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ORIGINAL RESEARCH

A study on ordered n-ary (semi)hypergroups and their subsets

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Abstract Starting with an n-ary (semi)hypergroup accompanied with a partial order, we define ordered n-ary (semi)hypergroup. The aim of this paper is to investigate some properties of the new defined concept. In this regard, we present non-trivial examples on ordered n-ary (semi)hypergroups and study their properties by studying some important subsets of them.

Keywords (semi)hypergroup · n-ary (semi)hypergroup · Partial order · n-ary ordered (semi)hypergroup · n-ary subsemihypergroup · n-ary subhypergroup · Hyperideal · Homomorphism.

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1 Introduction

The concept of hypergroup was introduced by Marty [15] in 1934 as a generalization of the group's concept. Marty's motivation was that the quotient of a group by any of it's subgroups is not necessarily a group, but it is a hypergroup. Since then, many generalizations of algebraic structures was done and different hyperstructures were introduced and studied by many authors. For details about hyperstructure theory and its applications, we refer to [1,3–5,7,9,18]. Inspired by ordered semigroups, Heidari and Davvaz [14] in 2011 introduced ordered semihypergroups by considering a semihypergroup $(S, \circ)$ besides a binary relation "$\leq$", where "$\leq$" is a partial order relation on S that satisfies the monotone property. This results in the introducing of ordered hyperstructures. For more details about ordered hyperstructures, we refer to [2,6].

n-ary algebraic systems are a generalization of algebraic systems and were first introduced and studied by Dornte [12] in 1928. Dornte observed that from a group $(G, \cdot)$, one can get the n-ary group (G, f) by taking $f(x_1, x_2, \ldots, x_n) = x_1x_2 \ldots x_n$ for all $x_1, \ldots, x_n \in G$. Davvaz and Vougioklis [8] in 2006 generalized n-ary groups to n-ary hypergroups and found their fundamental n-ary groups by defining a fundamental relation on them. Later in 2009, Davvaz et al. [10] considered a class of algebraic hypersystems which represents a generalization of semigroups, semihypergroups, and n-ary semigroups. They named the new class of hypersystems as n-ary semihypergroups and investigated its properties. After that, the concept of n-ary hyperstructures was generalized to include higher order hyperstructures like hyperrings by introducing (m, n)-hyperrings [11,13,16,17].

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In this paper, we consider a class of ordered algebraic hypersystems which represents a generalization of ordered semigroups and ordered semi(hypergroups), and we call it ordered n-ary (semi)hypergroups. More precisely, it is constructed as follows: after an Introduction, in Section2, we present basic definitions related to hyperstructures and n-ary hyperstructures that are used throughout the paper. In Section3, we introduce the notion of ordered n-ary semi(hypergroups), study its properties, and illustrate our new definition by some non-trivial examples. Finally, in Section4, we define n-ary sub(semi)hypergroups and hyperideals in ordered n-ary (semi)hypergroups, investigate their properties, and present some examples.

2 n-ary hyperstructures

In this section, we present some basics in the fields of hyperstructures and n-ary hyperstructures that are used throughout the paper.

Definition 2.1 [5] Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*.
If A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \ \ x \circ A = \{x\} \circ A \text{and} A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$, that is

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

Also, it is called a *quasi-hypergroup* if $x \circ H = H \circ x = H$ for all $x \in H$. A *hypergroup* is a quasi-hypergroup and semihypergroup. Moreover, it is commutative if $x \circ y = y \circ x$ for all $x, y \in H$.

Definition 2.2 [5] Let $(H, \circ)$ be a hypergroupoid and $e \in H$. Then e is an *identity* of H if $x \in (e \circ x) \cap (x \circ e)$ for all $x \in H$ and it is a *scalar identity* if $x \circ e = e \circ x = x$ for all $x \in H$.

Example 1 Let H be a non-empty set with $0 \in H$. Define "$\star$" on H as follows: For all $x, y \in H$,

$$x \star y = \begin{cases} H & \text{if } x \neq 0 \text{ and } y \neq 0; \\ x & \text{if } y = 0; \\ y & \text{if } x = 0. \end{cases}$$

Then $(H, \star)$ is a hypergroup with scalar identity "0".

Definition 2.3 [8] Let H be a non-empty set. Then $f : H^n \to \mathcal{P}^*(H)$ is called an *n-ary hyperoperation* and (H, f) is called an *n-ary hypergroupoid*.

