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ORIGINAL RESEARCH

Some calculus rules, generalized convexity via convexifactors and their applications

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Abstract In this paper, convexifactors of product and quotient of functions are computed. Generalized convexity of these functions in terms of convexifactors is studied. The obtained results are then applied to develop optimality conditions for fractional programming problem.

Keywords Convexifactors · Calculus rules · Generalized convexity · Optimality conditions

Mathematics Subject Classification 90C26 · 90C30 · 90C46

1 Introduction

Convexifactors were introduced by Demyanov [3] in 1994. They can be regarded as generalized subdifferentials for scalar valued functions. Unlike well known subdifferentials such as Clarke subdifferential and Michel-Penot which are always convex and compact, convexifactors though always closed sets but are not necessarily convex or compact. These relaxations allow their application to a large class of nonsmooth problems. Clarke subdifferential, Michel-Penot and other well known subdifferentials for locally Lipschitz functions are convexifactors and they often contain convex hull of a convexifactor [4, 8]. Using convexifactors, we obtain sharp optimality conditions and duality results as in the form of a convexifactor we get a smaller set. Because of these, they are explored by many researchers. Jeyakumar and Luc [8] studied relationships of convexifactors with subdifferentials, developed various calculus rules including mean value theorem and conditions of minimality of convexifactors. They used quasimonotonicity of convexifactors to characterize quasiconvexity of a continuous function. Motivated by them, Dutta and Chandra [4], gave characterization of quasiconvex and pseudoconvex functions in terms of convexifactors. They deduced Karush-Kuhn-Tucker (KKT) type necessary optimality condition for an inequality constrained optimization problem and an optimality condition for a minimization problem involving a set inclusion constraint. Convexifactors were further studied by Dutta and Chandra [5], Li and Zhang [14], Suneja and Kohli [21–23], Kohli [10, 11], Jayswal et al. [7], Luu [15], Ardali et al. [2], Kabgani and Soleimani-damaneh [9]. Kohli [12] established KKT type necessary strong stationarity condition under Guignard CQ (GCQ) and also obtained sufficient optimality conditions using generalized asymptotic convexity in terms of convexifactors for mathematical programs with equilibrium constraints. Recently, convexifactors have been explored by Khare and Nath [13] and Rimpi and Lalitha [16].

Fractional programming problems, where the objective function is the ratio of functions are of great importance in the areas of engineering, economics, modelling, information theory and numerical analysis etc. For study of fractional programming problems one can see [6, 17–20, 23, 24].

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In this paper, our aim is to compute convexifactors of product and quotient of functions. Generalized convexity of these functions has also been investigated. Results obtained thus are then applied to develop optimality conditions for fractional programming problem. Here, we have worked in finite dimensional spaces.

The paper is organized as follows. In Section 2, we give definitions of convexifactors. Convexifactors of some functions have been computed in Section 3. Section 4 establishes generalized convexity of the above mentioned functions. In Section 5, the results obtained are applied to fractional programming problem.

2 Convexifactors

We begin this section with the definitions of convexifactors given by Dutta and Chandra [4].

Definition 2.1 An extended real valued function $h : \mathrm{IR}^n \to \mathrm{IR} \cup \{\pm\infty\}$ with $h(x)$ finite at $x \in \mathrm{IR}^n$ is said to have

(i). an upper convexifactor (UCF) $\partial^u h(x)$ at x iff $\partial^u h(x) \subseteq \mathrm{IR}^n$ is a closed set and

$$(\bar{h})_d(x, v) \leq \sup_{x^* \in \partial^u h(x)} \langle x^*, v \rangle, \text{ for all } v \in \mathrm{IR}^n.$$

(ii). a lower convexifactor (LCF) $\partial_l h(x)$ at x iff $\partial_l h(x) \subseteq \mathrm{IR}^n$ is a closed set and

$$(h)_d^+(x, v) \geq \inf_{x^* \in \partial_l h(x)} \langle x^*, v \rangle, \text{ for all } v \in \mathrm{IR}^n.$$

(iii). a convexifactor (CF) $\partial^* h(x)$ at x iff $\partial^* h(x)$ is both (UCF) and (LCF) of h at x.

(iv). an upper semiregular convexifactor (USRCF) $\partial^{us} h(x)$ at x iff $\partial^{us} h(x) \subseteq \mathrm{IR}^n$ is a closed set and

$$(h)_d^+(x, v) \leq \sup_{x^* \in \partial^{us} h(x)} \langle x^*, v \rangle, \text{ for all } v \in \mathrm{IR}^n.$$

When equality holds in above, $\partial^{us} h(x)$ is called an upper regular convexifactor (URCF) of h at x.

(v). a lower semiregular convexifactor (LSRCF) $\partial_{ls} h(x)$ at x iff $\partial_{ls} h(x) \subseteq \mathrm{IR}^n$ is a closed set and

$$(\bar{h})_d^-(x, v) \geq \inf_{x^* \in \partial_{ls} h(x)} \langle x^*, v \rangle, \text{ for all } v \in \mathrm{IR}^n.$$

When equality holds in above, $\partial_{ls} h(x)$ is called a lower regular convexifactor (LRCF) of h at x.

Here, $(h)_d^+(x, v) := \limsup_{t \to 0^+} \frac{h(x+tv)-h(x)}{t}$ and $(h)_d^-(x, v) := \liminf_{t \to 0^+} \frac{h(x+tv)-h(x)}{t}$ are upper and lower Dini derivatives of h at x in direction v respectively.

