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ORIGINAL RESEARCH

Existence, stability, and numerical simulations of a fractal-fractional hepatitis B virus model

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Abstract This paper uses a new fractal-fractional operator with a power law-type kernel in the Riemann-Liouville sense to formulate the new fractal-fractional model of hepatitis B virus (HBV) transmission with asymptomatic carriers. The existence of the model's solutions is demonstrated using Schuder's fixed point theorem. The Banach fixed point theorem is utilized to prove the uniqueness of the solutions. Solutions' stability behaviors in the Ulam concept are also discussed. Further, using the newly created numerical scheme based on Newton's polynomial, the new numerical scheme for HBV is created. Numerical simulations show the accuracy of the approximate solutions of the new numerical method, along with the clear effect of the fractal dimension and fractional order on the spread of the HBV disease.

Keywords Fractal-Fractional order HBV Model · Stability · Numerical simulations · Newton polynomial

1 Introduction

Hepatitis B is a viral liver infection that can cause acute and chronic diseases. It is estimated that there are about 1.5 million new hepatitis B infections worldwide, and 25% die due to cirrhosis and liver cell carcinogenesis. Consequently, it is a threat to humanity and a major global problem. The Hepatitis B virus is transmitted in two ways, either vertically, from mother to infant, or by direct exposure to an infected person's bodily fluids. The incubation period for the virus lasts from 45 to 180 days, with an average of 120 per day, and most people do not show any symptoms. Asymptomatic carriers are people who have an infection without showing symptoms of the disease, and although symptoms do not appear, they transmit the disease to susceptible people. This group has a vital role in accelerating the spread of epidemics due to the absence of symptoms. The possibility of contact with susceptible individuals may increase; thus, the infection rate in the community increases. Studies have shown

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that this group has an influential role in the dynamics of transmission of the hepatitis B virus and participates in its transmission cycle between humans.

Mathematical models play a crucial role in providing a more profound understanding of epidemiological trends in infection control, particularly when investigating how infectious diseases spread over time, such as HBV. In recent years, there has been growing interest in utilizing fractional-order differential equations (FODEs) for modeling infectious diseases, as they provide significant advantages over traditional integer-order mathematical models. This is because FODEs are related to systems that possess a memory, a characteristic commonly found in biological and physical systems [3,4]. That is, when we calculate the time-fractional derivative of a function $f(t)$ at a specific moment $t = t_1$, it necessitates considering the entire prior history, encompassing all values of $f(t)$ from $t = 0$ to $t = t_1$. Additionally, these equations exhibit a close association with fractals, which are commonly observed in biological systems [5,6]. Moreover, utilizing FODEs can be advantageous in addressing errors arising from neglected parameters in modeling real-world phenomena. Furthermore, it's worth noting that FODEs offer stability characteristics that are, at the very least, as dependable as their integer-order counterparts [1]. Given the significant role of fractional calculus, numerous researchers have employed it in conducting research related to describing, predicting, and controlling the dynamics of epidemics. They have also introduced new fractional-order derivatives to advance this emerging field of calculus. For instance, the Caputo operator, which is frequently employed in physics, engineering, biology, and numerous other scientific disciplines [7, 14,17–20,23,25]. Additionally, the Caputo-Fabrizio fractional derivative, grounded in the exponential kernel as a non-local and non-singular kernel [32], has been applied in numerous significant research contributions [10–13]. Furthermore, Atangana and Baleanu expanded upon the Caputo-Fabrizio derivative by utilizing the generalized Mittag-Leffler function as a means of generalizing the exponential function [33], and it was effectively applied in the modeling of diverse real-world phenomena [9,15,16,21,22,24]. In addition, several other fractional derivatives have been successfully used in many scientific studies [8,34]. Recently, Atangana introduced a new concept of differentiation and integration known as the fractal-fractional derivative. The new differentiation is a combination of fractal differentiation and fractional differentiation, and it is constructed using the power law, the generalized Mittag-Leffler function, and also the exponential decay law [1]. Besides its reliance on a non-single and non-local kernel, this new derivative takes into account the fractal geometry of the dynamic system, which is widely found in biological systems. Furthermore, it considers the heterogeneity and viscoelasticity of the medium. These attributes render the fractal-fractional derivative superior to its counterparts, which solely rely on fractional differentiation in many different fields [27,35–38].

As we know, many differential equations have very complex analytical solutions that we are unable to make use of; therefore, it is crucial to use numerical approximations to understand the solutions better. Several researchers have proposed different methods of numerical approximation. The most widely used method in approximating solutions to nonlinear fractional differential equations that model epidemiological dynamics in the last few years has been the Adams-Bashforth numerical method, which is based on the Lagrange polynomial and has given very satisfactory results. Recently, Abdon Atangana and Seda Igret Araz found that Newton's polynomial is more accurate than Lagrange's, so they proposed a new numerical scheme based on Newton's polynomial [30,31]. Some researchers have used this new numerical method in their research, (See [26–29] and references therein).

Due to the severity of hepatitis B and its global prevalence, researchers have shown keen interest in improving mathematical models that describe its dynamics [40–43]. Gul et al. and others incorporated the category of asymptomatic carriers into their fractional model of HBV in [25], which has been ignored in many previous studies, even though it has a notable impact on HBV transmission, and more accurately in [39] by using the fractal-fractional order Atangana-Baleanu derivative, the HBV transmission model was investigated in the presence of the asymptomatic carrier group. However, in this latter work, the numerical simulation of the model was done using the Adams-Bashforth numerical method, which is less accurate than the new numerical method based on the Newton polynomial [30,31]. Motivated by what was mentioned above and due to the interesting features of the fractal-fractional operator, and to achieve the numerical accuracy provided by the numerical method based on Newton's polynomial, in this work, we aim to analyze the fractal-fractional Hepatitis B virus model with asymptomatic carriers and get a more accurate simulation for this model using the new numerical scheme based on Newton's polynomial. The main contribution of this paper is the combination of theoretical and numerical accuracy in analyzing the dynamics of the hepatitis B virus model with asymptomatic carriers.

The paper is organized as follows: In Section 2, basic concepts of fractal-fractional calculus are given. In Section 3, the formulation of the model using the fractal-fractional derivative of power-law type kernels in the sense of Riemann-Liouville derivatives is presented. In section 4, a qualitative analysis of the proposed HBV model, including a study of the existence, uniqueness, and stability of the system solutions is given. In section

5, a numerical method based on Newton's polynomial is constructed. Numerical simulations are provided in Section 6 to validate the results provided in previous sections while concluding remarks are given in Section 7.

2 Preliminaries

The fundamental fractal-fractional calculus principles will be demonstrated in this section. Here, $C([0,T],\mathbb{R})$ is the Banach space of continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm:

$$||\varphi||_\infty = \sup_{t\in[0,T]} |\varphi(t)|.$$

Definition 1 ([1]) Let $\mathcal{G}(t)$ be continuous in (a,b) and fractal-differentiable on (a,b) with order θ. Then, the fractal-fractional derivative of $\mathcal{G}$ of order γ in Riemann-Liouville sense with the power law kernel is:

$$^{FFP}D_{a,t}^{\gamma,\theta}\mathcal{G}(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d}{dt^\theta}\int_a^t (t-\tau)^{n-\gamma-1}\mathcal{G}(\tau)d\tau, \quad n-1<\gamma,\theta \leqslant n, \quad n\in\mathbb{N}. \tag{1}$$

Such that

$$\frac{d\mathcal{G}(\tau)}{d\tau^\theta} = \lim_{t\to\tau}\frac{\mathcal{G}(t)-\mathcal{G}(\tau)}{t^\theta-\tau^\theta}.$$

Definition 2 ([1]) Let $\mathcal{G}(t)$ be continuous on an open interval. Then, the fractal Laplace transform of $\mathcal{G}(t)$ of order γ is as follows:

$$^{F}L_p^\gamma(\mathcal{G}(t)) = \int_0^\infty \mathcal{G}(t)e^{-pt}t^{\gamma-1}dt, \qquad \gamma>0. \tag{2}$$

Definition 3 ([1]) Let $\mathcal{G}(t)$ be continuous in (a, b). Then, the fractal-fractional integral of $\mathcal{G}$ with order γ in the Riemann-Liouville sense with power law is as follows:

$$^{FFP}I_{a,t}^{\gamma,\theta}\mathcal{G}(t) = \frac{\theta}{\Gamma(\gamma)}\int_a^t \tau^{\theta-1}(t-\tau)^{\gamma-1}\mathcal{G}(\tau)d\tau. \tag{3}$$

Remark 1 It is known that

$$\frac{d\mathcal{G}(\tau)}{d\tau^\theta} = \lim_{t\to\tau}\frac{\mathcal{G}(t)-\mathcal{G}(\tau)}{t^\theta-\tau^\theta},$$

then

$$\begin{aligned}\frac{d\mathcal{G}(\tau)}{d\tau^\theta} &= \lim_{t\to\tau}[\frac{t-\tau}{t^\theta-\tau^\theta}\frac{\mathcal{G}(t)-\mathcal{G}(\tau)}{t-\tau}]\\ &= \lim_{t\to\tau}\frac{1}{\frac{t^\theta-\tau^\theta}{t-\tau}}\frac{d\mathcal{G}(\tau)}{d\tau} = \frac{1}{\theta\tau^{\theta-1}}\frac{d\mathcal{G}(\tau)}{d\tau}.\end{aligned}$$

So as a result:

$$^{FFP}D_{0,t}^{\gamma,\theta}\mathcal{G}(\tau) = \frac{1}{\theta\tau^{\theta-1}}\ ^{RL}D_{0,t}^{\gamma}\mathcal{G}(\tau). \tag{4}$$

3 Formulation of the model

The model from which we will take our fractal-fractional model is given below and can be seen in [25]

$$\begin{cases} \dfrac{d\mathfrak{S}}{dt} = \Lambda - \beta(\mathfrak{A} + \phi\mathfrak{A}_c + \epsilon\mathfrak{C})\mathfrak{S} - \xi\mathfrak{S} \\ \dfrac{d\mathfrak{E}}{dt} = \beta(\mathfrak{A} + \phi\mathfrak{A}_\mathfrak{c} + \epsilon\mathfrak{C})\mathfrak{S} - (\xi + \varkappa)\mathfrak{E} \\ \dfrac{d\mathfrak{A}}{dt} = \varkappa\zeta\mathfrak{E} - (\xi + \varrho + \eta + k)\mathfrak{A} \\ \dfrac{d\mathfrak{A}_c}{dt} = \varkappa(1-\zeta)\mathfrak{E} - (\xi + \varsigma + \rho)\mathfrak{A}_c \\ \dfrac{d\mathfrak{C}}{dt} = \eta\mathfrak{A} + \varsigma\mathfrak{A}_\mathfrak{c} - (\xi + \varepsilon + \sigma)\mathfrak{C} \\ \dfrac{d\mathfrak{R}}{dt} = k\mathfrak{A} + \sigma\mathfrak{C} + \zeta\mathfrak{A}_\mathfrak{c} - \xi\mathfrak{R} \end{cases}, \tag{5}$$

with

$$\mathfrak{S}(0) = \mathfrak{S}_0 \geqslant 0,\ \mathfrak{E}(0) = \mathfrak{E}_0 \geqslant 0,\ \mathfrak{A}(0) = \mathfrak{A}_0 \geqslant 0,$$
$$\mathfrak{A}_c(0) = \mathfrak{A}_{c_0} \geqslant 0,\ \mathfrak{C}(0) = \mathfrak{C}_0 \geqslant 0,\ \mathfrak{R}(0) = \mathfrak{R}_0 \geqslant 0.$$

where:

- N: total population size.
- $\mathfrak{S}$, $\mathfrak{E}$, $\mathfrak{A}$, $\mathfrak{A}_c$, $\mathfrak{C}$, $\mathfrak{R}$: are the susceptible individuals, exposed individuals, acute infected individuals, asymptomatic carriers, chronic infected individuals, and recovered individuals, respectively.
- Λ: the birth rate of the susceptible individuals.
- β: the effective contact rate.
- ξ: natural mortality rate in each component of the model.
- $\varkappa(1-\zeta)$: infection rate for exposed persons.
- $\varkappa\zeta$: transmission rate of people to acute infected individuals class. $\mathfrak{A}$
- η: a rate at which acute individuals become carriers of the disease.
- ς: a rate at which asymptomatic individuals become carriers of the disease.
- k: transmission rate of acute people to recovery class.
- ρ: transmission rate of asymptomatic people to the recovery class.
- σ: transmission rate of the carrier's people to the recovery class.
- γ: death rate due to the disease in acute.
- ε: death rate due to the disease in chronic.
- ϕ: the coefficient of asymptomatic individuals.
- ϵ: the coefficient of carriers individuals.