If (H, f) is an n-ary hypergroupoid and $X_1, \ldots, X_n$ are non-empty subsets of H then

$$f(A_1, \ldots, A_n) = \{f(a_1, \ldots, a_n) : a_i \in A_i \text{ for all } i = 1, \ldots, n\}.$$

Notation 1 If (H, f) is an n-ary hypergroupoid and $x_i, x_{i+1}, \ldots, x_j \in H$ with $i < j$ then the sequence $x_i, x_{i+1}, \ldots, x_j$ is denoted as x_i^j. Consequently,

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n) = f(x_1^i, y_{i+1}^j, z_{j+1}^n).$$

Definition 2.4 [8] Let (H, f) be an n-ary hypergroupoid. Then (H, f) is called

1. an n-ary semihypergroup if $f(x_1^{i-1}, f(x_i^{n+i-1}), x_{i+1}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{j+1}^{2n-1})$ for all $i, j \in \{1, 2, \ldots, 2n-1\}$.

2. an n-ary quasi-hypergroup if the equation $b \in f(x_1^{i-1}, x, x_{i+1}^n)$ has a solution $x \in H$ for all $x_1, \dots, x_n, b \in H$.
3. an n-ary hypergroup if it is an n-ary semihypergroup and n-ary quasi-hypergroup.

Remark 1 A (semi)hypergroup is a 2-ary (semi)hypergroup.

Definition 2.5 [8] Let (H, f) be an n-ary hypergroupoid and $e \in H$. Then e is an *identity* of H if $x \in f(\underbrace{e, \dots, e}_{i-1}, x, \underbrace{e, \dots, e}_{n-i})$ for all $x \in H$ and $i \in \{1, \dots, n\}$. Also, e is a *scalar identity* of H if $f(\underbrace{e, \dots, e}_{i-1}, x, \underbrace{e, \dots, e}_{n-i})$ for all $x \in H$ and $i \in \{1, \dots, n\} = x$.

Definition 2.6 [8] Let (H, f) be an n-ary hypergroupoid. Then H is commutative if for all $a_1^n \in H$ and $\sigma \in \mathbb{S}_n$, $f(a_1^n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)})$.

Example 2 Let H be a non-empty set, $n \in \mathbb{N}$, and $n \geq 2$. Define $f : H^n \to \mathcal{P}^*(H)$ as follows: For all $x_1, x_2, \dots, x_n \in H$,

$$f(x_1, x_2, \dots, x_n) = H.$$

Then (H, f) is a commutative n-ary hypergroup. Moreover, we name it **total n-ary hypergroup**.

Example 3 Let H be a non-empty set, $n \in \mathbb{N}$, and $n \geq 2$. Define $f : H^n \to \mathcal{P}^*(H)$ as follows: For all $x_1, x_2, \dots, x_n \in H$,

$$f(x_1, x_2, \dots, x_n) = \{x_1, x_2, \dots, x_n\}.$$

Then (H, f) is a commutative n-ary hypergroup. Moreover, we name it **n-set n-ary hypergroup**.

Example 4 Let H be a non-empty set with $0 \in H$. Define $f : H^3 \to \mathcal{P}^*(H)$ as follows: For all $x, y, z \in H$,

$$f(x, y, z) = \begin{cases} H & \text{if } (x, y) \neq (0, 0), or (x, z) \neq (0, 0), or (y, z) \neq (0, 0); \\ x & \text{if } (y, z) = (0, 0); \\ y & \text{if } (x, z) = (0, 0); \\ z & \text{if } (x, y) = (0, 0). \end{cases}$$

Then (H, f) is a ternary hypergroup with scalar identity “0".

Example 5 Let H be a non-empty set with $0 \in H$. Define $g : H^3 \to \mathcal{P}^*(H)$ as follows: For all $x, y, z \in H$,

$$g(x, y, z) = \begin{cases} \{x, y, z\} & x \neq 0, y \neq 0, \text{ and } z \neq 0; \\ \{x, y\} & x \neq 0, y \neq 0, \text{ and } z = 0; \\ \{x, z\} & x \neq 0, y = 0, \text{ and } z \neq 0; \\ \{y, z\} & x = 0, y \neq 0, \text{ and } z \neq 0; \\ x & \text{if } (y, z) = (0, 0); \\ y & \text{if } (x, z) = (0, 0); \\ z & \text{if } (x, y) = (0, 0). \end{cases}$$

Then (H, g) is a ternary hypergroup with scalar identity “0".

Definition 2.7 [8] Let (H, f) be an n-ary (semi)hypergroup and $S \neq \emptyset \subseteq H$. Then S is an n-ary sub(semi)hypergroup of H if (S, f) is an n-ary (semi)hypergroup.

Remark 2 Let H be a non-empty set and $S \neq \emptyset \subseteq H$.

