ORIGINAL RESEARCH

# Note on a class of polynomial sets

S. J. Rapeli[1,2] · A. K. Shukla[2]

Received: 16 June 2020 / Accepted: 22 April 2026 / Published online: 7 May 2026


**Abstract**
In this paper, we discuss the generalization of Sheffer polynomial, of type zero and an extension of multiple Appell polynomial sets.

**Keywords** Appell sets · Differential operator · Sheffer polynomials

**Mathematics Subject Classification** 33E30 · 44A45 · 46G25

## 1 Introduction

Let $\phi_n(x)$ be a simple set of polynomials and let $\phi_n(x)$ belong to the operator

$$J(x, D) = \sum_{k=0}^{\infty} T_k(x) D^{k+1},$$

with $T_k(x)$ of degree $\leq k$. If the maximum degree of the coefficients $T_k(x)$ is $m$ then the set $\phi_n(x)$ is of Sheffer A-type $m$. If the degree of $T_k(x)$ is unbounded as $k \to \infty$, we say that $\phi_n(x)$ is of Sheffer A-type $\infty$ [3, 13].

### 1.1 Polynomials of Sheffer A-type zero:

Let $\phi_n(x)$ be a Sheffer A-type zero polynomials and $\phi_n(x)$ belongs to the operator

$$J(D) = \sum_{k=0}^{\infty} c_k D^{k+1},$$

where $c_k$ is constant, $c_0 \neq 0$ *and* $J\phi_n = \phi_{n-1}$.

Furthermore, since $c_k$ is independent of $x$ for every $k$, a function $J(t)$ exists with the formal power-series expansion [5, 7, 15]

$$J(t) = \sum_{k=0}^{\infty} c_k t^{k+1}, \qquad c_0 \neq 0.$$

Dedicated to Professor (Late) R. K. Saxena

Communicated by S Ponnusamy.

 https://doi.org/10.1007/s13226-026-00968-w

Let $H(t)$ be the formal inverse of $J(t)$; that is,

$$J(H(t)) = H(J(t)) = t.$$

Al-Salam and Verma [1] discussed generalized Sheffer polynomials. Alidad [2] studied Sheffer polynomials A-type zero in two variables. Bretti et al.[4] discussed generalized Appell polynomials. Galiffa [8] obtained Sheffer B-type 1 orthogonal polynomial sequences. Khan et al.[10] obtained the properties of Hermite based Sheffer polynomials. Khan and Raza [9] discussed the 2-iterated Appell polynomials and related numbers. Rapeli and Shukla [14] obtained the $\alpha$ type polynomial sets.

Appell polynomial sets $\{p_n(x)\}$ are special case of Sheffer polynomials sets satisfying the following condition

$$p_n'(x) = p_{n-1}(x) \qquad n \geqslant 1.$$

Examples of Appell polynomial besides the trivial case $\{x^n\}_{n=0}^{\infty}$, include the Bernoulli polynomials, Euler polynomials, and Hermite polynomials. In particular, the Hermite polynomials form a unique sequence of Appell polynomials that are also orthogonal with respect to a positive measure.[6, 11, 12]

## 2 Generalization of Sequence of Appell polynomial sets

A polynomial sets $\{p_{n_1,n_2}(x)\}_{n=0}^{\infty}$ are said to be generalized multiple Appell polynomial sets, if there exists a generating function of the form

$$\sum_{i=1}^{r} A_i(t_1,t_2)\exp(x\epsilon_i(t_1+t_2)) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} p_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}, \tag{2.1}$$

where $\epsilon_1, \epsilon_2, \cdots, \epsilon_r$ are $r$ roots of unity and $A$ is a formal power series defined by

$$A_i(t_1,t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a_{n_1,n_2}^{(i)}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}, \tag{2.2}$$

where not all $a_{0,0}^{(s)}$ are zero.

**Theorem 1** *Let $\{p_{n_1,n_2}(x)\}_{n=0}^{\infty}$ be a sequence of class of Appell polynomial sets. Then there exists a sequence $\{a_{n_1,n_2}^{(i)}\}_{n_1,n_2=0}^{\infty}$ such that*

$$p_{n_1,n_2}(x) = \sum_{i=1}^{r}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}a_{n_1-k_1,n_2-k_2}^{(i)}(\epsilon_i x)^{k_1+k_2},$$

*where $\epsilon_1, \epsilon_2, \cdots, \epsilon_r$ are $r$ roots of unity and not all $\{a_{0,0}^{(i)}\}$ are zero.*

***Proof*** Let $\{p_{n_1,n_2}(x)\}_{n=0}^{\infty}$ be a sequence of class of Appell polynomial sets. From equation (2.1), we have

$$\sum_{i=1}^{r} A_i(t_1,t_2)\exp(x\epsilon_i(t_1+t_2)) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} p_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}.$$

