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ORIGINAL RESEARCH

The Bogomolov multiplier of Lie superalgebras

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Abstract In this paper, we extend the concept of the Bogomolov multiplier and the commutativity-preserving extension to Lie superalgebras. Additionally, we compute the Bogomolov multiplier of the Lie superalgebra associated with the Heisenberg and the real Lie superalgebras of dimension at most 4.

Keywords Commutativity preserving exterior product · $\tilde{B_0}$-pairing · Curly exterior product · Bogomolov multiplier · Heisenberg Lie superalgebra

Mathematics Subject Classification 14E08 · 17B01 · 17B05 · 19C09

1 Introduction and preliminaries

In the late nineteenth century, the study of Graded Lie algebras gained prominence within the physics field of "supersymmetries," particularly in the context of particles exhibiting various statistical properties. In [7], Kac introduced the theory of Lie superalgebras, which are commonly referred to as $\mathbb{Z}_2$-graded Lie algebras. Subsequently, this theory, akin to Lie's original work, has been developed to establish a connection between Lie superalgebras and Lie supergroups. This theory has made significant strides in recent years, resulting in numerous advancements, including many results obtained in representation theory and classification. It is noteworthy that a substantial portion of these results can be attributed to extensions of well-established facts within the realm of Lie algebras. For further information regarding Lie superalgebras, please refer to [5,7,9] and the references cited therein.

In this paper, we introduce the non-abelian commutativity-preserving exterior product and the Bogomolov multiplier for Lie superalgebras. The Bogomolov multiplier was initially defined for groups by Fedro Bogomolov [4] and serves as a group-theoretical invariant introduced as an obstruction to a problem in algebraic geometry known as the rationality problem or Noether's problem. Recently, in [1,2], we extended this concept to Lie algebras and established its connection with the Bogomolov multiplier of groups via the Lazard correspondence.

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We utilized a notion of the non-abelian exterior square $L \wedge L$ of a finite-dimensional Lie algebra L over a field $\mathbb{F}$ to provide a novel description of the Bogomolov multiplier. Furthermore, employing Hopf's formula, we demonstrated that if $0 \to R \to F \to L \to 0$ is a free presentation of the finite-dimensional Lie algebra over $\mathbb{F}$, then the Bogomolov multiplier is isomorphic to $\frac{R\cap[F,F]}{<K(F)\cap R>}$, where $K(F) = \{[x, y] \mid x, y \in F\}$.

At present, it is intriguing to develop an analogous theory of commutativity-preserving exterior products for Lie supertheory. The structure of the paper is outlined in the following sections: 2 and 3. In these sections, we introduce the non-abelian commutativity-preserving exterior product and Hopf's type formula for Lie superalgebras. Subsequently, in Sects. 4, we compute the Bogomolov multiplier of the Lie superalgebras of the Heisenberg type and the real Lie superalgebras of dimension at most 4.

Throughout this paper, all modules and algebras are defined over the unital commutative ring $\mathbb{K}$. Here, we introduce some terminology and notations on Lie superalgebras, which are defined in [5]. Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be a field, and we define $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. A $\mathbb{Z}_2$-graded module (or supermodule) M is $M = M_{\bar{0}} \oplus M_{\bar{1}}$, whose elements are called even and odd, respectively. Non-zero elements of $M_{\bar{0}} \cup M_{\bar{1}}$ are said to be homogeneous. For a homogeneous element $m \in M_{\bar{\alpha}}$ with $\alpha \in \mathbb{Z}_2$, $|m| = \bar{\alpha}$ is the degree of m. Therefore, whenever we have the notation $|m|$, m will be a homogeneous element. A submodule N of M is called a $\mathbb{Z}_2$-graded submodule (or sub supermodule) if $N = N_{\bar{0}} \oplus N_{\bar{1}}$ where $N_{\bar{0}} = N \cap M_{\bar{0}}$ and $N_{\bar{1}} = N \cap M_{\bar{1}}$.

Definition 1.1 [5, Definition 2.1] A Lie superalgebra is a superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with a multiplication denoted by [,], called the super bracket operation, satisfying the following identities:

(i) $[x, y] = -(-1)^{|x||y|}[y, x]$,
(ii) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$,
(iii) $[l_{\bar{0}}, l_{\bar{0}}] = 0$

for all homogeneous elements $x, y, z \in L$ and $l_{\bar{0}} \in L_{\bar{0}}$.

It is worth noting that the final equation can be readily derived from the initial equation, specifically when 2 is invertible in $\mathbb{K}$. The second equation is equivalent to the following graded Jacobi identity:

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.$$

Utilizing the aforementioned identities, it becomes evident that for a Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$, the even component $L_{\bar{0}}$ possesses the structure of a Lie algebra, while the odd component $L_{\bar{1}}$ serves as a module over $L_{\bar{0}}$. Consequently, if $L_{\bar{1}} = 0$, the superalgebra L is a Lie algebra. Conversely, if $L_{\bar{0}} = 0$, L transforms into an abelian Lie superalgebra, characterized by the property that $[x, y] = 0$ for all $x, y \in L$. However, it is important to note that in general, a Lie superalgebra does not necessarily possess the structure of a Lie algebra.

The sub-superalgebra K of L is a $\mathbb{Z}_2$-graded vector subspace that is closed under the bracket operation. Consider $[L, L]$, which represents a graded subalgebra of L and is denoted as L^2. A $\mathbb{Z}_2$-graded subspace I is referred to as a graded ideal of L if $[I, L] \subseteq I$ and for all $x \in L$, the ideal $Z(L) = \{z \in L; [z, x] = 0\}$ is also a graded ideal and is known as the center of L. If I is an ideal of L, the quotient Lie superalgebra L/I inherits a canonical Lie superalgebra structure, such that the natural projection map transforms into a homomorphism. The notions of epimorphisms, isomorphisms, and automorphisms are straightforwardly defined. According to the super dimension structure of Lie superalgebras over a field, we denote $L = L_{\bar{0}} \oplus L_{\bar{1}}$ as an $(m|n)$ Lie superalgebra if $\dim L_{\bar{0}} = m$ and $\dim L_{\bar{1}} = n$. Throughout, $A(m|n)$ represents an abelian Lie superalgebra with dimension $(m|n)$. The descending central sequence of a Lie superalgebra L is defined recursively as $L^1 = L$ and $L^{c+1} = [L^c, L]$ for all $c \geq 1$. If for some positive integer c, $L^{c+1} = 0$ and $L^c \neq 0$, L is characterized as nilpotent, with the nilpotency class denoted as c. Furthermore, it is worth noting that $|[m, n]| = |m| + |n|$.

Definition 1.2 [5] Let M and N be two Lie superalgebras. A linear map $f : M \to N$ is called a homomorphism of degree $|f| \in \mathbb{Z}_2$ (or Lie super homomorphism), if $f(M_{\bar{\alpha}}) \subseteq N_{\bar{\alpha}+|f|}$, for every $x, y \in M$, and $\bar{\alpha} \in \mathbb{Z}_2$.

Especially if $|f| = \bar{0}$, then the homomorphism f will be called an even linear map (or even grade). A Lie superalgebra homomorphism $f : M \to N$ is a homogeneuse linear map of even degree such that $f([x, y]) = [f(x), f(y)]$ holds for all $x, y \in M$.

Definition 1.3 Let L be a Lie algebra and M and N be graded ideals of L. The exterior product $M \wedge N$ is the Lie superalgebra generated by all symbols $m \wedge n$ subject to the following relations:

(i) $\lambda(m \wedge n) = \lambda m \wedge n = m \wedge \lambda n$,
(ii) $(m + m') \wedge n = m \curlywedge n + m' \wedge n$, where m, m' have the same grading,

(iii) $m \wedge (n + n') = m \curlywedge n + m \wedge n'$, where n, n' have the same grading,
(iv) $[m, m'] \wedge n = m \wedge [m', n] - (-1)^{|m||m'|} m' \wedge [m, n]$,
(v) $m \wedge [n, n'] = (-1)^{|n'|(|m|+|n|)} [n', m] \wedge n - (-1)^{|m||n|} [n, m] \wedge n'$,
(vi) $[(m \wedge n), (m' \wedge n')] = -(-1)^{|m||n|} [n, m] \wedge [m', n']$,

for all $\lambda \in \mathbb{K}$, $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$ and $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$.

It is noteworthy that if $M = M_{\bar{0}}$ and $N = N_{\bar{0}}$, then $M \wedge N$ can be regarded as a non-abelian exterior product of Lie algebras.

A more general structure $M \wedge N$ is presented in [5] for arbitrary crossed L-modules M and N.

Definition 1.4 Let L be a Lie superalgebra and M and N be graded ideals of L. A function $\rho : M \times N \to L$ is called a Lie super exterior pairing, if the following holds:

(i) $\rho(\lambda m, n) = \rho(m, \lambda n) = \lambda \rho(m, n)$,
(ii) $\rho(m + m', n) = \rho(m, n) + \rho(m', n)$, where m, m' have the same grading,
(iii) $\rho(m, n + n') = \rho(m, n) + \rho(m, n')$, where n, n' have the same grading,
(iv) $\rho([m, m'], n) = \rho(m, [m', n]) - (-1)^{|m||m'|} \rho(m', [m, n])$,
(v) $\rho(m, [n, n']) = (-1)^{|n'|(|m|+|n|)} \rho([n', m], n) - (-1)^{|m||n|} \rho([n, m], n')$,
(vi) $[\rho(m, n), \rho(m', n')] = -(-1)^{|m||n|} \rho([n, m], [m', n'])$,

for all $\lambda \in \mathbb{K}$, $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$ and $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$.

In this section, we introduce a non-abelian commutativity-preserving exterior product (CP exterior) of Lie superalgebras. This generalizes the non-abelian CP exterior product of Lie algebras introduced in [1]. We then present some basic results.

Definition 1.5 Let L be a Lie superalgebra and M and N be graded ideals of L. A function $\rho : M \times N \to L$, is called a Lie super $\tilde{B}_0$-pairing, if the following holds.

(i) $\rho(\lambda m, n) = \rho(m, \lambda n) = \lambda \rho(m, n)$,
(ii) $\rho(m + m', n) = \rho(m, n) + \rho(m', n)$, where m, m' have the same grading,
(iii) $\rho(m, n + n') = \rho(m, n) + \rho(m, n')$, where n, n' have the same grading,
(iv) $\rho([m, m'], n) = \rho(m, [m', n]) - (-1)^{|m||m'|} \rho(m', [m, n])$,
(v) $\rho(m, [n, n']) = (-1)^{|n'|(|m|+|n|)} \rho([n', m], n) - (-1)^{|m||n|} \rho([n, m], n')$,
(vi) $[\rho(m, n), \rho(m', n')] = -(-1)^{|m||n|} \rho([n, m], [m', n'])$,
(vii) If $[m, n] + (-1)^{|m'||n'|} [m', n'] = 0$, then $\rho(m, n) + (-1)^{|m'||n'|} \rho(m', n') = 0$,
If $[m_{\bar{0}}, n_{\bar{0}}] = 0$, then $\rho(m_{\bar{0}}, n_{\bar{0}}) = 0$,

for all $\lambda \in \mathbb{K}$, $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$, $m_{\bar{0}} \in M_{\bar{0}}$ and $n_{\bar{0}} \in N_{\bar{0}}$.

Definition 1.6 A Lie super $\tilde{B}_0$-pairing $\rho : M \times N \to L$ is called universal, if for any Lie super $\tilde{B}_0$-pairing $\rho' : M \times N \to L'$, there is a unique Lie homomorphism $\theta : L \to L'$ such that $\theta\rho = \rho'$.