(1) If (H, f) is an n-ary semihypergroup then to prove that S is an n-ary subsemihypergroup of H, it suffices to show that $f(x_1, x_2, \dots, x_n) \subseteq S$ for all $x_1, x_2, \dots, x_n \in S$.
(2) If (H, f) is an n-ary hypergroup then to prove that S is an n-ary subhypergroup of H, it suffices to show that $f(x_1, x_2, \dots, x_n) \subseteq S$ for all $x_1, x_2, \dots, x_n \in S$ and that S is an n-ary quasi-hypergroup.

Example 6 Every non-empty subset of the n-set n-ary hypergroup is an n-ary subhypergroup.

Example 7 A non-empty subset S of the total n-ary hypergroup is an n-ary subhypergroup if and only if $S = H$.

Remark 3 Let (H, f) be an n-ary (semi)hypergroup with a scalar identity $e \in H$. Then $\{e\}$ is an n-ary (subsemi-hypergroup)subhypergroup of H.

3 Ordered n-ary semi(hypergroups)

In this section, by using a partial order on n-ary (semi)hypergroups, we introduce the notion of ordered n-ary semi(hypergroups). Moreover, we illustrate our new definition by some non-trivial examples.

Definition 3.1 Let (H, f) be an n-ary (semi)hypergroup and "$\leq$" a partial order on H. If f satisfies the monotone property, i.e.,

$$\alpha \leq \beta \text{ implies } f(x_1^{i-1}, \alpha, x_{i+1}^n) \leq f(x_1^{i-1}, \beta, x_{i+1}^n) \text{ for all } x_i \in H, i \in \{1, 2, \ldots, n\}$$

then $(H, f, \leq)$ is an ordered n-ary (semi)hypergroup.

Remark 4 Let $(H, f, \leq)$ be an ordered n-ary (semi)hypergroup and A, B non-emty subsets of H. Then $A \leq B$ means that for every $b \in B$, there is $a \in A$ with $a \leq b$.

Remark 5 If (H, f) is a (semi)hypergroup (i.e., 2-ary (semi)hypergroup) and "$\leq$" a partial order on H satisfying the condition of Definition 3.1 then $(H, f, \leq)$ is the ordered (semi)hypergroup defined by Heidari and Davvaz in [14].

Definition 3.2 Let $(H, f, \leq)$ be an ordered n-ary (semi)hypergroup. If "$\leq$" is a total order on H then $(H, f, \leq)$ is a total ordered n-ary (semi)hypergroup.

Example 8 Let $H_0 = \{a, b, c\}$ and $(H_0, +)$ be defined by the following table.

$+$	a	b	c
a	a	$\{a, b\}$	$\{a, c\}$
b	a	$\{a, b\}$	$\{a, c\}$
c	a	$\{a, b\}$	$\{a, c\}$

By defining "$\leq_0$" on H_0 as follows:

$$\leq_0 = \{(a, a), (a, b), (b, b), (c, c)\},$$

we get that $(H_0, +, \leq_0)$ is an ordered semihypergroup.

Example 9 Let $H_0 = \{a, b, c\}$ and $(H_0, \circ)$ be defined by the following table.

$\circ$	a	b	c
a	H	H	H
b	H	$\{b, c\}$	$\{b, c\}$
c	H	$\{b, c\}$	c

By defining "$\leq_1$" on H_0 as follows:

$$\leq_1 = \{(a, a), (b, a), (b, b), (c, a), (c, c)\},$$

we get that $(H_0, \circ, \leq_1)$ is an ordered commutative hypergroup.

Remark 6 Let (H, f) be an n-ary (semi)hypergroup and "$\leq$" be the trivial order on H, i.e., $x \leq y$ if and only if $x = y$. Then $(H, f, \leq)$ is an ordered n-ary (semi)hypergroup.

Example 10 Let (H, f) be the total n-ary hypergroup and "$\leq$" be any partial order on H. Then $(H, f, \leq)$ is an ordered n-ary hypergroup.

Example 11 Let (H, g) be the n-set n-ary hypergroup defined in Example 3 with a fixed element $h_0 \in H$. Define "$\leq$" on H as follows.

$$x \leq y \text{ if and only if } x = h_0 \text{ or } x = y.$$

Then $(H, g, \leq)$ is an ordered n-ary hypergroup.

Example 12 Let $\mathbb{Z}$ be the set of integers and $h : \mathbb{Z}^3 \to \mathcal{P}^*(\mathbb{Z})$ be defined as follows.

$$h(m, n, p) = \begin{cases} 2\mathbb{Z} & \text{if } m + n + p \in 2\mathbb{Z}; \\ 2\mathbb{Z} + 1 & \text{otherwise.} \end{cases}$$

Then $(\mathbb{Z}, h, \leq_u)$ is a total ordered commutative n-ary hypergroup where "$\leq_u$" is the usual order on $\mathbb{Z}$. This is clear as $(2\mathbb{Z}] = (2\mathbb{Z} + 1] = \mathbb{Z}$.