Now we extend the ordinary model (5) to a new fractal-fractional model using the fractal-fractional derivative of order $0 < \gamma \leqslant 1$, and the fractal dimension $0 < \theta \leqslant 1$, in Riemann-Liouville sense with power law (denoted by ${}^{FFP}D_{0,t}^{\gamma,\theta}$), then we obtain the following model:

$$\begin{cases} {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{S} = \Lambda - \beta(\mathfrak{A} + \phi\mathfrak{A}_c + \epsilon\mathfrak{C})\mathfrak{S} - \xi\mathfrak{S} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{E} = \beta(\mathfrak{A} + \phi\mathfrak{A}_\mathfrak{c} + \epsilon\mathfrak{C})\mathfrak{S} - (\xi + \varkappa)\mathfrak{E} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{A} = \varkappa\zeta\mathfrak{E} - (\xi + \varrho + \eta + k)\mathfrak{A} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{A}_c = \varkappa(1-\zeta)\mathfrak{E} - (\xi + \varsigma + \rho)\mathfrak{A}_c \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{C} = \eta\mathfrak{A} + \varsigma\mathfrak{A}_\mathfrak{c} - (\xi + \varepsilon + \sigma)\mathfrak{C} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{R} = k\mathfrak{A} + \sigma\mathfrak{C} + \zeta\mathfrak{A}_\mathfrak{c} - \xi\mathfrak{R} \end{cases}. \tag{6}$$

The fractal model (6) provides a highly realistic result, thus allowing us to obtain essential predictions for disease behavior. Since the variable $\mathfrak{R}$ does not participate in the first five equations, we only consider the equations for $\mathfrak{S}$, $\mathfrak{E}$, $\mathfrak{A}$, $\mathfrak{A}_c$,and $\mathfrak{C}$, which are coupled, and leave out the equation for $\mathfrak{R}$ [2]. System (6) is simplified as follows:

$$\begin{cases} {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{S} = \Lambda - \beta(\mathfrak{A} + \phi\mathfrak{A}_c + \epsilon\mathfrak{C})\mathfrak{S} - \xi\mathfrak{S} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{E} = \beta(\mathfrak{A} + \phi\mathfrak{A}_{\mathfrak{c}} + \epsilon\mathfrak{C})\mathfrak{S} - (\xi + \varkappa)\mathfrak{E} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{A} = \varkappa\zeta\mathfrak{E} - (\xi + \varrho + \eta + k)\mathfrak{A} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{A}_c = \varkappa(1-\zeta)\mathfrak{E} - (\xi + \varsigma + \rho)\mathfrak{A}_c \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{C} = \eta\mathfrak{A} + \varsigma\mathfrak{A}_{\mathfrak{c}} - (\xi + \varepsilon + \sigma)\mathfrak{C} \end{cases}, \tag{7}$$

with the following initial conditions:

$$\mathfrak{S}(0) = \mathfrak{S}_0 \geqslant 0, \mathfrak{E}(0) = \mathfrak{E}_0 \geqslant 0, \mathfrak{A}(0) = \mathfrak{A}_0 \geqslant 0, \tag{8}$$
$$\mathfrak{A}_c(0) = \mathfrak{A}_{c_0} \geqslant 0, \mathfrak{C}(0) = \mathfrak{C}_0 \geqslant 0, \mathfrak{R}(0) = \mathfrak{R}_0 \geqslant 0.$$

Then, we restrict ourselves throughout the paper to the reduced model (7).

4 Qualitative analysis of the proposed HBV model

Let $E = [C([0, T], \mathbb{R}^+)]^5$ be a Banach space equipped with the norm

$$||\mathfrak{U}||_E = ||\mathfrak{S}(t)||_\infty + ||\mathfrak{E}(t)||_\infty + ||\mathfrak{A}(t)||_\infty + ||\mathfrak{A}_c(t)||_\infty + ||\mathfrak{C}(t)||_\infty.$$

Then, we reformulate the model (7) - (8) in the following form:

$$\begin{cases} {}^{FFP}D_{0,t}^{\gamma,\theta}\vartheta(t) = \mathbf{F}(t, \vartheta(t)), \quad \gamma, \theta \in (0, 1] \\ \vartheta(0) = \vartheta_0 \geqslant 0 \end{cases}, \tag{9}$$

where the vector $\vartheta \in E$

$$\vartheta(t) = (\vartheta_1(t), \vartheta_2(t), \vartheta_3(t), \vartheta_4(t), \vartheta_5(t)) = (\mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t)),$$

denotes the state variables, and

$$\vartheta(0) = (\vartheta_1(0), \vartheta_2(0), \vartheta_3(0), \vartheta_4(0), \vartheta_5(0)) = (\mathfrak{S}(0), \mathfrak{E}(0), \mathfrak{A}(0), \mathfrak{A}_c(0), \mathfrak{C}(0)),$$

denotes the corresponding initial condition. Moreover, $\mathbf{F} \in E$ defines a continuous vector as follows:

$$\mathbf{F}(t, \vartheta(t)) = \begin{pmatrix} f_1(t, \vartheta_1(t)) \\ f_2(t, \vartheta_2(t)) \\ f_3(t, \vartheta_3(t)) \\ f_4(t, \vartheta_4(t)) \\ f_5(t, \vartheta_5(t)) \end{pmatrix} = \begin{pmatrix} \Lambda - \beta(\mathfrak{A} + \phi\mathfrak{A}_c + \epsilon\mathfrak{C})\mathfrak{S} - \xi\mathfrak{S} \\ \beta(\mathfrak{A} + \phi\mathfrak{A}_{\mathfrak{c}} + \epsilon\mathfrak{C})\mathfrak{S} - (\xi + \varkappa)\mathfrak{E} \\ \varkappa\zeta\mathfrak{E} - (\xi + \varrho + \eta + k)\mathfrak{A} \\ \varkappa(1-\zeta)\mathfrak{E} - (\xi + \varsigma + \rho)\mathfrak{A}_c \\ \eta\mathfrak{A} + \varsigma\mathfrak{A}_{\mathfrak{c}} - (\xi + \varepsilon + \sigma)\mathfrak{C} \end{pmatrix}. \tag{10}$$

Applying the fractal-fractional integral on both sides of (9), we obtain [1]:

$$\vartheta(t) = \vartheta_0 + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1}\mathbf{F}(s, \vartheta(s))ds. \tag{11}$$

Then, Model (7) - (8) is equivalent to the following integral equation:

$$\vartheta(t) = \vartheta_0 + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1}\mathbf{F}(s, \vartheta(s))ds. \tag{12}$$

4.1 Existence and uniqueness results

Naturally, before analyzing any biological model, we ask whether such a dynamical problem really exists or not. To answer this question, we try to demonstrate the existence and uniqueness of the problem (7) - (8) using fixed point theory.

Theorem 1 *Let $\gamma, \theta \in (0, 1]$, $\vartheta_0 \in \mathbb{R}$,. The fractal-fractional problem (7) - (8) has a solution if the following assumption holds:*

$$T \leqslant \left(\frac{\Gamma(\gamma)}{5\theta\beta(\gamma,\theta)\varpi}\right)^{\frac{1}{\theta}},$$

where

$$\varpi = \max[(\xi + \beta N_0(1+\phi+\epsilon)+1), (\beta N_0(1+\phi+\epsilon)+(\xi+\varkappa)), (\varkappa\zeta-(\xi+\varrho-\eta+k)+3),$$
$$(\varkappa(1-\zeta)-(\xi+\varsigma+\rho)+3), (\eta+\tau-(\mu+\varepsilon+\sigma)+2)].$$

Lemma 1 *If the hypotheses of Theorem 1 are assumed. Then, a function $\vartheta \in E$ is a solution of the fractal-fractional problem (7) - (8) if and only if it is a solution of the fractal-fractional integral equation (11).*

Proof (Of theorem 1)
First, we transform Problem (7) - (8) into a fixed-point problem $H\vartheta(t) = \vartheta(t)$, such that:

$$H\vartheta(t) = (H_1\vartheta(t), H_2\vartheta(t), H_3\vartheta(t), H_4\vartheta(t), H_5\vartheta(t)),$$

and

$$H\vartheta(t) = \vartheta_0 + \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1}\mathbf{F}(s,\vartheta(s))ds. \tag{13}$$

If $\vartheta \in E$, then $H\vartheta \in E$, because $H\vartheta$ is an operator of a constant and a primitive of continuous functions. According to Lemma (1), it can be easily seen that the fixed points of H are solutions to the problem (7) - (8). We shall show that H satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