Both these Dini derivatives can be finite as well as infinite. If h is locally Lipschitz, both these are finite.

Throughout the paper, we denote by clA, convA and clconvA, for any set $A \subset \mathrm{IR}^n$, the closure, convex hull and the closed convex hull of A respectively.

3 Convexifactors of some functions

In this section, we shall develop some calculus rules using convexifactors.

Theorem 3.1. *Let $f, g : IR^n \to IR$ be continuous functions. Then, the function $fg : IR^n \to IR$ is defined by $(fg)(x) = f(x)g(x)$, for all $x \in IR^n$. Let $f(x) \geq 0$, $g(x) \geq 0$, for all $x \in IR^n$. Suppose that f and g admit USRCFs $\partial^{us} f(\overline{x})$ and $\partial^{us} g(\overline{x})$ respectively at $\overline{x}$. Then, $cl(f(\overline{x})\partial^{us} g(\overline{x}) + g(\overline{x})\partial^{us} f(\overline{x}))$ is an USRCF of fg at $\overline{x}$. Similarly, if f and g admit LSRCFs $\partial_{ls} f(\overline{x})$ and $\partial_{ls} g(\overline{x})$ respectively at $\overline{x}$, then, $cl(f(\overline{x})\partial_{ls} g(\overline{x}) + g(\overline{x})\partial_{ls} f(\overline{x}))$ is a LSRCF of fg at $\overline{x}$.*

If in the above, we take any one of the USRCF of f or g to be bounded, then $f(\overline{x})\partial^{us} g(\overline{x}) + g(\overline{x})\partial^{us} f(\overline{x})$ is an USRCF of fg at $\overline{x}$. Same is true for LSRCFs also.

Proof Consider

$$(fg)_d^+(\overline{x}, v) = \limsup_{t\downarrow 0} \frac{(fg)(\overline{x}+tv) - (fg)(\overline{x})}{t}.$$

$$\leq \limsup_{t\downarrow 0} \frac{f(\overline{x}+tv)[g(\overline{x}+tv) - g(\overline{x})]}{t} + \limsup_{t\downarrow 0} \frac{g(\overline{x})[f(\overline{x}+tv) - f(\overline{x})]}{t}$$

$$= f(\overline{x})(g)_d^+(\overline{x}, v) + g(\overline{x})(f)_d^+(\overline{x}, v).$$

$\leq \sup_{x^*\in\partial^{us} g(\overline{x})} \langle f(\overline{x})x^*, v\rangle + \sup_{x_1^*\in\partial^{us} f(\overline{x})} \langle g(\overline{x})x_1^*, v\rangle$, because $\partial^{us} g(\overline{x})$ and $\partial^{us} f(\overline{x})$ are USRCFs of g and f respectively at $\overline{x}$ and $f(\overline{x}) \geq 0$, $g(\overline{x}) \geq 0$.

$$= \sup_{z^*\in f(\overline{x})\partial^{us} g(\overline{x}) + g(\overline{x})\partial^{us} f(\overline{x})} \langle z^*, v\rangle = \sup_{z^*\in \mathrm{cl}(f(\overline{x})\partial^{us} g(\overline{x}) + g(\overline{x})\partial^{us} f(\overline{x}))} \langle z^*, v\rangle.$$

That is,

$$(fg)_d^+(\overline{x}, v) \leq \sup_{z^*\in \mathrm{cl}(f(\overline{x})\partial^{us} g(\overline{x}) + g(\overline{x})\partial^{us} f(\overline{x}))} \langle z^*, v\rangle.$$

Thus,
$\mathrm{cl}(f(\overline{x})\partial^{us} g(\overline{x}) + g(\overline{x})\partial^{us} f(\overline{x}))$ is an USRCF of fg at $\overline{x}$.
Similarly, we can prove that $\mathrm{cl}(f(\overline{x})\partial_{ls} g(\overline{x}) + g(\overline{x})\partial_{ls} f(\overline{x}))$ is a LSRCF of fg at $\overline{x}$. □

Theorem 3.2 *Let $f, g : IR^n \to IR$ be functions and g be continuous on IR^n . Then, the function $\frac{f}{g} : IR^n \to IR$ is defined by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, where $f(x) \geq 0$ and $g(x) > 0$, for all $x \in IR^n$, in particular for $\overline{x}$. Suppose that f admits an USRCF $\partial^{us} f(\overline{x})$ and $--g$ admits an UCF $\partial^u(-g)(\overline{x})$ at $\overline{x}$. Then, cl$\left(\frac{g(\overline{x})\partial^{us} f(\overline{x}) + f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$ is an USRCF of $\frac{f}{g}$ at $\overline{x}$. Similarly, if f admits a LSRCF $\partial_{ls} f(\overline{x})$ and $-g$ admits a LCF $\partial_l(-g)(\overline{x})$ at $\overline{x}$, then, cl $\left(\frac{g(\overline{x})\partial_{ls} f(\overline{x}) + f(\overline{x})\partial_l(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$ is a LSRCF of $\frac{f}{g}$ at $\overline{x}$. If in the above, we take any one of the USRCF, $\partial^{us} f(\overline{x})$ of f or UCF, $\partial^u(-g)(\overline{x})$ of g at $\overline{x}$ to be bounded, then $\frac{g(\overline{x})\partial^{us} f(\overline{x}) + f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}$ is an USRCF of $\frac{f}{g}$ at $\overline{x}$. Same is true for LSRCFs also.*