Further simplification yields,

$$
\begin{aligned}
&\sum_{i=1}^{r} A_i(t_1, t_2)\exp(x\epsilon_i(t_1+t_2)) \\
&= \sum_{i=1}^{r}\left\{\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{(i)}_{n_1,n_2}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}\right\}\left\{\sum_{k=0}^{\infty}\frac{(x\epsilon_i)^k}{k!}(t_1+t_2)^k\right\} \\
&= \sum_{i=1}^{r}\sum_{n_1=0}^{\infty}\sum_{k_1=0}^{\infty}\left\{\sum_{n_2=0}^{\infty}\sum_{k_2=0}^{\infty} a^{(i)}_{n_1,n_2}\frac{(x\epsilon_i)^{k_2}}{k_2!}\frac{t_2^{k_2+n_2}}{n_2!}\right\}\frac{(x\epsilon_i)^{k_1}}{k_1!}\frac{t_1^{k_1+n_1}}{n_1!} \\
&= \sum_{i=1}^{r}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left\{\sum_{k_2=0}^{n_2}\sum_{k_1=0}^{n_1}\binom{n_1}{k_1}\binom{n_2}{k_2} a^{(i)}_{n_1-k_1,n_2-k_2}(\epsilon_i x)^{k_1+k_2}\right\}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}.
\end{aligned}
$$

On comparing the coefficients of both the sides, we get

$$
p_{n_1,n_2}(x) = \sum_{i=1}^{r}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\binom{n_1}{k_1}\binom{n_2}{k_2} a^{(i)}_{n_1,n_2}(\epsilon_i x)^{k_1+k_2},
$$

where not all $\{a^{(i)}_{0,0}\}$ are zero. □

**Theorem 2** *Let $\{p_{n_1,n_2}(x)\}_{n=0}^{\infty}$ be a sequence of class of Appell polynomial sets and there exists a sequence $\{a^{(i)}_{n_1,n_2}\}_{n_1,n_2=0}^{\infty}$ such that*

$$
p_{n_1,n_2}(x) = \sum_{i=1}^{r}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2} a^{(i)}_{n_1-k_1,n_2-k_2}(\epsilon_i x)^{k_1+k_2}, \tag{2.3}
$$

*where $\epsilon_1, \epsilon_2, \cdots, \epsilon_r$ are $r$ roots of unity and not all $\{a^{(i)}_{0,0}\}$ are zero. Then*

$$
p'_{n_1,n_2}(x) = n_1 p_{n_1-1,n_2}(x) + n_2 p_{n_1,n_2-1}(x)
$$

*for $n_1 \geq 0$, $n_2 \geq 0$.*

***Proof*** Let

$$
p_{n_1,n_2}(x) = \sum_{i=1}^{r}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2} a^{(i)}_{n_1-k_1,n_2-k_2}(\epsilon_i x)^{k_1+k_2} \tag{2.4}
$$

Since,

$$
(k_1+k_2)\binom{n_1}{k_1}\binom{n_2}{k_2} = n_1\binom{n_1-1}{k_1-1}\binom{n_2}{k_2} + n_2\binom{n_1}{k_1}\binom{n_2-1}{k_2-1} \quad k_1, k_2 \geq 1.
$$

 https://doi.org/10.1007/s13226-026-00968-w

On differentiating equation (2.3) with respect to $x$, we get

$$
\begin{aligned}
p'_{n_1,n_2}(x) &= \sum_{i=1}^{r}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}a^{(i)}_{n_1-k_1,n_2-k_2}(\epsilon_i)^{k_1+k_2}(k_1+k_2)(x)^{k_1+k_2-1}\\
&= \sum_{i=1}^{r}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}a^{(i)}_{n_1-k_1,n_2-k_2}(\epsilon_i)^{k_1+k_2}\left\{n_1\binom{n_1-1}{k_1-1}\binom{n_2}{k_2}+n_2\binom{n_1}{k_1}\binom{n_2-1}{k_2-1}\right\}(x)^{k_1+k_2-1}\\
&= n_1\sum_{i=1}^{r}\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2}\binom{n_1-1}{k_1}\binom{n_2}{k_2}a^{(i)}_{(n_1-1)-k_1,n_2-k_2}(\epsilon_i x)^{k_1+k_2}\\
&\quad + n_2\sum_{i=1}^{r}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2-1}\binom{n_1}{k_1}\binom{n_2-1}{k_2}a^{(i)}_{n_1-k_1,(n_2-1)-k_2}(\epsilon_i x)^{k_1+k_2}\\
&= n_1 p_{n_1-1,n_2}(x)+n_2 p_{n_1,n_2-1}(x).
\end{aligned}
$$

This completes the proof. □

**Theorem 3** *Let $\{p_{n_1,n_2}(x)\}_{n=0}^{\infty}$ be a generalized polynomial sets and $\{p_{n_1,n_2}(x)\}$ satisfies*

$$
p_{n_1,n_2}(x+y)=\sum_{i=1}^{r}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}p_{n_1-k_1,n_2-k_2}(x)(\epsilon_i y)^{k_1+k_2},
$$