- **Step 1** $(H\vartheta)$ is continuous
  Let $(\vartheta_n)_{n\in\mathbb{N}} = (\mathfrak{S}_n, \mathfrak{E}_n, \mathfrak{A}_n, \mathfrak{A}_{c_n}, \mathfrak{C}_n)$ be five positive sequences that satisfy $\lim\limits_{n\to\infty}\vartheta_n = \vartheta$ in E. Then, for each $t \in [0, T]$, we have:

$$|H_i\vartheta_n(t) - H_i\vartheta(t)| \leqslant \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1}|f_i(s,\vartheta_n(s)) - f_i(s,\vartheta(s))|ds. \tag{14}$$

  where f_i is defined in (10).
  For the first equation of Problem (7) - (8), we have:

$$\begin{aligned}
|^{FFP}D^{\theta\gamma}\mathfrak{S}_n(t) - {}^{FFP}D^{\theta\gamma}\mathfrak{S}(t)| &= |f_1(t,\vartheta_n(t)) - f_1(t,\vartheta(t))| \\
&\leqslant \xi|\mathfrak{S}-\mathfrak{S}_n| + \beta|\mathfrak{A}\mathfrak{S}-\mathfrak{A}_n\mathfrak{S}_n| \\
&\quad + \beta\phi|\mathfrak{A}_c\mathfrak{S}-\mathfrak{A}_{c_n}\mathfrak{S}_n| + \epsilon\beta|\mathfrak{C}\mathfrak{S}-\mathfrak{C}_n\mathfrak{S}| \\
&\leqslant (\xi+\beta\mathfrak{A}_n)|\mathfrak{S}-\mathfrak{S}_n| + \beta\mathfrak{S}|\mathfrak{A}-\mathfrak{A}_n| + \beta\phi\mathfrak{A}_{c_n}|\mathfrak{S}-\mathfrak{S}_n| \\
&\quad + \beta\varkappa\mathfrak{S}|\mathfrak{A}_c-\mathfrak{A}_{c_n}| + \epsilon\beta\mathfrak{C}_n|\mathfrak{S}-\mathfrak{S}_n| + \epsilon\beta\mathfrak{S}|\mathfrak{C}-\mathfrak{C}_n| \\
&\leqslant (\xi+\beta\mathfrak{A}_n+\beta\phi\mathfrak{A}_{c_n}+\epsilon\beta\mathfrak{C}_n)|\mathfrak{S}-\mathfrak{S}_n| + \beta\mathfrak{S}|\mathfrak{A}-\mathfrak{A}_n| \\
&\quad + \beta\phi\mathfrak{S}|\mathfrak{A}_c-\mathfrak{A}_{c_n}| + \epsilon\beta\mathfrak{S}|\mathfrak{C}-\mathfrak{C}_n| \\
&\leqslant [\xi+\beta((\mathfrak{A}_n+\mathfrak{S})+\phi(\mathfrak{A}_{c_n}+\mathfrak{S})+\epsilon(\mathfrak{C}_n+\mathfrak{S}))]|X-X_n| \\
&\leqslant [\xi+\beta((\sup_{t\in[0,T]}\mathfrak{A}_n+\sup_{t\in[0,T]}\mathfrak{S})+\phi(\sup_{t\in[0,T]}\mathfrak{A}_{c_n}+\sup_{t\in[0,T]}\mathfrak{S}) \\
&\quad + \epsilon(\sup_{t\in[0,T]}\mathfrak{C}_n+\sup_{t\in[0,T]}\mathfrak{S}))]||\vartheta-\vartheta_n||_E.
\end{aligned}$$

Now, using the fact that the total population

$$N(t) = \mathfrak{S}(t) + \mathfrak{E}(t) + \mathfrak{A}(t) + \mathfrak{A}_c(t) + \mathfrak{C}(t),$$

and $N(t) \leqslant N(0)$ yields:

$$|^{FFP}D^{\theta\gamma}\mathfrak{S}_n(t) - ^{FFP}D^{\theta\gamma}\mathfrak{S}(t)| \leqslant [\xi + \beta N_0(1+\phi+\epsilon)+1]||\vartheta_n - \vartheta||_E. \tag{15}$$

For the other equations of (7) - (8) we get:

$$\begin{aligned}
|^{FFP}D^{\theta\gamma}\mathfrak{E}_n(t) - ^{FFP}D^{\theta\gamma}\mathfrak{E}(t)| &\leqslant [\beta N_0(1+\phi+\epsilon) + (\xi+\varkappa)]||\vartheta_n - \vartheta||_E,\\
|^{FFP}D^{\theta\gamma}\mathfrak{A}_n(t) - ^{FFP}D^{\theta\gamma}\mathfrak{A}(t)| &\leqslant [\varkappa\zeta - (\xi+\varrho-\eta+k)+3]||\vartheta_n - \vartheta||_E,\\
|^{FFP}D^{\theta\gamma}\mathfrak{A}_{c_n}(t) - ^{FFP}D^{\theta\gamma}\mathfrak{A}_c(t)| &\leqslant [\varkappa(1-\zeta) - (\xi+\varsigma+\rho)+3]||\vartheta_n - \vartheta||_E,\\
|^{FFP}D^{\theta\gamma}\mathfrak{C}_n(t) - ^{FFP}D^{\theta\gamma}\mathfrak{C}(t)| &\leqslant [\eta+\varsigma-(\xi+\varepsilon+\sigma)+2]||\vartheta_n - \vartheta||_E.
\end{aligned} \tag{16}$$

Since $\lim\limits_{n\to\infty}\vartheta_n = \vartheta$, then:

$$\begin{pmatrix} ^{FFP}D^{\theta\gamma}\mathfrak{S}_n(t)\\ ^{FFP}D^{\theta\gamma}\mathfrak{E}_n(t)\\ ^{FFP}D^{\theta\gamma}\mathfrak{A}_n(t)\\ ^{FFP}D^{\theta\gamma}\mathfrak{A}_{c_n}(t)\\ ^{FFP}D^{\theta\gamma}\mathfrak{C}_n(t)\end{pmatrix} \to \begin{pmatrix} ^{FFP}D^{\theta\gamma}\mathfrak{S}(t)\\ ^{FFP}D^{\theta\gamma}\mathfrak{E}(t)\\ ^{FFP}D^{\theta\gamma}\mathfrak{A}(t)\\ ^{FFP}D^{\theta\gamma}\mathfrak{A}_c(t)\\ ^{FFP}D^{\theta\gamma}\mathfrak{C}(t)\end{pmatrix}.$$

Hence, we obtain

$$\lim_{n\to\infty} f_i(t,\vartheta_n(t)) = f_i(t,\vartheta(t)), \quad i=\overline{1,5} \quad t\in[0,T].$$

Also, let $r>0$ be such that for each $t\in[0,T]$, we have

$$|f_i(t,\vartheta_n(t))| \leqslant r, |f_i(t,\vartheta(t))| \leqslant r, \quad i=\overline{1,5},$$

then,

$$\begin{aligned}
\frac{\theta}{\Gamma(\gamma)}s^{\theta-1}(t-s)^{\gamma-1}|f_i(s,\vartheta_n(s)) - f_i(s,\vartheta(s))| &\leqslant \frac{\theta}{\Gamma(\gamma)}s^{\theta-1}(t-s)^{\gamma-1}[|f_i(s,\vartheta_n(s))| + f_i(s,\vartheta(s))|]\\
&\leqslant \frac{2r\theta}{\Gamma(\gamma)}s^{\theta-1}(t-s)^{\gamma-1}.
\end{aligned}$$

For each $i=\overline{1,5}$, if the function $s\mapsto \frac{2r\theta}{\Gamma(\gamma)}s^{\theta-1}(t-s)^{\gamma-1}$ is integrable for all t in $[0,T]$, then the Lebesgue Dominated Convergence Theorem and (14) implies that

$$|H_i\vartheta_n(t) - H_i\vartheta(t)| \to 0 \text{ as } n\to\infty.$$

Hence,

$$\lim_{n\to\infty}||H\vartheta_n - H\vartheta||_E = 0.$$

Consequently, H is continuous.
Before proceeding to the next steps, let

$$k \geqslant \frac{5\Gamma(\gamma)}{\Gamma(\gamma) - 5\theta\beta(\theta,\gamma)T^\theta\varpi}\vartheta_i, \qquad \forall i=\overline{1,5},$$

and define the subset $Q_k = \{\vartheta\in E, \ \ ||\vartheta||_E \leqslant k\}$, note that Q_k is a bounded, closed, and convex subset of E, and let $H: Q_k \to E$.

- **Step2** : $H(Q_k) \subseteq Q_k$
  Firstly, from (15) and (16) we have:

$$\begin{aligned}
|^{FFP}D^{\theta\gamma}\mathfrak{S}(t)| &\leqslant [\xi + \beta N_0(1+\phi+\epsilon)+1]||\vartheta||_E,\\
|^{FFP}D^{\theta\gamma}\mathfrak{E}(t)| &\leqslant [\beta N_0(1+\phi+\epsilon)+(\xi+\varkappa)]||\vartheta||_E,\\
|^{FFP}D^{\theta\gamma}\mathfrak{A}(t)| &\leqslant [\varkappa\zeta-(\mu+\varrho-\eta+k)+3]||\vartheta||_E,\\
|^{FFP}D^{\theta\gamma}\mathfrak{A}_c(t)| &\leqslant [\varkappa(1-\zeta)-(\xi+\varsigma+\rho)+3]||\vartheta||_E,\\
|^{FFP}D^{\theta\gamma}\mathfrak{C}(t)| &\leqslant [\eta+\varsigma-(\xi+\varepsilon+\sigma)+2]||\vartheta||_E.
\end{aligned} \tag{17}$$

Put

$$\varpi = \max[(\xi+\beta N_0(1+\phi+\epsilon)+1), (\beta N_0(1+\phi+\epsilon)+(\xi+\varkappa)), (\varkappa\zeta-(\xi+\varrho-\eta+k)+3),\\ (\varkappa(1-\zeta)-(\xi+\varsigma+\rho)+3), (\eta+\tau-(\mu+\varepsilon+\sigma)+2)].$$

Thus, in each case, we get

$$|f_i(t,\vartheta(t))| \leqslant \varpi k, \qquad \forall i = \overline{1,5}.$$

Hence,

$$\begin{aligned}
|H_i\vartheta(t)| &\leqslant \vartheta_i + \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1}|f_i(s,\vartheta(s))|\\
&\leqslant \vartheta_i + \frac{\theta\beta(\theta,\gamma)\varpi t^{\theta+\gamma-1}}{\Gamma(\gamma)}k\\
&\leqslant \vartheta_i + \frac{\theta\beta(\theta,\gamma)\varpi T^{\theta}}{\Gamma(\gamma)}k\\
&\leqslant \frac{\Gamma(\gamma)-5\theta\beta(\theta,\gamma)\varpi T^{\theta}}{5\Gamma(\gamma)}k + \frac{\theta\beta(\theta,\gamma)\varpi T^{\theta}}{\Gamma(\gamma)}k,
\end{aligned}$$

This implies $|H_i\vartheta(t)| \leqslant \frac{1}{5}k$. Thus,

$$(||H_i\vartheta||_\infty)_{1\leqslant i\leqslant 5} \leqslant \frac{1}{5}k.$$

The above inequality implies,

$$||H\vartheta||_E = \sum_{i=1}^{5}||H_i\vartheta||_\infty \leqslant k.$$

as a result $H(Q_k) \subseteq Q_k$.
- **Step 3** The family $H(Q_k)$ is relatively compact.
  Let t_1, t_2 two numbers belonging to $[0,T]$, such that $0 \leqslant t_1 \leqslant t_2 \leqslant T$ and $\vartheta \in Q_k$, then for every $i = \overline{1,5}$, we get

$$\begin{aligned}
|H_i\vartheta(t_1) - H_i\vartheta(t_2)| &= |\vartheta_0 + \frac{\theta}{\Gamma(\gamma)}\int_0^{t_1} s^{\theta-1}(t_1-s)^{\gamma-1}f_i(s,\vartheta(s))ds - \vartheta_i\\
&\quad - \frac{\theta}{\Gamma(\gamma)}\int_0^{t_2} s^{\theta-1}(t_2-s)^{\gamma-1}f_i(s,\vartheta(s))ds|\\
&\leqslant \frac{\theta\varpi k}{\Gamma(\gamma)}[|\int_0^{t_1} s^{\theta-1}(t_1-s)^{\gamma-1}ds - \int_0^{t_2} s^{\theta-1}(t_2-s)^{\gamma-1}ds|]\\
&\leqslant \frac{\theta\varpi k}{\Gamma(\gamma)}(|I_1-I_2|).
\end{aligned}$$