Proof Consider

$$\left(\frac{f}{g}\right)_d^+(\overline{x}, v)$$

$$= \limsup_{t\downarrow 0} \frac{\left(\frac{f}{g}\right)(\overline{x}+tv) - \left(\frac{f}{g}\right)(\overline{x})}{t}$$

$$\leq \frac{g(\overline{x})(f)_d^+(\overline{x}, v)}{[g(\overline{x})]^2} + \frac{(-f(\overline{x}))(g)_d^+(\overline{x}, v)}{[g(\overline{x})]^2}.$$

$= \frac{g(\overline{x})(f)_d^+(\overline{x}, v)}{[g(\overline{x})]^2} + \frac{f(\overline{x})(-g)_d^-(\overline{x}, v)}{[g(\overline{x})]^2}$, using (b), Theorem 2.8 [1]

$\leq \sup_{z^*\in \frac{g(\overline{x})\partial^{us} f(\overline{x}) + f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}} \langle z^*, v\rangle$ as f admits an USRCF $\partial^{us} f(\overline{x})$ and -g admits an UCF $\partial^u(-g)(\overline{x})$ *at* $\overline{x}$,

$= \sup_{z^*\in \mathrm{cl}\left(\frac{g(\overline{x})\partial^{us} f(\overline{x}) + f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}\right)} \langle z^*, v\rangle.$

That is,

$$\left(\frac{f}{g}\right)_d^+(\overline{x}, v) \leq \sup_{z^*\in \mathrm{cl}\left(\frac{g(\overline{x})\partial^{us} f(\overline{x}) + f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}\right)} \langle z^*, v\rangle.$$

Thus, cl$\left(\frac{g(\overline{x})\partial^{us} f(\overline{x}) + f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$ is an USRCF of $\frac{f}{g}$ at $\overline{x}$. Similarly, we can prove that cl $\left(\frac{g(\overline{x})\partial_{ls} f(\overline{x}) + f(\overline{x})\partial_l(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$ is a LSRCF of $\frac{f}{g}$ at $\overline{x}$. □

Now we state Proposition 1(ii) from Hejazi and Nobakhtian [6] in the form of the following theorem.

Theorem 3.3. *Let $\partial^{us} f(\overline{x})$ and $\partial^{us} g(\overline{x})$ be USRCFs of the functions $f, g : IR^n \to IR$ at $\overline{x}$. Then, $cl(\partial^{us} f(\overline{x}) + \partial^{us} g(\overline{x}))$ is an USRCF of $f+g$ at $\overline{x}$.*

Remark 3.1 If in the above, we take any one of the USRCFs $\partial^{us} f(\overline{x})$ or $\partial^{us} g(\overline{x})$ to be bounded, then, $\partial^{us} f(\overline{x}) + \partial^{us} g(\overline{x})$ is an USRCF of $f+g$ at $\overline{x}$.

4 Generalized convexity of functions

In this section, we shall establish generalized convexity of the class of nonnegative fractional functions and sum of functions studied in the previous section.

We begin by defining $\partial_{ls}-$ pseudoconvex, $\partial^{us}-$ pseudoconvex and $\partial^{u}-$ quasiconvex functions in terms of LSRCF, USRCF and UCFs respectively as defined by Suneja and Kohli [19] via convexifactors.

Definition 4.1 f is said to be $\partial_{ls}-$ pseudoconvex at $\overline{x}$ iff for all $x \in \mathrm{IR}^n$

$$f(x) < f(\overline{x}) \Rightarrow \left\langle \xi^*, x - \overline{x} \right\rangle < 0 \text{ for all } \xi^* \in \partial^{us} f(\overline{x}).$$

Definition 4.2 f is said to be $\partial^{us}-$ pseudoconvex at $\overline{x}$ iff for all $x \in \mathrm{IR}^n$

$$f(x) < f(\overline{x}) \Rightarrow \left\langle \xi^*, x - \overline{x} \right\rangle < 0 \text{ for all } \xi^* \in \partial^{us} f(\overline{x}).$$

Definition 4.3 f is said to be $\partial^{u}-$ quasiconvex at $\overline{x}$ iff for all $x \in \mathrm{IR}^n$

$$f(x) \leq f(\overline{x}) \Rightarrow \left\langle \xi^*, x - \overline{x} \right\rangle \geq 0 \text{ for all } \xi^* \in \partial^{u} f(\overline{x}).$$

Similarly, we can define $\partial^{u}-$ pseudoconvex function.

Now we define $\partial_{ls}-$ convex and $\partial_{l}-$ convex functions.

Definition 4.4 f is said to be $\partial_{ls}-$ convex at $\overline{x}$ iff for all $x \in \mathrm{IR}^n$

$$f(x) - f(\overline{x}) \geq \left\langle \xi^*, x - \overline{x} \right\rangle \text{ for all } \xi^* \in \partial_{ls} f(\overline{x}).$$

Definition 4.5 f is said to be $\partial_{l}-$ convex at $\overline{x}$ iff for all $x \in \mathrm{IR}^n$

$$f(x) - f(\overline{x}) \geq \left\langle \xi^*, x - \overline{x} \right\rangle \text{ for all } \xi^* \in \partial_{l} f(\overline{x}).$$

