u-idempotent, i.e. $b^2 = ub$ and $b = ue$, e is an idempotent, then we have a direct sum decomposition of R:

$$R = eR \oplus (1 - e)R = bR \oplus (u - b)R$$

where $bR = eR$, $(u - b)R = (1 - e)R$ and $u - b = u(1 - e)$ is also a u-idempotent. Since R is regular by (2), every finitely generated right ideal, say aR, of R is generated by an idempotent. Hence $aR = eR = bR$ by (2).

(3) $\Rightarrow$ (4) Let $a \in R$. By (3), there exists a $\mathcal{G}$-projection $b := up$, where u is a self-adjoint unit-picker and p is a projection, such that $aR = bR = pR$. Let $a = br$ and $b = as$ for some $r, s \in R$. Hence

$$pa = p(br) = p(up)r = up^2r = upr = br = a$$

which implies that $Ra = Ra^*a$ and $a = pa = u^{-1}ba = u^{-1}b^*a$ by Proposition 2.4(2), i.e., $Ra \subseteq Rb^*a = Rba$.

(4) $\Rightarrow$ (1) The implication follows from [3, Proposition 2.7]. □

Proposition 3.7 *Let R be a $*$-ring. Assume that unit-pickers of R are self-adjoint. Then R is $*$-regular if and only if R is regular and every $\mathcal{G}$-idempotent of R is a $\mathcal{G}$-projection.*

Proof Let R is a $*$-regular ring. Then R is regular. Let q be a $\mathcal{G}$-idempotent, i.e., $q = ue$ where u is a self-adjoint unit-picker and e is an idempotent. Since R is a $*$-regular ring, the idempotent e is projection, i.e., $e = e^*$ by [7, Proposition 2.3]. Then $q^* = (ue)^* = e^*u^* = ue = q$ which implies that $uq = uq^*$. Hence $q^2 = uq = uq^*$, as desired.

For the converse, let $a \in R$. Since R is regular, there exists an idempotent e of R such that $aR = eR$. Then e is a $\mathcal{G}$-idempotent and so is a $\mathcal{G}$-projection by diagram (1). Let $e = up$ where u is a self-adjoint unit-picker and p is a projection. As $eR = pR$, we get $aR = eR = pR$. □

Proposition 3.8 *The following are equivalent for a $*$-ring R:*

(1) R is $*$-regular and $\mathcal{G}$-$PE(R) = \mathcal{G}(R) \cup \{0\} = \mathcal{G}(R)^* \cup \{0\}$,
(2) R is regular and $\mathcal{G}$-$E(R) = \mathcal{G}(R) \cup \{0\}$,
(3) R is a division ring.

Proof (1) $\Rightarrow$ (2) Clearly, R is a regular ring. Let $a \in \mathcal{G}$-$E(R)$, i.e., $a^2 = ua$ and $a = ue$ where $u \in \mathcal{G}(R)$ and $e \in E(R)$. By Lemma 3.6, there exists a $\mathcal{G}$-projection $b = vp$ where $v^* = v \in \mathcal{G}(R)$ and $p \in PE(R)$ such that $aR = bR = pR$. So $a = br$ and $b = as$ for some $r, s \in R$. Since $b \in \mathcal{G}(R) \cup \{0\}$ by hypothesis, we have the following two cases.
Case 1.: If $b = 0$, then $br = a = 0$.
Case 2.: If $b = vp \in \mathcal{G}(R)$, then $p = 1$. Hence $v = b = as$ implies

$$av = a^2s = (ua)s = u(as) = ub = bu$$

which gives us that $a = u \in \mathcal{G}(R)$ and so $a \in \mathcal{G}(R) \cup \{0\}$.

The reverse implication, $\mathcal{G}(R) \cup \{0\} \subseteq \mathcal{G}\text{-}E(R)$, is clear.

(2) $\Rightarrow$ (3) Let $x \in R \setminus \{0\}$. Since R is regular, $x = xyx$ for some $y \in R$. As $0 \neq xy \in E(R)$ and $0 \neq yx \in E(R)$, we get $xy, yx \in \mathcal{G}\text{-}E(R) = \mathcal{G}(R) \cup \{0\}$ and so $xy = 1 = yx$. Hence $x \in U(R)$.