The following definition extends the concept of CP exterior product in [1] to the theory of Lie superalgebras.

Definition 1.7 Let L be a Lie superalgebra and M and N be two graded ideals of L. The CP exterior product $M \curlywedge N$ is the Lie superalgebra generated by all symbols $m \curlywedge n$ subject to the following relations

(i) $\lambda(m \curlywedge n) = \lambda m \curlywedge n = m \curlywedge \lambda n$,
(ii) $(m + m') \curlywedge n = m \curlywedge n + m' \curlywedge n$, where m, m' have the same graded,
(iii) $m \curlywedge (n + n') = m \curlywedge n + m \curlywedge n'$, where n, n' have the same graded,
(iv) $[m, m'] \curlywedge n = m \curlywedge [m', n] - (-1)^{|m||m'|} m' \curlywedge [m, n]$,
(v) $m \curlywedge [n, n'] = (-1)^{|n'|(|m|+|n|)} [n', m] \curlywedge n - (-1)^{|m||n|} [n, m] \curlywedge n'$,
(vi) $[(m \curlywedge n), (m' \curlywedge n')] = -(-1)^{|m||n|} [n, m] \curlywedge [m', n']$,
(vii) If $[m, n] + (-1)^{|m'||n'|} [m', n'] = 0$, then $m \curlywedge n + (-1)^{|m'||n'|} m' \curlywedge n' = 0$,
If $[m_{\bar{0}}, n_{\bar{0}}] = 0$, then $m_{\bar{0}} \curlywedge n_{\bar{0}} = 0$,

for all $\lambda \in \mathbb{K}$, $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$, $m_{\bar{0}} \in M_{\bar{0}}$ and $n_{\bar{0}} \in N_{\bar{0}}$.

In the case $M = N = L = L_{\bar{0}} \oplus L_{\bar{1}}$, we call $L \curlywedge L$ the curly exterior product of L and for any $x, y \in L_{\bar{0}} \cup L_{\bar{1}}$ and $x_{\bar{0}} \in L_{\bar{0}}$, since $[x, y] + (-1)^{|x||y|}[y, x] = 0$ and $[x_{\bar{0}}, x_{\bar{0}}] = 0$, we have

$$x \curlywedge y + (-1)^{|x||y|} y \curlywedge x = 0 \quad , \quad x_{\bar{0}} \curlywedge x_{\bar{0}} = 0.$$

Remark. Let $m = m_{\bar{0}} + m_{\bar{1}}$ and $n = n_{\bar{0}} + n_{\bar{1}}$ be arbitrary elements of M and N respectively, then according to the Definition 1.7 we have

$$m \curlywedge n = m_{\bar{0}} \curlywedge n_{\bar{0}} + m_{\bar{0}} \curlywedge n_{\bar{1}} + m_{\bar{1}} \curlywedge n_{\bar{0}} + m_{\bar{1}} \curlywedge n_{\bar{1}}.$$

If $M = M_{\bar{0}}$ and $N = N_{\bar{0}}$ then the $M \curlywedge N$ is called the non-abelian CP exterior product of Lie algebras which is introduced in [1].

Lemma 1.8 *The function $\rho : M \times N \to M \curlywedge N$ given by $(m, n) \longmapsto m \curlywedge n$ is a universal Lie super $\tilde{B}_0$-pairing.*

Proof The proof is straightforward. □

Theorem 1.9 *Let L be a Lie superalgebra and M and N be two ideals of L. Then we have*

$$M \curlywedge N \cong \frac{M \wedge N}{\mathcal{M}_0(M, N)},$$

where $\mathcal{M}_0(M, N)$ is the graded submodule of $M \wedge N$ generated by the elements

(i) $m \wedge n + (-1)^{|m'||n'|} m' \wedge n'$, where $[m, n] + (-1)^{|m'||n'|}[m', n'] = 0$,
(ii) $m_{\bar{0}} \wedge n_{\bar{0}}$, where $[m_{\bar{0}}, n_{\bar{0}}] = 0$,

with $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$, $m_{\bar{0}} \in M_{\bar{0}}$ and $n_{\bar{0}} \in N_{\bar{0}}$.

Proof By using Definition 1.4, the function $\rho : M \times N \to M \curlywedge N$ given by $(m, n) \longmapsto (m \curlywedge n)$ is a Lie super exterior pairing. So it induces a Lie super homomorphism $\tilde{\rho} : M \wedge N \to M \curlywedge N$, given by $(m \wedge n) \longmapsto m \curlywedge n$, for all $m \in M$ and $n \in N$. Clearly $\mathcal{M}_0(M, N) \subseteq \ker \tilde{\rho}$, so we have the Lie super homomorphism $\rho^* : (M \wedge N)/\mathcal{M}_0(M, N) \to M \curlywedge N$ given by $(m \wedge n) + \mathcal{M}_0(M, N) \longmapsto (m \curlywedge n)$. On the other hand, the map $l^* : M \curlywedge N \to (M \wedge N)/\mathcal{M}_0(M, N)$ given by $(m \curlywedge n) \longmapsto (m \wedge n) + \mathcal{M}_0(M, N)$ is induced by the Lie super exterior pairing $l : M \times N \to (M \wedge N)/\mathcal{M}_0(M, N)$ given by $(m, n) \longmapsto (m \wedge n) + \mathcal{M}_0(M, N)$. Now it is easy to see that $\rho^* l^* = l^* \rho^* = 1$. Thus l^* is an isomorphism. □

It is known that $\kappa : M \times N \to [M, N]$ given by $(m, n) \longmapsto [m, n]$ is a Lie super exterior pairing. So for all $m \in M$ and $n \in N$, it induces a Lie super homomorphism $\tilde{\kappa} : M \wedge N \to [M, N]$, such that $\tilde{\kappa}(m \wedge n) = [m, n]$. Moreover, the kernel of $\tilde{\kappa}$ is denoted by $\mathcal{M}(M, N)$. It can easily be seen that $\mathcal{M}_0(M, N)$ is a graded submodule of $\mathcal{M}(M, N)$, thus there is a Lie super homomorphism $\kappa^* : M \wedge N/\mathcal{M}_0(M, N) \to [M, N]$ given by $m \wedge n + \mathcal{M}_0(M, N) \longmapsto [m, n]$, with $\ker \kappa^* \cong \mathcal{M}(M, N)/\mathcal{M}_0(M, N)$. Similar to the theory of Lie algebras, we denote $\mathcal{M}(M, N)/\mathcal{M}_0(M, N)$ by $\tilde{B}_0(M, N)$, and we call it the Bogomolov multiplier of the pair of Lie superalgebras (M, N). Therefore, we have an exact sequence

$$0 \to \tilde{B}_0(M, N) \to M \curlywedge N \to [M, N] \to 0.$$

In the case $M = N = L$, we denote $\mathcal{M}_0(L, L)$ by $\mathcal{M}_0(L)$.

Similar to Lie algebras, we denote $\mathcal{M}(L)/\mathcal{M}_0(L)$ by $\tilde{B}_0(L)$, and we call it the Bogomolov multiplier of the Lie superalgebra L. So we have an exact sequence

$$0 \to \tilde{B}_0(L) \to L \curlywedge L \to L^2 \to 0.$$

Proposition 1.10 *Let L be a Lie superalgebra and M, N and K be graded ideals of L, such that $K \subseteq M \cap N$. Then there is an isomorphism*

$$M/K \curlywedge N/K \cong (M \curlywedge N)/T,$$

where T is the ideal of $M \curlywedge N$ generated by the following elements:

(i) $m \curlywedge n + (-1)^{|m'||n'|} m' \curlywedge n'$, where $[m, n] + (-1)^{|m'||n'|}[m', n'] \in K$,
(ii) $m_{\bar{0}} \curlywedge n_{\bar{0}}$, where $[m_{\bar{0}}, n_{\bar{0}}] \in K$,

with $m, m' \in M_{\bar{0}} \cup M_{\bar{1}}$, $n, n' \in N_{\bar{0}} \cup N_{\bar{1}}$, $m_{\bar{0}} \in M_{\bar{0}}$ *and* $n_{\bar{0}} \in N_{\bar{0}}$.

Proof The function $\phi : M \times N \to M/K \curlywedge N/K$ given by $(m, n) \to (m+K) \curlywedge (n+K)$ is a well-defined Lie super $\tilde{B_0}$-pairing. Thus there is a Lie super homomorphism $\phi^* : M \curlywedge N \to M/K \curlywedge N/K$ with $m \curlywedge n \longmapsto (m+K) \curlywedge (n+K)$. Clearly $T \subseteq \ker \phi^*$ and since T is an ideal, we have the homomorphism $\psi : (M \curlywedge N)/T \to M/K \curlywedge N/K$ given by $(m \curlywedge n)+T \longmapsto (m+K) \curlywedge (n+K)$. On the other hand, the map $\varphi^* : M/K \curlywedge N/K \to (M \curlywedge N)/T$ given by $(m+K) \curlywedge (n+K) \longmapsto (m \curlywedge n)+T$ is induced by the Lie super $\tilde{B_0}$-pairing $\varphi : M/K \times N/K \to (M \curlywedge N)/T$ given by $(m+K, n+K) \longmapsto (m \curlywedge n)+T$. One can check that, $\varphi^*\psi = \psi\varphi^* = 1$. Thus, φ^* is an isomorphism, and the proof is completed. □

Now, we give the behaviour of the CP exterior product with respect to a direct sum of Lie superalgebras.

Proposition 1.11 *Let M and N be two graded ideals of a Lie superalgebra L. Then*

$$(M \oplus N) \curlywedge (M \oplus N) \cong (M \curlywedge M) \oplus (N \curlywedge N).$$

Proof By using Theorem 3.9 in [13], we have

$$\begin{aligned}(M \oplus N) \curlywedge (M \oplus N) &\cong (M \oplus N) \wedge (M \oplus N) + M_0(M \oplus N)\\ &= (M \wedge M) \oplus (N \wedge N) \oplus \frac{M}{M^2} \otimes \frac{N}{N^2} + M_0(M \oplus N).\end{aligned}$$

Since

$$M_0(M \oplus N) = M_0(M) \oplus M_0(N) \oplus \frac{M}{M^2} \otimes \frac{N}{N^2},$$

we have $(M \oplus N) \curlywedge (M \oplus N) \cong (M \curlywedge M) \oplus (N \curlywedge N)$, as follows. □

2 Hopf's type formula for the Bogomolov multiplier of Lie superalgebras

This section is dedicated to providing the same Hopf's type formula for the Bogomolov multiplier of a super algebra as it is for Lie algebras. For reference, we recall the following definitions from [9,10].

Definition 2.1 [10, Definition 2.1] The free Lie superalgebra on a $\mathbb{Z}_2$-graded set $X = X_{\bar{0}} \cup X_{\bar{1}}$ is a Lie superalgebra $F(X)$ together with a degree zero map $i : X \to F(X)$ such that if M is any Lie superalgebra and $j : X \to M$ is a degree zero map, then there is a unique Lie super homomorphism $h : F(X) \to M$ with $j = hoi$.

The existence of a free Lie superalgebra is guaranteed by an analogue of Witt's theorem (see [9, Theorem 6.2.1]).