Theorem 4.1 *Let f admits a LSRCF $\partial_{ls} f(\overline{x})$ and $-g$ admits a LCF $\partial_l(-g)(\overline{x})$ at $\overline{x}$. Suppose that f and $-g$ are $\partial_{ls}-$ convex and ∂_l- convex with respect to $\partial_{ls} f(\overline{x})$ and $\partial_l(-g)(\overline{x})$ respectively at $\overline{x}$. Then, $\frac{f}{g}$ is $\partial_{ls}-$ pseudoconvex with respect to $\partial_{ls}\left(\frac{f}{g}\right)(\overline{x})$. Similarly, we can prove that $\frac{f}{g}$ is $\partial^{us}-$ pseudoconvex with respect to $\partial^{us}\left(\frac{f}{g}\right)(\overline{x})$. Let $\partial_{ls}\left(\frac{f}{g}\right)(\overline{x}) = cl\left(\frac{g(\overline{x})\partial_{ls} f(\overline{x}) + f(\overline{x})\partial_l(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$ and $\partial^{us}\left(\frac{f}{g}\right)(\overline{x}) = cl\left(\frac{g(\overline{x})\partial^{us} f(\overline{x}) + f(\overline{x})\partial^{u}(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$, where $f(x) \geq 0$ and $g(x) > 0$, for all $x \in IR^n$, in particular for $\overline{x}$.*

Proof We have to show that $\frac{f}{g}$ is $\partial_{ls}-$ pseudoconvex with respect to $\partial_{ls}\left(\frac{f}{g}\right)(\overline{x}) = \mathrm{cl}\left(\frac{g(\overline{x})\partial_{ls} f(\overline{x}) + f(\overline{x})\partial_l(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$ where $f(\overline{x}) \geq 0$, $g(\overline{x}) > 0$.

Let us suppose that

$$\left(\frac{f}{g}\right)(x) < \left(\frac{f}{g}\right)(\overline{x}). \tag{1}$$

To show that $\langle \xi, x - \overline{x} \rangle < 0$, for all $\xi \in \partial_{ls}\left(\frac{f}{g}\right)(\overline{x}) = \mathrm{cl}\left(\frac{g(\overline{x})\partial_{ls} f(\overline{x}) + f(\overline{x})\partial_l(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$

That is, there exist $\xi_n \in \frac{g(\overline{x})\partial_{ls} f(\overline{x}) + f(\overline{x})\partial_l(-g)(\overline{x})}{[g(\overline{x})]^2}$ such that $\xi = \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \frac{g(\overline{x})\xi_{1n} + f(\overline{x})\xi_{2n}}{[g(\overline{x})]^2}$ for some $\xi_{1n} \in \partial_{ls} f(\overline{x})$, $\xi_{2n} \in \partial_l(-g)(\overline{x})$.

As convexifactors are closed sets, $\lim_{n\to\infty} \xi_{1n} = \xi_1 \in \partial_{ls} f(\overline{x})$, $\lim_{n\to\infty} \xi_{2n} = \xi_2 \in \partial_l(-g)(\overline{x})$.

Therefore, $\xi = \frac{g(\overline{x})\xi_1 + f(\overline{x})\xi_2}{[g(\overline{x})]^2}$

Since f and $-g$ are $\partial_{ls}-$ convex and ∂_l- convex respectively, we have

$$f(x) - f(\overline{x}) \geq \langle \xi_1, x - \overline{x} \rangle, \text{for all}\, \xi_1 \in \partial_{ls} f(\overline{x}) \tag{2}$$

and

$$-g(x) + g(\overline{x}) \geq \langle \xi_2, x - \overline{x} \rangle, \text{for all } \xi_2 \in \partial_l(-g)(\overline{x}) \tag{3}$$

As $g(x) > 0$ and $f(x) \geq 0$, for all $x \in \mathrm{IR}^n$, in particular for $\overline{x}$, we have from (2) and (3) respectively

$$f(x)g(\overline{x}) - f(\overline{x})g(\overline{x}) \geq \langle g(\overline{x})\xi_1, x - \overline{x} \rangle, \text{ for all}\, \xi_1 \in \partial_{ls} f(\overline{x}) \tag{4}$$

$$-f(\overline{x})g(x) + f(\overline{x})g(\overline{x}) \geq \langle f(\overline{x})\xi_2, x - \overline{x} \rangle, \text{ for all}\, \xi_2 \in \partial_l - g(\overline{x}) \tag{5}$$

Adding (4), (5) and using (1), we get

$$\left\langle \frac{g(\overline{x})\xi_1 + f(\overline{x})\xi_2}{[g(\overline{x})]^2}, x - \overline{x} \right\rangle < 0, \text{ for all}\, \xi_1 \in \partial_{ls} f(\overline{x}),\ \xi_2 \in \partial_l(-g)(\overline{x}),$$

$$\Rightarrow \langle \xi, x - \overline{x} \rangle < 0, \text{ for all } \xi \in \partial_{ls}\left(\frac{f}{g}\right)(\overline{x}) = \mathrm{cl}\left(\frac{g(\overline{x})\partial_{ls} f(\overline{x}) + f(\overline{x})\partial_l(-g)(\overline{x})}{[g(\overline{x})]^2}\right).$$

Hence proved.