(3) $\Rightarrow$ (1) By [7, Proposition 3.5], R is a $*$-regular ring and 0, 1 are only projections. Let a be a $\mathcal{G}$-projection. Then there exists a $u^* = u \in \mathcal{G}(R)$ and $p \in PE(R)$ such that $a = up$. If $p = 0$, then $a = 0$ as desired. If $p = 1$, then $a = u \in \mathcal{G}(R)$. □

A ring is *Abelian* if all of its idempotents are central, and a $*$-ring is $*$-*Abelian* if every projection is central ([17]). If every idempotent of $*$-ring R is projection, then R is Abelian ([13]). Clearly, Abelian $*$-rings are $*$-Abelian. Recall that Abelian regular rings are strongly regular, and a $*$-ring R is $*$-strongly regular if and only if R is regular and every idempotent of R is projection ([18, Corollary 3.5]).

Corollary 3.9 *A $*$-ring R is $*$-strongly regular if and only if R is regular and every $\mathcal{G}$-idempotent of R is $\mathcal{G}$-projection.*

Proof (:$\Rightarrow$) This implication follows from Theorem 3.1 and [18, Corollary 3.5].

($\Leftarrow$:) This implication follows from Proposition 2.4, Theorem 3.7 and [18, Corollary 3.5]. □

Recall from [1], the involution of a $*$-ring is said to be proper if $a^*a = 0$ implies $a = 0$.

Theorem 3.10 *The following are equivalent for a $*$-ring R:*

(1) R is $*$-strongly regular,
(2) R is strongly regular and the involution is proper,
(3) For each $a \in R$, there exists a central $\mathcal{G}$-projection $b = up$, where u is a self-adjoint unit-picker and p is a projection, such that $aR = bR (= pR)$,
(4) For each $a \in R$, $Ra = Raa^* \subseteq aR$,
(5) R is $*$-regular, R is Abelian and every $\mathcal{G}$-projection is central.

Proof (1) $\Rightarrow$ (2) Because of Theorem 3.1, we only prove that the involution $*$ is proper. Let $a \in R$ and $a^*a = 0$. By Corollary 3.4, there exist a unit, say u, and a $\mathcal{G}$-projection, say b, such that $ab = a = ub = bu$. Since b is a $\mathcal{G}$-projection, we have $b^2 = vb$ where v is a self-adjoint unit-picker and $b = b^*$ by Proposition 2.4(2). Hence

$$0 = a^*a = (bu)^*(bu) = u^*b^*bu = u^*b^2u = u^*(vb)u = u^*va$$

which implies that $a = 0$ as $U(R) = U(R)^*$ (i.e., u^* is also a unit).

(2) $\Rightarrow$ (3) We again recall that, since R is strongly regular, R is regular and Abelian and hence R is $*$-strongly regular. Let $a \in R$. By Lemma 3.6, there exists a $\mathcal{G}$-projection $b = up$ where u is a self-adjoint unit-picker by Proposition 2.4(1) and p is a projection such that $aR = bR = pR$. Now we show that b is central. Since R is Abelian, the projection p (by Lemma 2.8(1)) is also a central. Hence, for any $r \in R$,

$$br = (uq)r = u(qr) = u(rq) = r(uq) = rb$$

and $r^*b = br^*$ as $(br)^* = (rb)^*$.

(3) $\Rightarrow$ (4) Let $a \in R$. By Lemma 3.6, we have $Ra = Raa^*$. By (3), there exists a central $\mathcal{G}$-projection $b = up$, where u is a self-adjoint unit-picker and p is a projection, such that $aR = bR = pR$. Then $a = ps$ and $b = ar$ for some $s, r \in R$. Note that $pa = p^2s = ps = a$ since p is a projection. Let $e := u^{-1}ra$. Then

$$ae = a(u^{-1}ra) = u^{-1}(ara) = u^{-1}(ba) = (u^{-1}b)a = pa = a$$

and $e^2 = e$. Now, there exists a central $\mathcal{G}$-projection $c = vt$, where v is a central self-adjoint unit-picker and t is a projection, such that $eR = cR = tR$ by (3). By Lemma 2.8(1), the projection t is central. As $e = te = et = t$, we get e is a central projection and so $ue = ra = eb$ is a central projection. On the other hand, $b^2 = ub$ implies that

$$b = u^{-1}b^2 = u^{-1}(arar) = a(u^{-1}ra)r = aer = are = be = eb.$$

Thus $b = eb = be = ue = up$ which gives $e = p$. So

$$Ra = Re = Rb = Rb^* \subseteq Ra^*.$$

Similarly, $Ra^* \subseteq Ra$. Therefore

$$Ra = Raa^* = Ra^*.$$

It is also easy to see that

$$Ra = Re = Rb = bR = arR \subseteq aR.$$

By Lemma 3.6,

$$Ra = Rb^*a = Rba \subseteq aR$$

and so

$$Ra = Rb^*a = Rba = bRa \subseteq bR = b^*R.$$

(4) $\Rightarrow$ (5) By [18, Theorem 3.13], R is $*$-regular and R is Abelian. Let a be a $\mathcal{G}$-projection, i.e., $a = up$, where u is a self-adjoint unit-picker and p is a projection. Since R is Abelian, the projection p is central. By Lemma 2.8(1), the $\mathcal{G}$-projection a is central.