Let L be a Lie superalgebra generated by a $\mathbb{Z}_2$-graded set $X = X_{\bar{0}} \cup X_{\bar{1}}$ and $\phi : X \to L$ be a degree zero map, then there is a free Lie superalgebra F and $\psi : F \to L$ expanding ϕ. Let $R = \ker(\psi)$, the extension $0 \to R \to F \to L \to 0$ is named a free presentation of L and is denoted by (F, ψ).

According to the well-known Hopf's type formula [10], we have an isomorphism $\mathcal{M}(L) \cong (R \cap F^2)/[R, F]$. Now we want to introduce the similar formula for the Bogomolov multiplier of a Lie superalgebra L.

The ideas of the proofs of two next propositions are exactly the same as in [1], so we omit the proofs.

Let $K(F)$ be used to denote $\{[x, y] \mid x, y \in F\}$.

Proposition 2.2 *Let L be a Lie superalgebra with the free presentation $L \cong F/R$, then*

$$\tilde{B_0}(L) \cong \frac{R \cap F^2}{< K(F) \cap R >}.$$

Proposition 2.3 *Let L be a Lie superalgebra with the free presentation $0 \to R \to F \xrightarrow{\pi} L \to 0$. If T is an ideal in F with $M = T/R$, then the following sequences are exact.*

(i) $\tilde{B_0}(L) \to \tilde{B_0}(\frac{L}{M}) \to \frac{M}{< K(L) \cap M >} \to \frac{L}{L^2} \to \frac{\frac{L}{M}}{(\frac{L}{M})^2} \to 0,$

(ii) $0 \to \frac{R \cap < K(F) \cap T >}{< K(F) \cap R >} \to \tilde{B_0}(L) \to \tilde{B_0}(\frac{L}{M}) \to \frac{M \cap L^2}{< K(L) \cap M >} \to 0.$

We know for Lie algebras, the Schur multiplier is a universal object of central extensions. Recently, parallel to the classical theory of central extensions, we in [1,2] developed a version of extensions that preserve commutativity and showed that the Bogomolov multiplier is the universal object parametrizing such extensions for a given Lie algebra. Here, we want to introduce a similar notion for Lie superalgebras.

Definition 2.4 Let L, M, and C be Lie superalgebras. An exact sequence of Lie superalgebras $0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ is called a commutativity preserving extension (CP extension) of M by L, if commuting pairs of elements of L have commuting lifts in C. A special type of CP extension with the central kernel is named a central CP extension.

Proposition 2.5 *Let $e : 0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ be a central extension. Then e is a CP extension if and only if $\chi(M) \cap K(C) = 0$.*

Proof Suppose that e is a CP central extension. Let $[c_1, c_2] \in \chi(M) \cap K(C)$, then there is a commuting lift $(c_1', c_2') \in C \times C$ of the commuting pair $(\pi(c_1), \pi(c_2))$, such that $\pi(c_1') = \pi(c_1)$ and $\pi(c_2') = \pi(c_2)$. So for some $a, b \in \chi(M)$, we have $c_1' = c_1 + a$, $c_2' = c_2 + b$. Therefore $0 = [c_1', c_2'] = [c_1, c_2 + b] = [c_1, c_2]$. Hence $\chi(M) \cap K(C) = 0$. Conversely, suppose that $\chi(M) \cap K(C) = 0$. Choose $x, y \in L$ with $[x, y] = 0$, we have $x = \pi(c_1)$ and $y = \pi(c_2)$ for some $c_1, c_2 \in C$. Therefore $\pi([c_1, c_2]) = 0$. Hence $[c_1, c_2] \in \chi(M) \cap K(C) = 0$, so $[c_1, c_2] = 0$. Thus the central extension e is a CP extension. □

Definition 2.6 An abelian ideal M of a Lie superalgebra L is called a CP Lie sub superalgebra of L if the extension $0 \to M \to L \to \dfrac{L}{M} \to 0$ is a CP extension.

Also, by using Proposition 2.5, an abelian ideal M of a Lie superalgebra L is a CP Lie sub superalgebra of L if $M \cap K(L) = 0$.

Example 2.7 Let $0 \to R \to F \to L \to 0$ be a free presentation of L. Then the extension

$$0 \to \frac{R \cap F^2}{< K(F) \cap R >} \to \frac{R}{< K(F) \cap R >} \to \frac{R}{R \cap F^2} \to 0$$

is a CP extension. So, the abelian ideal $\frac{R \cap F^2}{<K(F) \cap R>}$ of $\frac{R}{<K(F)\cap R>}$ is a CP Lie sub-superalgebra.

Example 2.8 We know in general that a Lie algebra is a Lie superalgebra. Thus, when L is a cyclic Lie algebra and C is a non-abelian Lie algebra, then $e : 0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ is a CP extension that is not central.

Example 2.9 Let L be a Lie algebra in which for every commuting pair x, y the subgroup $< x, y >$ is cyclic. Then every commuting pair of elements in L has a commuting lift. Thus every extension of L is central CP.

Proposition 2.10 *Let L be a Lie superalgebra with the free presentation $0 \to R \to F \xrightarrow{\pi} L \to 0$. If T is an ideal in F with $N = T/R$, then the following are equivalent.*

(1) N is a CP sub Lie superalgebra of L,
(2) The canonical map $\psi : \mathcal{M}_0(L) \to \mathcal{M}_0(L/N)$ is surjective.,
(3) The canonical map $\varphi : L \curlywedge L \to L/N \curlywedge L/N$ is an isomorphism..

Proof We know there is a Lie superalgebra homomorphism

$$\begin{aligned} \psi : &\frac{< K(F) \cap R >}{[R, F]} \to \frac{< K(F) \cap T >}{[T, F]} \\ &x + [R, F] \mapsto x + [T, F]. \end{aligned}$$

Now, as N is a CP sub Lie superalgebra of L, then by using the Proposition 2.5 and Definition 2.6, $N \cap K(L) = 0$. So, $T \cap K(F) \subseteq R$. Hence, $< K(F) \cap R >=< K(F) \cap T >$. Thus im$\psi =< K(F) \cap T > /[T, F]$ and ψ is surjective. On the other hand, if ψ is surjective, then $< K(F) \cap T >=< K(F) \cap R >$. So, $K(F) \cap T \subseteq R$. Thus $N \cap K(L) = 0$ and N is a CP Lie sub algebra of L. Therefore (1) and (2) are equivalent.

Now, let N be a CP sub Lie algebra of L, then $N \cap K(L) = 0$. So, $T \cap K(F) \subseteq R$. By using Proposition 2.2, $L \curlywedge L \cong F^2/ < K(F) \cap R >$ and for all $x \in F^2$, we have

$$\begin{aligned} \varphi : &\frac{F^2}{< K(F) \cap R >} \to \frac{F^2}{< K(F) \cap T >} \\ &x+ < K(F) \cap R > \mapsto x+ < K(F) \cap T > . \end{aligned}$$

Also,

$$\ker\varphi = \{x+ < K(F)\cap R > \mid x \in < K(F)\cap T >\} = \frac{< K(F)\cap T >}{< K(F)\cap R >}.$$

Since $T \cap K(F) \subseteq R$, then $< T \cap K(F) > \subseteq < R \cap K(F) >$. So $\ker\varphi = 0$ and φ is injective. Also, $\mathrm{im}\varphi = F^2/ < K(F)\cap T >$. Thus, φ is surjective. Hence, φ is an isomorphism. On the other hand, if φ is an isomorphism, then $< K(F)\cap T > \subseteq < K(F)\cap R >$ and $K(F)\cap T \subseteq R$. Thus, $N \cap K(L) = 0$. Hence, N is a CP sub Lie superalgebra of L. □

At present, our objective is to derive an explicit formula for the Bogomolov multiplier of a direct product of two Lie superalgebras. The following lemma provides a free presentation for $L_1 \oplus L_2$, expressed in terms of the provided free presentation for L_1 and L_2. This lemma will serve as a crucial component in the subsequent calculations.

Lemma 2.11 [12, Lemma 5.5] *Let F_1 and F_2 be two free Lie superalgebras freely generated by X and Y, respectively, and $F = F_1 * F_2$ be the free product of F_1 and F_2. Then $0 \to R \to F \xrightarrow{\delta}\!\!\to L_1 \oplus L_2$ is a free presentation for $L_1 \oplus L_2$ in which $R = R_1 + R_2 + [F_1, F_2]$.*

Proposition 2.12 *Let L_1 and L_2 be Lie superalgebras. Then*

$$\tilde{B_0}(L_1 \oplus L_2) \cong \tilde{B_0}(L_1) \oplus \tilde{B_0}(L_2).$$

Proof $L_1 \oplus L_2$ is a Lie superalgebra with even part $(L_1 \oplus L_2)_{\bar{0}} = (L_1)_{\bar{0}} + (L_2)_{\bar{0}}$ and odd part $(L_1 \oplus L_2)_{\bar{1}} = (L_1)_{\bar{1}} + (L_2)_{\bar{1}}$. Now, by using Lemma 2.11, we have

$$\tilde{B_0}(L_1 \oplus L_2) = \frac{(R_1 + R_2 + [F_2, F_1]) \cap (F_1 * F_2)^2}{< K(F_1 * F_2) \cap (R_1 + R_2 + [F_1, F_2]) >}.$$

Let $F = F_1 * F_2$, then the Lie super epimorphism $F \to F_1 \times F_2$ induces the following Lie super epimorphism

$$\alpha : \frac{R \cap F^2}{< K(F)\cap R >} \to \frac{R_1 \cap {F_1}^2}{< K(F_1)\cap R_1 >} \oplus \frac{R_2 \cap {F_2}^2}{< K(F_2)\cap R_2 >}$$
$$x+ < K(F)\cap R > \longmapsto (x_1+ < K(F_1)\cap R_1 >, x_2+ < K(F_2)\cap R_2 >)$$

where $x = x_1 + x_2$, $x_1 \in R_1 \cap {F_1}^2$ and $x_2 \in R_2 \cap {F_2}^2$.
On the other hand,

$$\beta : \frac{R_1 \cap {F_1}^2}{< K(F_1)\cap R_1 >} \oplus \frac{R_2 \cap {F_2}^2}{< K(F_2)\cap R_2 >} \to \frac{R \cap F^2}{< K(F)\cap R >}$$

given by

$$(x_1+ < K(F_1)\cap R_1 >, x_2+ < K(F_2)\cap R_2 >) \longmapsto x+ < K(F)\cap R >$$

is a well-defined Lie super homomorphism. It is easy to check that β is a right inverse to α, so α is a Lie super epimorphism. Now, we show that α is a Lie super monomorphism. Let $x+ < K(F)\cap R > \in \ker\alpha$, such that $x = t_1 + t_2$. So we have $t_1 \in < K(F_1)\cap R_1 >$ and $t_2 \in < K(F_2)\cap R_2 >$. Since $t_1, t_2 \in < R \cap K(F) >$, then $x \in < K(F)\cap R >$. Thus α is a Lie super monomorphism. □

3 Computing the Bogomolov multiplier of Heisenberg Lie superalgebras

Heisenberg Lie superalgebras hold significant interest for physicists as they provide an isomorphic representation of the canonical commutation relations in quantum mechanics. Conversely, in the realm of geometry, the study of nil manifolds commences with the exploration of certain elementary nilpotent Lie groups, akin to the Heisenberg group.

The primary focus of this section is on the computation of the Bogomolov multiplier for Heisenberg Lie superalgebras.

Heisenberg Lie superalgebra is a Lie superalgebra $H = H_{\bar{0}} \oplus H_{\bar{1}}$ such that its first derived ideal is equal to its 1-dimensional homogeneous center, i.e. $Z(H) = H^2$ and $\dim H^2 = 1$.