Similarly, we can prove that $\frac{f}{g}$ is $\partial^{us}-$ pseudoconvex with respect to $\partial^{us}\left(\frac{f}{g}\right)(\overline{x}) = \mathrm{cl}\left(\frac{g(\overline{x})\partial^{us} f(\overline{x}) + f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$. □

Theorem 4.2 *Let f admits a LSRCF $\partial_{ls} f(\overline{x})$ and g admits a LSRCF $\partial_{ls} g(\overline{x})$ at $\overline{x}$. Suppose that f and g are $\partial_{ls}-$ convex with respect to $\partial_{ls} f(\overline{x})$ and $\partial_{ls} g(\overline{x})$ respectively at $\overline{x}$. Then, $f+g$ is $\partial_{ls}-$ pseudoconvex with respect to $\partial_{ls}(f+g)(\overline{x})$. Similarly, we can prove that $f+g$ is $\partial^{us}-$ pseudoconvex with respect to $\partial^{us}(f+g)(\overline{x})$. Let $\partial_{ls}(f+g)(\overline{x}) = cl(\partial_{ls} f(\overline{x}) + \partial_{ls} g(\overline{x}))$ and $\partial^{us}(f+g)(\overline{x}) = cl(\partial^{us} f(\overline{x}) + \partial^{us} g(\overline{x}))$.*

Proof To show that $f+g$ is $\partial_{ls}-$ pseudoconvex with respect to $\partial_{ls}(f+g)(\overline{x})$.

Suppose that

$$(f+g)(x) < (f+g)(\overline{x}). \tag{6}$$

To show that $\langle \xi, x - \overline{x} \rangle < 0$ for all $\xi \in \partial_{ls}(f+g)(\overline{x}) = \mathrm{cl}(\partial_{ls} f(\overline{x}), +\partial_{ls} g(\overline{x}))$.

That is, there exists $\xi_n \in \partial_{ls} f(\overline{x}) + \partial_{ls}(g)(\overline{x})$ such that $\xi = \lim_{n\to\infty} \xi_n = \lim_{n\to\infty}(\xi_{1n} + \xi_{2n})$ for some $\xi_{1n} \in \partial_{ls} f(\overline{x})$, $\xi_{2n} \in \partial_{ls} g(\overline{x})$.

As convexifactors are closed sets, $\lim_{n\to\infty} \xi_{1n} = \xi_1 \in \partial_{ls} f(\overline{x})$, $\lim_{n\to\infty} \xi_{2n} = \xi_2 \in \partial_{ls} g(\overline{x})$.

Therefore, $\xi = \xi_1 + \xi_2$

Since f and g are $\partial_{ls}-$ convex, we have

$$f(x) - f(\overline{x}) \geq \langle \xi_1, x - \overline{x} \rangle, \text{for all } \xi_1 \in \partial_{ls} f(\overline{x}) \tag{7}$$

and

$$g(x) - g(\overline{x}) \geq \langle \xi_2, x - \overline{x} \rangle, \text{ for all } \xi_2 \in \partial_{ls} g(\overline{x}). \tag{8}$$

Adding (7), (8) and using (6), we get

$\langle \xi_1 + \xi_2, x - \overline{x} \rangle < 0$, for all $\xi_1 \in \partial_{ls} f(\overline{x})$ and $\xi_2 \in \partial_{ls} g(\overline{x})$.

That is, $\langle \xi, x - \overline{x} \rangle < 0$, for all $\xi \in \partial_{ls}(f+g)(\overline{x}) = \mathrm{cl}(\partial_{ls} f(\overline{x}) + \partial_{ls} g(\overline{x}))$.

Hence proved. □

Similarly, we can prove that $f+g$ is $\partial^{us}-$ pseudoconvex with respect to $\partial^{us}(f+g)(\overline{x}) = \mathrm{cl}(\partial^{us} f(\overline{x}) + \partial^{us} g(\overline{x}))$.

Remark 4.1. If f admits a LSRCF $\partial_{ls} f(\overline{x})$ and $-g$ admits a LCF $\partial_l(-g)(\overline{x})$ at $\overline{x}$. Then, by Theorem 3.2, cl $\left(\frac{g(\overline{x})\partial_{ls} f(\overline{x}) + f(\overline{x})\partial_l(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$ is a LSRCF of $\frac{f}{g}$ at $\overline{x}$. Similarly, we have for USRCFs also. Suppose that f admits a LSRCF $\partial_{ls} f(\overline{x})$ and g admits a LSRCF $\partial_{ls} g(\overline{x})$ at $\overline{x}$. Then, by Theorem 3.3, $\mathrm{cl}(\partial_{ls} f(\overline{x}) + \partial_{ls} g(\overline{x}))$ is a LSRCF of $f+g$ at $\overline{x}$. Similarly, we have for USRCFs also.

5 Application to optimization problems

In this section, the results established in previous sections are applied to develop optimality conditions for fractional programming problem (P) given as below:

$$\text{(P)} \qquad \text{Minimize}\left(\frac{f}{g}\right)(x)$$
$$\text{subject to} \qquad h_i(x) \leq 0 \; i = 1, 2, ..., m, \; x \in \mathrm{IR}^n$$

where $f, g, h_i : \mathrm{IR}^n \to \mathrm{IR}$, $i = 1, 2, ..., m$.

Before proceeding further, we shall now give Slater-type weak constraint qualification [23] which will be used for the development of our results.

Definition 5.1 The function h is said to satisfy the Slater-type weak constraint qualification at a feasible point $\overline{x}$ if h_I is ∂^u- pseudoconvex at $\overline{x}$, and there exists an $x_0 \in \mathrm{IR}^n$, such that $h_I(x_0) < 0$ where $I = \{i \in \{1, 2, ..., m\} : h_i(\overline{x}) = 0\}$.