Where

$$I_1 = \int_0^{t_1} s^{\theta-1}(t_1 - s)^{\gamma-1} ds,$$
$$I_2 = \int_0^{t_2} s^{\theta-1}(t_2 - s)^{\gamma-1} ds.$$

Putting $s = t_1 n$, to estimate I_1 and for evaluating I_2 we put $s = t_2 m$, then we obtain

$$\begin{aligned}|H_i\vartheta(t_1) - H_i\vartheta(t_2)| &\leqslant \frac{\theta\varpi k}{\Gamma(\gamma)}[|t_1^{\theta+\gamma-1}\beta(\gamma,\theta) - t_2^{\theta+\gamma-1}\beta(\gamma,\theta)|]\\ &\leqslant \frac{\theta\varpi k\beta(\gamma,\theta)}{\Gamma(\gamma)}|t_1 - t_2|,\end{aligned}$$

therefore, if $t_1 \to t_2$ we get $|H_i\vartheta(t_1) - H_i\vartheta(t_2)| \to 0$ for all $i = \overline{1,5}$.

As a consequence of steps 1 to 3 along with the Arzelá-Ascoli theorem, we conclude that $H : Q_k \to Q_k$ fulfills Schauder's fixed point theorem conditions, so we deduce that H has a fixed point, which is a solution to the problem (7) - (8). □

Lemma 2 *Let $\vartheta_i, \vartheta_i^* \in E$, and let $\Xi_i > 0, i = \overline{1,5}$ be constants satisfying:*

$$||\vartheta_i|| \leqslant \Xi_i, \quad i = \overline{1,5},$$

then the functions $f_i(t, \vartheta(t)), i = \overline{1,5}$ defined in (9) fulfill the Lipschitz condition given by

$$||f_i(t, \vartheta_i(t)) - f_i(t, \vartheta_i^*(t))||_E \leqslant \Pi_i||\vartheta_i - \vartheta_i^*||_E, \quad i = \overline{1,5},$$

such that

$$\vartheta_i^*(t) = (\vartheta_1^*(t), \vartheta_2^*(t), \vartheta_3^*(t), \vartheta_4^*(t), \vartheta_5^*(t)) = (\mathfrak{S}^*(t), \mathfrak{E}^*(t), \mathfrak{A}^*(t), \mathfrak{A}_c^*(t), \mathfrak{C}^*(t)),$$

and

$$\Pi_1 = \xi + \beta(\Xi_3 + \phi\Xi_4 + \epsilon\Xi_5);\ \Pi_2 = \xi + \varkappa;\ \Pi_3 = \xi + \varrho - \eta + k;\ \Pi_4 = \xi + \tau + \upsilon;\ \Pi_5 = \xi + \varepsilon + \sigma.$$

Proof For the first equation of the fractal-fractional problem (7) - (8) we have:

$$\begin{aligned}|f_1(t, \vartheta_1(t)) - f_1(t, \vartheta_1^*(t))| &= |\Lambda - \beta(\mathfrak{A} + \phi\mathfrak{A}_c + \epsilon\mathfrak{C})\mathfrak{S} - \xi\mathfrak{S} - \Lambda + \beta(\mathfrak{A} + \phi\mathfrak{A}_c + \epsilon\mathfrak{C})\mathfrak{S}^* + \xi\mathfrak{S}^*|\\ &\leqslant \beta(\mathfrak{A} + \phi\mathfrak{A}_c + \epsilon\mathfrak{C})\mathfrak{S}|\mathfrak{S} - \mathfrak{S}^*| + \xi|\mathfrak{S} - \mathfrak{S}^*|\\ &\leqslant (\beta(\Xi_3 + \phi\Xi_4 + \epsilon\Xi_5) + \xi)||\mathfrak{S} - \mathfrak{S}^*||.\end{aligned}$$

Similarly, we get:

$$\begin{aligned}|f_2(t, X_2(t)) - f_2(t, X_2^*(t))| &\leqslant (\xi + \varkappa)||\mathfrak{E} - \mathfrak{E}^*||,\\ |f_3(t, X_3(t)) - f_3(t, X_3^*(t))| &\leqslant (\xi + \varrho - \eta + k)||\mathfrak{A} - \mathfrak{A}^*||,\\ |f_4(t, X_4(t)) - f_4(t, X_4^*(t))| &\leqslant (\xi + \varsigma + \rho)||\mathfrak{A}_c - \mathfrak{A}_c^*||,\\ |f_5(t, X_5(t)) - f_5(t, X_5^*(t))| &\leqslant (\xi + \varepsilon + \sigma)||\mathfrak{C} - \mathfrak{C}^*||.\end{aligned}$$

□

Theorem 2 *(uniqueness) the solution of problem (7) - (8) is unique under the condition:*

$$G = \frac{\theta T^{\theta}\beta(\gamma,\theta)}{\Gamma(\gamma)}\Pi_i < 1, \quad i = \overline{1,5}.$$

Proof Let $\vartheta, \vartheta^* \in E$, for $t \in [0, T]$, and using Lemma 2 for each $i = \overline{1, 5}$, we have:

$$\begin{aligned}
|H_i\vartheta_i(t) - H_i\vartheta_i^*(t)| &= |\frac{\theta}{\Gamma(\gamma)}(\int_0^t s^{\theta-1}(t-s)^{\gamma-1}(f_i(s,\vartheta_i(s)) - f_i(s,\vartheta_i^*(s)))ds| \\
&\leqslant \frac{\theta\Pi_i}{\Gamma(\gamma)}||\vartheta_i(s) - \vartheta_i^*(s)||_E \int_0^t s^{\theta-1}(t-s)^{\gamma-1}ds \\
&\leqslant \frac{\theta\Pi_i\beta(\theta,\gamma)t^{\theta+\gamma-1}}{\Gamma(\gamma)}||\vartheta_i(s) - \vartheta_i^*(s)||_E \\
&\leqslant \frac{\theta\Pi_i\beta(\theta,\gamma)t^{\theta}}{\Gamma(\gamma)}||\vartheta_i(s) - \vartheta_i^*(s)||_E \\
&\leqslant \frac{\theta\Pi_i\beta(\theta,\gamma)T^{\theta}}{\Gamma(\gamma)}||\vartheta_i(s) - \vartheta_i^*(s)||_E.
\end{aligned}$$

Therefore,

$$||H_iX_i(t) - H_iX_i^*(t)||_\infty \leqslant G||\vartheta_1 - \vartheta_2||_E, \quad \forall i = \overline{1, 5},$$

where $G = \frac{\theta\Pi_i\beta(\theta,\gamma)T^{\theta}}{\Gamma(\gamma)}$.

If $G = \frac{\theta\Pi_i\beta(\theta,\gamma)T^{\theta}}{\Gamma(\gamma)} < 1$, then H is a contraction. Therefore, it fulfills the assumption of Banach's fixed point theorem. Hence, H has a unique fixed point, so as a result, the problem (7) - (8) has a unique solution. □

4.2 Stability results

In this part of the paper, we study the stability of the considered model (7) - (8) in the frame of Ulam-Hyers and Ulam-Hyers-Rassias stability.

Starting with Ulam-Hyers stability, we will consider small perturbations $\sigma_i \in E (i \in \{1, ..., 5\})$ that satisfy:

(C1) $\sigma_i(0) = 0$.
(C2) $|\sigma_i(t)| \leqslant \epsilon_i \forall\epsilon_i > 0, t \in [0, T], i = \overline{1, 5}$.
(C3) ${}^{FFP}D^{\gamma,\theta}\Upsilon(t) = f_i(t, \Upsilon(t)) + \sigma_i(t), \quad t \in [0, T]$.

Such that $\Upsilon(t) = (\mathfrak{S}^*, \mathfrak{E}^*, \mathfrak{A}^*, \mathfrak{A}_c^*, \mathfrak{C}^*) \in E$ is a solution to the following inequality

$$|{}^{FFP}D^{\gamma,\theta}\Upsilon(t) - f_i(t, \Upsilon(t))| \leqslant \epsilon_i \quad \epsilon_i > 0 \quad t \in [0, T], \quad i = \overline{1, 5}. \tag{18}$$

Definition 4 The HBV fractal-fractional model is Ulame Hyers stable if there exists real constants $M_{f_i} > 0$, $(i \in \{1, ..., 5\})$, such that $\forall\epsilon_i > 0$, and $\forall\Upsilon \in E$, which satisfies Inequality (18), there exists a solution $\vartheta \in E$ of our fractal-fractional problem (7) - (8) with:

$$|\Upsilon(t) - \vartheta(t)| \leqslant \epsilon_i M_{f_i} \quad \forall t \in [0, T].$$

Lemma 3 *Solutions to the perturbed problem*

$$\begin{cases} {}^{FFP}D^{\gamma,\theta}\Upsilon(t) = f_i(t, \Upsilon(t)) + \sigma_i(t) \\ \Upsilon(0) = \Upsilon_0 \end{cases}, \tag{19}$$

satisfy:

$$|\Upsilon(t) - \Upsilon(0) - {}^{FFP}I_{0,t}^{\gamma} f_i(t, \Upsilon(t))| \leqslant (\frac{\theta\beta(\gamma,\theta)}{\Gamma(\gamma)}t^{\theta+\gamma-1})\epsilon_i, \quad i = \overline{1, 5}, \tag{20}$$

where $\Upsilon_0 = (\Upsilon_1(0), \Upsilon_2(0), \Upsilon_3(0), \Upsilon_4(0), \Upsilon_5(0)) = (\mathfrak{S}^*(0), \mathfrak{E}^*(0), \mathfrak{A}^*(0), \mathfrak{A}_c^*(0), \mathfrak{C}^*(0))$.

Proof Let $i \in \{1, .., 5\}$, then for each case, the solution of perturbed problems (C3) is given by

$$\begin{aligned}\mathsf{Y}(t) &= \mathsf{Y}(0) + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1}[f_i(s, \mathsf{Y}(s)) + \sigma_i(s)]ds \\ &= \mathsf{Y}(0) + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \mathsf{Y}(s))ds + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1}\sigma_i(s)ds \\ &= \mathsf{Y}(0) + {}^{FFP} I_{0,t}^{\gamma} f_i(t, \mathsf{Y}(t)) + {}^{FFP} I_{0,t}^{\gamma}\sigma_i(s).\end{aligned}$$

Thus,

$$\begin{aligned}|\mathsf{Y}(t) - \mathsf{Y}(0) - {}^{FFP} I_{0,t}^{\gamma} f_i(t, \mathsf{Y}(t))| &= |\frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1}\sigma_i(s)ds| \\ &\leqslant \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1}|\sigma_i(s)|ds \\ &\leqslant (\frac{\theta\beta(\theta, \gamma)}{\Gamma(\gamma)} t^{\theta+\gamma-1})\epsilon_i.\end{aligned}$$

□

Theorem 3 *(Ulam-Hyers stability) Under the assumptions in Lemma 2 and Theorem 2, with*

$$(\frac{\theta\Pi_i\beta(\gamma, \theta)}{\Gamma(\gamma)} t^{\theta+\gamma-1}) < 1,$$

then System (7) - (8) is Ulam-Hyers stable.