According to this definition, there are two cases. (1) if the 1-dimensional center is even, then H_0 is the well-known Heisenberg Lie algebra,

(2) if the 1-dimensional center is odd, then H_0 is an abelian Lie algebra.

Definition 3.1 [10, Definition 4.1] A special Heisenberg Lie superalgebra is a Heisenberg Lie superalgebra with an even center.

Theorem 3.2 [10, Theorem 4.2] *Every special Heisenberg Lie superalgebra has dimension* $(2m+1|n)$*, and it is isomorphic to* $H_{(m,n)} = H_{\bar{0}} \oplus H_{\bar{1}}$*, where*

$$H_{\bar{0}} =< x_1, x_2, ..., x_m, x_{m+1}, ..., x_{2m}, z \mid [x_i, x_{m+i}] = z \ ; \ \ i = 1, ..., m >$$

and

$$H_{\bar{1}} =< y_1, y_2, ..., y_n \mid [y_j, y_j] = z \ ; \ \ j = 1, ..., n > .$$

Theorem 3.3 [10] *Every Heisenberg Lie superalgebra with an odd center has dimension* $(m|m+1)$*, and it is isomorphic to* $H_m = H_{\bar{0}} \oplus H_{\bar{1}}$*, where*

$$H_m =< x_1, x_2, ..., x_m, y_1, y_2, ..., y_m, z \mid [x_j, y_j] = z \ ; \ \ j = 1, ..., m > .$$

Theorem 3.4 *Every Heisenberg Lie superalgebra with odd center has a trivial Bogomolov multiplier.*

Proof By using Theorem 3.3, we have

$$H_m =< x_1, x_2, ..., x_m, y_1, y_2, ..., y_m, z \mid [x_j, y_j] = z \ ; \ \ j = 1, ..., m > .$$

According to the Definition 1.3, we have

$$z \wedge z = z \wedge [x_j, y_j] = (-1)^{|y_j|(|z|+|x_j|)}([y_j, z] \wedge x_j) = 0 \wedge x_j = 0.$$

Thus

$$\begin{aligned} H_m \wedge H_m = &< x_i \wedge x_j \ ; \ \ i, j = 1, ..., m \ , \ i \neq j > \\ &\cup < x_i \wedge y_j, x_i \wedge z, y_i \wedge y_j, y_i \wedge z \ ; \ \ i, j = 1, ..., m > . \end{aligned}$$

Now for all $w \in \mathcal{M}(H_m) \leq H_m \wedge H_m$, there exist $\alpha_{i,j}, \beta_{i,j}, \beta'_j, \gamma_i, \eta_{i,j}, \delta_i \in \mathbb{K}$ $(i, j = 1, ..., m)$ such that

$$\begin{aligned} w = &\sum_{\substack{i,j=1 \\ j\neq i}}^{m} \alpha_{i,j}(x_i \wedge x_j) + \sum_{\substack{i,j=1 \\ j\neq i}}^{m} \beta_{i,j}(x_i \wedge y_j) + \sum_{j=1}^{m} \beta'_j(x_j \wedge y_j) \\ &+ \sum_{i=1}^{m} \gamma_i(x_i \wedge z) + \sum_{i,j=1}^{m} \eta_{i,j}(y_i \wedge y_j) + \sum_{i=1}^{m} \delta_i(y_i \wedge z). \end{aligned}$$

Let $\tilde{\kappa} : H_m \wedge H_m \to [H_m, H_m]$ be given by $x \wedge y \mapsto [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have

$$\begin{aligned} w = & \sum_{\substack{i,j=1\\ j\neq i}}^{m} \alpha_{i,j}[x_i, x_j] + \sum_{\substack{i,j=1\\ j\neq i}}^{m} \beta_{i,j}[x_i, y_j] + \sum_{j=1}^{m}\beta'_j[x_j, y_j] \\ & + \sum_{i=1}^{m}\gamma_i[x_i, z] + \sum_{i,j=1}^{m} \eta_{i,j}[y_i, y_j] + \sum_{i=1}^{m}\delta_i[y_i, z] \\ = & \ 0. \end{aligned}$$

So $\sum_{j=1}^{m}\beta'_j z = 0$. Hence, $\sum_{j=1}^{m}\beta'_j = 0$ and $\beta'_m = -\sum_{j=1}^{m-1}\beta'_j$. Thus

$$\begin{aligned} w = & \sum_{\substack{i,j=1\\ j\neq i}}^{m} \alpha_{i,j}(x_i \wedge x_j) + \sum_{\substack{i,j=1\\ j\neq i}}^{m} \beta_{i,j}(x_i \wedge y_j) + \sum_{j=1}^{m-1}\beta'_j(x_j \wedge y_j - x_m \wedge y_m) \\ & + \sum_{i=1}^{m}\gamma_i(x_i \wedge z) + \sum_{i,j=1}^{m} \eta_{i,j}(y_i \wedge y_j) + \sum_{i=1}^{m}\delta_i(y_i \wedge z). \end{aligned}$$

On the other hand, we have

$$x_i \wedge z = x_i \wedge z + (-1)^{|x_i||x_i|}(x_i \wedge x_i) \ ; \ [x_i, z] + (-1)^{|x_i||x_i|}[x_i, x_i] = 0.$$

Therefore $x_i \wedge z \in \mathcal{M}_0(H_m)$. Similarly $y_i \wedge z$, $y_i \wedge y_j$ and for $i \neq j$, $x_i \wedge y_j$, belong to $\mathcal{M}_0(H_m)$.

Also we have

$$x_j \wedge y_j - x_m \wedge y_m = (-x_m) \wedge y_m + x_j \wedge y_j = (-x_m) \wedge y_m + (-1)^{|x_j||y_j|}x_j \wedge y_j,$$

and

$$[-x_m, y_m] + (-1)^{|x_j||y_j|}[x_j, y_j] = -[x_m, y_m] + [x_j, y_j] = -z + z = 0.$$

So, for all $i, j = 1...m-1$, $(x_i \wedge y_j - x_m \wedge y_m) \in \mathcal{M}_0(H_m)$. Thus $\mathcal{M}(H_m) \subseteq \mathcal{M}_0(H_m)$ and $\tilde{B}_0(H_m) = 0$. □

Theorem 3.5 *Every special Heisenberg Lie superalgebra has a trivial Bogomolov multiplier.*

Proof By using Theorem 3.2, $H_{(m,n)} = H_{\bar{0}} \oplus H_{\bar{1}}$ and

$$H_{\bar{0}} =< x_1, x_2, ..., x_m, x_{m+1}, ..., x_{2m}, z \ | \ [x_i, x_{m+i}] = z \ ; \ i = 1, ..., m >$$

and

$$H_{\bar{1}} =< y_1, y_2, ..., y_n \ | \ [y_j, y_j] = z \ ; \ j = 1, ..., n > .$$

We can see that

$$\begin{aligned} H_{(m,n)} \wedge H_{(m,n)} =& < x_i \wedge x_j \ ; \ i = 1, ..., 2m \ , \ j = 1, ..., n \ , \ i \neq j > \\ & \cup < x_i \wedge z, x_i \wedge y_j, y_i \wedge y_j \ ; \ i = 1, ..., 2m \ , \ j = 1, ..., n > . \end{aligned}$$

By using Definition 1.3, we have

$$x_i \wedge z = x_i \wedge [y_j, y_j] = (-1)^{|y_j|(|x_i||y_j|)}([y_j, x_i] \wedge y_j) - (-1)^{|x_i||y_j|}([y_j, x_i] \wedge y_j) = 0,$$

and $y_i \wedge z = y_i \wedge [x_i, x_m + i] = 0$. So

$$\begin{aligned} H_{(m,n)} \wedge H_{(m,n)} =& < x_i \wedge x_j \ ; \ i = 1, ..., 2m \ , \ j = 1, ..., n \ , \ i \neq j > \\ & \cup < x_i \wedge y_j, y_i \wedge y_j \ ; \ i = 1, ..., 2m \ , \ j = 1...n > . \end{aligned}$$

Now for all $w \in \mathcal{M}(H_{(m,n)}) \subseteq H_{(m,n)} \wedge H_{(m,n)}$, there exist $\alpha_{i,j}, \beta_{i,j}, \alpha'_i, \gamma_j \in \mathbb{K}$ with $(i = 1, ..., 2m\ ,\ j = 1, ..., n)$ such that

$$w = \sum_{i=1}^{2m} \alpha'_i (x_i \wedge x_{m+i}) + \sum_{i=1}^{2m} \sum_{\substack{j=1 \\ j \neq m+i}}^{n} \alpha_{i,j} (x_i \wedge x_j) + \sum_{i=1}^{2m} \sum_{j=1}^{n} \beta_{i,j} (x_i \wedge y_j)$$
$$+ \sum_{j=1}^{n} \gamma_j (y_j \wedge y_j) + \sum_{i=1}^{2m} \sum_{\substack{j=1 \\ i \neq j}}^{n} \gamma'_{i,j} (y_i \wedge y_j).$$

Let $\tilde{\kappa} : H_{(m,n)} \wedge H_{(m,n)} \to [H_{(m,n)}, H_{(m,n)}]$ be given by $x \wedge y \mapsto [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have

$$w = \sum_{i=1}^{2m} \alpha'_i [x_i, x_{m+i}] + \sum_{i=1}^{2m} \sum_{\substack{j=1 \\ j \neq m+i}}^{n} \alpha_{i,j} [x_i, x_j] + \sum_{i=1}^{2m} \sum_{j=1}^{n} \beta_{i,j} [x_i, y_j]$$
$$+ \sum_{j=1}^{n} \gamma_j [y_j, y_j] + \sum_{i=1}^{2m} \sum_{\substack{j=1 \\ i \neq J}}^{n} \gamma'_{i,j} [y_i, y_j] = 0.$$

So $(\sum_{i=1}^{2m} \alpha'_i + \sum_{j=1}^{n} \gamma_j) z = 0$. Hence, $\sum_{i=1}^{2m} \alpha'_i + \sum_{j=1}^{n} \gamma_j = 0$ and $\gamma_n = -\sum_{i=1}^{2m} \alpha'_i - \sum_{j=1}^{n-1} \gamma_j$. Thus

$$w = \sum_{i=1}^{2m} \alpha'_i (x_i \wedge x_{m+i}) + \sum_{i=1}^{2m} \sum_{\substack{j=1 \\ j \neq m+i}}^{n} \alpha_{i,j} (x_i \wedge x_j) + \sum_{i=1}^{2m} \sum_{\substack{j=1 \\ i \neq j}}^{n} \beta_{i,j} (x_i \wedge y_j)$$
$$+ \sum_{j=1}^{n} \gamma_j (y_j \wedge y_j) - \left(\sum_{i=1}^{2m} \alpha'_i + \sum_{j=1}^{n-1} \gamma_j \right) (y_m \wedge y_m).$$

Therefore

$$w = \sum_{i=1}^{2m} \alpha'_i (x_i \wedge x_{m+i} - y_m \wedge y_m) + \sum_{i=1}^{2m} \sum_{\substack{j=1 \\ j \neq m+i}}^{n} \alpha_{i,j} (x_i \wedge x_j)$$
$$+ \sum_{i=1}^{2m} \sum_{j=1}^{n} \beta_{i,j} (x_i \wedge y_j) + \sum_{j=1}^{n-1} \gamma_j (y_j \wedge y_j - y_m \wedge y_m).$$

According to the definition $\mathcal{M}_0(H_{(m,n)})$, for all $i = 1, ..., 2m\ ,\ j = 1, ..., n$ such that $j \neq m + i$, $x_i \wedge x_j \in \mathcal{M}_0(H_{(m,n)})$ and for all $i = 1, ..., 2m\ ,\ j = 1, ..., n$ where $i \neq j$, $x_i \wedge y_j \in \mathcal{M}_0(H_{(m,n)})$.