Theorem 5.1 *Let $\overline{x}$ be a local optimal solution to problem (P) and $f(x) \geq 0, g(x) > 0$ for all $x \in IR^n$. Suppose that f admits a bounded USRCF $\partial^{us} f(\overline{x})$, $-g$ and h_i, $i = 1, 2, ..., m$ admit bounded UCFs $\partial^u(-g)(\overline{x}), \partial^u h_i(\overline{x})$, $i = 1, 2, ..., m$ respectively at $\overline{x}$. Let f, g, h_i, $i = 1, 2, ..., m$ be continuous functions on IR^n and $\partial^{us} f, \partial^u(-g)$ and $\partial^u h_i$, $i = 1, 2, ..., m$ be upper semicontinuous at $\overline{x}$. Let $\partial^{us}\left(\frac{f}{g}\right)(\overline{x}) = cl \left(\frac{g(\overline{x})\partial^{us} f(\overline{x})+f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}\right) = \frac{g(\overline{x})\partial^{us} f(\overline{x})+f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}$. Suppose that Slater-type weak constraint qualification holds at $\overline{x}$. Then, there exist scalars $\lambda_1 > 0$, $\mu_i \geq 0$, $i = 1, 2, ..., m$ with $\lambda_1 + \sum_{i=1}^{m} \mu_i = 1$ such that*

(i) $0 \in \frac{\lambda_1 g(\bar{x}) conv \partial^{us} f(\bar{x}) + \lambda_1 f(\bar{x}) conv \partial^u(-g)(\bar{x})}{[g(\bar{x})]^2} + \sum_{i=1}^{m} \mu_i conv \partial^u h_i(\bar{x})$.

(ii) $\mu_i h_i(\bar{x}) = 0, \; i = 1, 2, ..., m$.

Similar is the case when f admits a bounded LSRCF $\partial_{ls} f(\overline{x})$, $-g$ and h_i, $i = 1, 2, ..., m$ admit bounded LCFs $\partial_l(-g)(\overline{x})$, $\partial_l h_i(\overline{x})$, $i = 1, 2, ..., m$ respectively at $\overline{x}$.

Proof Since $\overline{x}$ is a local optimal solution to (P), it solves the following unconstrained problem

$$\text{Minimize} \quad F(x), \; x \in \mathrm{IR}^n$$

where $F(x) = \max\left\{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(\overline{x}), h_1(x), ..., h_m(x)\right\}$,

Using Theorem 3.2 [4], we get

$$\partial^{us} F(\overline{x}) = \text{conv}\left\{\partial^{us}\left(\frac{f}{g}\right)(\overline{x}) \cup \left(\bigcup_{i \in I} \partial^u h_i(\overline{x})\right)\right\},$$

where $I = \{i \in \{1, 2, ..., m\} : h_i(\overline{x}) = 0\}$.

Then, by Proposition 4.1 [8],

$$0 \in \text{clconv}\partial^{us} F(\overline{x}) = \text{clconv}\left\{\partial^{us}\left(\frac{f}{g}\right)(\overline{x}) \cup \left(\bigcup_{i \in I} \partial^u h_i(\overline{x})\right)\right\}.$$

As $\text{clconv}(S) = \text{conv}(\text{clS})$, for a bounded set S in IR^n, we have

$$0 \in \text{conv}\left\{\partial^{us}\left(\frac{f}{g}\right)(\overline{x}) \cup \left(\bigcup_{i \in I} \partial^u h_i(\overline{x})\right)\right\}.$$

Convexifactors are in general non convex sets and there exist scalars $\lambda_1 \geq 0, \mu_i \geq 0, i \in I$ with $\lambda_1 + \sum_{i \in I} \mu_i = 1$ such that

$0 \in \lambda_1 \text{conv}\partial^{us}\left(\frac{f}{g}\right)(\overline{x}) + \sum_{i \in I} \mu_i \text{conv}\partial^u h_i(\overline{x})$ and hence

$$0 \in \lambda_1 \text{conv}\left(\frac{g(\overline{x})\partial^{us} f(\overline{x}) + f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}\right) + \sum_{i \in I} \mu_i \text{conv}\partial^u h_i(\overline{x}).$$

Using convex hull properties, $\text{conv}(S_1 + S_2) = \text{conv}S_1 + \text{conv}S_2$ and $\text{conv}(\lambda S) = \lambda\text{conv}S$, of subsets S_1, S_2 and S of IR^n, we have

$$0 \in \frac{\lambda_1 g(\overline{x})\text{conv}\partial^{us} f(\overline{x}) + \lambda_1 f(\overline{x})\text{conv}\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2} + \sum_{i \in I} \mu_i \text{conv}\partial^u h_i(\overline{x}),$$

with $\lambda_1 + \sum_{i \in I} \mu_i = 1$

Now $h_i(\overline{x}) = 0$, $i \in I$, therefore $\mu_i h_i(\overline{x}) = 0$ for $i \in I$. For $i \notin I$, $h_i(\overline{x}) < 0$ so let $\mu_i = 0$, $i \notin I$, then $\mu_i h_i(\overline{x}) = 0$, $i = 1, 2, ..., m$.