Example 13 Let $H_1 = \{0, 1\}$ and $k : H_1^4 \to \mathcal{P}^*(H_1)$ be defined as follows.

$$k(x, y, z, w) = \begin{cases} 1 & \text{if } x = y = z = w = 1; \\ H_1 & \text{otherwise.} \end{cases}$$

Let $\leq_1 = \{(0, 0), (1, 0), (1, 1)\}$ be the partial order on H_1. Then $(H_1, k, \leq_1)$ is a commutative total ordered 4-ary hypergroup.

The next theorem presents a way to construct an ordered n-ary semihypergroup from an already existing ordered semihypergroup.

Theorem 3.3 *Let $(H, \cdot, \leq)$ be an ordered semihypergroup and $n \geq 2$ be an integer. Let $f : H^n \to \mathcal{P}^*(H)$ be defined as follows.*

$$f(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n.$$

Then $(H, f, \leq)$ is an ordered n-ary semihypergroup.

Proof First, we prove that (H, f) is associative. Let $h_1, h_2, \ldots, h_n, x_2, \ldots, x_n \in H$. Then

$$f(f(h_1, \ldots, h_n), x_2, \ldots, x_n) = \bigcup_{a \in h_1 h_2 \ldots h_n} f(a, x_2, \ldots, x_n) = (h_1 h_2 \ldots h_n) x_2 \ldots x_n.$$

Having $(H, \cdot)$ associative implies that

$$f(f(h_1, \ldots, h_n), x_2, \ldots, x_n) = h_1(h_2 \ldots h_n x_2) \ldots x_n = f(h_1, f(h_2, \ldots, h_n, x_2), \ldots, x_n).$$

Continuing on this way, we can get that

$$f(f(h_1, \ldots, h_n), x_2, \ldots, x_n) = f(h_1, f(h_2, \ldots, h_n, x_2), \ldots, x_n) = f(h_1, \ldots, h_{n-1}, f(h_n, x_2, \ldots, x_n)).$$

Thus, (H, f) is an n-ary semihypergroup.

Let $\alpha \leq \beta$ and $x_2, \ldots, x_n \in H$. We need to prove that $f(x_2^i, \alpha, x_{i+1}^n) \leq f(x_2^i, \beta, x_{i+1}^n)$ for all $i = 1, \ldots, n$. We prove it for $i = 1$ and the other cases are done similarly. We have that $f(\alpha, x_2, \ldots, x_n) = \alpha(x_2 \ldots x_n)$ and $f(\beta, x_2, \ldots, x_n) = \beta(x_2 \ldots x_n)$. Having $(H, \cdot, \leq)$ an ordered semihypergroup implies that $\alpha x \leq \beta x$ for all $x \in H$. Let $x \in x_2 \ldots x_n$. Then $\alpha x \leq \beta x$. The latter implies that for every $y \in \alpha x \subseteq \alpha(x_2 \ldots x_n)$ there exists $z \in \beta x \subseteq \beta(x_2 \ldots x_n)$ with $y \leq z$. Therefore, $(H, f, \leq)$ is an ordered n-ary semihypergroup. □

Example 14 Let $(H_0, +, \leq_0)$ be the ordered semihypergroup defined in Example 8. Theorem 3.3 asserts that $(H_0, f_0, \leq_0)$ is an ordered ternary semihypergroup. Here, $f_0 : H_0^3 \to \mathcal{P}^*(H_0)$ is defined as follows.

$$f_0(x, y, z) = \begin{cases} a & \text{if } z = a, \\ \{a, b\} & \text{if } z = b, \\ \{a, c\} & \text{otherwise.} \end{cases}$$

Example 15 Let $H_1 = \{0, 1\}$ and $(H_1, \odot)$ be defined by the following table.

$\odot$	0	1
0	0	0
1	0	H_1

By defining "$\leq_2$" on H_1 as follows.

$$\leq_2 = \{(0, 0), (0, 1), (1, 1)\},$$

we get that $(H_1, \odot, \leq_2)$ is an ordered commutative semihypergroup. Using Theorem 3.3, we get that $(H_1^n, f_1', \leq_2)$ is an ordered commutative n-ary semihypergroup. Here, $n > 2$ is an integer and $f_1' : H_1^n \to \mathcal{P}^*(H_1)$ is defined as follows.

$$f_1'(x_1, x_2, \ldots, x_n) = \begin{cases} H & \text{if } (x_1, x_2, \ldots, x_n) = (1, 1, \ldots, 1); \\ 0 & \text{otherwise.} \end{cases}$$

The next theorem presents a way to construct an ordered n-ary hypergroup from an already existing ordered hypergroup.