Also since

$$x_i \wedge x_{m+i} - y_m \wedge y_m = x_i \wedge x_{m+i} + (-1)^{|y_m||y_m|} (y_m \wedge y_m),$$

and

$$[x_i, x_{m+i}] + (-1)^{|y_m||y_m|} [y_m, y_m] = z - z = 0.$$

Then $x_i \wedge x_{m+i} - y_m \wedge y_m \in \mathcal{M}_0(H_{(m,n)})$. Similarly, we can see that $(y_j \wedge y_j - y_m \wedge y_m) \in \mathcal{M}_0(H_{(m,n)})$. Thus $\mathcal{M}(H_{(m,n)}) \subseteq \mathcal{M}_0(H_{(m,n)})$ and $\tilde{B}_0(H_{(m,n)}) = 0$. □

Theorem 3.6 *Let L be a nilpotent Lie superalgebra of dimension $(k|l)$ with* $\dim L^2 = (r|s)$, *where $r + s = 1$. Then we have*

(i) *if* $r = 1$, $s = 0$ *then* $\tilde{B_0}(L) = 0$,
(ii) *if* $r = 0$, $s = 1$ *then* $\tilde{B_0}(L) = 0$.

Proof By Proposition 5.5 in [13], if $r = 1$ and $s = 0$ then

$$L \cong H_{(m,n)} \oplus A(k - 2m - 1 \mid l - n) \;\; ; \;\; m + n \geq 1.$$

Now using Proposition 2.12 and Theorem 3.5, we have

$$\begin{aligned}\tilde{B_0}(L) &\cong \tilde{B_0}(H_{(m,n)} \oplus A(k - 2m - 1 \mid l - n)) \\ &\cong \tilde{B_0}(H_{(m,n)}) \oplus \tilde{B_0}(A(k - 2m - 1 \mid l - n)) = 0.\end{aligned}$$

Since $\tilde{B_0}(H_{(m,n)}) = \tilde{B_0}(A(k - 2m - 1 \mid l - n)) = 0$, the result follows. Also for $r = 0$ and $s = 1$, by using Proposition 3.4 in [11], $L \cong H_m \oplus A(k - m \mid l - m - 1)$. Similarly by using Proposition 2.12 and Theorem 3.5, we have

$$\begin{aligned}\tilde{B_0}(L) &\cong \tilde{B_0}(H_m \oplus A(k - m \mid l - m - 1)) \\ &\cong \tilde{B_0}(H_m) \oplus \tilde{B_0}(A(k - m \mid l - m - 1)) = 0,\end{aligned}$$

as required. □

One of the significant results on the Schur multiplier of Lie superalgebras was presented by Nayak in [10,11]. He demonstrated that for a nilpotent Lie superalgebra L of super dimension $(m|n)$, $\dim \mathcal{M}(L) = \frac{1}{2}[(m + n)^2 + n - m] - t(L)$, for some $t(L) \geq 0$ that is called **corank**. Their results suggest an interesting question, "can we classify Lie superalgebras of super dimension $(m|n)$ by corank?" He answered to this question for $t(L) = 0, \ldots, 4$ in [11]. Now, according to this classification, we will investigate the Bogomolov multiplier for some Lie superalgebras.

Theorem 3.7 *Let L be a $(m|n)$-super dimensional nilpotent Lie superalgebra and $t(L) \leq 3$. Then $\tilde{B_0}(L) = 0$.*

Proof By Theorem 1.4 in [11], Theorem 3.5, and Proposition 2.12, $\tilde{B_0}(L) = 0$. □

Theorem 3.8 *Let L be a $(m|n)$-super dimensional nilpotent Lie superalgebra and $t(L) = 4$. Then $\tilde{B_0}(L) \neq 0$ if and only if*

$$L \cong < a, b, c, d, e \;\mid\; [a, b] = c, [a, c] = d, [a, d] = [b, c] = e > .$$

Proof By using Theorem 5.11 in [13], Theorems 6.1 and 6.4 in [1], Theorem 3.5, and Proposition 2.12, the proof is obtained. □

In the following section, we want to classify all real Lie superalgebras L of dimension at most 4 such that $\tilde{B_0}(L) = 0$.

4 Computing the Bogomolov multiplier of real Lie superalgebras of dimension at most 4

This section is dedicated to obtaining the Bogomolov multiplier for the real Lie superalgebras of dimension at most four that are not Lie algebras. The classification of these Lie superalgebras is provided in [3]. Nigel Backhouse in [3] classified these Lie superalgebras into two categories: trivial and non-trivial. It is noteworthy that the Lie superalgebra L is considered trivial if $[L_{\bar{1}}, L_{\bar{1}}] = 0$. Otherwise, L is non-trivial. According to the notations used in [3], we also denote the elements of $L_{\bar{0}}$ (resp. $L_{\bar{1}}$) by Latin letters (resp. Greek letters) taken from the beginning of the alphabet.

Theorem 4.1 *Let $L_{(0,1)}$ be a trivial Lie superalgebra of dimension 1 with a basis $\{\alpha\}$ and the only Lie super bracket $[\alpha, \alpha] = 0$. Then $\tilde{B_0}(L_{(0,1)}) = 0$.*

Proof $L_{(0,1)}$ is abelian, so its Bogomolov multiplier is trivial. □

From [3], there is one trivial Lie superalgebra of dimension 2 with a basis $\{a, \alpha\}$ and the only non-zero Lie super bracket $[a, \alpha] = \alpha$.

Theorem 4.2 *Let $L_{(1,1)}$ be the trivial Lie superalgebra of dimension* 2. *Then* $\tilde{B_0}(L_{(1,1)}) = 0$.

Proof We can see that $L_{(1,1)} \wedge L_{(1,1)} =< a \wedge \alpha, \alpha \wedge \alpha >$. Hence, for all $w \in \mathcal{M}(L_{(1,1)}) \subseteq L_{(1,1)} \wedge L_{(1,1)}$, there exist $\alpha_1, \alpha_2 \in \mathbb{R}$, such that $w = \alpha_1(a \wedge \alpha) + \alpha_2(\alpha \wedge \alpha)$. Now, consider $\tilde{\kappa} : L_{(1,1)} \wedge L_{(1,1)} \to [L_{(1,1)}, L_{(1,1)}]$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[a, \alpha] + \alpha_2[\alpha, \alpha] = 0$. Thus, $\alpha_1\alpha = 0$. So $\alpha_1 = 0$. Hence $w = \alpha_2(\alpha \wedge \alpha)$. Also, we can see that $\alpha \wedge \alpha = \alpha \wedge \alpha + (-1)^{|a||a|} a \wedge a$, such that $[\alpha, \alpha] + (-1)^{|a||a|}[a, a] = 0$. Thus $\alpha \wedge \alpha \in \mathcal{M}_0(L_{(1,1)})$ and $\mathcal{M}(L_{(1,1)}) \subseteq \mathcal{M}_0(L_{(1,1)})$. Thus $\tilde{B_0}(L_{(1,1)}) = 0$. □

From [3], there are two trivial Lie superalgebras of dimension 3, which are denoted as $L_{(1,2)}$ and $L_{(2,1)}$. The Lie superalgebra $L_{(2,1)}$ has the basis $\{a, b, \alpha\}$, with the only non-zero Lie super brackets $[a, b] = b, [a, \alpha] = p\alpha$, where $p \neq 0$. Also, we have the following trivial Lie superalgebras of types (1, 2) that we showed them with $L^i_{(1,2)}$, where $i \in I = \{1, 2, 3, 4\}$.

- $L^1_{(1,2)} \cong< a, \alpha, \beta \mid [a, \alpha] = \alpha, [a, \beta] = p\beta \ ; \ 0 < |p| \leq 1 >$,
- $L^2_{(1,2)} \cong< a, \alpha, \beta \mid [a, \beta] = \alpha >$,
- $L^3_{(1,2)} \cong< a, \alpha, \beta \mid [a, \alpha] = \alpha, [a, \beta] = \alpha + \beta >$,
- $L^4_{(1,2)} \cong< a, \alpha, \beta \mid [a, \alpha] = p\alpha - \beta, [a, \beta] = \alpha + p\beta \ ; \ p \geq 0 >$.

Theorem 4.3 *Let L be a trivial Lie superalgebra of dimension* 3. *Then* $\tilde{B_0}(L) = 0$.

Proof Let $L \cong L_{(2,1)} =< a, b, \alpha \mid [a, b] = b, [a, \alpha] = p\alpha \ ; \ p \neq 0 >$. We have $L_{(2,1)} \wedge L_{(2,1)} =< a \wedge b, a \wedge \alpha, b \wedge \alpha, \alpha \wedge \alpha >$. So, for all $w \in \mathcal{M}(L_{(2,1)}) \leq L_{(2,1)} \wedge L_{(2,1)}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \in \mathbb{R}$, such that $w = \alpha_1(a \wedge b) + \alpha_2(a \wedge \alpha) + \alpha_3(b \wedge \alpha) + \alpha_4(\alpha \wedge \alpha)$. Now, let $\tilde{\kappa} : L_{(2,1)} \wedge L_{(2,1)} \to [L_{(2,1)}, L_{(2,1)}]$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[a, b] + \alpha_2[a, \alpha] + \alpha_3[b, \alpha] + \alpha_4[\alpha, \alpha] = 0$. So $\alpha_1 b + \alpha_2(p\alpha) = 0$ and so $\alpha_1 = \alpha_2 = 0$. Thus, $w = \alpha_3(b \wedge \alpha) + \alpha_4(\alpha \wedge \alpha)$. Also we can see that

$$b \wedge \alpha = b \wedge \alpha + (-1)^{|a||a|} a \wedge a \quad , \quad \alpha \wedge \alpha = \alpha \wedge \alpha + (-1)^{|a||a|} a \wedge a.$$

So $w \in \mathcal{M}_0(L_{(2,1)})$ and $\mathcal{M}(L_{(2,1)}) \subseteq \mathcal{M}_0(L_{(2,1)})$. Hence $\tilde{B_0}(L_{(2,1)}) = 0$.