Thus, we have

$$0 \in \frac{\lambda_1 g(\overline{x})\text{conv}\partial^{us} f(\overline{x}) + \lambda_1 f(\overline{x})\text{conv}\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2} + \sum_{i=1}^{m} \mu_i \text{conv}\partial^u h_i(\overline{x}).$$

$$\mu_i h_i(\overline{x}) = 0, \ i = 1, 2, ..., m, \ \text{with } \lambda_1 + \sum_{i=1}^{m} \mu_i = 1.$$

Suppose that $\lambda_1 = 0$, then using above, we get that there exists $\xi_i \in \text{conv}\partial^u h_i(\overline{x})$, $i = 1, 2, ..., m$ such that

$$\sum_{i=1}^{m} \mu_i \xi_i = 0. \tag{9}$$

Since h satisfies Slater-type weak constraint qualification at $\overline{x}$, there exists an $x_0 \in \text{IR}^n$, such that $h_i(x_0) < h_i(\overline{x})$, $i \in I$, which on using ∂^u- pseudoconvexity of h_i, $i \in I$, gives

$\left\langle \sum_{i=1}^{m} \mu_i \xi_i, x_0 - \overline{x} \right\rangle < 0$ for all $\xi_i \in \partial^u h_i(\overline{x})$, $\mu_i \geq 0$, $i \in I$ and $\mu_i = 0$, $i \notin I$.

This is contradiction to (9).

Hence result. □

Remark 5.1 If f admits an USRCF $\partial^{us} f(\overline{x})$ and $-g$ admits an UCF $\partial^u(-g)(\overline{x})$ at $\overline{x}$. Then, by Theorem 3.2, cl $\left(\frac{g(\overline{x})\partial^{us} f(\overline{x})+f(\overline{x})\partial^u(-g)(\overline{x})}{[g(\overline{x})]^2}\right)$ is an USRCF of $\frac{f}{g}$ at $\overline{x}$. Similarly, we have for LSRCFs also.

Following example illustrates above theorem.

Example 5.1 Consider the problem

$$\begin{aligned} \text{Minimize} \quad & \left(\frac{f}{g}\right)(x) \\ \text{subject to} \quad & h(x) \leq 0 \end{aligned}$$

where $f, g, h : \text{IR} \to \text{IR}$ are the functions defined as follows:

$$f(x) := 1 + \frac{|x|}{2}, \ g(x) := 1 + |x|, \ h(x) := x^2 + x$$

$$\left(\frac{f}{g}\right)(x) = \frac{1 + \frac{|x|}{2}}{1 + |x|}.$$

$\overline{x} = 0$ is an optimal solution of the problem.

It can be seen that f admits an USRCF $\partial^{us} f(0) = \{\frac{-1}{2}, \frac{1}{2}\}$, $-g$ admits an UCF $\partial^{u}(-g)(0) = \{-1, 1\}$, and h admits an UCF $\partial^{u} h(0) = \{1, 2\}$ at 0.

We can see that $\frac{f}{g}$ admits an USRCF $\partial^{us}\left(\frac{f}{g}\right)(0) = \frac{g(0)\partial^{us} f(0)+f(0)\partial^{u}(-g)(0)}{[g(0)]^2} = \left\{\frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}\right\}$.

Slater-type weak constraint qualification holds at $\overline{x} = 0$.

Then, there exist scalars $\lambda_1 = \frac{1}{2}$, $\mu = \frac{1}{2}$ with $\lambda_1 + \mu = 1$ such that

(i) $0 \in \frac{\lambda_1 g(0)\text{conv}\partial^{us} f(0)+\lambda_1 f(0)\text{conv}\partial^{u}(-g)(0)}{[g(0)]^2} + \mu\text{conv}\partial^{u} h(0)$

(ii) $\mu h(0) = 0$

Theorem 5.2 *Let $\overline{x}$ be a feasible solution to the problem (P). Suppose that f admits a bounded USRCF $\partial^{us} f(\overline{x})$, $-g$ and h_i, $i = 1, 2, ..., m$ admit bounded UCFs $\partial^{u}(-g)(\overline{x})$, $\partial^{u} h_i(\overline{x})$, $i = 1, 2, ..., m$ respectively at $\overline{x}$. Let $\partial^{us}\left(\frac{f}{g}\right)(\overline{x}) = \frac{g(\overline{x})\partial^{us} f(\overline{x})+f(\overline{x})\partial^{u}(-g)(\overline{x})}{[g(\overline{x})]^2}$. Assume that $\frac{f}{g}$ is $\partial^{us}-$ pseudoconvex, h_i,$i \in I$ are $\partial^{u}-$ quasi-convex functions. Suppose that there exist scalars $\lambda_1 > 0$, $\mu_i \geq 0$, $i = 1, 2, ..., m$ such that the conditions (i) and (ii) (Theorem 5.1) are satisfied. Then, $\overline{x}$ is a local optimal solution to the problem (P).*

Proof Suppose that $\overline{x}$ is not a local optimal solution to the problem (P), then there exists a feasible solution x such that

$$\left(\frac{f}{g}\right)(x) < \left(\frac{f}{g}\right)(\overline{x})$$

Using $\partial^{us}-$ pseudoconvexity of $\frac{f}{g}$ at $\overline{x}$, we have

$\langle \xi\prime, x - \overline{x}\rangle < 0$ for all $\xi\prime \in \partial^{us}\left(\frac{f}{g}\right)(\overline{x})$.

