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ORIGINAL RESEARCH

On some properties of ω-uniqueness in tensor complementarity problem

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Abstract In this article, we introduce column adequate tensor in the context of tensor complementarity problem and consider some important properties. The tensor complementarity problem is a class of nonlinear complementarity problem with the involved function being defined by a tensor. We establish the inheritance property and invariance property of column adequate tensors. We study TCP(q, $\boldsymbol{A}$) in the frame of LCP($\bar{q}$, $\bar{A}$). We show that TCP(q, $\boldsymbol{A}$) has ω-unique solution under some assumptions.

Keywords Tensor complementarity problem · Column adequate tensor · Column sufficient tensor · P_0-tensor · ω-unique solution

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1 Introduction

The tensor complementarity problem is a class of nonlinear complementarity problem with the involved function being a special polynomial defined by a tensor. The tensor complementarity problem was initially studied by Song et al. [1]. During the last several years, the tensor complementarity problem attains much attraction and has been studied extensively with respect to theory, solution methods and applications. In recent times various tensors with special structures are studied. For details see [1–7]. Now we define the complementarity problem: For a given mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ the complementarity problem is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \text{and } x^T F(x) = 0. \tag{1}$$

If F is nonlinear mapping, then the problem (1) is called a nonlinear complementarity problem [8], and if F is linear function, then the problem (1) reduces to a linear complementarity problem [9]. The linear complementarity problem may be defined as follows:

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Given a matrix $M \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem [9], denoted by LCP(q, M), is to find a pair of vectors $w, z \in \mathbb{R}^n$ such that

$$z \geq 0, \quad w = Mz + q \geq 0, \quad z^T w = 0. \tag{2}$$

A pair of vectors (w, z) satisfying (2) is called a solution of the LCP(q, M). A vector z is called a z-solution if there exists a vector w such that (w, z) is a solution of the LCP(q, M). Similarly vector w is called a w-solution if there exists a vector z such that (w, z) is a solution of the LCP(q, M). The solution (w, z) is said to be w-unique if all solution pairs (w, z), have the same w, for a given q. Ingleton [10], [11] studied the w-uniqueness of solutions to linear complementarity problem.
The algorithm based on principal pivot transform namely Lemke's algorithm, Criss-cross algorithm which are used to find the solutions of linear complementarity problem are studied extensively considering several matrix classes. For details, see [12–19]. In this context properties of matrix classes ensure the processability of LCP(q, M). For details see [14,20–24]. Complementarity theory helps to establish the connection between KKT point and the optimal point for structured stochastic game and multiobjective programming problem. For details see [25–32].
Now we consider the case $F(x) = \mathcal{A}x^{m-1} + q$ with $\mathcal{A} \in T_{m,n}$ and $q \in \mathbb{R}^n$ then the problem (1) becomes

$$x \geq 0, \quad \mathcal{A}x^{m-1} + q \geq 0, \quad \text{and} \quad x^T(\mathcal{A}x^{m-1} + q) = 0 \tag{3}$$

which is called a tensor complementarity problem, denoted by the TCP$(q, \mathcal{A})$. Given $\omega = \mathcal{A}x^{m-1} + q$, the tensor complementarity problem is to find x such that

$$x \geq 0, \quad \omega = \mathcal{A}x^{m-1} + q \geq 0, \quad \text{and} \quad x^T\omega = 0. \tag{4}$$

A pair of vectors (ω, x) satisfying (4) is called a solution of the TCP$(q, \mathcal{A})$. A vector x is called a x-solution if there exists a vector ω such that (ω, x) is a solution of the TCP$(q, \mathcal{A})$. Then we define the vector ω as ω-solution if there exists a vector x such that (ω, x) is a solution of the TCP$(q, \mathcal{A})$. The solution (ω, x) is said to be ω-unique if all solution pairs (ω, x), have the same ω, for a given q.
Motivated by the discussion on positive definiteness of multivariate homogeneous polynomial forms [33–35], Qi [36] introduced the concept of symmetric positive definite (positive semidefinite) tensors. Song et al. [6] studied $P(P_0)$-tensors.
Bai et al. [2] established the global uniqueness of solution of TCP$(q, \mathcal{A})$ for strong P-tensor. Ding et al. [37] introduced a class of P-tensor and studied the properties of TCP$(q, \mathcal{A})$. The properties of the several classes of Q-tensors were presented by Huang et al. [38]. In this paper we introduce column adequate tensor in the context of tensor complementarity problem and study different properties of this tensor.

The paper is organized as follows. Section 2 presents some basic notations and results. In section 3, we introduce the column adequate tensors and study tensor theoretic properties as well as the properties of ω-solution of TCP$(q, \mathcal{A})$. In this context, we consider auxiliary matrix formation. We establish the inheritance property and invariance property of column adequate tensors.

2 Preliminaries

We consider tensors, matrices and vectors with real entries. For any positive integer n, $[n]$ denotes set $\{1, 2, ..., n\}$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x > 0\}$. $\mathbb{Z}$ denotes the set of integers and $\mathbb{Z}^n_+ = \{x \in \mathbb{Z}^n : x \geq 0\}$. Any vector $x \in \mathbb{R}^n$ is a column vector and x^T denotes the row transpose of x. A^c denotes the complement of set A. For $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Z}^n_+$, $|\alpha|$ be defined as $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. For $x \in \mathbb{R}^n$, let $x^{[m]} \in \mathbb{R}^n$ with its ith component being x_i^m for all $i \in [n]$. If V is a finite-dimensional vector space, an ordered basis for V is a finite sequence of vectors that is linearly independent and spans V. For details see [39]. A diagonal matrix $D = [d_{ij}]_{n\times n} = diag(d_1,\ d_2,\ ...,\ d_n)$ is defined as

$$d_{ij} = \begin{cases} d_i \ ; \ \forall\ i = j, \\ 0 \ ; \ \forall\ i \neq j. \end{cases}$$

Given a matrix $M \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, we define the feasible set FEA$(q, M) = \{z \in \mathbb{R}^n : z \geq 0,\ Mz + q \geq 0\}$ and the solution set of LCP(q, M) by SOL$(q, M) = \{z \in$ FEA$(q, M) : z^T(q + Mz) = 0\}$.
Adequate matrices play an important role in the study of linear complementarity problem. Cottle [40], Eaves [41] and Ingleton [10,11] studied several properties of column adequate matrices in context of linear complementarity problem. We start with the definition of column adequate matrices.

Definition 1 [10] A matrix $M \in \mathbb{R}^{n\times n}$ is said to be a column adequate matrix if

$$x_i(Mx)_i \leq 0 \text{ for all } i = 1, 2, ..., n \implies Mx = 0.$$

The following theorem gives an equivalent condition for the w-unique solution of LCP(q, M).

Theorem 1 *[10,11] Let M be a matrix of order n. Then the following conditions are equivalent:*

(a) M is a column adequate matrix
(b) For all $q \in \mathbb{R}^n$, for which (2) is solvable, the solution is w-unique.

Now we consider some ordering on monomials. $\mathbb{R}[x_1, ..., x_n]$ denotes the set of all polynomials in $x_1,\ x_2\ ...\ x_n$ with coefficients in $\mathbb{R}$. Any monomial in $\mathbb{R}[x_1, ..., x_n]$ is denoted by $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where the n-tuple of exponents $\alpha = (\alpha_1 \cdots \alpha_n) \in \mathbb{Z}^n_+$. There is a one-to-one correspondence between the monomials in $\mathbb{R}[x_1, ..., x_n]$ and $\mathbb{Z}^n_+$. Furthermore, any ordering $>$ on the space $\mathbb{Z}^n_+$ provides an ordering on monomials i.e. for $\alpha, \beta \in \mathbb{Z}^n_+$ if $\alpha > \beta$ according to this ordering implies $x^\alpha > x^\beta$.

Definition 2 [42] (Lexicographic Order). Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ be in $\mathbb{Z}^n_+$. We say $\alpha >_{lex} \beta$ if the leftmost non-zero entry of the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive. We write $x^\alpha >_{lex} x^\beta$ if $\alpha >_{lex} \beta$.

Example 1 Let $\alpha = (1, 2, 0)$ and $\beta = (0, 3, 4)$. Then $(1, 2, 0) >_{lex} (0, 3, 4)$, since $\alpha - \beta = (1, -1, -4)$.

Example 2 Let $\alpha = (3, 2, 4)$ and $\beta = (3, 2, 1)$. Then $(3, 2, 4) >_{lex} (3, 2, 1)$, since $\alpha - \beta = (0, 0, 3)$.

Example 3 Naturally $x_1 >_{lex} x_2 >_{lex} \cdots >_{lex} x_n$.

Definition 3 [42] (Graded Lexicographic Order). Let $\alpha, \beta \in \mathbb{Z}^n_+$. We say $\alpha >_{grlex} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \text{ or } |\alpha| = |\beta| \text{ and } \alpha >_{lex} \beta.$$

Example 1 Let $\alpha = (1, 2, 3)$ and $\beta = (3, 2, 0)$. Then $(1, 2, 3) >_{grlex} (3, 2, 0)$, since $|(1, 2, 3)| = 6 > |(3, 2, 0)| = 5$.

Example 2 Let $\alpha = (1, 2, 4)$ and $\beta = (1, 1, 5)$. Then $(1, 2, 4) >_{grlex} (1, 1, 5)$, since $|(1, 2, 4)| = |(1, 1, 5)|$ and $(1, 2, 4) >_{lex} (1, 1, 5)$.