Let $L \cong L^4_{(1,2)} =< a, \alpha, \beta \mid [a, \alpha] = p\alpha - \beta, [a, \beta] = \alpha + p\beta \ ; \ p \geq 0 >$. We can check that $L^4_{(1,2)} \wedge L^4_{(1,2)} =< a \wedge \alpha, a \wedge \beta, \alpha \wedge \beta, \alpha \wedge \alpha, \beta \wedge \beta >$. Hence, for all $w \in \mathcal{M}(L^4_{(1,2)}) \leq L^4_{(1,2)} \wedge L^4_{(1,2)}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{R}$, such that $w = \alpha_1(a \wedge \alpha) + \alpha_2(a \wedge \beta) + \alpha_3(\alpha \wedge \beta) + \alpha_4(\alpha \wedge \alpha) + \alpha_5(\beta \wedge \beta)$. Now, considering $\tilde{\kappa} : L^4_{(1,2)} \wedge L^4_{(1,2)} \to [L^4_{(1,2)}, L^4_{(1,2)}]$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[a, \alpha] + \alpha_2[a, \beta] + \alpha_3[\alpha, \beta] + \alpha_4[\alpha, \alpha] + \alpha_5[\beta, \beta] = 0$. So $\alpha_1(p\alpha - \beta) + \alpha_2(\alpha + p\beta) = 0$ and $(\alpha_1 p + \alpha_2)\alpha + (-\alpha_1 + p\alpha_2)\beta = 0$. So, $\alpha_1 p + \alpha_2 = -\alpha_1 + p\alpha_2 = 0$ and $\alpha_1 = \alpha_2 = 0$. Thus, $w = \alpha_3(\alpha \wedge \beta) + \alpha_4(\alpha \wedge \alpha) + \alpha_5(\beta \wedge \beta)$. On the other hand,

$$\alpha \wedge \beta = \alpha \wedge \beta + (-1)^{|a||a|} a \wedge a \quad , \quad \alpha \wedge \alpha = \alpha \wedge \alpha + (-1)^{|a||a|} a \wedge a$$
$$\beta \wedge \beta = \beta \wedge \beta + (-1)^{|a||a|} a \wedge a.$$

Thus $w \in \mathcal{M}_0(L^4_{(1,2)})$. So $\mathcal{M}(L^4_{(1,2)}) \subseteq \mathcal{M}_0(L^4_{(1,2)})$. Hence $\tilde{B_0}(L^4_{(1,2)}) = 0$. Similarly, we can see that $\tilde{B_0}(L^1_{(1,2)}) = \tilde{B_0}(L^2_{(1,2)}) = \tilde{B_0}(L^3_{(1,2)}) = 0$. □

From [3], there are three trivial Lie superalgebras of dimension 4, which are denoted as $L_{(3,1)}$, $L_{(2,2)}$, and $L_{(1,3)}$. We have the following presentations for trivial Lie superalgebras of types (3, 1), (2, 2), and (1, 3) that we denote with $L^i_{(3,1)}$, $L^i_{(2,2)}$, and $L^i_{(1,3)}$ where $i \in I = \{1, ..., 6\}$.

- $L^1_{(3,1)} \cong< a, b, c, \alpha \mid [b, c] = a, [b, \alpha] = \alpha >$,
- $L^2_{(3,1)} \cong< a, b, c, \alpha \mid [a, c] = a, [b, c] = a + b, [c, \alpha] = q\alpha \ ; \ pq \neq 0 >$,
- $L^3_{(3,1)} \cong< a, b, c, \alpha, \mid [a, c] = pa - b, [b, c] = a + pb, [c, \alpha] = q\alpha \ ; \ q \neq 0 >$,
- $L^1_{(2,2)} \cong< a, b, \alpha, \beta \mid [a, \alpha] = \alpha, [a, \beta] = \beta, [b, \beta] = \alpha >$,
- $L^2_{(2,2)} \cong< a, b, \alpha, \beta \mid [a, \alpha] = \alpha, [a, \beta] = \beta, [b, \alpha] = -\beta, [b, \beta] = \alpha >$,
- $L^3_{(2,2)} \cong< a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = p\alpha, [a, \beta] = q\beta \ ; \ pq \neq 0 \ , \ p \geq q >$,
- $L^4_{(2,2)} \cong< a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = p\alpha, [a, \beta] = \alpha + p\beta \ ; \ p \neq 0 >$,
- $L^5_{(2,2)} \cong< a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = p\alpha - q\beta, [a, \beta] = q\alpha + p\beta \ ; \ q > 0 >$,

- $L^6_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = (p+1)\alpha, [a, \beta] = p\beta, [b, \beta] = \alpha >$,
- $L^1_{(1,3)} \cong < a, \alpha, \beta, \gamma \mid [a, \alpha] = \alpha, [a, \beta] = p\beta, [a, \gamma] = q\gamma >$,
- $L^2_{(1,3)} \cong < a, \alpha, \beta, \gamma \mid [a, \alpha] = \alpha, [a, \gamma] = \beta >$,
- $L^3_{(1,3)} \cong < a, \alpha, \beta, \gamma \mid [a, \alpha] = p\alpha, [a, \beta] = \beta, [a, \gamma] = b + \gamma \;\; ; \;\; p \neq 0 >$,
- $L^4_{(1,3)} \cong < a, \alpha, \beta, \gamma \mid [a, \alpha] = p\alpha, [a, \beta] = q\beta - \gamma, [a, \gamma] = \beta + q\gamma \;\; ; \;\; q \geq 0\,,\;\; p \neq 0 >$,
- $L^5_{(1,3)} \cong < a, \alpha, \beta, \gamma \mid [a, \beta] = \alpha, [a, \gamma] = \beta >$,
- $L^6_{(1,3)} \cong < a, \alpha, \beta, \gamma \mid [a, \alpha] = \alpha, [a, \beta] = \alpha + \beta, [a, \gamma] = \beta + \gamma >$.

Theorem 4.4 *Let L be a trivial Lie superalgebra of dimension* 4. *Then* $\tilde{B}_0(L) = 0$.

Proof Let $L \cong L^3_{(3,1)} =< a, b, c, \alpha, \mid [a, c] = pa - b, [b, c] = a + pb, [c, \alpha] = q\alpha \;\; ; \;\; q \neq 0 >$. By using Definition 1.3, we have

$$\begin{aligned} a \wedge \alpha &= ([b, c] - pb) \wedge \alpha = [b, c] \wedge \alpha - p(b \wedge \alpha) \\ &= b \wedge [c, \alpha] - (-1)^{|b||c|}(c \wedge [b, \alpha]) - p(b \wedge \alpha) \\ &= (q - p)(b \wedge \alpha). \end{aligned}$$

Thus, we have

$$L^3_{(3,1)} \wedge L^3_{(3,1)} =< b \wedge \alpha, c \wedge \alpha, a \wedge b, a \wedge c, b \wedge c, \alpha \wedge \alpha > .$$

Hence, for all $w \in \mathcal{M}(L^3_{(3,1)}) \subseteq L^3_{(3,1)} \wedge L^3_{(3,1)}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{R}$, such that $w = \alpha_1(b \wedge \alpha) + \alpha_2(c \wedge \alpha) + \alpha_3(a \wedge b) + \alpha_4(a \wedge c) + \alpha_5(b \wedge c) + \alpha_6(\alpha \wedge \alpha)$. Now, let $\tilde{\kappa} : L^3_{(3,1)} \wedge L^3_{(3,1)} \to [L^3_{(3,1)}, L^3_{(3,1)}]$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[b, \alpha] + \alpha_2[c, \alpha] + \alpha_3[a, b] + \alpha_4[a, c] + \alpha_5[b, c] + \alpha_6[\alpha, \alpha] = 0$. So $\alpha_2(q\alpha) + \alpha_4(pa - b) + \alpha_5(a + pb) = 0$ and $(\alpha_2 q)\alpha + (\alpha_4 p + \alpha_5)a + (-\alpha_4 + \alpha_5 p)b = 0$. Since $q \neq 0$, $\alpha_2 = 0$ and $\alpha_4 = \alpha_5 = 0$. Thus, $w = \alpha_1(b \wedge \alpha) + \alpha_3(a \wedge b) + \alpha_6(\alpha \wedge \alpha)$. Similar to the previous one, we can see that $b \wedge \alpha, a \wedge b, \alpha \wedge \alpha \in \mathcal{M}_0(L^3_{(3,1)})$ and $w \in \mathcal{M}_0(L^3_{(3,1)})$. Thus $\mathcal{M}(L^3_{(3,1)}) \subseteq \mathcal{M}_0(L^3_{(3,1)})$. Hence $\tilde{B}_0(L^3_{(3,1)}) = 0$. Similarly, we have $\tilde{B}_0(L^1_{(3,1)}) = \tilde{B}_0(L^2_{(3,1)}) = 0$.

Let $L \cong L^6_{(2,2)} =< a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = (p+1)\alpha, [a, \beta] = p\beta, [b, \beta] = \alpha >$. By using Definition 1.3, we have

$$\begin{aligned} a \wedge \alpha &= a \wedge [b, \beta] = (-1)^{|\beta|(|a|+|b|)}([\beta, a] \wedge b) - (-1)^{|a||b|}([b, a] \wedge \beta) \\ &= -(-1)^{|\beta||a|}([a, \beta] \wedge b) + (-1)^{|b||a|}([a, b] \wedge \beta) \\ &= -p\beta \wedge b + b \wedge \beta = -p(\beta \wedge b) + b \wedge \beta \\ &= -p(-(-1)^{|\beta||b|} b \wedge \beta) + b \wedge \beta \\ &= (p+1) b \wedge \beta, \end{aligned}$$

and

$$\alpha \wedge \alpha = \alpha \wedge [b, \beta] = (-1)^{|\beta|(|\alpha|+|b|)}([\beta, \alpha] \wedge b) - (-1)^{|\alpha||b|}([b, \alpha] \wedge \beta) = 0$$

Therefore, we have

$$L \cong L^6_{(2,2)} \wedge L \cong L^6_{(2,2)} =< a \wedge \beta, b \wedge \alpha, b \wedge \beta, \alpha \wedge \beta, \beta \wedge \beta, a \wedge b > .$$

Hence, for all $w \in \mathcal{M}(L^6_{(2,2)}) \subseteq L^6_{(2,2)} \wedge L^6_{(2,2)}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{R}$, such that

$$w = \alpha_1(a \wedge \beta) + \alpha_2(b \wedge \alpha) + \alpha_3(b \wedge \beta) + \alpha_4(\alpha \wedge \beta) + \alpha_5(\beta \wedge \beta) + \alpha_6(a \wedge b).$$

Now, considering $\tilde{\kappa} : L^6_{(2,2)} \wedge L^6_{(2,2)} \to [L^6_{(2,2)}, L^6_{(2,2)}]$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[a, \beta] + \alpha_2[b, \alpha] + \alpha_3[b, \beta] + \alpha_4[\alpha, \beta] + \alpha_5[\beta, \beta] + \alpha_6[a, b] = 0$. So $\alpha_1(p\beta) + \alpha_3\alpha + \alpha_6 b = 0$. Thus if $p \neq 0$, then $\alpha_1 = \alpha_3 = \alpha_6 = 0$. Thus $w = \alpha_2(b \wedge \alpha) + \alpha_4(\alpha \wedge \beta) + \alpha_5(\beta \wedge \beta)$ and $b \wedge \alpha, \alpha \wedge \beta, \beta \wedge \beta \in \mathcal{M}_0(L^6_{(2,2)})$. Therefore $w \in \mathcal{M}_0(L^6_{(2,2)})$ and $\mathcal{M}(L^6_{(2,2)}) \subseteq \mathcal{M}_0(L^6_{(2,2)})$. Hence $\tilde{B}_0(L^6_{(2,2)}) = 0$.

But in case $p = 0$, we have

$$L^6_{(2,2)} =< a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \alpha, [b, \beta] = \alpha > .$$

So

$$w = \alpha_1(a \wedge \beta) + \alpha_2(b \wedge \alpha) + \alpha_4(\alpha \wedge \beta) + \alpha_5(\beta \wedge \beta).$$

Same as before, $a \wedge \beta, b \wedge \alpha, \alpha \wedge \beta, \beta \wedge \beta \in \mathcal{M}_0(L^6_{(2,2)})$ and finally, in this case, $\tilde{B}_0(L^6_{(2,2)}) = 0$. Similarly, we have $\tilde{B}_0(L^i_{(2,2)}) = 0$, where $i = 1, ..., 5$.