It follows that

$\langle \xi, x - \overline{x}\rangle < 0$ for all $\xi \in \text{conv}\partial^{us}\left(\frac{f}{g}\right)(\overline{x})$, as ξ can be expressed as convex combination of finite number of elements of $\partial^{us}\left(\frac{f}{g}\right)(\overline{x})$, which gives on using convex hull properties, $\text{conv}(S_1 + S_2) = \text{conv}S_1 + \text{conv}S_2$ and $\text{conv}(\lambda S) = \lambda\text{conv}S$, of subsets S_1, S_2 and S of IR^n, and multiplying above by $\lambda_1 > 0$, that

$$\left\langle \frac{\lambda_1 g(\overline{x})\xi_1 + \lambda_1 f(\overline{x})\xi_2}{[g(\overline{x})]^2}, x - \overline{x}\right\rangle < 0, \text{ for all } \xi_1 \in \text{conv}\partial^{us} f(\overline{x}),\ \xi_2 \in \text{conv}\partial^{u}(-g)(\overline{x}). \tag{10}$$

Since, $\overline{x}$ is feasible for (P), by given condition (i), there exist $\xi_1 \in \text{conv}\partial^{us} f(\overline{x})$, $\xi_2 \in \text{conv}\partial^{u}(-g)(\overline{x})$, $\xi_{3i} \in \text{conv}\partial^{u} h_i(\overline{x})$, $i = 1, 2, ..., m$ such that for any feasible solution x of (P), we have

$$\left\langle \frac{\lambda_1 g(\overline{x})\xi_1 + \lambda_1 f(\overline{x})\xi_2}{[g(\overline{x})]^2} + \sum_{i=1}^{m} \mu_i \xi_{3i}, x - \overline{x}\right\rangle = 0. \tag{11}$$

Now, since, $\overline{x}$ is feasible for (P),

$$h_i(x) \leq h_i(\overline{x}), i \in I.$$

$\partial^{u}-$ quasiconvexity of h_i, $i \in I$, gives

$\langle \xi_{3i}, x - \overline{x}\rangle \leq 0$ for all $\xi_{3i} \in \text{conv}\partial^{u} h_i(\overline{x})$, $i \in I$, as ξ_{3i} can be expressed as convex combination of finite number of elements of $\partial^{u} h_i(\overline{x})$, $i \in I$.

Multiplying above by $\mu_i \geq 0$, $i \in I$, adding and since $\mu_i h_i(\overline{x}) = 0$, $i = 1, 2, ..., m$, $h_i(\overline{x}) = 0$, $i \in I$ and for $i \notin I$, $h_i(\overline{x}) < 0$, taking $\mu_i = 0$, $i \notin I$, we get

$$\left\langle \sum_{i=1}^{m} \mu_i \xi_{3i}, x - \overline{x}\right\rangle \leq 0 \text{ for all } \xi_{3i} \in \text{conv}\partial^{u} h_i(\overline{x}),\ i = 1, 2, ..., m. \tag{12}$$

Adding (10) and (12), we get

$$\left\langle \frac{\lambda_1 g(\overline{x})\xi_1 + \lambda_1 f(\overline{x})\xi_2}{[g(\overline{x})]^2} + \sum_{i=1}^{m} \mu_i \xi_{3i}, x - \overline{x} \right\rangle < 0$$

which contradicts (11).

Hence $\overline{x}$ is a local optimal solution to the problem (P). □

Now we give following example to illustrate above theorem.

Example 5.2. Consider the problem

$$\begin{array}{ll} \text{Minimize} & \left(\frac{f}{g}\right)(x) \\ \text{subject to} & h(x) \leq 0 \end{array}$$

where $f, g, h : \mathrm{IR} \to \mathrm{IR}$ are the functions defined as follows:

$$f(x) := 1 + x^2 + |x|,\ g(x) := 1 + \frac{|x|}{2},\ h(x) := x^2 + x$$

$$\left(\frac{f}{g}\right)(x) = \frac{1 + x^2 + |x|}{1 + \frac{|x|}{2}}$$

It can be seen that f admits an USRCF $\partial^{us} f(0) = \{-1, 1\}$, $-g$ admits an UCF $\partial^u(-g)(0) = \left\{\frac{-1}{2}, \frac{1}{2}\right\}$, and h admits an UCF $\partial^u h(0) = \{1, 2\}$ at 0.

We can see that $\frac{f}{g}$ admits an USRCF $\partial^{us}\left(\frac{f}{g}\right)(0) = \frac{g(0)\partial^{us} f(0) + f(0)\partial^u(-g)(0)}{[g(0)]^2} = \left\{\frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}\right\}$.

$\frac{f}{g}$ is $\partial^{us}-$ pseudoconvex w.r.t. $\partial^{us}\left(\frac{f}{g}\right)(0) = \left\{\frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}\right\}$,$h$ is ∂^u- quasiconvex w.r.t. $\partial^u h(0) = \{1, 2\}$ at $\overline{x} = 0$.

Then, there exist scalars $\lambda_1 = \frac{1}{2}$, $\mu = \frac{1}{2}$ with $\lambda_1 + \mu = 1$ such that

(i) $0 \in \frac{\lambda_1 g(0)\mathrm{conv}\partial^{us} f(0) + \lambda_1 f(0)\mathrm{conv}\partial^u(-g)(0)}{[g(0)]^2} + \mu\mathrm{conv}\partial^u h(0)$

(ii) $\mu h(0) = 0$

Then, $\overline{x} = 0$ is an optimal solution of the problem.

6 Conclusions

In this paper, we have obtained some calculus rules for computing convexifactors of product and quotient of functions. After that, generalized convexity of these functions has been established. At the end, we have applied these results to develop necessary and sufficient optimality conditions for fractional programming problems.

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