Example 3 Consider the monomials $x_1^2,\ x_2^2,\ x_3^2,\ x_1x_2,\ x_2x_3,\ x_3x_1$ which represent the vectors $(2, 0, 0),\ (0, 2, 0),\ (0, 0, 2),\ (1, 1, 0),\ (0, 1, 1),\ (1, 0, 1)$. Since $(2, 0, 0) >_{grlex} (1, 1, 0) >_{grlex} (1, 0, 1) >_{grlex} (0, 2, 0) >_{grlex} (0, 1, 1) >_{grlex} (0, 0, 2)$. Then in grlex order they are arranged as

$$x_1^2 >_{grlex} x_1x_2 >_{grlex} x_1x_3 >_{grlex} x_2^2 >_{grlex} x_2x_3 >_{grlex} x_3^2.$$

An mth order n dimensional real tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is a multidimensional array of entries $a_{i_1 i_2 \ldots i_m} \in \mathbb{R}$ where $i_j \in [n]$ with $j \in [m]$. $T_{m,n}$ denotes the set of real tensors of order m and dimension n. Any $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in T_{m,n}$ is called a symmetric tensor, if the entries $a_{i_1 i_2 \ldots i_m}$ are invariant under any permutation of their indices. $S_{m,n}$ denotes the collection of all symmetric tensors of order m and dimension n where m and n are two given positive integers with $m, n \geq 2$. An identity tensor $\mathcal{I}_m = (\delta_{i_1 \ldots i_m}) \in T_{m,n}$ is defined as follows:

$$\delta_{i_1 \ldots i_m} = \begin{cases} 1\ ;\ i_1 = \ldots = i_m, \\ 0\ ;\ else. \end{cases}$$

For $\mathcal{A} \in T_{m,n}$ and $x \in \mathbb{R}$, $\mathcal{A}x^{m-1} \in \mathbb{R}^n$ is a vector defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, i_3, \ldots i_m = 1}^{n} a_{i i_2 i_3 \ldots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}, \quad \text{for all } i \in [n]$$

and $\mathcal{A}x^m \in \mathbb{R}$ is a scalar defined by

$$x^T \mathcal{A}x^{m-1} = \mathcal{A}x^m = \sum_{i_1, i_2, \ldots i_m = 1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

The general product of tensors was introduced by Shao [43].

Definition 4 [43] Let $\mathcal{A}$ and $\mathcal{B}$ be order $m \geq 2$ and order $k \geq 1$, n-dimensional tensor, respectively. The product $\mathcal{A} \cdot \mathcal{B}$ is a tensor $\mathcal{C}$ of order $((m-1)(k-1))+1$ and n-dimensional with entries

$$c_{i\alpha_1\cdots\alpha_{m-1}} = \sum_{i_2,\cdots,i_m\in[n]} a_{ii_2\cdots i_m} b_{i_2\alpha_1}\cdots b_{i_m\alpha_{m-1}},$$

where $i \in [n]$, $\alpha_1, \cdots, \alpha_{m-1} \in [n]^{k-1}$.

Definition 5 [6] Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in T_{m,n}$. Let $J \subseteq [n]$ with $|J| = r,\ 1 \leq r \leq n$. Then a principal subtensor of $\mathcal{A}$ is denoted by $\mathcal{A}_r^J$ and is defined as

$$\mathcal{A}_r^J = (a_{i_1i_2\ldots i_m}),\ \forall\, i_1, i_2, \ldots i_m \in J.$$

Example 1 Let $\mathcal{A} \in T_{3,3}$ be such that $a_{111} = 2,\ a_{222} = -1,\ a_{333} = 2,\ a_{223} = -2,\ a_{232} = 1,\ a_{233} = -1$ and all other entries of $\mathcal{A}$ are zeros.
For $r = 1$, there are three principal subtensors of one dimension. Let $J_1 = \{1\}$, $J_2 = \{2\}$ and $J_3 = \{3\}$. Then principal subtensors of one dimension are $\mathcal{A}_1^{J_1} = (a_{111}) = (2)$, $\mathcal{A}_1^{J_2} = (a_{222}) = (-1)$, $\mathcal{A}_1^{J_3} = (a_{333}) = (3)$.

For $r = 2$, there are three principal subtensors of two dimension. Let $J_4 = \{1, 2\}$, $J_5 = \{2, 3\}$, $J_6 = \{1, 3\}$. The principal subtensors of two dimension are $\mathcal{A}_2^{J_4},\ \mathcal{A}_2^{J_5}$, and $\mathcal{A}_2^{J_6}$. Here $\mathcal{A}_2^{J_4} = (a_{ijk}),\ \forall\, i, j, k \in J_4$. Then $\mathcal{A}_2^{J_4}$ is given by $a_{111} = 2,\ a_{222} = -1$ and all other entries of $\mathcal{A}_2^{J_4}$ are zeros. Here $\mathcal{A}_2^{J_5} = (a_{ijk}),\ \forall\, i, j, k \in J_5$. Then $\mathcal{A}_2^{J_5}$ is given by $a_{222} = -1,\ a_{333} = 2,\ a_{223} = -2,\ a_{232} = 1,\ a_{233} = -1$ and all other entries of $\mathcal{A}_2^{J_5}$ are zeros. Here $\mathcal{A}_2^{J_6} = (a_{ijk}),\ \forall\, i, j, k \in J_6$. Then $\mathcal{A}_2^{J_6}$ is given by $a_{222} = -1,\ a_{333} = 2$ and all other entries of $\mathcal{A}_2^{J_6}$ are zeros.

For $r = 3$, there is only one principal subtensor of three dimension which is $\mathcal{A}$.

Definition 6 [7] Given $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in T_{m,n}$ and $q \in \mathbb{R}^n$, a vector x is said to be (strictly) feasible solution of TCP$(q, \mathcal{A})$ defined by (3) if $x(>) \geq 0$ and $\mathcal{A}x^{m-1} + q(>) \geq 0$. TCP$(q, \mathcal{A})$ defined by (3) is said to be (strictly) feasible if a (strictly) feasible vector exists. The set of feasible solutions of TCP$(q, \mathcal{A})$ is denoted by FEA$(q, \mathcal{A}) = \{x \in \mathbb{R}^n : x \geq 0,\ \mathcal{A}x^{m-1} + q \geq 0\}$.

Definition 7 [7] Given $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in T_{m,n}$ and $q \in \mathbb{R}^n$, then TCP$(q, \mathcal{A})$ defined by (3) is said to be solvable if there is a feasible vector x satisfying $x^T(\mathcal{A}x^{m-1} + q) = 0$ and x is a solution of the TCP$(q, \mathcal{A})$. The solution set of TCP$(q, \mathcal{A})$ is denoted by SOL$(q, \mathcal{A}) = \{x \in \text{FEA}(q, \mathcal{A}) : x^T(\mathcal{A}x^{m-1} + q) = 0\}$.

Definition 8 [36] A tensor $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in T_{m,n}$, is said to be a positive definite tensor, if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n\backslash\{0\}$, provided m is even. Furthermore if m is even, $\mathcal{A} \in S_{m,n}$ and $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n\backslash\{0\}$ then $\mathcal{A}$ is said to be a symmetric positive definite tensor.

Definition 9 [36] A tensor $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in T_{m,n}$ is said to be a positive semidefinite tensor, if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n$. Furthermore if $\mathcal{A} \in S_{m,n}$ and $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n$ then $\mathcal{A}$ is said to be a symmetric positive semidefinite tensor.

Definition 10 [6] A tensor $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in T_{m,n}$ is said to be a $P(P_0)$-tensor, if for each $x \in \mathbb{R}^n\backslash\{0\}$, there exists an index $i \in [n]$ such that $x_i \neq 0$ and $x_i(\mathcal{A}x^{m-1})_i > (\geq 0)$.

Definition 11 [44] A tensor $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in T_{m,n}$ is said to be weak P-tensor, if for each $x \in \mathbb{R}^n\backslash\{0\}$, there exists an index $i \in [n]$ such that $x_i \neq 0$ and $x_i^{m-1}(\mathcal{A}x^{m-1})_i > 0$.

Definition 12 [38] A tensor $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in T_{m,n}$ is said to be P_0'-tensor, if for each $x \in \mathbb{R}^n\backslash\{0\}$, there exists an index $i \in [n]$ such that $x_i \neq 0$ and $x_i^{m-1}(\mathcal{A}x^{m-1})_i \geq 0$.

Definition 13 [45] A tensor $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in T_{m,n}$ is said to be a column sufficient tensor, if for $x \in \mathbb{R}^n$, $x_i(\mathcal{A}x^{m-1})_i \leq 0,\ \forall\, i \in [n] \implies x_i(\mathcal{A}x^{m-1})_i = 0,\ \forall\, i \in [n]$.

Definition 14 [46] Let $\mathcal{A} \in T_{m,n}$ and $R_i(\mathcal{A}) = (r_{ii_2\ldots i_m})_{i_2\ldots i_m}^n \in T_{m-1,n}$ such that $r_{ii_2\ldots i_m} = a_{ii_2\ldots i_m}$. $\mathcal{A}$ is said to be row subtensor diagonal, or simply row diagonal, if all its row subtensors $R_i(\mathcal{A}),\ i = 1, 2, \ldots n$ are diagonal tensors, i.e. $a_{ii_2\ldots i_m}$ take non-zero values only when $i_2 = \cdots = i_m$.

Example 2 Let $\mathcal{A} \in T_{4,2}$ such that $a_{1111} = 3,\ a_{2222} = 1,\ a_{1222} = -2,\ a_{2111} = 1$ and all other entries of $\mathcal{A}$ are zeros. Then $\mathcal{A}$ is a row diagonal tensor.

Definition 15 [47] Let $\mathcal{A} \in T_{m,n}$. The majorization matrix $M(\mathcal{A})$ of $\mathcal{A}$ is the $n \times n$ matrix with entries $M(\mathcal{A})_{ij} = a_{ijj\ldots j}$, where $i, j = 1, 2, \ldots, n$.

Example 3 Let $\mathcal{A} \in T_{4,2}$, be such that $a_{1111} = 1,\ a_{1222} = -1,\ a_{2112} = 3,\ a_{2111} = 0,\ a_{1112} = -2,\ a_{2222} = 2$ and all other entries of $\mathcal{A}$ are zeros. Then the majorization matrix of $\mathcal{A}$ is $M(\mathcal{A}) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Now we state some results on the tensor complementarity problem.

Theorem 2 *[46] Let $\mathcal{A}$ be an order m and dimension n tensor. Then $\mathcal{A}$ is row diagonal if and only if $\mathcal{A} = M(\mathcal{A})\mathcal{I}_m$, where $\mathcal{I}_m$ is the identity tensor of order m and dimension n.*

Theorem 3 *[2] For any $q \in \mathbb{R}^n$ and a P-tensor $\mathcal{A} \in T_{m,n}$, the solution set of TCP$(q, \mathcal{A})$ is nonempty and compact.*

Theorem 4 *[45] Let $\mathcal{A}$ be a column sufficient tensor with order m dimension n. Then the TCP$(q, \mathcal{A})$ has unique zero solution for $q \in \mathbb{R}^n_{++}$.*

3 Main results

Now we introduce column adequate tensor which is defined as follows:

Definition 16 A tensor $\mathcal{A} \in T_{m,n}$ is said to be column adequate tensor if for $x \in \mathbb{R}^n$, $x_i(\mathcal{A}x^{m-1})_i \le 0,\ \forall\ i \in [n]$ implies $\mathcal{A}x^{m-1} = 0$.