Theorem 3.4 *Let $(H, \cdot, \leq)$ be a commutative ordered hypergroup and $n > 2$ is an integer. Let $f : H^n \to \mathcal{P}^*(H)$ be defined as follows.*

$$f(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n.$$

Then $(H, f, \leq)$ is a commutative ordered n-ary hypergroup.

Proof If $(H, \cdot)$ is a commutative hypergroup, it is easy to see that (H, f) is a commutative n-ary hypergroupoid. Theorem 3.3 asserts that $(H, f, \leq)$ is an ordered n-ary semihypergroup. We need to prove that (H, f) is an n-ary quasi-hypergroup, i.e., we need to show that the equation

$$b \in f(h_2^i, x, h_{i+1}^n)$$

is solvable in H, for all $x, h_2, \ldots, h_n$. Since $(H, \cdot)$ is commutative, it suffices to show that the equation $b \in f(x, h_2, \ldots, h_n)$ is solvable for all $x, h_2, \ldots, h_n$. Let $b \in f(x, h_2, \ldots, h_n) = x h_2 \ldots h_n$. Then there exists $y \in h_2 \ldots h_n$ such that $b \in xy$. Since $(H, \cdot)$ is a quasihypergroup, it follows that the equation $b \in xy$ is solvable in H and hence, the equation $b \in f(x, h_2, \ldots, h_n)$ is solvable in H. Therefore, $(H, f, \leq)$ is a commutative ordered n-ary hypergroup. □

Example 16 Let $(H_0, \circ, \leq_1)$ be the ordered hypergroup defined in Example 9. Theorem 3.4 asserts that $(H_0, f_1, \leq_1)$ is an ordered commutative ternary hypergroup. Here, $f_1 : H_0^3 \to \mathcal{P}^*(H_0)$ is defined as follows.

$$f_1(x, y, z) = \begin{cases} c & \text{if } x = y = z = c, \\ H & \text{if } x = a or y = a or z = a, \\ \{b, c\} & \text{otherwise.} \end{cases}$$

Example 17 Let $H_2 = \{0, d, e\}$ and $(H_2, \oplus)$ be defined by the following table.

$\oplus$	0	d	e
0	0	$\{0, d\}$	H
d	$\{0, d\}$	H	H
e	H	H	H

By defining "$\leq_3$" on H_2 as follows.

$$\leq_3 = \{(0, 0), (0, d), (0, e), (d, d)(d, e), (e, e)\},$$

we get that $(H_2, \oplus, \leq_3)$ is a total ordered commutative hypergroup. Theorem 3.4 asserts that $(H_2, g_2, \leq_3)$ is a total order commutative 4-ary hypergroup. Here, $g_2 : H_2^4 \to \mathcal{P}^*(H_2)$ is defined as follows.

$$g_2(x, y, z, w) = \begin{cases} 0 & \text{if } (x, y, z, w) = (0, 0, 0, 0), \\ \{0, d\} & \text{if all are 0 except one of them is } d, \\ H & \text{otherwise.} \end{cases}$$

Let $(H_1, f_1, \leq_1)$, $(H_2, f_2, \leq_2)$ be ordered n-ary (semi)hypergroups and let $(H_1 \times H_2, f, \leq)$ be defined as follows. For all $a_1, \ldots, a_n \in H_1, b_1, \ldots, b_n \in H_2$,

$$f((a_1, b_1), \ldots, (a_n, b_n)) = (f_1(a_1, \ldots, a_n), f_2(b_1, \ldots, b_n))$$

and

$$(a_1, b_1) \leq (a_2, b_2) \qquad \text{if and only if} \qquad a_1 \leq_1 a_2 \text{ and } b_1 \leq_2 b_2.$$

Theorem 3.5 *Let $(H_1, f_1, \leq_1)$, $(H_2, f_2, \leq_2)$ be ordered n-ary (semi)hypergroups. Then $(H_1 \times H_2, f, \leq)$ is an ordered n-ary (semi)hypergroup.*

Proof The proof is straightforward. □

Corollary 3.6 *Let $(H_i, f_i, \leq_i)$ be an ordered n-ary (semi)hypergroup for $i \in \{1, \ldots, k\}$. Then $(H_1 \times \ldots \times H_k, f, \leq)$ is an ordered n-ary (semi)hypergroup.*