Proof Let $\mathsf{Y} \in E$ be a solution to inequality (18), and $\vartheta \in E$ be a unique solution of the following problem

$$\begin{cases} {}^{FFP}D^{\gamma,\theta}\vartheta(t) = f_i(t, \vartheta(t)), \quad i = \overline{1,5} \\ \vartheta(0) = \mathsf{Y}_0 \end{cases}, \tag{21}$$

then,

$$\begin{aligned}|\mathsf{Y} - \vartheta| &= |\mathsf{Y}(t) - (\vartheta_0 + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \vartheta(s))ds| \\ &\leqslant |\mathsf{Y}(t) - (\vartheta_0 + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \vartheta(s))ds \\ &\quad - (\mathsf{Y}_0 + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \mathsf{Y}(s))ds) \\ &\quad + (\mathsf{Y}_0 + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \mathsf{Y}(s))ds)| \\ &\leqslant |\mathsf{Y}(t) - (\mathsf{Y}_0 + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \mathsf{Y}(s))ds| \\ &\quad + |(\mathsf{Y}_0 + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \mathsf{Y}(s))ds \\ &\quad - (\vartheta_0(t) + \frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \vartheta(s))ds| \\ &\leqslant (\frac{\theta\beta(\gamma, \theta)}{\Gamma(\gamma)} t^{\theta+\gamma-1})\epsilon_i + |\frac{\theta}{\Gamma(\gamma)} \int_0^t s^{\theta-1}(t-s)^{\gamma-1}[f_i(s, \mathsf{Y}(s) - f_i(s, \vartheta(s))]ds| \\ &\leqslant (\frac{\theta\beta(\gamma, \theta)}{\Gamma(\gamma)} t^{\theta+\gamma-1})\epsilon_i + \frac{\theta\Pi_i\beta(\theta, \gamma)}{\Gamma(\gamma)} t^{\theta+\gamma-1}|\mathsf{Y} - \vartheta|.\end{aligned}$$

Which yields there is $M_{f_i} = \frac{(\frac{\theta\beta(\gamma,\theta)}{\Gamma(\gamma)} t^{\theta+\gamma-1})\epsilon_i}{1-(\frac{\theta\Pi_i\beta(\gamma,\theta)}{\Gamma(\gamma)} t^{\theta+\gamma-1})}$, $i = \overline{1,5}$, such that

$$||\mathsf{Y} - \vartheta|| \leqslant M_{f_i}\epsilon_i.$$

□

Next, for Ulam-Hyers-Rassias stability, let $\varphi_i \in E$, and we consider $\sigma_i^* \in E$ as a small perturbation that satisfies:

1. $|\sigma_i^*(t)| \leqslant \epsilon_i \varphi_i(t)$, for $t \in [0, T], i = \overline{1, 5}$.
2. ${}^{FFP}D_{0,t}^{\gamma\theta}\Upsilon(t) = f_i(t, \Upsilon(t)) + \sigma_i^*(t), t \in [0, T], i = \overline{1, 5}$.

This allows us to say that $\Upsilon(t)$ is a solution to the following inequality:

$$|{}^{FFP}D_{0,t}^{\gamma\theta}\Upsilon(t) - f_i(t, \Upsilon(t))| \leqslant \epsilon_i \varphi_i(t) \quad \forall t \in [0, T], i = \overline{1, 5}, \tag{22}$$

for each $i = \overline{1, 5}$ and $f_i \in E$.

Definition 5 The HBV fractal-fractional problem is Hyers-Ulam-Rassias stable with regard to functions φ_i, ($i \in 1, ..., 5$), if there exists $M_{f_i,\varphi_i} > 0$, such that for each $\Upsilon \in E$ which satisfies the inequality (22), there exists a solution $\vartheta \in E$ of (7)-(8) with:

$$|\Upsilon(t) - \vartheta(t)| \leqslant M_{f_i,\varphi_i}\epsilon_i \varphi_i(t), \quad \forall t \in [0, T], i = \overline{1, 5}.$$

Theorem 4 *Let $f_i \in E$ satisfies Lemma 2, and $\varphi_i \in E$ be an increasing function, and $M_{\varphi_i} > 0$ $(i = 1, .., 5)$, such that for each $t \in [0, T]$*

$${}^{FFP}I_{0,t}^{\gamma,\theta}\varphi_i(t) \leqslant M_{\varphi_i}\varphi_i(t), \quad (i = \overline{1, 5}). \tag{23}$$

Then, the given fractal-fractional model (7) - (8) is the Ulam-Hyers-Rassias stable.

Proof Let $\Upsilon \in E$ satisfy Inequality (22) and $\vartheta \in E$ be a unique solution of (18), then we get

$$\Upsilon(t) = \Upsilon_0 + \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \vartheta(s))ds + \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1}\sigma_i^*(s)ds,$$

and

$$\vartheta(t) = \vartheta_0 + \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \vartheta(s))ds, \quad t \in [0, T].$$

Then,

$$\begin{aligned}
|\Upsilon(t) - \vartheta(t)| = {} & |\Upsilon_0 + \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \Upsilon(s))ds) \\
& + \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1}\sigma_i^*(s)ds - \vartheta_0 - \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1} f_i(s, \vartheta(s))ds| \\
\leqslant {} & \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1}|f_i(s, \Upsilon) - f_i(s, \vartheta(s))|ds \\
& + \frac{\theta}{\Gamma(\gamma)}\int_0^t s^{\theta-1}(t-s)^{\gamma-1}\sigma_i^*(s)ds \\
\leqslant {} & \epsilon_i M_{\varphi_i}\varphi_i(t) + \frac{\theta\Pi_i\beta(\gamma, \theta)}{\Gamma(\gamma)}t^{\theta+\gamma-1}|\Upsilon(t) - \vartheta(t)|.
\end{aligned}$$

Hence,

$$||\Upsilon(t) - \vartheta(t)|| \leqslant \frac{\epsilon_i M_{\varphi_i}\varphi_i(t)}{1 - \frac{\theta\Pi_i\beta(\gamma,\theta)}{\Gamma(\gamma)}t^{\theta+\gamma-1}}.$$

Introducing the notation $M_{f_i,\varphi_i} = \frac{M_{\varphi_i}}{1 - \frac{\theta\Pi_i\beta(\gamma,\theta)}{\Gamma(\gamma)}t^{\theta+\gamma-1}}$, yields

$$|\Upsilon(t) - \vartheta(t)| \leqslant \epsilon_i M_{f_i,\varphi_i}\varphi_i(t).$$

□

5 Numerical scheme

We simulate the proposed HBV disease model (6) using a numerical method based on a Newton polynomial. First, let's write the model (6) as:

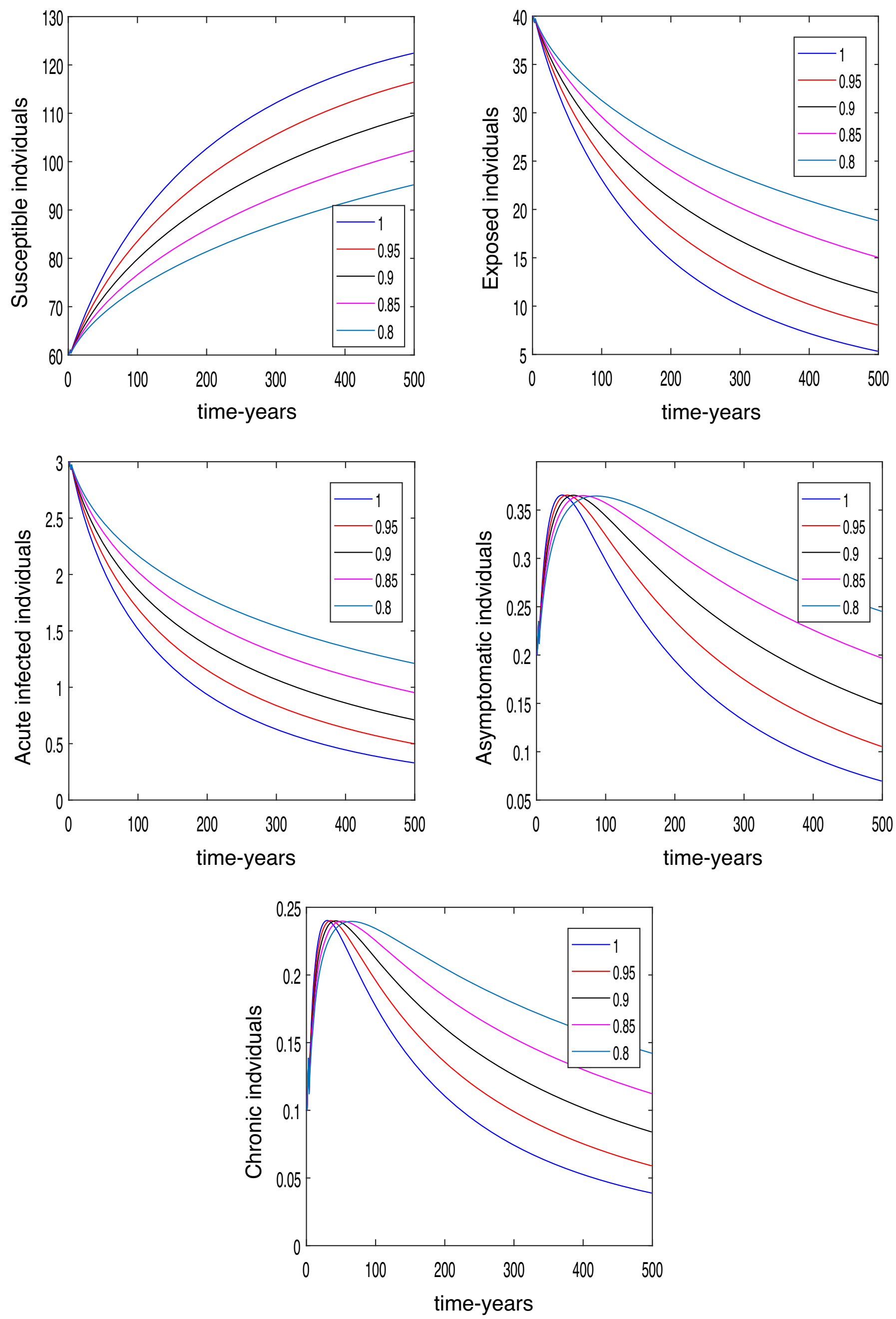

Fig. 1 Numerical solutions of Model (7) - (8) for $\theta = 0.9$, and various dimensions γ