Remark 1 Let $\mathcal{A}$ be a P-tensor. Then the condition $x_i(\mathcal{A}x^{m-1})_i \le 0,\ \forall\ i \in [n]$ holds only for $x = 0$. Therefore a P-tensor is trivially a column adequate tensor.

Example 4 Let $\mathcal{A} \in T_{4,2}$, such that $a_{1111} = 2,\ a_{1112} = 1,\ a_{2122} = 4,\ a_{2222} = 2$ and all other entries of $\mathcal{A}$ are zeros. Then for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ we have, $\mathcal{A}x^3 = \begin{pmatrix} (2x_1 + x_2)x_1^2 \\ 2(2x_1 + x_2)x_2^2 \end{pmatrix}$. Let $x_i(\mathcal{A}x^3)_i \le 0$ for $i = 1, 2$. Consider the following system of inequations

$$\begin{aligned} (2x_1 + x_2)x_1^3 &\le 0, \\ (2x_1 + x_2)2x_2^3 &\le 0. \end{aligned} \tag{5}$$

Note that the system has solutions for $x = k\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ where $k \in \mathbb{R}$. This implies $\mathcal{A}x^3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $\mathcal{A}$ is a column adequate tensor.
For $x = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ we have $x_i(\mathcal{A}x^3)_i = 0$ for $i = 1, 2$. This shows that $\mathcal{A}$ is not a P-tensor.
For $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ we have $\mathcal{A}x^4 = -1$. This shows that $\mathcal{A}$ is not a positive semidefinite tensor.

Example 5 Let $\mathcal{A} \in T_{4,2}$, such that $a_{1111} = 1,\ a_{1112} = -1,\ a_{2111} = 1$ and all other entries of $\mathcal{A}$ are zeros. Then for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, we have $\mathcal{A}x^4 = x_1^4 \ge 0$. Then $\mathcal{A}$ is a non-symmetric positive semidefinite tensor. Again $\mathcal{A}x^3 = \begin{pmatrix} x_1^2(x_1 - x_2) \\ x_1^3 \end{pmatrix}$. Then we have $x_1(\mathcal{A}x^3)_1 = x_1^3(x_1 - x_2)$, and $x_2(\mathcal{A}x^3)_2 = x_1^3x_2$. Now $x_i(\mathcal{A}x^3)_i \le 0,\ \forall\ i \in [2]$ implies $x = \begin{pmatrix} 0 \\ k \end{pmatrix}$ where $k \in \mathbb{R}$. For $x = \begin{pmatrix} 0 \\ k \end{pmatrix}$ we obtain $\mathcal{A}x^3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence $\mathcal{A}$ is column adequate tensor.

Theorem 5 *A column adequate tensor is a column sufficient tensor.*

Proof Let $\mathcal{A} \in T_{m,n}$ be column adequate tensor. Then for $x \in \mathbb{R}^n$, $x_i(\mathcal{A}x^{m-1})_i \le 0, \forall\ i \in [n] \implies \mathcal{A}x^{m-1} = 0 \implies x_i(\mathcal{A}x^{m-1})_i = 0,\ \forall\ i \in [n]$. This implies that $\mathcal{A}$ is column sufficient tensor. □

Remark 2 Not all column sufficient tensors are column adequate tensors.

Below we give an example of a column sufficient tensor which is not a column adequate tensor.

Example 6 Here we consider the example given in page 261 of [45] for $\mathcal{A} \in T_{4,2}$. Here $a_{1112} = -2,\ a_{2111} = a_{2222} = 1$ and all other entries of $\mathcal{A}$ are zeros. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. $\mathcal{A}$ is a column sufficient tensor. Now $\mathcal{A}x^3 = \begin{pmatrix} -2x_1^2x_2 \\ x_1^3 + x_2^3 \end{pmatrix}$ and $x_1(\mathcal{A}x^3)_1 = -2x_1^3x_2, x_2(\mathcal{A}x^3)_2 = x_1^3x_2 + x_2^4$. Then for $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have $x_i(\mathcal{A}x^3)_i = 0,\ \forall\ i \in [2]$ but $\mathcal{A}x^3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $\mathcal{A}$ is not a column adequate tensor.

In the following result, we prove the inheritance property of column adequate tensors in the context of principal subtensor.

Theorem 6 *Suppose that $\mathcal{A} \in T_{m,n}$ is a column adequate tensor then all principal subtensors of $\mathcal{A}$ are column adequate tensors.*

Proof Let $J \subseteq [n], |J| = r$ and $\mathcal{A}_r^J$ be the principal subtensor of $\mathcal{A}$. For some $x \in \mathbb{R}^r$, if $x_i(\mathcal{A}_r^J x^{m-1})_i \leq 0,\ \forall\ i \in J$, we construct $y \in \mathbb{R}^n$ such that $y_i = \begin{cases} x_i\ ;\ \forall\ i \in J \\ 0\ ;\ \forall\ i \in J^c \end{cases}$. Then we have $(\mathcal{A}y^{m-1})_i = (\mathcal{A}_r^J x^{m-1})_i,\ \forall\ i \in J$, which follows

$$y_i(\mathcal{A}y^{m-1})_i = \begin{cases} x_i(\mathcal{A}_r^J x^{m-1})_i \leq 0 & ;\ \forall\ i \in J \\ 0 & ;\ \forall\ i \in J^c \end{cases}.$$

Thus $y_i(\mathcal{A}y^{m-1})_i \leq 0,\ \forall\ i \in [n]$. Since $\mathcal{A}$ is column adequate tensor, $y_i(\mathcal{A}y^{m-1})_i \leq 0,\ \forall\ i \in [n] \implies (\mathcal{A}y^{m-1})_i = 0\ , \forall\ i \in [n] \implies (\mathcal{A}_r^J x^{m-1})_i = 0,\ \forall\ i \in J$. Hence $\mathcal{A}_r^J$ is a column adequate tensor. □

Theorem 7 *Let $\mathcal{A} \in T_{m,n}$ and $P = diag(p_1, p_2, ..., p_n)$, $Q = diag(q_1, q_2, ..., q_n)$ be two diagonal matrices of order n where $p_iq_i > 0,\ \forall\ i \in [n]$. $\mathcal{A}$ is column adequate tensor if and only if $P\mathcal{A}Q$ is column adequate tensor.*

Proof Let $P = diag(p_1, p_2, ..., p_n)$ and $Q = diag(q_1, q_2, ..., q_n)$ be two diagonal matrices of order n with $p_iq_i > 0,\ \forall\ i \in [n]$. Let $\mathcal{A} = (a_{i_1i_2...i_m}) \in T_{m,n}$ be column adequate tensor with order m and dimension n. Let $\mathcal{B} = P\mathcal{A}Q$, assume $\mathcal{B} = (b_{i_1i_2...i_m})$. By the definition of multiplication for all $i_1, i_2, ..., i_m \in [n]$, we have,

$$\begin{aligned} b_{i_1i_2...i_m} &= \sum_{j_1, j_2, ..., j_m \in [n]} p_{i_1j_1}a_{j_1j_2...j_m}q_{j_2i_2}q_{j_3i_3}\cdots q_{j_mi_m} \\ &= p_{i_1}a_{i_1i_2...i_m}q_{i_2}q_{i_3}\cdots q_{i_m}. \end{aligned}$$

For $x \in \mathbb{R}^n$ and $i \in [n]$, we obtain,

$$\begin{aligned} x_i(\mathcal{B}x^{m-1})_i &= x_i \sum_{i_2,...,i_m \in [n]} b_{ii_2...i_m}x_{i_2}x_{i_3}\cdots x_{i_m} \\ &= x_i \sum_{i_2,...,i_m \in [n]} p_i a_{ii_2...i_m}q_{i_2}q_{i_3}\cdots q_{i_m}x_{i_2}x_{i_3}\cdots x_{i_m} \\ &= \frac{p_i}{q_i}(q_ix_i) \sum_{i_2,...,i_m \in [n]} a_{ii_2...i_m}(q_{i_2}x_{i_2})(q_{i_3}x_{i_3})\cdots(q_{i_m}x_{i_m}) \\ &= \frac{p_i}{q_i}y_i \sum_{i_2,...,i_m \in [n]} a_{ii_2...i_m}y_{i_2}y_{i_3}\cdots y_{i_m} \\ &= \frac{p_i}{q_i}y_i(\mathcal{A}y^{m-1})_i \end{aligned}$$

where $y = Qx$. Since $p_iq_i > 0,\ \forall\ i \in [n]$, $x_i(\mathcal{B}x^{m-1})_i \leq 0 \iff y_i(\mathcal{A}y^{m-1})_i \leq 0, \forall\ i \in [n]$. Since $\mathcal{A}$ is column adequate tensor $\forall\ i \in [n]$, $y_i(\mathcal{A}y^{m-1})_i \leq 0 \implies \mathcal{A}y^{m-1} = 0$. Again $(\mathcal{B}x^{m-1})_i = p_i(\mathcal{A}y^{m-1})_i,\ \forall\ i \in [n]$. Therefore $\mathcal{A}y^{m-1} = 0 \implies \mathcal{B}x^{m-1} = 0$. Thus $x_i(\mathcal{B}x^{m-1})_i \leq 0,\ \forall\ i \in [n] \implies \mathcal{B}x^{m-1} = 0$. Hence, $\mathcal{B} = P\mathcal{A}Q$ is column adequate tensor.

Conversely, let $P\mathcal{A}Q$ be column adequate tensor. Since the entries of P and Q are such that $p_iq_i > 0,\ \forall\ i \in [n]$, then P and Q are invertible with $P^{-1} = diag(\frac{1}{p_1}, \cdots, \frac{1}{p_n})$ and $Q^{-1} = diag(\frac{1}{q_1}, \cdots, \frac{1}{q_n})$, where $\frac{1}{p_iq_i} > 0,\ \forall\ i \in [n]$. Therefore $P^{-1}(P\mathcal{A}Q)Q^{-1} = \mathcal{A}$ which is column adequate tensor. □

Now we prove an invariant property of column adequate tensors under rearrangement of subscripts.