(5) $\Rightarrow$ (1) This implication follows from [18, Theorem 3.13]. □

Corollary 3.11 *Let R be a $*$-ring. Every $\mathcal{G}$-idempotent of R is $*$-strongly regular if and only if $\mathcal{G}\text{-}PE(R) = \mathcal{G}\text{-}E(R)$.*

According to [6], a subring S of a $*$-ring R is called *$*$-invariant* if $S^* \subseteq S$. Then S is a $*$-ring (possibly without the identity of R).

Theorem 3.12 *Let R be a $*$-ring.*

(1) If R is $*$-regular and a is a $\mathcal{G}$-projection of R, then aRa is $*$-regular.
(2) If R is $*$-strongly regular and a is a $\mathcal{G}$-projection of R, then aRa is $*$-strongly regular.
(3) R is $*$-strongly regular if and only if $\mathcal{G}\text{-}PE(R) = \mathcal{G}\text{-}E(R)$ and aRa is a $*$-strongly regular ring for each $a \in \mathcal{G}\text{-}PE(R)$.

Proof (1) Let R be $*$-regular ring and $a = up$ be a $\mathcal{G}$-projection of R where u is a self-adjoint unit-picker and p is a projection. Let $x \in aRa$. Then $x = ara$ for some $r \in R$. Now $x = ara = (up)r(up)$. Since R is a regular ring, we choose $y \in R$ as $x = xyx$. Hence

$$xp = (ara)p = ((up)r(up))p = uprup^2 = (up)r(up) = ara = x$$

and, similarly, $px = x$. On the other hand,

$$xyx = (xp)y(px) = x(pyp)x = x(u^{-1}ayau^{-1})x.$$

Replacing y by $u^{-1}ayau^{-1}$, we get $y \in aRa$. Since aRa is $*$-ring, aRa is a $*$-invariant.

(2) Let R be a $*$-regular ring. Then R is Let R be $*$-regular and Abelian by Theorem 3.10. By (1), aRa is $*$-regular and Abelian. Hence aRa is $*$-strongly regular.

(3) (:$\Rightarrow$) This implication follows from (2), Theorem 3.1 and Corollary 3.11.

($\Leftarrow$:) Let $x \in R$ and a be a $\mathcal{G}$-idempotent. Then a is a $\mathcal{G}$-projection, i.e., $a = up$ where u is a self-adjoint unit-picker and p is a projection. Since $p = u^{-1}a \in PE(R) \subseteq \mathcal{G}\text{-}PE(R) = \mathcal{G}\text{-}E(R)$. By hypothesis, pRp and $(1-p)R(1-p)$ are $*$-strongly regular. By Proposition 2.4(7), the elements $u^{-1}aRau^{-1}$ and $(1-u^{-1}e)R(1-u^{-1}e)$ are $*$-strongly regular. By Corollary 3.4, we obtain

$$u^{-1}axau^{-1} = vp_1 = p_1v$$

where $v \in U(u^{-1}aRau^{-1})$ and $p_1 \in \mathcal{G}\text{-}PE(u^{-1}aRau^{-1})$, and

$$(1-u^{-1}a)x(1-u^{-1}a) = wp_2 = p_2w$$

where $w \in U((1-u^{-1}a)R(1-u^{-1}a))$ and $p_1 \in \mathcal{G}\text{-}PE((1-u^{-1}a)R(1-u^{-1}a))$. Since $\mathcal{G}\text{-}PE(R) = \mathcal{G}\text{-}E(R)$, by Lemma 2.8(2),

$$\begin{aligned} x &= xp + x(1-p) = pxp + (1-p)x(1-p) \\ &= (u^{-1}a)x(u^{-1}a) + (1-(u^{-1}a))x(1-(u^{-1}a)) \\ &= vp_1 + wp_2 \\ &= (p_1+p_2)(v+w) \end{aligned}$$

where $p_1 + p_2 \in \mathcal{G}\text{-}PE(R)$ and $v + w \in U(R)$. By Corollary 3.4, we get R is $*$-strongly regular. □

Acknowledgements The author is thankful to the referee for his/her valuable comments and suggestions to improve the article.

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