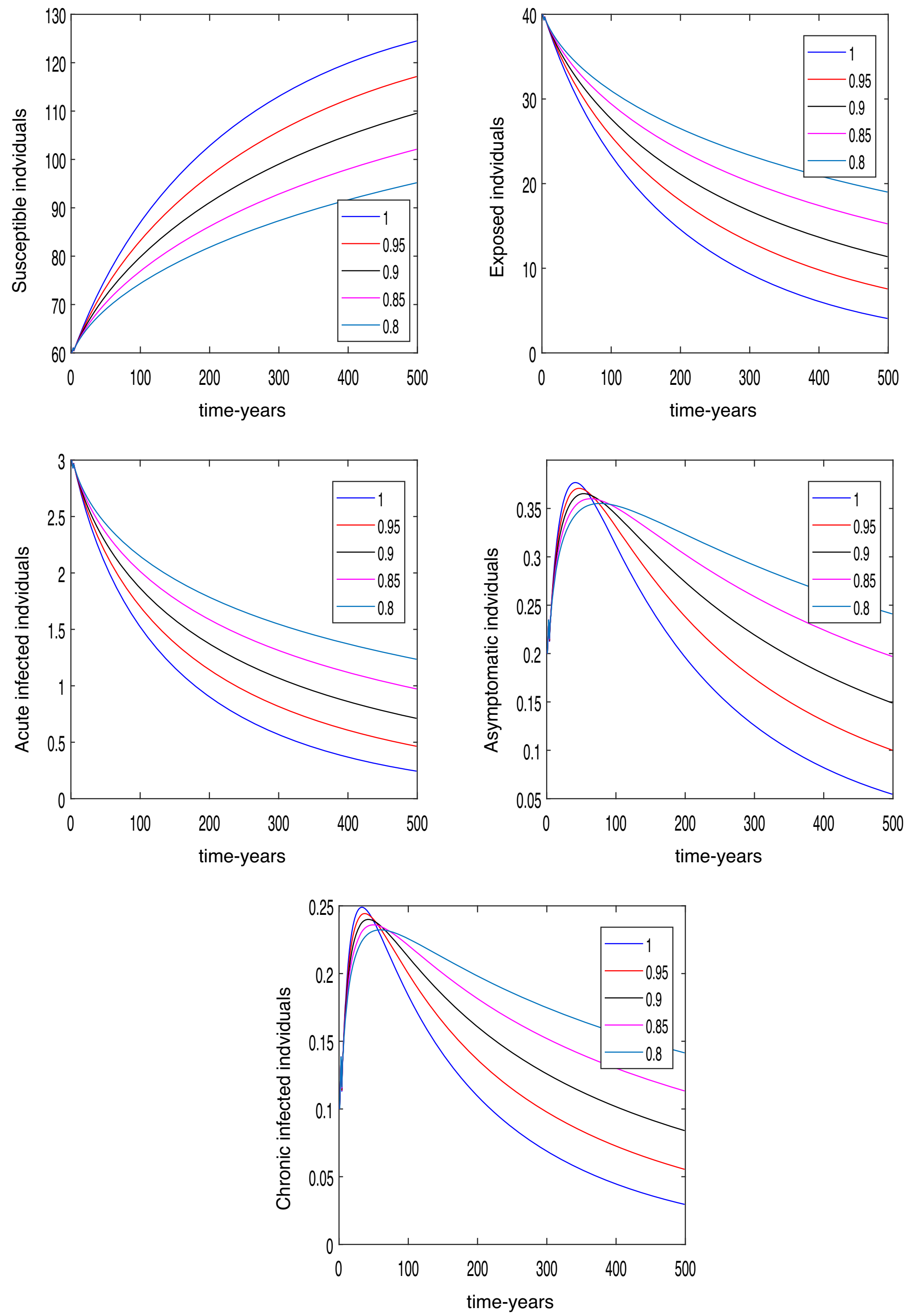

Fig. 2 Numerical solutions of Model (7) - (8) for $\gamma = 0.9$ and various θ

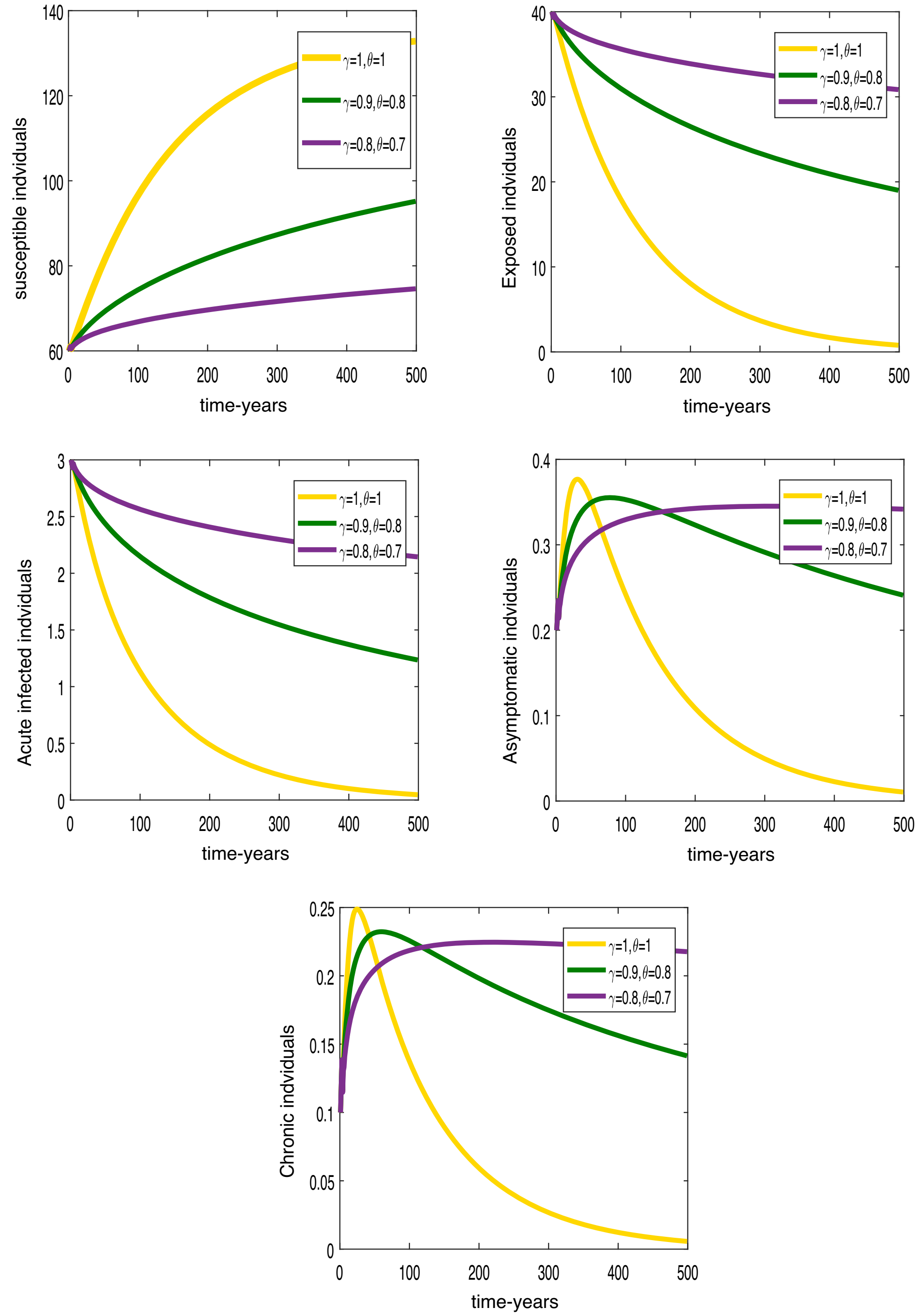

Fig. 3 Numerical solutions of Model (7) - (8) for different $\theta = 1, 0.8, 0.7$ and $\gamma = 1, 0.9, 0.8$,respectively

$$\begin{cases} {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{S}(t) = f_1(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) & = \Lambda - \beta(\mathfrak{A} + \phi\mathfrak{A}_c + \epsilon\mathfrak{C})\mathfrak{S} - \xi\mathfrak{S} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{E}(t) = f_2(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) & = \beta(\mathfrak{A} + \phi\mathfrak{A}_\mathfrak{c} + \epsilon\mathfrak{C})\mathfrak{S} - (\xi + \varkappa)\mathfrak{E} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{A}(t) = f_3(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) & = \varkappa\zeta\mathfrak{E} - (\xi + \varrho + \eta + k)\mathfrak{A} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{A}_c(t) = f_4(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) & = \varkappa(1 - \zeta)\mathfrak{E} - (\xi + \varsigma + \rho)\mathfrak{A}_c \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{C}(t) = f_5(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) & = \eta\mathfrak{A} + \varsigma\mathfrak{A}_\mathfrak{c} - (\xi + \varepsilon + \sigma)\mathfrak{C} \\ {}^{FFP}D_{0,t}^{\gamma,\theta}\mathfrak{R}(t) = f_6(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) & = k\mathfrak{A} + \sigma\mathfrak{C} + \zeta\mathfrak{A}_\mathfrak{c} - \xi\mathfrak{R} \end{cases}.$$

Since the integral is differentiable ,using Remark (1) we can reformulate the above fractal-fractional problem as follows:

$$\begin{cases} {}^{RL}D_{0,t}^{\gamma}\mathfrak{S}(t) = \theta t^{\theta-1} f_1(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ {}^{RL}D_{0,t}^{\gamma}\mathfrak{E}(t) = \theta t^{\theta-1} f_2(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ {}^{RL}D_{0,t}^{\gamma}\mathfrak{A}(t) = \theta t^{\theta-1} f_3(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ {}^{RL}D_{0,t}^{\gamma}\mathfrak{A}_c(t) = \theta t^{\theta-1} f_4(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ {}^{RL}D_{0,t}^{\gamma}\mathfrak{C}(t) = \theta t^{\theta-1} f_5(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ {}^{RL}D_{0,t}^{\gamma}\mathfrak{R}(t) = \theta t^{\theta-1} f_6(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \end{cases}.$$

Then, we replace the RL derivative with the Caputo derivative; next, we apply the Riemann-Liouville fractional integral on both sides to get the following Volterra integral equations:

$$\begin{cases} \mathfrak{S}(t) = \mathfrak{S}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^t \tau^{\theta-1}(t-\tau)^{\gamma-1} f_1(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ \mathfrak{E}(t) = \mathfrak{E}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^t \tau^{\theta-1}(t-\tau)^{\gamma-1} f_2(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ \mathfrak{A}(t) = \mathfrak{A}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^t \tau^{\theta-1}(t-\tau)^{\gamma-1} f_3(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ \mathfrak{A}_c(t) = \mathfrak{A}_c(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^t \tau^{\theta-1}(t-\tau)^{\gamma-1} f_4(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ \mathfrak{C}(t) = \mathfrak{C}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^t \tau^{\theta-1}(t-\tau)^{\gamma-1} f_5(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ \mathfrak{R}(t) = \mathfrak{R}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^t \tau^{\theta-1}(t-\tau)^{\gamma-1} f_6(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \end{cases}. \tag{24}$$