Theorem 8 *Let $\mathcal{A} \in T_{m,n}$ and $P = [p_{ij}]_{n\times n}$ be a permutation matrix of order n. $\mathcal{A}$ is column adequate tensor if and only if $P^T\mathcal{A}P$ is column adequate tensor.*

Proof Let P be a permutation matrix and $\mathcal{A} \in T_{m,n}$ be a column adequate tensor. Now consider the tensor $\mathcal{B} = P^T\mathcal{A}P$. Then

$$b_{i_1 i_2 \dots i_m} = \sum_{j_1, j_2, \dots, j_m \in [n]} p_{j_1 i_1} a_{j_1 j_2 \dots j_m} p_{j_2 i_2} p_{j_3 i_3} \cdots p_{j_m i_m}.$$

Without loss of generality, for any $i \in [n]$, suppose $p_{i'i} = 1$ and $p_{ji} = 0,\ j \neq i',\ j \in [n]$. Then we have

$$\begin{aligned} b_{i_1 i_2 \dots i_m} &= \sum_{j_1, j_2, \dots, j_m \in [n]} p_{j_1 i_1} a_{j_1 j_2 \dots j_m} p_{j_2 i_2} p_{j_3 i_3} \cdots p_{j_m i_m} \\ &= p_{i'_1 i_1} a_{i'_1 i'_2 \dots i'_m} p_{i'_2 i_2} p_{i'_3 i_3} \cdots p_{i'_m i_m}. \end{aligned} \tag{6}$$

Consider $x \in \mathbb{R}^n$ such that $x_i(\mathcal{B}x^{m-1})_i \leq 0,\ \forall\, i \in [n]$. Then by (6),

$$\begin{aligned} x_i(\mathcal{B}x^{m-1})_i &= x_i \sum_{i'_2, \dots, i'_m \in [n]} p_{i'i} a_{i' i'_2 \dots i'_m} p_{i'_2 i_2} p_{i'_3 i_3} \cdots p_{i'_m i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \\ &= p_{i'i} x_i \sum_{i'_2, \dots, i'_m \in [n]} a_{i' i'_2 \dots i'_m} p_{i'_2 i_2} p_{i'_3 i_3} \cdots p_{i'_m i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \\ &= y_{i'} \sum_{i'_2, \dots, i'_m \in [n]} a_{i' i'_2 \dots i'_m} y_{i_{2'}} y_{i_{3'}} \cdots y_{i_{m'}} \\ &= y_{i'}(\mathcal{A}y^{m-1})_{i'} \\ &\leq 0. \end{aligned}$$

Then for $y = Px \in \mathbb{R}^n$ we have $x_i(\mathcal{B}x^{m-1})_i = y_{i'}(\mathcal{A}y^{m-1})_{i'} \leq 0,\ \forall\ i \in [n]$. This implies $\mathcal{A}y^{m-1} = 0$, since $\mathcal{A}$ is column adequate tensor. Then $(\mathcal{B}x^{m-1})_i = p_{i'i}(\mathcal{A}y^{m-1})_{i'} = 0, \forall\ i \in [n]$, since $\mathcal{A}y^{m-1} = 0$. Thus $x_i(\mathcal{B}x^{m-1})_i \leq 0,\ \forall\, i \in [n] \implies \mathcal{B}x^{m-1} = 0$. Hence, $\mathcal{B} = P^T\mathcal{A}P$ is column adequate tensor.

Conversely, let $\mathcal{B} = P^T\mathcal{A}P$ be column adequate tensor. Since P is a permutation matrix, then so is P^{-1} and we have $P^{-1} = P^T$ and $(P^{-1})^T = P$. Then by the first part of the proof, $(P^{-1})^T\mathcal{B}P^{-1}$ is column adequate tensor. Now $(P^{-1})^T\mathcal{B}P^{-1} = (P^T)^{-1}(P^T\mathcal{A}P)P^{-1} = \mathcal{A}$. Hence $\mathcal{A}$ is a column adequate tensor. $\square$

Theorem 9 *Let $\mathcal{A} \in T_{m,n}$ be a column adequate tensor. Then $\mathcal{A}$ is a P_0-tensor.*

Proof Let $\mathcal{A} \in T_{m,n}$ be column adequate tensor. Then for $x \in \mathbb{R}^n,\ x_i(\mathcal{A}x^{m-1})_i \leq 0,\ \forall\ i \in [n]$, implies $\mathcal{A}x^{m-1} = 0$. Let $\mathcal{A} \in T_{m,n}$ be a tensor which is not a P_0-tensor. Then $\exists\ y \in \mathbb{R}^n$ with $J \subseteq [n],\ J \neq \phi$ such that $y_i \neq 0,\ \forall\ i \in J$ and $y_i = 0,\ \forall\ i \in J^c$ for which $y_i(\mathcal{A}y^{m-1})_i < 0,\ \forall\ i \in J$ and $y_i(\mathcal{A}y^{m-1})_i = 0,\ \forall\ i \in J^c$. Then $y_i(\mathcal{A}y^{m-1})_i \leq 0,\ \forall\ i \in [n]$ but $y_i(\mathcal{A}y^{m-1})_i < 0,\ \forall\ i \in J$ implies that $\mathcal{A}y^{m-1} \neq 0$. This contradicts our assumption that $\mathcal{A}$ is a column adequate tensor. Hence $\mathcal{A} \in P_0$. $\square$

Remark 3 Not all P_0-tensors are column adequate tensor.

Now we illustrate the remark by following examples.

Example 7 Let $\mathcal{A} \in T_{4,2}$ such that $a_{1111} = 1,\ a_{1222} = -1,\ a_{2221} = 1$ and all other entries of $\mathcal{A}$ are zeros. Then for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, we have $\mathcal{A}x^4 = x_1^4 \geq 0$. This implies $\mathcal{A}$ is a non-symmetric positive semidefinite tensor as well as P_0-tensor.
Now $\mathcal{A}x^3 = \begin{pmatrix} x_1^3 - x_2^3 \\ x_1 x_2^2 \end{pmatrix}$ and $x_1(\mathcal{A}x^3)_1 = x_1(x_1^3 - x_2^3),\ x_2(\mathcal{A}x^3)_2 = x_1 x_2^3$. Then for $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have $x_1(\mathcal{A}x^3)_1 = 0,\ x_2(\mathcal{A}x^3)_2 = 0$ but $\mathcal{A}x^3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Therefore $\mathcal{A}$ is not a column adequate tensor.

Example 8 Consider $\mathcal{A} \in T_{4,2}$ such that $a_{1111} = 1,\ a_{1222} = -1,\ a_{2111} = 4,\ a_{2121} = 2,\ a_{2112} = -2$, and all other entries of $\mathcal{A}$ are zeros. For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, we have $\mathcal{A}x^4 = x_1^4 - x_1x_2^3 + 4x_1^3x_2$ and $\mathcal{A}x^3 = \begin{pmatrix} x_1^3 - x_2^3 \\ 4x_1^3 \end{pmatrix}$. Then $\mathcal{A}$ is a P_0-tensor which is not a positive semidefinite tensor. For $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have $x_i(\mathcal{A}x^3)_i \leq 0,\ \forall\, i \in [n]$ but $\mathcal{A}x^3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Therefore $\mathcal{A}$ is not a column adequate tensor.

Now we introduce weak column adequate tensor.

Definition 17 A tensor $\mathcal{A} \in T_{m,n}$ is said to be weak column adequate if

$$x_i^{m-1}(\mathcal{A}x^{m-1})_i \leq 0,\ \forall\, i \in [n] \text{ implies } \mathcal{A}x^{m-1} = 0. \tag{7}$$

Example 9 Let $\mathcal{A} \in T_{m,n}$, where $a_{111} = 1$ and all other entries of $\mathcal{A}$ are zeros. For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, we have $\mathcal{A}x^2 = \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$. Now $x_i^2(\mathcal{A}x^2)_i \leq 0,\ \forall\, i \in [2]$ implies $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This follows that $\mathcal{A}x^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $\mathcal{A}$ is a weak column adequate tensor. To show that $\mathcal{A}$ is not a column adequate tensor consider $x = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Remark 4 For even order tensors the set of column adequate tensors and the set of weak column adequate tensors are the same.

Theorem 10 *Let $\mathcal{A} \in T_{m,n}$ be weak column adequate tensor. Then $\mathcal{A}$ is P_0'-tensor.*

Proof Let $\mathcal{A} \in T_{m,n}$ be weak column adequate tensor. Then for $x \in \mathbb{R}^n$, $x_i^{m-1}(\mathcal{A}x^{m-1})_i \leq 0,\ \forall\, i \in [n]$, implies $\mathcal{A}x^{m-1} = 0$. Let $\mathcal{A} \in T_{m,n}$ be a tensor which is not a P_0'-tensor. Then $\exists\ y \in \mathbb{R}^n$ with $J \subseteq [n],\ J \neq \phi$ such that $y_i \neq 0,\ \forall\, i \in J$ and $y_i = 0,\ \forall\, i \in J^c$ for which $y_i^{m-1}(\mathcal{A}y^{m-1})_i < 0,\ \forall\, i \in J$ and $y_i^{m-1}(\mathcal{A}y^{m-1})_i = 0,\ \forall\, i \in J^c$. Then $y_i^{m-1}(\mathcal{A}y^{m-1})_i \leq 0,\ \forall\, i \in [n]$ but $y_i^{m-1}(\mathcal{A}y^{m-1})_i < 0,\ \forall\, i \in J$ implies that $\mathcal{A}y^{m-1} \neq 0$. This contradicts our assumption that $\mathcal{A}$ is a weak column adequate tensor. Hence $\mathcal{A}$ is a P_0'-tensor. □

3.1 Auxiliary matrix formation

Consider a tensor $\mathcal{A} \in T_{m,n}$ of order m and dimension n. Then for any vector $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ the equation $\mathcal{A}x^{m-1} = b$ gives a system of homogeneous polynomial equations in $x_1, x_2, ..., x_n$ of degree $(m-1)$. Here the number of equations are n and in each polynomial equation, the number of monomials are $N = \binom{m+n-2}{n-1}$ which is equal to number of monomials in $x_1,\ x_2,\ \ldots,\ x_n$ of degree $(m-1)$. Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_n}) \in T_{m,n}$. Then $(\mathcal{A}x^{m-1})_i$ is a homogeneous polynomial of degree $(m-1)$ in n variables $x_1, x_2, ..., x_n$, which can be represented as,

$$\begin{aligned}
(\mathcal{A}x^{m-1})_i &= \sum_{i_2, i_3, \ldots, i_n = 1}^{n} a_{i i_2 \ldots i_n} x_{i_2} x_{i_3} \cdots x_{i_n} \\
&= \sum_{\alpha_1 + \alpha_2 + \cdots + \alpha_n = m-1} C^i_{(\alpha_1, \alpha_2, \cdots, \alpha_n)} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\
&= \sum_{|\alpha| = m-1} C^i_\alpha x^\alpha
\end{aligned}$$

where $\alpha_i \in [m-1] \cup \{0\},\ i = 1, 2, ...n$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ such that $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = m-1$.