Example 18 Let $(H_1, \odot, \leq_2)$ be the ordered n-ary semihypergroup in Example 15. Theorem 3.5 asserts that $(H_1 \times H_1, f, \leq)$ is an ordered n-ary semihypergroup. Here,

$$f((x_1, y_1), \ldots, (x_n, y_n)) = \begin{cases} H_1 \times H_1 & \text{if } x_1 = \ldots = x_n = y_1 = \ldots = y_n = 1, \\ H_1 \times \{0\} & \text{if } x_1 = \ldots = x_n = 1 \text{ and there exists } i \in [1, n] \text{ with } y_i = 0, \\ \{0\} \times H_1 & \text{if } y_1 = \ldots = y_n = 1 \text{ and there exists } i \in [1, n] \text{ with } x_i = 0, \\ \{(0, 0)\} & \text{otherwise.} \end{cases}$$

and "$\leq$" is defined as follows.

$$\{((0,0),(0,0)),((0,0),(0,1)),((0,0),(1,0)),((0,0),(1,1)),((1,0),(1,0)),((1,0),(1,1)),((0,1),(0,1)),$$
$$((0,1),(1,1)),((1,1),(1,1))\}.$$

4 Subsets of ordered *n*-ary ordered (semi)hypergroups

In this section, we define n-ary sub(semi)hypergroups and hyperideals in ordered n-ary (semi)hypergroups, investigate their properties, and present some examples.

Definition 4.1 Let (H, f) be an n-ary ordered (semi)hypergroup and $S \neq \emptyset \subseteq H$. Then S is an n-ary subsemihypergroup (subhypergroup) of H if (S, f) is an n-ary ordered (semi)hypergroup and $(S] = \{x \in H : x \leq y \text{ for some } y \in S\} \subseteq S$.

Proposition 4.2 *Let $(H_i, f_i, \leq_i)$ be an ordered n-ary (semi)hypergroup for $i \in \{1, \ldots, k\}$ and $S_i \neq \emptyset \subseteq H_i$ for all $i = 1, \ldots, k$. Then*

$$(S_1 \times \ldots \times S_k] = (S_1] \times \ldots \times (S_k].$$

Proof The proof is straightforward. □

Proposition 4.3 *A non-empty subset of an n-ary ordered semihypergroup is an n-ary subsemihypergroup of H if and only if the following conditions are satisfied.*

(1) If $a_1, \ldots, a_n \in S$ then $f(a_1, \ldots, a_n) \subseteq S$;
(2) $(S] \subseteq S$.

Proof The proof is straightforward. □

Proposition 4.4 *A non-empty subset of an n-ary ordered hypergroup is an n-ary subhypergroup of H if and only if the following conditions are satisfied.*

(1) If $a_1, \ldots, a_n \in S$ then $f(a_1, \ldots, a_n) \subseteq S$;
(2) If $b, a_2, \ldots, a_n \in S$ then the equation $b \in f(a_2^i, x, a_{i+1}^n)$ is solvable in S.
(3) $(S] \subseteq S$.

Proof The proof is straightforward. □

Definition 4.5 Let (H, f) be an n-ary ordered (semi)hypergroup and S be an n-ary sub(semi)hypergroup of H. Then S is an i-hyperideal of H if $f(x_1^{i-1}, S, x_{i+1}^n) \subseteq S$ for all $x_1, \ldots, x_n \in H$ and $(S] = \{x \in H : x \leq y, y \in S\} \subseteq S$.

Definition 4.6 Let (H, f) be an n-ary ordered (semi)hypergroup and $S \neq \emptyset \subseteq H$. Then S is a hyperideal of H if S is an i-hyperideal of H for all $i = 1, \ldots, n$.

Next, we present an example on an n-ary subhypergroup of an ordered n-ary hypergroup that is not a hyperideal.

Example 19 Let $H = \{m, a, d\}$ and $(H, f, \leq)$ be the ordered 3-set hypergroup defined in Example 3 with $\leq = \{(m, m), (m, a), (m, d), (a, a), (a, d), (d, d)\}$. Then $\{m, a\}$ is a ternary subhypergroup of H but it is not an i-hyperideal of H for all $i = 1, 2, 3$. Moreover, $\{m\}$ and $\{m, a, d\}$ are the only hyperideals of H.