Now to provide the numerical schemes for our fractal-fractional model for the HBV dynamics based on Newton polynomial approximation, we try to calculate the approximations at t_{n+1} through the following expression:

$$\begin{cases} \mathfrak{S}(t_{n+1}) = \mathfrak{S}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^{t_{n+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_1(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ \mathfrak{E}(t_{n+1}) = \mathfrak{E}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^{t_{n+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_2(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ \mathfrak{A}(t_{n+1}) = \mathfrak{A}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^{t_{n+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_3(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ \mathfrak{A}_c(t_{n+1}) = \mathfrak{A}_c(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^{t_{n+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_4(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ \mathfrak{C}(t_{n+1}) = \mathfrak{C}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^{t_{n+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_5(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\ \mathfrak{R}(t_{n+1}) = \mathfrak{R}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\int_0^{t_{n+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_6(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \end{cases} \tag{25}$$

Approximating the above integrals by:

$$
\begin{cases}
\mathfrak{S}(t_{n+1}) = \mathfrak{S}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\sum_{p=2}^{n} \int_{t_p}^{t_{p+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_1(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\
\mathfrak{E}(t_{n+1}) = \mathfrak{E}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\sum_{p=2}^{n} \int_{t_p}^{t_{p+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_2(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\
\mathfrak{A}(t_{n+1}) = \mathfrak{A}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\sum_{p=2}^{n} \int_{t_p}^{t_{p+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_3(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\
\mathfrak{A}_c(t_{n+1}) = \mathfrak{A}_c(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\sum_{p=2}^{n} \int_{t_p}^{t_{p+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_4(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\
\mathfrak{C}(t_{n+1}) = \mathfrak{C}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\sum_{p=2}^{n} \int_{t_p}^{t_{p+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_5(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t)) \\
\mathfrak{R}(t_{n+1}) = \mathfrak{R}(0) + \dfrac{\theta}{\Gamma(\gamma)} \displaystyle\sum_{p=2}^{n} \int_{t_p}^{t_{p+1}} \tau^{\theta-1}(t_{n+1}-\tau)^{\gamma-1} f_6(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t))
\end{cases}.
$$

Also, in the finite interval $[t_j, t_{j+1}]$, we approximate the functions $\tau^{\theta-1} f_i(t, \mathfrak{S}(t), \mathfrak{E}(t), \mathfrak{A}(t), \mathfrak{A}_c(t), \mathfrak{C}(t), \mathfrak{R}(t))$, $i = 1, 2, 3, 4, 5, 6$, using the two steps of the Newton polynomial and get:

$$
\mathfrak{S}(t_{n+1}) = \mathfrak{S}(0) + \frac{\theta}{\Gamma(\gamma)} \sum_{p=2}^{n} \int_{t_p}^{t_{p+1}}
\left\{
\begin{array}{c}
f_1(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2}) \\
+\left[\frac{f_1(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{h}\right. \\
\left.-\frac{f_1(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{h}\right](\tau - t_{p-2}) \\
+\left[\frac{f_1(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)}{2h^2}\right. \\
-\frac{2 f_1(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{2h^2} \\
\left.+\frac{f_1(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{2h^2}\right] \\
(\tau - t_{p-2})(\tau - t_{p-1})
\end{array}
\right\}
$$

$$
\times (t_{n+1}-\tau)^{\gamma-1} d\tau,
$$

$$
\mathfrak{E}(t_{n+1}) = \mathfrak{E}(0) + \frac{\theta}{\Gamma(\gamma)} \sum_{p=2}^{n} \int_{t_p}^{t_{p+1}}
\left\{
\begin{array}{c}
f_2(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2}) \\
+\left[\frac{f_2(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{h}\right. \\
\left.-\frac{f_2(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{h}\right](\tau - t_{p-2}) \\
+\left[\frac{f_2(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)}{2h^2}\right. \\
-\frac{2 f_2(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{2h^2} \\
\left.+\frac{f_1(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{2h^2}\right] \\
(\tau - t_{p-2})(\tau - t_{p-1})
\end{array}
\right\}
$$

$$
\times (t_{n+1}-\tau)^{\gamma-1} d\tau,
$$

$$
\begin{aligned}
\mathfrak{A}(t_{n+1}) = \mathfrak{A}(0) + \frac{\theta}{\Gamma(\gamma)} \sum_{p=2}^{n} \int_{t_p}^{t_{p+1}} & \left\{ \begin{array}{c}
f_3(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2}) \\
+ \left[\frac{f_3(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{h} \right. \\
\left. - \frac{f_3(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{h} \right] (\tau - t_{p-2}) \\
+ \left[\frac{f_3(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)}{2h^2} \right. \\
- \frac{2f_1(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{2h^2} \\
\left. + \frac{f_3(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{2h^2} \right] \\
(\tau - t_{p-2})(\tau - t_{p-1})
\end{array} \right\} \\
& \times (t_{n+1} - \tau)^{\gamma-1} d\tau,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\mathfrak{A}_c(t_{n+1}) = \mathfrak{A}_c(0) + \frac{\theta}{\Gamma(\gamma)} \sum_{p=2}^{n} \int_{t_p}^{t_{p+1}} & \left\{ \begin{array}{c}
f_4(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2}) \\
+ \left[\frac{f_4(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{h} \right. \\
\left. - \frac{f_4(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{h} \right] (\tau - t_{p-2}) \\
+ \left[\frac{f_4(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)}{2h^2} \right. \\
- \frac{2f_4(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{2h^2} \\
\left. + \frac{f_4(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{2h^2} \right] \\
(\tau - t_{p-2})(\tau - t_{p-1})
\end{array} \right\} \\
& \times (t_{n+1} - \tau)^{\gamma-1} d\tau,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C}(t_{n+1}) = \mathfrak{C}(0) + \frac{\theta}{\Gamma(\gamma)} \sum_{p=2}^{n} \int_{t_p}^{t_{p+1}} & \left\{ \begin{array}{c}
f_5(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2}) \\
+ \left[\frac{f_5(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{h} \right. \\
\left. - \frac{f_5(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{h} \right] (\tau - t_{p-2}) \\
+ \left[\frac{f_5(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)}{2h^2} \right. \\
- \frac{2f_5(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{2h^2} \\
\left. + \frac{f_5(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{2h^2} \right] \\
(\tau - t_{p-2})(\tau - t_{p-1})
\end{array} \right\} \\
& \times (t_{n+1} - \tau)^{\gamma-1} d\tau,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{R}(t_{n+1}) = \mathfrak{R}(0) + \frac{\theta}{\Gamma(\gamma)} \sum_{p=2}^{n} \int_{t_p}^{t_{p+1}} & \left\{ \begin{array}{c}
f_6(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2}) \\
+ \left[\frac{f_6(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{h} \right. \\
\left. - \frac{f_6(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{h} \right] (\tau - t_{p-2}) \\
+ \left[\frac{f_6(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)}{2h^2} \right. \\
- \frac{2f_6(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})}{2h^2} \\
\left. + \frac{f_6(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})}{2h^2} \right] \\
(\tau - t_{p-2})(\tau - t_{p-1})
\end{array} \right\} \\
& \times (t_{n+1} - \tau)^{\gamma-1} d\tau.
\end{aligned} \tag{27}
$$

Then, taking into consideration ($t_p = ph$), we calculate the integrals:

$$\int_{t_p}^{t_{p+1}} (t_{n+1}-\tau)^{\gamma-1} d\tau = \frac{h^\gamma}{\gamma}[(n-p+1)^\gamma - (n-p)^\gamma] \tag{28}$$

$$\int_{t_p}^{t_{p+1}} (\tau - t_{p-2})(t_{n+1}-\tau)^{\gamma-1} d\tau$$
$$= \frac{h^{\gamma+1}}{\gamma(\gamma+1)}\Big[(n-p+1)^\gamma(n-p+3+2\gamma) - (n-p)^\gamma(n-p+3+3\gamma)\Big] \tag{29}$$

$$\int_{t_p}^{t_{p+1}} (\tau - t_{p-2})(\tau - t_{p-1})(t_{n+1}-\tau)^{\gamma-1} d\tau$$
$$= \frac{h^{\gamma+2}}{\gamma(\gamma+1)(\gamma+2)}\Big[(n-p+1)^\gamma[2(n-p)^2 + (3\gamma+10)(n-p) + 2\gamma^2 + 9\gamma + 12]$$
$$-(n-p)^\gamma[2(n-p)^2 + (5\gamma+10)(n-p) + 6\gamma^2 + 18\gamma + 12]\Big] \tag{30}$$

We substitute (28), (29),and (30) in (26) to obtain the numerical methods for the Model (6) of the hepatitis B virus epidemic model as follows:

$$\begin{aligned}
\mathfrak{S}(t_{n+1}) &= \mathfrak{S}_0 + \frac{h^\gamma}{\Gamma(\gamma+1)}\sum_{p=2}^{n} \theta t_{p-2}^{\theta-1} f_1(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\\
&\times \Big[(n-p+1)^\gamma - (n-p)^\gamma\Big]\\
&+ \frac{h^\gamma}{\Gamma(\gamma+2)}\sum_{p=2}^{n}\Big[\theta t_{p-1}^{\theta-1} f_1(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})\\
&- \theta t_{p-2}^{\theta-1} f_1(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}},\\
&\times \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^\gamma(n-p+3+2\gamma) - (n-p)^\gamma(n-p+3+3\gamma)\Big]\\
&+ \frac{h^\gamma}{2\Gamma(\gamma+3)}\sum_{p=2}^{n}\Big[\theta t_{p}^{\theta-1} f_1(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)\\
&- 2\theta t_{p-1}^{\theta-1} f_1(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1}) + \theta t_{p-2}^{\theta-1} f_1(t_{p-2}, \mathfrak{S}_{p-2},\\
&\times \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^\gamma[2(n-p)^2\\
&+ (3\gamma+10)(n-p) + 2\gamma^2 + 9\gamma + 12]\\
&- (n-p)^\gamma[2(n-p)^2 + (5\gamma+10)(n-p) + 6\gamma^2 + 18\gamma + 12]\Big],\\
\times\mathfrak{E}(t_{n+1}) &= \mathfrak{E}_0 + \frac{h^\gamma}{\Gamma(\gamma+1)}\sum_{p=2}^{n} \theta t_{p-2}^{\theta-1} f_2(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big[(n-p+1)^\gamma - (n-p)^\gamma\Big]\\
&+ \frac{h^\gamma}{\Gamma(\gamma+2)}\sum_{p=2}^{n}\Big[\theta t_{p-1}^{\theta-1} f_2(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1}) - \theta t_{p-2}^{\theta-1} f_2(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}},\\
&\times \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^\gamma(n-p+3+2\gamma) - (n-p)^\gamma(n-p+3+3\gamma)\Big] + \frac{h^\gamma}{2\Gamma(\gamma+3)}\sum_{p=2}^{n}\Big[\theta t_p^{\theta-1}\\
&\times f_2(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p) - 2\theta t_{p-1}^{\theta-1} f_2(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1}) + \theta t_{p-2}^{\theta-1} f_2(t_{p-2}, \mathfrak{S}_{p-2},\\
&\times \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^\gamma[2(n-p)^2 + (3\gamma+10)(n-p) + 2\gamma^2 + 9\gamma + 12]\\
&- (n-p)^\gamma[2(n-p)^2 + (5\gamma+10)(n-p) + 6\gamma^2 + 18\gamma + 12]\Big],\\
\times\mathfrak{A}(t_{n+1}) &= \mathfrak{A}_0 + \frac{h^\gamma}{\Gamma(\gamma+1)}\sum_{p=2}^{n} \theta t_{p-2}^{\theta-1} f_3(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big[(n-p+1)^\gamma - (n-p)^\gamma\Big]\\
&+ \frac{h^\gamma}{\Gamma(\gamma+2)}\sum_{p=2}^{n}\Big[\theta t_{p-1}^{\theta-1} f_3(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1}) - \theta t_{p-2}^{\theta-1} f_3(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}},
\end{aligned}$$