Let

$$L \cong L^4_{(1,3)} =< a, \alpha, \beta, \gamma \mid [a, \alpha] = p\alpha, [a, \beta] = q\beta - \gamma, [a, \gamma] = \beta + q\gamma \;\; ; \;\; q \geq 0\,, \;\; p \neq 0 > .$$

By using Definition 1.3, we have

$$\alpha \wedge \beta = -p(\alpha \wedge \gamma) \;\; , \;\; \beta \wedge \beta = -q(\beta \wedge \gamma) + \gamma \wedge \gamma.$$

Therefore,

$$L^4_{(1,3)} \wedge L^4_{(1,3)} =< a \wedge \alpha, a \wedge \beta, a \wedge \gamma, \alpha \wedge \alpha, \alpha \wedge \gamma, \beta \wedge \gamma, \gamma \wedge \gamma > .$$

Hence, for all $w \in \mathcal{M}(L^4_{(1,3)}) \subseteq L^4_{(1,3)} \wedge L^4_{(1,3)}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \in \mathbb{R}$, such that

$$\begin{aligned} w = \alpha_1(a \wedge \alpha) + \alpha_2(a \wedge \beta) + \alpha_3(a \wedge \gamma) + \alpha_4(\alpha \wedge \alpha) \\ + \alpha_5(\alpha \wedge \gamma) + \alpha_6(\beta \wedge \gamma) + \alpha_7(\gamma \wedge \gamma). \end{aligned}$$

Now, let $\tilde{\kappa} : L^4_{(1,3)} \wedge L^4_{(1,3)} \to [L^4_{(1,3)}, L^4_{(1,3)}]$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have

$$\alpha_1[a, \alpha] + \alpha_2[a, \beta] + \alpha_3[a, \gamma] + \alpha_4[\alpha, \alpha] + \alpha_5[\alpha, \gamma] + \alpha_6[\beta, \gamma] + \alpha_7[\gamma, \gamma] = 0.$$

So $\alpha_1(p\alpha) + \alpha_2(q\beta - \gamma) + \alpha_3(\beta + q\gamma) = 0$. Hence $(\alpha_1 p)\alpha + (\alpha_2 q + \alpha_3)\beta + (-\alpha_2 + \alpha_3 q)\gamma = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Also we know

$$(\alpha \wedge \alpha), (\alpha \wedge \gamma), (\beta \wedge \gamma), (\gamma \wedge \gamma) \in \mathcal{M}_0(L^4_{(1,3)}).$$

Therefore $\mathcal{M}(L^4_{(1,3)}) \subseteq \mathcal{M}_0(L^4_{(1,3)})$ and $\tilde{B}_0(L^4_{(1,3)}) = 0$. □

Corollary 4.5 *All trivial real Lie superalgebras of dimension at most* 4 *have trivial Bogomolov multiplier.*

4.1 Computing the Bogomolov multiplier of real nontrivial Lie superalgebras of dimension at most 4

We now perform the same calculations for all nontrivial real Lie superalgebras of dimension at most four. From [3], there exists a unique superalgebra of dimension 2 whose basis is a, and α. The sole non-zero Lie superbracket is $[\alpha, \alpha] = a$.

Theorem 4.6 *Let L be a nontrivial Lie superalgebra of dimension* 2*. Then $\tilde{B}_0(L) = 0$.*

Proof Let $L \cong L_{(1,1)}$. We can see that $L_{(1,1)} \wedge L_{(1,1)} =< a \wedge \alpha, \alpha \wedge \alpha >$. Hence, for all $w \in \mathcal{M}(L_{(1,1)}) \subseteq L_{(1,1)} \wedge L_{(1,1)}$, there exist $\alpha_1, \alpha_2 \in \mathbb{R}$, such that $w = \alpha_1(a \wedge \alpha) + \alpha_2(\alpha \wedge \alpha)$. Now, considering $\tilde{\kappa} : L_{(1,1)} \wedge L_{(1,1)} \to [L_{(1,1)}, L_{(1,1)}]$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[a, \alpha] + \alpha_2[\alpha, \alpha] = 0$. Thus, $\alpha_1 a = 0$ and so $\alpha_1 = 0$. Hence $w = \alpha_2(\alpha \wedge \alpha) \in \mathcal{M}_0(L_{(1,1)})$ and $\mathcal{M}(L_{(1,1)}) \subseteq \mathcal{M}_0(L_{(1,1)})$. Thus $\tilde{B}_0(L_{(1,1)}) = 0$. □

From [3], there are two nontrivial Lie superalgebras of dimension 3, which are denoted by $L_{(1,2)}$ and $L_{(2,1)}$. The Lie superalgebra $L_{(2,1)}$ has the basis $\{a, b, \alpha\}$, with the only non-zero Lie super brackets $[a, b] = b$, $[a, \alpha] = 1/2\alpha$ and $[\alpha, \alpha] = b$. Also, we have the following nontrivial Lie superalgebras of types $(1, 2)$ that we denote by $L^i_{(1,2)}$, where $i \in I = \{1, 2\}$.

- $L^1_{(1,2)} \cong< a, \alpha, \beta \mid [\alpha, \alpha] = a, [\beta, \beta] = a >$,

- $L^2_{(1,2)} \cong < a, \alpha, \beta \mid [\alpha, \alpha] = a, [\beta, \beta] = -a >$.

Theorem 4.7 *Let L be a nontrivial Lie superalgebra of dimension* 3. *Then* $\tilde{B}_0(L) = 0$.

Proof Let $L \cong L^1_{(1,2)} = < a, \alpha, \beta \mid [\alpha, \alpha] = a, [\beta, \beta] = a >$. By using Definition 1.3, $a \wedge \alpha, a \wedge \beta = 0$. So we have $L^1_{(1,2)} \wedge L^1_{(1,2)} = < \alpha \wedge \alpha, \alpha \wedge \beta, \beta \wedge \beta >$. Hence, for all $w \in \mathcal{M}(L^1_{(1,2)}) \subseteq L^1_{(1,2)} \wedge L^1_{(1,2)}$, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, such that $w = \alpha_1(\alpha \wedge \alpha) + \alpha_2(\alpha \wedge \beta) + \alpha_3(\beta \wedge \beta)$. Now, let $\tilde{\kappa} : L^1_{(1,2)} \wedge L^1_{(1,2)} \to [L^1_{(1,2)}, L^1_{(1,2)}]$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[\alpha, \alpha] + \alpha_2[\alpha, \beta] + \alpha_3[\beta, \beta] = 0$. So $(\alpha_1 + \alpha_3)a = 0$ and $\alpha_3 = -\alpha_1$. Thus, $w = \alpha_1(\alpha \wedge \alpha - \beta \wedge \beta) + \alpha_2(\alpha \wedge \beta)$.

Similar to the previous one, we can see that $\alpha \wedge \beta \in \mathcal{M}_0(L^1_{(1,2)})$. On the other hand, we have

$$\alpha \wedge \alpha - \beta \wedge \beta = \alpha \wedge \alpha + (-1)^{|\beta||\beta|}\beta \wedge \beta \quad ; \quad [\alpha, \alpha] + (-1)^{|\beta||\beta|}[\beta, \beta] = 0.$$

Therefore $\alpha \wedge \alpha + (-1)^{|\beta||\beta|}\beta \wedge \beta \in \mathcal{M}_0(L^1_{(1,2)})$ and $w \in \mathcal{M}_0(L^1_{(1,2)})$. Thus $\mathcal{M}(L^1_{(1,2)}) \subseteq \mathcal{M}_0(L^1_{(1,2)})$. Hence $\tilde{B}_0(L^1_{(1,2)}) = 0$. Similarly, $\tilde{B}_0(L^2_{(1,2)}) = 0$. □

From [3], there are three nontrivial Lie superalgebras of dimension 4, which are denoted by $L_{(3,1)}$, $L_{(2,2)}$, and $L_{(1,3)}$. We have the following presentations for nontrivial Lie superalgebras of types (3, 1), (2, 2), and (1, 3) that we denote by $L^i_{(3,1)}$, $L^i_{(2,2)}$, and $L^i_{(1,3)}$ where $i \in I = \{1, ..., 17\}$.

- $L^1_{(3,1)} \cong < a, b, c, \alpha \mid [b, c] = a, [\alpha, \alpha] = a >$,
- $L^2_{(3,1)} \cong < a, b, c, \alpha \mid [a, b] = b, [a, c] = pc, [a, \alpha] = \frac{1}{2}\alpha, [\alpha, \alpha] = b \;\; ; \;\; p \neq 0 >$,
- $L^3_{(3,1)} \cong < a, b, c, \alpha, \mid [a, b] = b, [a, c] = b + c, [a, \alpha] = \frac{1}{2}\alpha, [\alpha, \alpha] = b >$,
- $L^4_{(3,1)} \cong < a, b, c, \alpha, \mid [a, b] = b, [a, c] = -b + c, [a, \alpha] = \frac{1}{2}\alpha, [\alpha, \alpha] = b >$,
- $L^1_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \frac{1}{2}\beta, [\alpha, \alpha] = b, [\beta, \beta] = b >$,
- $L^2_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \frac{1}{2}\beta, [\alpha, \alpha] = b, [\beta, \beta] = -b >$,
- $L^3_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \frac{1}{2}\beta, [\alpha, \alpha] = b >$,
- $L^4_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = p\alpha, [a, \beta] = (1 - p)\beta, [\alpha, \beta] = b \;\; ; \;\; p \leq \frac{1}{2} >$,
- $L^5_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \alpha + \frac{1}{2}\beta, [\beta, \beta] = b >$,
- $L^6_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \frac{1}{2}\alpha - p\beta, [a, \beta] = p\alpha + \frac{1}{2}\beta, [\alpha, \alpha] = b, [\beta, \beta] = b \;\; ; \;\; p > 0 >$,
- $L^7_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \alpha, [b, \beta] = \alpha, [\beta, \beta] = a, [\alpha, \beta] = -\frac{1}{2}b >$,
- $L^8_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \alpha, [b, \beta] = \alpha, [\beta, \beta] = -a, [\alpha, \beta] = \frac{1}{2}b >$,
- $L^9_{(2,2)} \cong < a, b, \alpha, \beta \mid [\alpha, \alpha] = a, [\beta, \beta] = b >$,
- $L^{10}_{(2,2)} \cong < a, b, \alpha, \beta \mid [\alpha, \alpha] = a, [\beta, \beta] = b, [\alpha, \beta] = a >$,
- $L^{11}_{(2,2)} \cong < a, b, \alpha, \beta \mid [\alpha, \alpha] = a, [\beta, \beta] = b, [\alpha, \beta] = p(a + b) \;\; ; \;\; p > 0 >$,
- $L^{12}_{(2,2)} \cong < a, b, \alpha, \beta \mid [\alpha, \alpha] = a, [\beta, \beta] = b, [\alpha, \beta] = p(a - b) \;\; ; \;\; p > 0 >$,
- $L^{13}_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \alpha, [\alpha, \beta] = b >$,
- $L^{14}_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \frac{1}{2}\alpha, [\alpha, \alpha] = b >$,
- $L^{15}_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, \alpha] = \alpha, [a, \beta] = -\beta, [a, \beta] = b >$,
- $L^{16}_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, \beta] = \alpha, [\beta, \beta] = b >$,
- $L^{17}_{(2,2)} \cong < a, b, \alpha, \beta \mid [a, \alpha] = -\beta, [a, \beta] = \alpha, [\alpha, \alpha] = b, [\beta, \beta] = b >$,
- $L^1_{(1,3)} \cong < a, \alpha, \beta, \gamma \mid [\alpha, \alpha] = a, [\beta, \beta] = a, [\gamma, \gamma] = a >$,
- $L^2_{(1,3)} \cong < a, \alpha, \beta, \gamma \mid [\alpha, \alpha] = a, [\beta, \beta] = a, [\gamma, \gamma] = -a >$.