Here $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and

$$\begin{aligned}
C^i_{(\alpha_1, \alpha_2, \cdots, \alpha_n)} &= C^i_\alpha \\
&= \text{Coefficient of the monomial } x^\alpha \text{ in } (\mathcal{A}x^{m-1})_i \\
&= \text{Coefficient of the monomial } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \text{ in } (\mathcal{A}x^{m-1})_i.
\end{aligned}$$

Clearly C^i_α = Sum of all the elements of the form $a_{i\pi\{(\alpha_1\ times\ 1)(\alpha_2\ times\ 2)\cdots(\alpha_n\ times\ n)\}}$ from $\mathcal{A}$ where $\pi\{(\alpha_1\ times\ 1)(\alpha_2\ times\ 2)\cdots(\alpha_n\ times\ n)\}$ represents all possible permutation of $1, 1, ...\alpha_1$ times , $2, 2, ...\alpha_2$ times , $...n, n, ...\alpha_n$ times . Note that for any $k \in [n]$, $\alpha_k = 0$, then 0 times k has no impact on C^i_α.
Now we introduce modified graded lexicographic order which is defined as follows.

Definition 18 Let $\alpha, \beta \in \mathbb{Z}^n_+$. We say $\alpha >_{mglo} \beta$ if any one of the followings holds:

(a) For α such that $\alpha_i = \begin{cases} |\alpha| \ ; \ \forall\ i \in [n] \\ 0 \ \ ; \ otherwise \end{cases}$

and for β such that $\beta_i = \begin{cases} |\beta| \ ; \ \forall\ i \in [n] \\ 0 \ \ ; \ otherwise \end{cases}$ and $\alpha >_{lex} \beta$.

(b) For α such that $\alpha_i = \begin{cases} |\alpha| \ ; \ \forall\ i \in [n] \\ 0 \ \ ; \ otherwise \end{cases}$ and for β such that β contains atleast two non-zero components.

(c) For α, β such that α, β contains at least two non-zero components and $\alpha >_{grlex} \beta$.

Example 1 Let $\alpha = (2, 0, 0)$ and $\beta = (0, 2, 0)$. Then $(2, 0, 0) >_{mglo} (0, 2, 0)$, since $(2, 0, 0) >_{grlex} (0, 2, 0)$.

Example 2 Let $\alpha = (2, 0, 0)$ and $\beta = (2, 1, 0)$. Then $(2, 0, 0) >_{mglo} (2, 1, 0)$, since $(2, 1, 0)$ has two non-zero components and $(2, 0, 0)$ has only one non-zero component, though $(2, 1, 0) >_{grlex} (2, 0, 0)$.

Example 3 Let $\alpha = (2, 2, 0)$ and $\beta = (2, 1, 1)$. Then $(2, 2, 0) >_{mglo} (2, 1, 1)$, since $(2, 2, 0) >_{grlex} (2, 1, 1)$.

Example 4 Consider the monomials $x_1^2,\ x_2^2,\ x_3^2,\ x_1x_2,\ x_2x_3,\ x_3x_1$. Since $(2, 0, 0) >_{mglo} (2, 0, 0) >_{mglo} (0, 0, 2) >_{mglo} (1, 1, 0) >_{mglo} (1, 0, 1) >_{mglo} (0, 1, 1)$, in modified graded lexicographic order the monomials are arranged as

$$x_1^2 >_{mglo} x_2^2 >_{mglo} x_3^2 >_{mglo} x_1x_2 >_{mglo} x_1x_3 >_{mglo} x_2x_3.$$

3.1.1 Auxiliary matrix

The polynomials involved in $\mathcal{A}x^{m-1}$ are elements of the vector space of homogeneous polyniomials of degree $(m-1)$ in n variables $x_1,\ x_2,\ ...,\ x_n$. A basis of this vector space is $\{x_1^{m-1},\ \cdots,\ x_n^{m-1},\ x_1^{m-2}x_2,\ \cdots,\ x_1^{m-2}x_n,\ \cdots,\ x_1x_n^{m-2}\}$ which contains N number of monomials as its basis vectors. Consider the ordered basis B of the space of homogeneous polynomials with modified graded lexicographic order as $B = \{x_1^{m-1},\ x_2^{m-1},\ \cdots, x_n^{m-1},\ x_1^{m-2}x_2, \cdots,\ x_1x_n^{m-2}\}$.
Now if we assign $y_1 = x_1^{m-1},\ y_2 = x_2^{m-1},\ \cdots,\ y_n = x_n^{m-1}$ and assign $y_{n+1},\ y_{n+2},\ \cdots,\ y_N$ with the respective ordered vectors from B, then B can be considered as $B' = \{y_1,\ y_2,\ \cdots, y_n, y_{n+1}, \cdots, y_N\}$. Let for some $j \in [N]$, $y_j = x^\alpha$ for the j-th monomial x^α in B. Then the polynomial $(\mathcal{A}x^{m-1})_i$ can be expressed as $\sum_{j=1}^N a_{ij}y_j$ where $a_{ij} = C^i_\alpha$ = coefficient of j-th monomial in $(\mathcal{A}x^{m-1})_i$. Then $\mathcal{A}x^{m-1}$ can be expresseed as Ay where $y = (y_1,\ y_2,\ \cdots,\ y_n,\ y_{n+1},\ \cdots, y_N)$ and A is the coefficient matrix $A = (a_{ij})_{n\times N}$. Therefore $\mathcal{A}x^{m-1} = Ay$.
Consider the tensor complementarity problem TCP$(q, \mathcal{A})$ which is to find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad \omega = \mathcal{A}x^{m-1} + q \geq 0, \quad x^T\omega = 0.$$

Since $\mathcal{A}x^{m-1} = Ay$, so with the help of the matrix A, we define an auxiliary LCP. The matrix A is a rectangular matrix of order $n \times N$ where $n < N$. We define a new square matrix say auxiliary matrix $\bar{A}$ by conjoining $N-n$ number of zero rows below the matrix A, that is $\bar{A} = \begin{pmatrix} A \\ O \end{pmatrix}$ where O is a null matrix of order $(N-n) \times N$.

Let $\bar{q} \in \mathbb{R}^N$ be defined as $\bar{q}_i = \begin{cases} q_i \ ; \ \forall\ i \in [n] \\ 0 \ ; \ \ \forall\ i \in [N]\backslash[n] \end{cases}$, i.e. $\bar{q} = \begin{pmatrix} q \\ 0 \end{pmatrix}$ where 0 is a null vector of order $(N-n)$.
Then the LCP$(\bar{q}, \bar{A})$ is to find $y \in \mathbb{R}^N$ such that,

$$y \geq 0, \quad w = \bar{A}y + \bar{q} \geq 0, \quad y^Tw = 0, \tag{8}$$

is said to be the auxiliary LCP to the TCP$(q, \mathcal{A})$.
Now we establish some relation between solution of TCP$(q, \mathcal{A})$ and solution of LCP$(\bar{q}, \bar{A})$.

Theorem 11 *Let $\mathcal{A} \in T_{m,n}$ and for some $q \in \mathbb{R}^n$, LCP($\bar{q}, \bar{A}$) be the corresponding auxiliary LCP defined by (8) and $x = (x_1, x_2, ..., x_n)^T \in SOL(q, \mathcal{A})$. Then $y = (y_1, y_2, ..., y_n, y_{n+1}, y_{n+2}, ..., y_N)^T$ where each y_i is associated with the monomials of degree $(m-1)$ in $x_1, x_2, ...x_n$, by modified graded lexicographic order is a solution of LCP($\bar{q}, \bar{A}$). i.e. $x \in SOL(q, \mathcal{A}) \implies y \in SOL(\bar{q}, \bar{A})$.*

Proof Let $x \in \mathrm{SOL}(q, \mathcal{A})$, then $x \geq 0, \ \ \omega = \mathcal{A}x^{m-1} + q \geq 0$, and $x^T\omega = 0$. Therefore for $x \in \mathrm{SOL}(q, \mathcal{A})$,

$$x_i \geq 0, \ \forall \ i \in [n] \tag{9}$$
$$\omega_i = (\mathcal{A}x^{m-1})_i + q_i \geq 0, \ \forall \ i \in [n] \tag{10}$$

and

$$x_i\omega_i = x_i((\mathcal{A}x^{m-1})_i + q_i) = 0, \ \forall \ i \in [n]. \tag{11}$$

Following the construction of the vector $y \in \mathbb{R}^N$, we have

$$y_i \geq 0, \ \forall \ i \in [N], \ \text{by (9)}. \tag{12}$$

Now $w_i = (\bar{A}y)_i + \bar{q}_i = \begin{cases} (Ay)_i + q_i \ ; \ \forall \ i \in [n] \\ 0 \qquad\qquad ; \ \forall \ i \in [N]\backslash[n] \end{cases}$. Since $Ay = \mathcal{A}x^{m-1}$, $(Ay)_i = (\mathcal{A}x^{m-1})_i, \ \forall \ i \in [n]$. This implies

$$w_i = \begin{cases} (\mathcal{A}x^{m-1})_i + q_i \ ; \ \forall \ i \in [n] \\ 0 \qquad\qquad\qquad ; \ \forall \ i \in [N]\backslash[n] \end{cases} = \begin{cases} \omega_i \ ; \ \forall \ i \in [n] \\ 0 \ \ ; \ \forall \ i \in [N]\backslash[n] \end{cases}.$$

Therefore by (10)

$$w_i = (\bar{A}y)_i + \bar{q}_i \geq 0, \ \forall \ i \in [N]. \tag{13}$$

Finally,

$$\begin{aligned} y_iw_i = y_i((\bar{A}y)_i + \bar{q}_i) &= \begin{cases} y_i((Ay)_i + q_i) \ ; \ \forall \ i \in [n] \\ 0 \qquad\qquad\qquad ; \ \forall \ i \in [N]\backslash[n] \end{cases} \\ &= \begin{cases} x_i^{m-1}((\mathcal{A}x^{m-1})_i + q_i) \ ; \ \forall \ i \in [n] \\ 0 \qquad\qquad\qquad\qquad\qquad ; \ \forall \ i \in [N]\backslash[n] \end{cases}. \end{aligned}$$

Therefore by (11) we obtain,

$$y_iw_i = y_i((\bar{A}y)_i + \bar{q}_i) = 0, \ \forall \ i \in [N]. \tag{14}$$

Hence by (12), (13) and (14) we conclude $y \in \mathrm{SOL}(\bar{q}, \bar{A})$. □

We illustrate Theorem 11 with the help of following two examples.