$$\times\, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^{\gamma}(n-p+3+2\gamma)-(n-p)^{\gamma}(n-p+3+3\gamma)\Big]+\frac{h^{\gamma}}{2\Gamma(\gamma+3)}\sum_{p=2}^{n}\Big[\theta t_{p}^{\theta-1}$$
$$\times\, f_3(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)-2\theta t_{p-1}^{\theta-1}f_3(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})+\theta t_{p-2}^{\theta-1}f_3(t_{p-2}, \mathfrak{S}_{p-2},$$
$$\times\, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^{\gamma}[2(n-p)^2+(3\gamma+10)(n-p)+2\gamma^2+9\gamma+12]$$
$$-(n-p)^{\gamma}[2(n-p)^2+(5\gamma+10)(n-p)+6\gamma^2+18\gamma+12]\Big],$$

$$\times\mathfrak{A}_c(t_{n+1})=\mathfrak{A}_{c0}+\frac{h^{\gamma}}{\Gamma(\gamma+1)}\sum_{p=2}^{n}\theta t_{p-2}^{\theta-1}f_4(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big[(n-p+1)^{\gamma}-(n-p)^{\gamma}\Big]$$
$$+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\sum_{p=2}^{n}\Big[\theta t_{p-1}^{\theta-1}f_4(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})-\theta t_{p-2}^{\theta-1}f_4(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}},$$
$$\times\, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^{\gamma}(n-p+3+2\gamma)-(n-p)^{\gamma}(n-p+3+3\gamma)\Big]+\frac{h^{\gamma}}{2\Gamma(\gamma+3)}\sum_{p=2}^{n}\Big[\theta t_{p}^{\theta-1}$$
$$\times\, f_4(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)-2\theta t_{p-1}^{\theta-1}f_4(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})+\theta t_{p-2}^{\theta-1}f_4(t_{p-2}, \mathfrak{S}_{p-2},$$
$$\times\, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^{\gamma}[2(n-p)^2+(3\gamma+10)(n-p)+2\gamma^2+9\gamma+12]$$
$$-(n-p)^{\gamma}[2(n-p)^2+(5\gamma+10)(n-p)+6\gamma^2+18\gamma+12]\Big],$$

$$\times\mathfrak{C}(t_{n+1})=\mathfrak{C}_{0}+\frac{h^{\gamma}}{\Gamma(\gamma+1)}\sum_{p=2}^{n}\theta t_{p-2}^{\theta-1}f_5(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big[(n-p+1)^{\gamma}-(n-p)^{\gamma}\Big]$$
$$+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\sum_{p=2}^{n}\Big[\theta t_{p-1}^{\theta-1}f_5(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})-\theta t_{p-2}^{\theta-1}f_5(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}},$$
$$\times\, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^{\gamma}(n-p+3+2\gamma)-(n-p)^{\gamma}(n-p+3+3\gamma)\Big]+\frac{h^{\gamma}}{2\Gamma(\gamma+3)}\sum_{p=2}^{n}\Big[\theta t_{p}^{\theta-1}$$
$$\times\, f_5(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)-2\theta t_{p-1}^{\theta-1}f_5(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})+\theta t_{p-2}^{\theta-1}f_5(t_{p-2}, \mathfrak{S}_{p-2},$$
$$\times\, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^{\gamma}[2(n-p)^2+(3\gamma+10)(n-p)+2\gamma^2+9\gamma+12]$$
$$-(n-p)^{\gamma}[2(n-p)^2+(5\gamma+10)(n-p)+6\gamma^2+18\gamma+12]\Big],$$

$$\times\mathfrak{R}(t_{n+1})=\mathfrak{R}_{0}+\frac{h^{\gamma}}{\Gamma(\gamma+1)}\sum_{p=2}^{n}\theta t_{p-2}^{\theta-1}f_6(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big[(n-p+1)^{\gamma}-(n-p)^{\gamma}\Big]$$
$$+\frac{h^{\gamma}}{\Gamma(\gamma+2)}\sum_{p=2}^{n}\Big[\theta t_{p-1}^{\theta-1}f_6(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})-\theta t_{p-2}^{\theta-1}f_6(t_{p-2}, \mathfrak{S}_{p-2}, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}},$$
$$\times\, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^{\gamma}(n-p+3+2\gamma)-(n-p)^{\gamma}(n-p+3+3\gamma)\Big]+\frac{h^{\gamma}}{2\Gamma(\gamma+3)}\sum_{p=2}^{n}\Big[\theta t_{p}^{\theta-1}$$
$$\times\, f_6(t_p, \mathfrak{S}_p, \mathfrak{E}_p, \mathfrak{A}_p, \mathfrak{A}_{c_p}, \mathfrak{C}_p, \mathfrak{R}_p)-2\theta t_{p-1}^{\theta-1}f_6(t_{p-1}, \mathfrak{S}_{p-1}, \mathfrak{E}_{p-1}, \mathfrak{A}_{p-1}, \mathfrak{A}_{c_{p-1}}, \mathfrak{C}_{p-1}, \mathfrak{R}_{p-1})+\theta t_{p-2}^{\theta-1}f_6(t_{p-2}, \mathfrak{S}_{p-2},$$
$$\times\, \mathfrak{E}_{p-2}, \mathfrak{A}_{p-2}, \mathfrak{A}_{c_{p-2}}, \mathfrak{C}_{p-2}, \mathfrak{R}_{p-2})\Big]\Big[(n-p+1)^{\gamma}[2(n-p)^2+(3\gamma+10)(n-p)+2\gamma^2+9\gamma+12]$$
$$-(n-p)^{\gamma}[2(n-p)^2+(5\gamma+10)(n-p)+6\gamma^2+18\gamma+12]\Big].$$

6 Simulations

In this section, we use the numerical scheme presented in the previous section to simulate and analyze the stability and convergence behavior of the HBV model (7) - (8). In view of [25], we let the following parameters be used: $\Lambda = 2$, $\xi = 1/67.7$, $\beta = 0.00042$, $\phi = 0.002$, $\epsilon = 0.002$, $\varkappa = 0.004$, $\zeta = 0.6$, $\varrho = 0.001$, $\eta = 0.02$, $k = 0.02$, $\varsigma = 0.02$, $\rho = 0.1$, $\varepsilon = 0.003$, $\sigma = 0.2$, and the initial values for the existing six functions

$$\mathfrak{S}(0) = 60,\ \mathfrak{E}(0) = 40,\ \mathfrak{A}(0) = 3,\ \mathfrak{A}_c(0) = 0.2,\ \mathfrak{C}(0) = 0.1,\ \mathfrak{R}(0) = 0.$$

Using the MATLAB program, we get different figures that display the behavior of susceptible, exposed, acute, asymptomatic, and chronic individuals in the population over time.

- **Case 1** ($\theta = 0.9$ and $\gamma = 1,\ 0.95,\ 0.9,\ 0.85,\ 0.8$)
Figure 1 shows the behavior of the five groups for the fractal order $\theta = 0.9$ and various fractional dimensions $\gamma = 1, 0.95, 0.9, 0.85, 0.8$. Over time, we see that the group of susceptible individuals increases while the groups of exposed and acute individuals decrease. Moreover, the population in asymptomatic and chronic groups increases and then decreases.
- **Case 2** ($\gamma = 0.9$, and $\theta = 1,\ 0.95,\ 0.9,\ 0.85,\ 0.8$)
Figure 2 shows the dynamics of the proposed model (7) - (8) for fixed fractional dimension = 0.9, and various fractal orders $\theta = 1,\ 0.95,\ 0.9,\ 0.85,\ 0.8$, we see that all groups of the population have the same behavior as in the first case. In the initial scenario, as seen in Figure 1, the increase in the fractional order γ results in a more rapid rise in the number of susceptible people. Additionally, it influences how fast the decline in the remaining four groups exhibit more speed, and this observation holds when the fractal values θ increase as well as seen in Figure 2.
- **Case 3** ($\theta = 1,\ 0.8,\ 0.7$, and $\gamma = 1,\ 0.9,\ 0.8$)
Finally, in Figure 3, we try to observe the five groups' behavior when simultaneously changing the fractional order and the fractal dimension. We take values of $\theta = 1,\ 0.8,\ 0.7$ and $\gamma = 1,\ 0.9,\ 0.8$, respectively. In each case, the groups of exposed and acute people have the same behavior of convergence and stability. Also, the groups of asymptomatic and chronic individuals have the same behavior in each case.

7 Conclusion

In this research, a developed model of hepatitis B virus transmission between humans was presented. We used fractal-fractional derivatives with a power law kernel. In the qualitative analysis of the proposed model, we used Schuder's fixed point theorem to demonstrate the existence of solutions to the system. Then, Banach's fixed point theorem was used to prove the solutions' uniqueness. We also presented the conditions for the stability of the proposed system's solutions in the concepts of Ulam-Hyers and Ulam-Hyers-Rassias. Moreover, a numerical scheme for the model was provided using the newly developed Newton polynomial. Then, through the change in the values of the fractal dimension and fractional order, we graphically analyzed the convergence and stability of the six solutions of the system. The solutions' graphs assisted us in predicting the dynamics of hepatitis B virus spread in the presence of an asymptomatic carrier group and its impact on disease spread in society when the fractional order is fixed. Moreover, with the change in the fractal dimension values, we noted that the fractal dimension reduces the spread of the disease by accelerating the reduction of infected people and carriers without symptoms in the community. Furthermore, when fixing the fractal dimension and changing the values of the fractional order, we also noted that the fractal arrangement reduces the speed of disease spread. Still, when changing the fractional order and fractal dimension at the same time, we saw in the third figure that 500 days after the disease appeared in the community, and when the fractal order and the fractal dimension are one at the same time, acute infected and asymptomatic groups are eliminated, thus eliminating the disease forever from society. This research demonstrated the high efficiency of simulating the fractal-fractional HBV model through the innovative scheme based on Newton's polynomial for forecasting the disease's future behavior. The precision of these results motivates us to contemplate employing this novel numerical scheme in the numerical simulation of fractal-fractional models for exploring optimal control of HBV spread and in further investigations related to disease transmission dynamics.

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