Theorem 4.8 *Let L be a nontrivial Lie superalgebra of dimension* 4. *Then* $\tilde{B}_0(L) = 0$.

Proof Let $L \cong L^3_{(3,1)} = < a, b, c, \alpha, \mid [a, b] = b, [a, c] = b + c, [a, \alpha] = \frac{1}{2}\alpha, [\alpha, \alpha] = b >$. According to the Definition 1.3

$$a \wedge b = \alpha \wedge \alpha \;\; , \;\; b \wedge c = 0 \;\; , \;\; c \wedge \alpha = -\frac{2}{3}(b \wedge \alpha),$$

We have $L^3_{(3,1)} \wedge L^3_{(3,1)} =< a \wedge b, a \wedge c, a \wedge \alpha, b \wedge \alpha >$. Hence, for all $w \in \mathcal{M}(L^3_{(3,1)}) \subseteq L^3_{(3,1)} \wedge L^3_{(3,1)}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$, such that $w = \alpha_1(a \wedge b) + \alpha_2(a \wedge c) + \alpha_3(a \wedge \alpha) + \alpha_4(b \wedge \alpha)$. Now, let $\tilde{\kappa} : L^3_{(3,1)} \wedge L^3_{(3,1)} \to [L^3_{(3,1)}, L^3_{(3,1)}]$ be given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have $\alpha_1[a, b] + \alpha_2[a, c] + \alpha_3[a, \alpha] + \alpha_4[b, \alpha] = 0$ and so $(\alpha_1 + \alpha_2)b + \alpha_2 c + \frac{1}{2}\alpha_3\alpha = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus, $w = \alpha_4(b \wedge \alpha)$ and $b \wedge \alpha \in \mathcal{M}_0(L^3_{(3,1)})$. Hence $\mathcal{M}(L^3_{(3,1)}) \subseteq \mathcal{M}_0(L^3_{(3,1)})$. Thus $\tilde{B}_0(L^3_{(3,1)}) = 0$.

Now, let $L \cong L^1_{(2,2)} =< a, b, \alpha, \beta \mid [a, b] = b, [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \frac{1}{2}\beta, [\alpha, \alpha] = b, [\beta, \beta] = b >$. Since

$$a \wedge b = \alpha \wedge \alpha = \beta \wedge \beta \ \ , \ \ b \wedge \alpha = b \wedge \beta = 0,$$

we have

$$L^1_{(2,2)} \wedge L^1_{(2,2)} =< a \wedge b, a \wedge \alpha, a \wedge \beta, \alpha \wedge \beta > .$$

For all $w \in \mathcal{M}(L^1_{(2,2)}) \subseteq L^1_{(2,2)} \wedge L^1_{(2,2)}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ such that

$$w = \alpha_1(a \wedge b) + \alpha_2(a \wedge \alpha) + \alpha_3(a \wedge \beta) + \alpha_4(\alpha \wedge \beta).$$

Now let $\tilde{\kappa} : L^1_{(2,2)} \wedge L^1_{(2,2)} \to [L^1_{(2,2)}, L^1_{(2,2)}]$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have

$$\alpha_1[a, b] + \alpha_2[a, \alpha] + \alpha_3[a, \beta] + \alpha_4[\alpha, \beta] = 0.$$

Thus $\alpha_1 b + \frac{1}{2}\alpha_2\alpha + \frac{1}{2}\alpha_3\beta = 0$, so $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence $w = \alpha_4(\alpha \wedge \beta)$. So, $w \in \mathcal{M}_0(L^1_{(2,2)})$ and $\mathcal{M}(L^1_{(2,2)}) \subseteq \mathcal{M}_0(L^1_{(2,2)})$. Therefore $\tilde{B}_0(L^1_{(2,2)}) = 0$.

Let $L \cong L^{11}_{(2,2)} =< a, b, \alpha, \beta \mid [\alpha, \alpha] = a, [\beta, \beta] = a, [\alpha, \beta] = p(a + b)\ ;\ p > 0 >$. By using Definition 1.3, we have

$$a \wedge b = [\alpha, \alpha] \wedge b = \alpha \wedge [\alpha, b] - (-1)^{|\alpha||\alpha|}(\alpha \wedge [\alpha, b]) = 0,$$

and

$$a \wedge \beta = 2p(\alpha \wedge a + \alpha \wedge b) \ \ , \ \ b \wedge \alpha = 2p(\beta \wedge a + \beta \wedge b).$$

Hence, we see that

$$L^{11}_{(2,2)} \wedge L^{11}_{(2,2)} =< a \wedge \alpha, b \wedge \alpha, \alpha \wedge \alpha, \alpha \wedge \beta, \beta \wedge \beta > .$$

So for all $w \in \mathcal{M}(L^{11}_{(2,2)}) \subseteq L^{11}_{(2,2)} \wedge L^{11}_{(2,2)}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{R}$ such that

$$\begin{aligned} w = \alpha_1(a \wedge \alpha) + \alpha_2(b \wedge \alpha) + \alpha_3(\alpha \wedge \alpha) \\ + \alpha_4(\alpha \wedge \beta) + \alpha_5(\beta \wedge \beta). \end{aligned}$$

By using $\tilde{\kappa} : L^{11}_{(2,2)} \wedge L^{11}_{(2,2)} \to [L^{11}_{(2,2)}, L^{11}_{(2,2)}]$ given by $x \wedge y \to [x, y]$. Since $\tilde{\kappa}(w) = 0$, we have

$$\alpha_1[a, \alpha] + \alpha_2[b, \alpha] + \alpha_3[\alpha, \alpha] + \alpha_4[\alpha, \beta] + \alpha_5[\beta, \beta] = 0.$$

Thus $(\alpha_3 + p\alpha_4)a + (\alpha_5 + p\alpha_4)b = 0$ and so, $\alpha_3 = \alpha_5 = -p\alpha_4$. Hence

$$w = \alpha_1(a \wedge \alpha) + \alpha_2(b \wedge \alpha) + \alpha_4(\alpha \wedge \beta - p\alpha \wedge \alpha - p\beta \wedge \beta).$$

On the other hand, we have

$$\alpha \wedge \beta - p\alpha \wedge \alpha - p\beta \wedge \beta = ((\alpha - p\beta) \wedge \beta) - (p\alpha \wedge \alpha) = ((\alpha - p\beta) \wedge \beta) + (-1)^{|p\alpha||\alpha|}(p\alpha \wedge \alpha),$$

and

$$[\alpha - p\beta, \beta] + (-1)^{|p\alpha||\alpha|}[p\alpha, \alpha] = 0.$$

Thus $\alpha \wedge \beta - p\alpha \wedge \alpha - p\beta \wedge \beta \in \mathcal{M}_0(L^{11}_{(2,2)})$. Also, $a \wedge \alpha, b \wedge \alpha \in \mathcal{M}_0(L^{11}_{(2,2)})$. Therefore, $w \in \mathcal{M}_0(L^{11}_{(2,2)})$. So $\mathcal{M}(L^{11}_{(2,2)}) \subseteq \mathcal{M}_0(L^{11}_{(2,2)})$. Hence $\tilde{B}_0(L^{11}_{(2,2)}) = 0$. Similarly, $\tilde{B}_0(L^{12}_{(2,2)}) = 0$.

Let $L \cong L^1_{(1,3)} =< a, \alpha, \beta, \gamma \mid [\alpha, \alpha] = a, [\beta, \beta] = a, [\gamma, \gamma] = a >$. According to the Definition 1.3, we have

$$a \wedge \alpha = a \wedge \beta = a \wedge \gamma = 0.$$

Hence, we see that

$$L^1_{(1,3)} \wedge L^1_{(1,3)} =< \alpha \wedge \alpha, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \beta, \beta \wedge \gamma, \gamma \wedge \gamma > .$$

So for all $w \in \mathcal{M}(L^1_{(1,3)}) \subseteq L^1_{(1,3)} \wedge L^1_{(1,3)}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{R}$ such that

$$\begin{aligned} w = &\alpha_1(\alpha \wedge \alpha) + \alpha_2(\alpha \wedge \beta) + \alpha_3(\alpha \wedge \gamma) \\ &+ \alpha_4(\beta \wedge \beta) + \alpha_5(\beta \wedge \gamma) + \alpha_6(\gamma \wedge \gamma). \end{aligned}$$

By using a Lie super homomorphism $\tilde{\kappa} : L^1_{(1,3)} \wedge L^1_{(1,3)} \to [L^1_{(1,3)}, L^1_{(1,3)}]$ given by $x \wedge y \to [x, y]$, since $\tilde{\kappa}(w) = 0$, we have

$$\alpha_1[\alpha, \alpha] + \alpha_2[\alpha, \beta] + \alpha_3[\alpha, \gamma] + \alpha_4[\beta, \beta] + \alpha_5[\beta, \gamma] + \alpha_6[\gamma, \gamma] = 0.$$

Thus $(\alpha_1 + \alpha_4 + \alpha_6)a = 0$ and $\alpha_6 = -\alpha_1 - \alpha_4$. Hence

$$\begin{aligned} w = &\alpha_1(\alpha \wedge \alpha - \gamma \wedge \gamma) + \alpha_2(\alpha \wedge \beta) + \alpha_3(\alpha \wedge \gamma) \\ &+ \alpha_4(\beta \wedge \beta - \gamma \wedge \gamma) + \alpha_5(\beta \wedge \gamma). \end{aligned}$$

Now, since

$$\alpha \wedge \alpha - \gamma \wedge \gamma = \alpha \wedge \alpha + (-1)^{|\gamma||\gamma|}(\gamma \wedge \gamma) \ , \ [\alpha, \alpha] + (-1)^{|\gamma||\gamma|}[\gamma, \gamma] = 0,$$

and

$$\beta \wedge \beta - \gamma \wedge \gamma = \beta \wedge \beta + (-1)^{|\gamma||\gamma|}(\gamma \wedge \gamma) \ , \ [\beta, \beta] + (-1)^{|\gamma||\gamma|}[\gamma, \gamma] = 0,$$

we have $(\alpha \wedge \alpha - \gamma \wedge \gamma), (\beta \wedge \beta - \gamma \wedge \gamma) \in \mathcal{M}_0(L^1_{(1,3)})$.

Also, $\alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma \in \mathcal{M}_0(L^1_{(1,3)})$. Thus, $w \in \mathcal{M}_0(L^1_{(1,3)})$. So $\mathcal{M}(L^1_{(1,3)}) \subseteq \mathcal{M}_0(L^1_{(1,3)})$. Hence $\tilde{B}_0(L^1_{(1,3)}) = 0$. Similarly, $\tilde{B}_0(L^2_{(1,3)}) = 0$.

In general, for $i = 1, 2, 4$, $\tilde{B}_0(L^i_{(3,1)}) = 0$, for $i = 2, ..., 17$, $\tilde{B}_0(L^i_{(2,2)}) = 0$ and $\tilde{B}_0(L^2_{(1,3)}) = 0$. □

Corollary 4.9 *All nontrivial real Lie superalgebras of dimension at most* 4 *have trivial Bogomolov multiplier.*

Declarations

Competing interests The authors declare no competing interests.

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