Example 10 Let $\mathcal{A} \in T_{3,2}$ such that $a_{111} = a_{222} = 1$ and all other entries of $\mathcal{A}$ are zeros. Then $\mathcal{A}x^2$ consists of homogeneous polynomials in x_1, x_2 of degree 2. Then the monomials in modified graded lexicographic order are $x_1^2, \ x_2^2, \ x_1x_2$. Then we represent $\mathcal{A}x^2$ as

$$\mathcal{A}x^2 = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where $y_1 = x_1^2, \ \ y_2 = x_2^2$, and $y_3 = x_1x_2$. Therefore $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and for $y = (y_1, y_2, y_3)^T$, we have $\mathcal{A}x^2 = Ay$.

Now we construct $\bar{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and for $q = (0, -1)^T$ we construct $\bar{q} = (0, -1, 0)^T$. The solution set of TCP($q, \mathcal{A}$) $= \mathrm{SOL}(q, \mathcal{A}) = \{(0, 1)^T\}$ and solution set of LCP($\bar{q}, \bar{A}$) is $\mathrm{SOL}(\bar{q}, \bar{A}) = \{(0, 1, y_3)^T, y_3 \geq 0\}$. Here for $x = (0, 1)^T \in \mathrm{SOL}(q, \mathcal{A})$, we have $y = (0, 1, 0)^T \in \mathrm{SOL}(\bar{q}, \bar{A})$.

Example 11 Let $\mathcal{A} \in T_{4,2}$ such that $a_{1111} = 1, a_{1112} = -2, a_{1122} = 1, a_{2222} = 1$ and all other entries of $\mathcal{A}$ are zeros. Then $\mathcal{A}x^3$ consist of homogeneous polynomials in x_1, x_2 of degree 3. The monomials in modified graded lexicographic order are $x_1^3,\ x_2^3,\ x_1^2x_2,\ x_1x_2^2$. We can represent $\mathcal{A}x^3$ as

$$\begin{aligned}\mathcal{A}x^3 &= \begin{pmatrix} x_1^3 + 0\cdot x_2^3 - 2x_1^2x_2 + x_1x_2^2 \\ 0\cdot x_1^3 + x_2^3 + 0\cdot x_1^2x_2 + 0\cdot x_1x_2^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1^3 \\ x_2^3 \\ x_1^2x_2 \\ x_1x_2^3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}\end{aligned}$$

where $y_1 = x_1^3,\ y_2 = x_2^3,\ y_3 = x_1^2x_2,\ y_4 = x_1x_2^2$. Let $A = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ then for $y = (y_1, y_2, y_3, y_4)^T$ we have $\mathcal{A}x^3 = Ay$.

Now we construct $\bar{A} = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and for $q = (0, -1)^T$ we construct $\bar{q} = (0, -1, 0, 0)^T$. Then solution set of the $TCP(q, \mathcal{A}) = SOL(q, \mathcal{A}) = \{(0, 1)^T, (1, 1)^T\}$. See Example 3.2 of [2]. The LCP$(\bar{q}, \bar{A})$ is to find $y \in \mathbb{R}^4$ such that $y \geq 0,\ w = \bar{A}y + \bar{q} \geq 0$ and $y^T w = 0$, where $w = \bar{A}y + \bar{q} = \begin{pmatrix} y_1 + 0 - 2y_3 + y_4 \\ y_2 - 1 \\ 0 + 0 \\ 0 + 0 \end{pmatrix}$ $= \begin{pmatrix} y_1 - 2y_3 + y_4 \\ y_2 - 1 \\ 0 \\ 0 \end{pmatrix}$.

The solution set of LCP$(\bar{q}, \bar{A}) = SOL(\bar{q}, \bar{A}) = \{(2y_3 - y_4, 1, y_3, y_4)^T, 2y_3 \geq y_4 \geq 0\} \cup \{(0, 1, y_3, y_4) : y_4 \geq 2y_3 \geq 0\}$. Here for $x = (0, 1)^T \in$ SOL$(q, \mathcal{A})$, we have $y = (0, 1, 0, 0)^T \in$ SOL$(\bar{q}, \bar{A})$ and for $x = (1, 1)^T \in$ SOL$(q, \mathcal{A})$, we have $y = (1, 1, 1, 1)^T \in$ SOL$(\bar{q}, \bar{A})$.

Next, we prove the following result to establish the ω-uniqueness of $TCP(q, \mathcal{A})$.

Theorem 12 *Let LCP$(\bar{q}, \bar{A})$ be the auxiliary LCP of TCP$(q, \mathcal{A})$. If LCP$(\bar{q}, \bar{A})$ has unique w-solution for all $\bar{q} \in \mathbb{R}^N$ then TCP$(q, \mathcal{A})$ has unique ω-solution (if exists) for all $q \in \mathbb{R}^n$.*

Proof Let SOL$(\bar{q}, \bar{A})$ be the solution set of LCP$(\bar{q}, \bar{A})$. Then SOL$(\bar{q}, \bar{A}) = \{y \in \mathbb{R}^N : y \geq 0,\ w = \bar{A}y + \bar{q} \geq 0,\ y^T w = 0\}$. If LCP$(\bar{q}, \bar{A})$ has unique w-solution, then $\{w = \bar{A}y + \bar{q} : y \in \text{SOL}(\bar{q}, \bar{A})\}$ is singleton and w_i's are uniquely determined for each $i \in [N]$. Now we show that the set $\{\omega = \mathcal{A}x^{m-1} + q : x \in \text{SOL}(q, \mathcal{A})\}$ is singleton. For any $x \in$ SOL$(q, \mathcal{A})$ by Theorem 11, $\exists\ y \in$ SOL$(\bar{q}, \bar{A})$ such that

$$\begin{aligned} w_i &= \begin{cases} (\bar{A}y + \bar{q})_i\ ; & \forall\ i \in [n] \\ 0 & ;\ \forall\ i \in [N]\backslash[n] \end{cases} \\ &= \begin{cases} (\mathcal{A}x^{m-1})_i + q_i\ ; & \forall\ i \in [n] \\ 0 & ;\ \forall\ i \in [N]\backslash[n] \end{cases} \\ &= \begin{cases} \omega_i\ ; & \forall\ i \in [n] \\ 0\ ; & \forall\ i \in [N]\backslash[n] \end{cases}. \end{aligned}$$

Thus the values of ω_i's are exactly same as that of $w_i,\ \forall\ i \in [n]$. Since w_i's are uniquely determined for each $i \in [N]$ and for all $y \in$ SOL$(\bar{q}, \bar{A})$, ω_i's are uniquely determined for each $i \in [n]$ and for all $x \in$ SOL$(q, \mathcal{A})$. Hence $\{\omega = \mathcal{A}x^{m-1} + q : x \in \text{SOL}(q, \mathcal{A})\}$ is singleton. Therefore TCP$(q, \mathcal{A})$ has unique ω-solution. □

Now we illustrate Theorem 12 with the help of the following example.

Example 12 Let $\mathcal{A} \in T_{3,2}$ such that $a_{111} = a_{222} = 1$ and all other entries of $\mathcal{A}$ are zeros. Then the corresponding $\bar{A}$ is obtained as $\bar{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Then for $q \geq 0$, $SOL(\bar{q}, \bar{A}) = \{(0, 0, y_3)^T, y_3 \geq 0\}$. Then the w-solution is uniquely determined with $w = (q_1, q_2, 0)^T$.

For $q = (q_1, q_2)^T \leq 0$, $SOL(\bar{q}, \bar{A}) = \{(-q_1, -q_2, y_3)^T, y_3 \geq 0\}$. Then the w-solution is uniquely determined with $w = (0, 0, 0)^T$.

For $q = (q_1, q_2)^T$ such that $q_1 > 0, q_2 < 0$, $SOL(\bar{q}, \bar{A}) = \{(0, -q_2, y_3)^T, y_3 \geq 0\}$. Then the w-solution is uniquely determined with $w = (q_1, 0, 0)^T$.

For $q = (q_1, q_2)^T$ such that $q_1 < 0, q_2 > 0$, $SOL(\bar{q}, \bar{A}) = \{(-q_1, 0, y_3)^T, y_3 \geq 0\}$. Then the w-solution is uniquely determined with $w = (0, q_2, 0)^T$.

Thus whatever the case may be, w-solution of LCP$(\bar{q}, \bar{A})$ is uniquely determined.

Here the immediate conclusion is that ω-solution of TCP$(q, \mathcal{A})$ is also unique for $q \in \mathbb{R}^n$.

We establish a sufficient condition for column adequate tensor.

Theorem 13 *Let $\mathcal{A} \in T_{m,n}$ where m is even. If $\bar{A}$ is a column adequate matrix then $\mathcal{A}$ is a column adequate tensor.*

Proof Let $\bar{A}$ be a column adequate matrix. Then for $y \in \mathbb{R}^N$, $y_i(\bar{A}y)_i \leq 0, \ \forall\, i \in [N] \implies \bar{A}y = 0$. Let for some $x \in \mathbb{R}^n$, $x_i(\mathcal{A}x^{m-1})_i \leq 0, \ \forall\, i \in [n]$. This implies $x_i^{m-1}(\mathcal{A}x^{m-1})_i \leq 0, \ \forall\, i \in [n]$, since m is even. Now constructing y with the help of x, we obtain $(\mathcal{A}x^{m-1})_i = (\bar{A}y)_i$ and $y_i(\bar{A}y)_i = x_i^{m-1}(\mathcal{A}x^{m-1})_i, \ \forall\, i \in [n]$. Therefore $x_i^{m-1}(\mathcal{A}x^{m-1})_i \leq 0 \iff y_i(\bar{A}y)_i \leq 0, \ \forall\, i \in [n]$ and by the construction of $\bar{A}$ we have $y_i(\bar{A}y)_i = 0, \ \forall\, i \in [N]\backslash[n]$. Therefore we obtain $y_i(\bar{A}y)_i \leq 0, \ \forall\, i \in [N] \implies \bar{A}y = 0$, since $\bar{A}$ is column adequate matrix. Now $\bar{A}y = 0 \implies (\bar{A}y)_i = 0, \ \forall\, i \in [N] \implies (\mathcal{A}x^{m-1})_i = 0, \ \forall\, i \in [n]$. Thus $x_i(\mathcal{A}x^{m-1})_i \leq 0, \ \forall\, i \in [n] \implies \mathcal{A}x^{m-1} = 0$. Hence $\mathcal{A}$ is a column adequate tensor. □

Now we establish the condition for tensor $\mathcal{A}$ under which $\bar{A}$ is always column adequate matrix. Consider the majorization matrix $M(\mathcal{A})$ of $\mathcal{A}$. Then it is easy to observe that $A = (M(\mathcal{A}) \ B)$ for some $B \in \mathbb{R}^{(N-n)\times(N-n)}$ and then $\bar{A} = \begin{pmatrix} M(\mathcal{A}) & B \\ O & O \end{pmatrix}$.