Theorem 4.7 *Let (H, f) be an n-ary ordered (semi)hypergroup and $S_1, \ldots, S_k$ be n-ary sub(semi)hypergroups of H. If $\bigcap_{1\leq \imath\leq k} S_i \neq \emptyset$ then $\bigcap_{1\leq i\leq k} S_i$ is an n-ary subsemihypergroup (subhypergroup) of H.*

Proof The proof is straightforward. □

Theorem 4.8 *Let (H, f) be an n-ary ordered (semi)hypergroup and $S_1, \ldots, S_k$ be an i-hyperideal of H. If $\bigcap_{1\leq j\leq k} S_i \neq \emptyset$ then $\bigcap_{1\leq \jmath\leq \mathrm{k}} S_i$ is an i-hyperideal of H.*

Proof The proof is straightforward. □

Corollary 4.9 *Let (H, f) be an n-ary ordered (semi)hypergroup and $S_1, \ldots, S_k$ be a hyperideal of H. If $\bigcap_{1\leq j\leq k} S_i \neq \emptyset$ then $\bigcap_{1\leq j\leq k} S_i$ is a hyperideal of H.*

Proof The proof follows from Theorem 4.8 and the definition of hyperideals. □

Definition 4.10 Let $(H_1, f, \leq_1), (H_2, g, \leq_2)$ be ordered n-ary (semi)hypergroups and $\phi : H_1 \to H_2$ be a function. Then ϕ is a homomorphism if for all $a, b \in H_1$, the following conditions hold.

1. $\phi(f(a_1^n)) = g(\phi(a_1), \ldots, \phi(a_n))$;
2. If $a \leq_1 b$ then $\phi(a) \leq_2 \phi(b)$.

Definition 4.11 Let $(H_1, f, \leq_1), (H_2, g, \leq_2)$ be ordered n-ary (semi)hypergroups. Then H_1, H_2 are isomorphic ordered n-ary (semi)hypergroups if there exists a bijective homomorphism $\phi : H_1 \to H_2$. In this case, we write $H_1 \cong H_2$.

Example 20 Let $(H, f, \leq)$ be any ordered n-ary (semi)hypergroup and S an n-ary subsemihypergroup (subhypergroup) of H. Then the inclusion map $\phi : S \to H$ is a homomorphism.

Example 21 Let $(H, f, \leq)$ be any ordered n-ary (semi)hypergroup and $(K, g, \leq')$ be any ordered n-ary (semi)hypergroup with scalar identity $0 \in K$. Then the map $\phi : H \to K$, with $\phi(x) = 0$ for all $x \in H$, is a homomorphism.

Theorem 4.12 *Let $(H_1, f, \leq_1), (H_2, g, \leq_2)$ be ordered n-ary semihypergroups and $\phi : H_1 \to H_2$ be a homomorphism. If S is an n-ary subsemihypergroup of H_2 and $\phi^{-1}(S) \neq \emptyset$ then $\phi^{-1}(S)$ is an n-ary subsemihypergroup of H_1.*

Proof Let $x_1, \ldots, x_n \in \phi^{-1}(S) \neq \emptyset$. Then $\phi(x_1), \ldots, \phi(x_n) \in S$. Having S an n-ary subsemihypergroup of H_2 implies that $g(\phi(x_1), \ldots, \phi(x_n)) \subseteq S$. Since ϕ is a homomorphism, it follows that $\phi(f(x_1, \ldots, x_n)) = g(\phi(x_1), \ldots, \phi(x_n)) \subseteq S$ and hence, $f(x_1, \ldots, x_n) \subseteq \phi^{-1}(S)$. Let $x \in \phi^{-1}(S)$ and $a \leq_1 x$. The latter implies that $\phi(x) \in S$ and $\phi(a) \leq_2 \phi(x)$. Since S is an n-ary subsemihypergroup of H_2 and $\phi(a) \leq_2 \phi(x)$, it follows that $\phi(a) \in S$ and hence, $a \in \phi^{-1}(S)$. □

Theorem 4.13 *Let $(H_1, f, \leq_1), (H_2, g, \leq_2)$ be ordered n-ary semihypergroups and $\phi : H_1 \to H_2$ be a homomorphism. If S is an i-hyperideal of H_2 and $\phi^{-1}(S) \neq \emptyset$ then $\phi^{-1}(S)$ is an i-hyperideal of H_1.*

Proof Theorem 4.12 asserts that $\phi^{-1}(S)$ is an n-ary subsemihypergroup of H_1. Let $x \in \phi^{-1}(S) \neq \emptyset$ and $x_2, \ldots, x_n \in H_1$. Then $\phi(x) \in S$. Having S an i-hyperideal of H_2 implies that $g(\phi(x_1), \ldots, \phi(x_{i-1}), x, \phi(x_{i+1}), \ldots, \phi(x_n)) \subseteq S$. Since ϕ is a homomorphism, it follows that

$$\phi(f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)) = g(\phi(x_1), \ldots, \phi(x_{i-1}), x, \phi(x_{i+1}), \ldots, \phi(x_n)) \subseteq S$$

and hence, $f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \subseteq \phi^{-1}(S)$. □