*for $n_1 \geq 0$, $n_2 \geq 0$. Then $\{p_{n_1,n_2}(x)\}_{n=0}^{\infty}$ is a sequence of generalized Appell polynomial sets.*

***Proof*** Let

$$
p_{n_1,n_2}(x+y)=\sum_{i=1}^{r}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}p_{n_1-k_1,n_2-k_2}(x)(\epsilon_i y)^{k_1+k_2}.
$$

Therefore,

$$
\begin{aligned}
&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}p_{n_1,n_2}(x+y)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}\\
&\quad=\sum_{i=1}^{r}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left\{\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}p_{n_1-k_1,n_2-k_2}(x)(\epsilon_i y)^{k_1+k_2}\right\}\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!},\\
&\quad=\sum_{i=1}^{r}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}p_{n_1,n_2}(x)(\epsilon_i y)^{k_1+k_2}\frac{t_1^{n_1+k_1}t_2^{n_2+k_2}}{k_1!k_2!n_1!n_2!},\\
&\quad=\sum_{i=1}^{r}\left\{\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}p_{n_1,n_2}(x)\frac{t_1^{n_1}t_2^{n_2}}{n_1!n_2!}\right\}\left\{\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(\epsilon_i y)^{k_1+k_2}\frac{t_1^{k_1}t_2^{k_2}}{k_1!k_2!},\right\}
\end{aligned}
$$

This gives

$$
\sum_{i=1}^{r}A_i(t_1,t_2)\exp((x+y)\epsilon_i(t_1+t_2)).
$$

This completes the proof. □

**Acknowledgements** This work was carried out under the Post Doctoral Fellowship (Award letter No. 2/40(48)/2015/R&D-II/11575) awarded to the first author by the National Board of Higher Mathematics (NBHM), Department of Atomic Energy (DAE), Government of India, Mumbai.

## Declarations

**Conflicts of Interest** The authors declare no potential conflicts of interest.

## References

1. W. A. Al-Salam and A. Verma, Generalized Sheffer polynomials, Duke Math. J., Vol. 37 (1970), 361-365.
2. B. Alidad, On some problems of special functions and structural matrix analysis, Ph.D. diss., Aligarh Muslim University, 2008.
3. R. P. Boas, Jr. and R. C. Buck, Polynomial expansions of analytic functions, Rev. Ed., Academic Press, New York, 1964.
4. G. Bretti, C. Cesarano and P.E. Ricci, Laguerre-type exponentials and generalized Appell polynomials, Comp. Math. Appl., Vol. 48 (2004), 833-839.
5. J. W. Brown, On the Sheffer A-type of certain modified polynomial sets, Proc. Amer. Math. Soc., Vol. 23 (1969), 718-722.
6. T. S. Chihara, An Introduction to Orthogonal polynomials, Gordon and Breach, 1978.
7. W. N. Huff and E. D. Rainville, On the Sheffer A-type of polynomials generated by $\phi(t) f(xt)$, Proc. Amer. Math. Soc., Vol. 3 (1952), 296-299.
8. D. J. Galiffa , The Sheffer B-type 1 orthogonal polynomial sequences, Ph.D. diss., University of Central Florida, USA, 2009.
9. Subuhi Khan and Nusrat Raza, 2-iterated Appell polynomials and related numbers, Appl. Math. Comput., Vol. 219 (2013), 9469-9483.
10. Subuhi Khan, M.W. Al-Saad and Ghazala Yasmin, Some properties of Hermite-based Sheffer polynomials, Appl. Math. Comput., Vol. 217 (2010), 2169-2183.
11. D. W. Lee, On multiple Appell polynomials, Proc. Amer. Math. Soc., Vol. 139(6) (2011), 2133-2141.
12. V. B. Osegove, Some extremal properties of generalized Appell polynomials, Soviet Math. Vol. 5 (1964), 1651-1653.
13. E. D. Rainville, Special Functions, The Macmillan Company, New York, 1960.
14. S. J. Rapeli and A. K. Shukla, A note on $\alpha$ -type polynomial sets, Integral Transforms Spec. Funct., Vol. 25 (3) (2014), 249-253.
15. I. M. Sheffer, Some properties of polynomial sets of type zero, Duke Math. J., Vol. 5 (1939), 590-622.



## Authors and Affiliations

**S. J. Rapeli[1,2] · A. K. Shukla[2]**

A. K. Shukla
ajayshukla2@rediffmail.com; aks@amhd.svnit.ac.in

S. J. Rapeli
shrinu0711@gmail.com

[1] Department of Mathematics, M. B. Patel Science College, Anand 388001, Gujarat, India

[2] Department of Mathematics, Sardar Vallabhbhai National Institute of Technology, Surat 395007, Gujarat, India