Theorem 14 *For an $\mathcal{A} \in T_{m,n}$ let $\bar{A} = \begin{pmatrix} M(\mathcal{A}) & B \\ O & O \end{pmatrix}$. $\bar{A}$ is a column adequate matrix if and only if the following two conditions hold:*

(a) $M(\mathcal{A})$ is column adequate matrix
(b) $B = O$.

Proof Let $\bar{A}$ be a column adequate matrix. Then for $y \in \mathbb{R}^N$,

$$y_i(\bar{A}y)_i \leq 0, \ \forall\, i \in [N] \implies (\bar{A}y)_i = 0, \ \forall\, i \in [N].$$

If $B \neq O$ we consider the vector $y \in \mathbb{R}^N$ such that $y = \begin{pmatrix} 0_{n,1} \\ e_{(N-n),1} \end{pmatrix}$, where $e_{(N-n),1} = (1, \cdots, 1)^T$, then $y_i(\bar{A}y)_i = 0, \ \forall\, i \in [N]$. However,

$$\bar{A}y = \begin{pmatrix} M(\mathcal{A})_{n,n} & B_{N-n,N-n} \\ O_{n,n} & O_{N-n,N-n} \end{pmatrix} \begin{pmatrix} 0_{n,1} \\ e_{N-n,1} \end{pmatrix} = \begin{pmatrix} (Be)_{n,1} \\ O_{N-n,1} \end{pmatrix} \neq O_{N,1}.$$

This contradicts the fact that $\bar{A}$ is a column adequate matrix. Therefore $B = O$.

Now $B = O$ gives $\bar{A} = \begin{pmatrix} M(\mathcal{A}) & O \\ O & O \end{pmatrix}$. We prove that $M(\mathcal{A})$ is a column adequate matrix. Consider $z \in \mathbb{R}^n$ for which $z_i(M(\mathcal{A})z)_i \leq 0, \ \forall\, i \in [n]$. We choose $y \in \mathbb{R}^N$ such that $y_i = \begin{cases} z_i \ ; \ \forall\, i \in [n] \\ 0 \ ; \ \forall\, i \in [N]\backslash[n] \end{cases}$. Then

$(\bar{A}y)_i = \begin{cases} (M(\mathcal{A})z)_i \ ; \ \forall\, i \in [n] \\ 0 \qquad\qquad ; \ \forall\, i \in [N]\backslash[n] \end{cases}$. Then $y_i(\bar{A}y)_i \leq 0, \ \forall\, i \in [N] \implies (\bar{A}y)_i = 0, \ \forall\, i \in [N]$, since $\bar{A}$ is column adequate matrix. This implies $(M(\mathcal{A})z)_i = 0, \ \forall\, i \in [n]$. Hence $M(\mathcal{A})$ is column adequate matrix.

Conversely, let $M(\mathcal{A})$ be column adequate matrix and $B = O$. Then for $y \in \mathbb{R}^N$, we define $z \in \mathbb{R}^n$ such that $z_i = y_i, \ \forall\, i \in [n]$. Then $(\bar{A}y)_i = \begin{cases} (M(\mathcal{A})z)_i \ ; \ \forall\, i \in [n] \\ 0 \qquad\qquad ; \ \forall\, i \in [N]\backslash[n] \end{cases}$. Then

$$\begin{aligned} y_i(\bar{A}y)_i \leq 0, \ \forall\, i \in [N] &\implies z_i(M(\mathcal{A})z)_i \leq 0, \ \forall\, i \in [n] \\ &\implies M(\mathcal{A})z = 0, \ \text{since } M(\mathcal{A}) \text{ is column adequate matrix.} \end{aligned}$$

Therefore for $y \in \mathbb{R}^N, \ y_i(\bar{A}y)_i \leq 0, \ \forall\, i \in [N] \implies \bar{A}y = 0$. Hence $\bar{A}$ is column adequate matrix. □

Remark 5 If $B \neq O$, then $\bar{A}$ is not a column adequate matrix.

Lemma 15 *Let $\mathcal{A} \in T_{m,n}$ and $\bar{A}$ be the auxiliary matrix of $\mathcal{A}$ such that $\bar{A} = \begin{pmatrix} M(\mathcal{A}) & O \\ O & O \end{pmatrix}$. Then for all $y \in SOL(\bar{q}, \bar{A})$ there exists $x \in \mathbb{R}^n$ with $x_i = y_i^{\frac{1}{m-1}}, \ \forall\, i \in [n]$ such that $x \in SOL(q, \mathcal{A})$.*

Proof Let $y \in SOL(\bar{q}, \bar{A})$ where $y = (y_1, \ \cdots \ y_n, \ y_{n+1}, \ \cdots, \ y_N)$. Then we consider $\bar{y}$ such that $\bar{y}_i = \begin{cases} y_i \ ; \ \forall\, i \in [n] \\ 0 \ \ ; \ \text{ otherwise.} \end{cases}$. First we prove that $y \in SOL(\bar{q}, \bar{A})$ implies $\bar{y} \in SOL(\bar{q}, \bar{A})$. Since $y \in SOL(\bar{q}, \bar{A})$ so we have $y \geq 0, \ w = \bar{A}y + \bar{q} \geq 0$ and $y^T w = 0$. Now by the construction of $\bar{y}$ we have $\bar{y} \geq 0, \bar{A}y = \bar{A}\bar{y}$ and $\bar{y}^T w = \begin{cases} (y^T w)_i \ ; \ \forall\, i \in [n] \\ 0 \qquad\quad ; \ \forall\, i \in [N]\backslash[n] \end{cases}$. Thus we obtain $\bar{y} \geq 0, \ \bar{A}\bar{y} + \bar{q} = w \geq 0$ and $\bar{y}^T w = 0$. Hence $\bar{y} \in SOL(\bar{q}, \bar{A})$. Since $\bar{y} \geq 0$, there exists $x \in \mathbb{R}^n$ such that $x_i = \bar{y}_i^{\frac{1}{m-1}} = y_i^{\frac{1}{m-1}} \geq 0, \ \forall\, i \in [n]$. Again for all $i \in [n]$,

$$\begin{aligned} (\mathcal{A}x^{m-1} + q)_i &= (M(\mathcal{A})\mathcal{I}_n x^{m-1} + q)_i \\ &= (M(\mathcal{A})x^{[m-1]} + q)_i \\ &= (\bar{A}\bar{y} + \bar{q})_i \\ &= (\bar{A}y + \bar{q})_i \geq 0. \end{aligned}$$

Also for all $i \in [n]$ we obtain $0 = y_i(\bar{A}y + \bar{q})_i = x_i^{m-1}(\mathcal{A}x^{m-1} + q)_i$, which implies $x_i(\mathcal{A}x^{m-1} + q)_i = 0 \ \forall\, i \in [n]$. Therefore $x \in SOL(q, \mathcal{A})$. □

Lemma 16 *Let $\mathcal{A} \in T_{m,n}$ be even order row diagonal. $\mathcal{A}$ is column adequate tensor if and only if $M(\mathcal{A})$ is column adequate matrix.*

Proof Let $\mathcal{A}$ be row diagonal. Then $\mathcal{A} = M(\mathcal{A})\mathcal{I}_m$. Then $\mathcal{A}x^{m-1} = M(\mathcal{A})\mathcal{I}_m x^{m-1} = M(\mathcal{A})y$, where $x = (x_1, x_2, ..., x_n)^T$ and $y = x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, ..., x_n^{m-1})^T$.
Let $\mathcal{A}$ be column adequate tensor, i.e. $x_i(\mathcal{A}x^{m-1})_i \leq 0 \ \forall\, i \in [n] \implies \mathcal{A}x^{m-1} = 0$. Suppose for some $y \in \mathbb{R}^n$ we have $y_i(M(\mathcal{A})y)_i \leq 0$. Since m is even, $\exists\, x = y^{[\frac{1}{m-1}]} = (y_1^{\frac{1}{m-1}}, \ y_2^{\frac{1}{m-1}}, \ ..., \ y_n^{\frac{1}{m-1}})^T$ such that

$$y_i(M(\mathcal{A})y)_i = x_i^{m-1}(M(\mathcal{A})\mathcal{I}_m x^{m-1})_i = x_i^{m-1}(\mathcal{A}x^{m-1})_i.$$

Thus $y_i(M(\mathcal{A})y)_i = x_i^{m-1}(\mathcal{A}x^{m-1})_i \leq 0 \ \ \forall\, i \in [n] \implies \mathcal{A}x^{m-1} = 0 \implies M(\mathcal{A})y = 0$. Therefor, $M(\mathcal{A})$ is column adequate matrix.

Conversely, let $M(\mathcal{A})$ be column adequate matrix then

$$\begin{aligned} x_i^{m-1}(\mathcal{A}x^{m-1})_i = y_i(M(\mathcal{A})y)_i \leq 0, \ \ &\forall\, i \in [n] \\ \implies M(\mathcal{A})y = 0& \\ \implies \mathcal{A}x^{m-1} = 0.& \end{aligned}$$

Therefore $\mathcal{A}$ is a column adequate tensor. □

Below we find equivalent conditions for column adequate tensor which ensures the ω-uniqueness for TCP$(q, \mathcal{A})$.

Theorem 17 *Let $\mathcal{A} \in T_{m,n}$ be even order row diagonal. Then the following are equivalent:*

(a) $M(\mathcal{A})$ is a column adequate matrix
(b) $\mathcal{A}$ is a column adequate tensor
(c) For $q \in \mathbb{R}^n$, TCP$(q, \mathcal{A})$ has ω-unique solution (if exists).