Theorem 4.14 *Let $(H_1, f, \leq_1), (H_2, g, \leq_2)$ be ordered n-ary semihypergroups and $\phi : H_1 \to H_2$ be a homomorphism. If S is a hyperideal of H_2 and $\phi^{-1}(S) \neq \emptyset$ then $\phi^{-1}(S)$ is a hyperideal of H_1.*

Proof The proof follows from Theorem 4.13 and the definition of hyperideals. ◻

Theorem 4.15 *Let $(H_1, f, \leq_1), (H_2, g, \leq_2)$ be ordered n-ary hypergroups and $\phi : H_1 \to H_2$ be a homomorphism. If S is an n-ary subhypergroup of H_2 and $\phi^{-1}(S) \neq \emptyset$ then $\phi^{-1}(S)$ is an n-ary subhypergroup of H_1.*

Proof Theorem 4.12 asserts that $\phi^{-1}(S)$ is an n-ary subsemihypergroup of H_1. Let $b, x_2, \ldots, x_n \in \phi^{-1}(S) \neq \emptyset$ and $b \in f(x_2^i, x, x_{i+1}^n)$. Then $\phi(x_2), \ldots, \phi(x_n) \in S$ and $\phi(b) \in \phi(f(x_2^i, x, x_{i+1}^n)) = g(\phi(x_2), \ldots, \phi(x_i), \phi(x), \phi(x_{i+1}), \ldots, \phi(x_n))$. Having S an n-ary subhypergroup of H_2 implies that there exists a solution $y_0 \in S$ for the equation $\phi(b) \in g(\phi(x_2), \ldots, \phi(x_i), \phi(x), \phi(x_{i+1}), \ldots, \phi(x_n))$. The latter implies that there exists $x_0 \in \phi^{-1}(S)$ such that $\phi(x_0) = y_0$ and x_0 a solution for the equation $b \in f(x_2^i, x, x_{i+1}^n)$. Thus, $\phi^{-1}(S)$ is an n-ary subhypergroup of H_1. ◻

Theorem 4.16 *Let $(H_1, f, \leq_1), (H_2, g, \leq_2)$ be ordered n-ary hypergroups and $\phi : H_1 \to H_2$ be a homomorphism. If S is an i-hyperideal of H_2 and $\phi^{-1}(S) \neq \emptyset$ then $\phi^{-1}(S)$ is an i-hyperideal of H_1.*

Proof The proof follows from Theorems 4.13 and 4.15. ◻

Corollary 4.17 *Let $(H_1, f, \leq_1), (H_2, g, \leq_2)$ be ordered n-ary hypergroups and $\phi : H_1 \to H_2$ be a homomorphism. If S is a hyperideal of H_2 and $\phi^{-1}(S) \neq \emptyset$ then $\phi^{-1}(S)$ is a hyperideal of H_1.*

Proof The proof follows from Theorem 4.16 and the definition of hyperideals. ◻

Proposition 4.18 *Let $(H_i, f_i, \leq_i)$ be an ordered n-ary (semi)hypergroup for $i \in \{1, \ldots, k\}$. Then a non-empty subset of $H_1 \times \ldots \times H_k$ is an n-ary (subsemihypergroup)subhypergroup of $H_1 \times \ldots \times H_k$ if and only if there exists $S_i \neq \emptyset \subseteq H_i$ for all $i = 1, \ldots, k$ such that $S = S_1 \times \ldots \times S_k$ and S_i is an n-ary (subsemihypergroup)subhypergroup of H_i.*

Proof The proof is straightforward. ◻

Proposition 4.19 *Let $(H_i, f_i, \leq_i)$ be an ordered n-ary (semi)hypergroup for $i \in \{1, \ldots, k\}$. Then a non-empty subset of $H_1 \times \ldots \times H_k$ is a hyperideal of $H_1 \times \ldots \times H_k$ if and only if there exist $S_i \neq \emptyset \subseteq H_i$ for all $i = 1, \ldots, k$ such that $S = S_1 \times \ldots \times S_k$, S_i is a hyperideal of H_i.*

Proof The proof is straightforward. ◻

5 Conclusion

This paper contributed to the study of n-ary hyperstructures by introducing, for the first time, ordered n-ary hyperstructures as a class of ordered algebraic hypersystems which represents a generalization of ordered semigroups and ordered semi(hypergroups). More precisely, it discussed ordered n-ary (semi)hypergroups and investigated some of their properties by studying their subsets.

For future work, it will be interesting to introduce a partial order on higher n-ary hyperstructures such as (m, n)-ary hyperrings.

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