Proof (a) $\Longleftrightarrow$ (b): By Lemma 16.
(a) $\Longrightarrow$ (c): Since $\mathcal{A}$ is row diagonal, we have $\mathcal{A}x^{m-1} = M(\mathcal{A})x^{[m-1]}$. Then the TCP$(q, \mathcal{A})$ is equivalent to the following LCP$(q, M(\mathcal{A}))$

$$y \geq 0, \quad M(\mathcal{A})y + q \geq 0, \quad y^T(M(\mathcal{A})y + q)^T = 0,$$

where $y = x^{[m-1]} = (x_1^{m-1},\ x_2^{m-1},\ \ldots,\ x_n^{m-1})^T$. Since $M(\mathcal{A})$ is column adequate matrix, by Theorem 1 for $q \in \mathbb{R}^n$, LCP$(q, M(\mathcal{A}))$ has w-unique solution (if exists). Hence for $q \in \mathbb{R}^n$, TCP$(q, \mathcal{A})$ has ω-unique solution (if exists).
(c) $\Longrightarrow$ (a) Let $\mathcal{A}$ be row diagonal tensor and for $q \in \mathbb{R}^n$, TCP$(q, \mathcal{A})$ has ω-unique solution. This implies LCP$(q, M(\mathcal{A}))$ has w-unique solution for $q \in \mathbb{R}^n$. Then by Theorem 1 we obtain $M(\mathcal{A})$ is column adequate matrix. $\square$

Theorem 18 *Let $\mathcal{A} \in T_{m,n}$ where m is even and the auxiliary matrix $\bar{A}$ is of the form $\bar{A} = \begin{pmatrix} M(\mathcal{A}) & O \\ O & O \end{pmatrix}$. Then the following are equivalent:*

(a) $M(\mathcal{A})$ is column adequate matrix
(b) $\bar{A}$ is column adequate matrix
(c) $\mathcal{A}$ is column adequate tensor.

Proof (a) $\Longleftrightarrow$ (b): By Theorem 14.
(b) $\Longrightarrow$ (c): By Theorem 13.
(c) $\Longrightarrow$ (b): Let $\mathcal{A}$ be a tensor such that the auxiliary matrix $\bar{A}$ is of the form $\bar{A} = \begin{pmatrix} M(\mathcal{A}) & O \\ O & O \end{pmatrix}$. Then for $i \in [n]$ and $x \in \mathbb{R}^n$ we can construct $y \in \mathbb{R}^N$ for which

$$(\mathcal{A}x^{m-1})_i = (\bar{A}y)_i = (M(\mathcal{A})\mathcal{I}_m x^{m-1})_i. \tag{15}$$

Let $\mathcal{A}$ be column adequate tensor and m be even. Then $\forall\ i \in [n]$, $x_i^{m-1}(\mathcal{A}x^{m-1})_i \leq 0 \iff x_i(\mathcal{A}x^{m-1})_i \leq 0 \implies \mathcal{A}x^{m-1} = 0$. For the auxiliary matrix $\bar{A}$, if $y \in \mathbb{R}^N$ and $y_i(\bar{A}y)_i \leq 0,\ \forall\ i \in [N]$ then we define $z \in \mathbb{R}^n$ such that $z_i = y_i,\ \forall\ i \in [n]$. Since m is even, for $z \in \mathbb{R}^n\ \exists\ x \in \mathbb{R}^n$ such that $x = z^{[\frac{1}{m-1}]}$. Then we have

$$\begin{aligned} y_i(\bar{A}y)_i &= \begin{cases} z_i(M(\mathcal{A})z)_i\ ; & \forall\ i \in [n] \\ 0\ ; & \forall\ i \in [N]\backslash[n] \end{cases} \\ &= \begin{cases} x_i^{m-1}(\mathcal{A}x^{m-1})_i\ ; & \forall\ i \in [n] \\ 0\ ; & \forall\ i \in [N]\backslash[n] \end{cases}. \end{aligned}$$

Then $y_i(\bar{A}y)_i \leq 0,\ \forall\ i \in [N] \implies x_i^{m-1}(\mathcal{A}x^{m-1})_i \leq 0,\ \forall\ i \in [n]$. This implies $\mathcal{A}x^{m-1} = 0$, since $\mathcal{A}$ is column adequate tensor. This implies $M(\mathcal{A})\mathcal{I}_n x^{m-1} = 0 \implies (\bar{A}y)_i = 0,\ \forall\ i \in [n]$, by (15). By the construction of $\bar{A}$ for all $y \in \mathbb{R}^N$, we have $(\bar{A}y)_i = 0,\ \forall\ i \in [N]\backslash[n]$. Thus $y_i(\bar{A}y)_i \leq 0,\ \forall\ i \in [N] \implies (\bar{A}y)_i = 0$. Hence $\bar{A}$ is a column adequate matrix. $\square$

Theorem 19 *Let $\mathcal{A} \in T_{m,n}$ where m is even and the auxiliary matrix $\bar{A}$ is of the form $\bar{A} = \begin{pmatrix} M(\mathcal{A}) & O \\ O & O \end{pmatrix}$. Then the following are equivalent:*

(a) $\bar{A}$ is column adequate matrix
(b) For $\bar{q} \in \mathbb{R}^n$, LCP$(\bar{q}, \bar{A})$ has w-unique solution (if exists)
(c) For $q \in \mathbb{R}^n$, TCP$(q, \mathcal{A})$ has ω-unique solution (if exists).

Proof (a) $\Longrightarrow$ (b): By Theorem 1.
(b) $\Longrightarrow$ (c): By Theorem 12.
(c) $\Longrightarrow$ (b): Let TCP$(q,\mathcal{A})$ has ω-unique solution for $q\in\mathbb{R}^n$. We prove that LCP$(\bar{q},\bar{A})$ has w-unique solution. Consider the solution set of LCP$(\bar{q},\bar{A})$ as SOL$(\bar{q},\bar{A})$. Then for $y\in$ SOL$(\bar{q},\bar{A})$ we get $w=\bar{A}y+\bar{q}$. If possible let w be not unique. Nonuniqueness of w implies there exist $y^1, y^2\in$ SOL$(\bar{q},\bar{A})$ such that $w^1\neq w^2$, where $w^1=\bar{A}y^1+\bar{q}$ and $w^2=\bar{A}y^2+\bar{q}$. Since $w^1\neq w^2$ and $w_i=0,\ \forall\ i\in[N]\backslash[n]$, then $\exists\ k\in[n]$ such that $w_k^1\neq w_k^2$. Since $y^1, y^2\in$ SOL$(\bar{q},\bar{A})$ so by Lemma 15 $\exists\ x^1,\ x^2\in$ SOL$(q,\mathcal{A})$ such that $x_i^1=(y^1)_i^{\frac{1}{m-1}}$ and $x_i^2=(y^2)_i^{\frac{1}{m-1}}\ \forall\ i\in[n]$. Now,

$$\begin{aligned} w_k^1 &= (\bar{A}y^1+\bar{q})_k \\ &= (M(\mathcal{A})x^{1^{[m-1]}}+q)_k \\ &= (\mathcal{A}x^{1^{m-1}}+q)_k \end{aligned}$$

and

$$\begin{aligned} w_k^2 &= (\bar{A}y^2+\bar{q})_k \\ &= (M(\mathcal{A})x^{2^{[m-1]}}+q)_k \\ &= (\mathcal{A}x^{2^{m-1}}+q)_k. \end{aligned}$$

Since TCP$(q,\mathcal{A})$ has ω-unique solution, $(\mathcal{A}x^{1^{m-1}}+q)_k=(\mathcal{A}x^{2^{m-1}}+q)_k$. This contradicts the fact that $w_k^1\neq w_k^2$. Therefore w is unique.
(b) $\Longrightarrow$ (a): Assume for all vector $\bar{q}$, LCP$(\bar{q},\bar{A})$ has unique w-solutions. Then for $q\in\mathbb{R}^n$, LCP$(q,M(\mathcal{A}))$ has unique w-solutions. Then by Theorem 1, $M(\mathcal{A})$ is column adequate matrix. By Theorem 14, $\bar{A}$ is column adequate matrix. □

Example 13 Let $\mathcal{A}\in T_{4,2}$ such that $a_{1111}=a_{2222}=1,\ a_{1222}=a_{2111}=-1,\ a_{1121}=3,\ a_{1112}=-2,\ a_{1211}=-1$ and all other entries of $\mathcal{A}$ are zeros. Then $\mathcal{A}x^3$ consists of homogeneous polynomials in $x_1,\ x_2$ of degree 3. Then the monomials in modified graded lexicographic order are $x_1^3,\ x_2^3,\ x_1^2x_2,\ x_1x_2^2$. We can represent $\mathcal{A}x^3$ as

$$\begin{aligned} \mathcal{A}x^3 &= \begin{pmatrix} 1\cdot x_1^3+-1\cdot x_2^3+0\cdot x_1^2x_2+0\cdot x_1x_2^2 \\ -1\cdot x_1^3+1\cdot x_2^3+0\cdot x_1^2x_2+0\cdot x_1x_2^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1^3 \\ x_2^3 \\ x_1^2x_2 \\ x_1x_2^3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \end{aligned}$$

where $y_1=x_1^3,\ y_2=x_2^3,\ y_3=x_1^2x_2$ and $y_4=x_1x_2^2$. Let $A=\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$. For $y=(y_1,y_2,y_3,y_4)^T$ we have $\mathcal{A}x^3=Ay$.

Now we construct $\bar{A}=\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $M(\mathcal{A})=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $\bar{A}$ is column adequate matrix, since $M(\mathcal{A})$ is column adequate matrix. Therefore $\mathcal{A}$ is a column adequate tensor and TCP$(q,\mathcal{A})$ has unique ω-solution by Theorem 19.

4 Conclusion

In this paper, we introduce column adequate tensor in the context of tensor complementarity problem. We establish several tensor theoretic properties. We define an auxiliary matrix of the tensor and propose an auxiliary linear complementarity problem related to the tensor complementarity problem. We study the ω-solution of tensor complementarity problem in the context of the corresponding auxiliary linear complementarity problem. We show that TCP$(q, \mathcal{A})$ has ω-unique solution under some assumptions. We illustrate the results with some examples.

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Declarations

Conflicts of Interest The authors declare that there is no conflicts of interest.

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