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ORIGINAL RESEARCH

On the sum of powers of distance Laplacian eigenvalues in terms of Wiener index and complement of a graph

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Abstract Let G be a connected graph with n vertices, m edges and having distance Laplacian eigenvalues $\partial_1 \geq \partial_2 \geq \dots \partial_{n-1} > \partial_n = 0$. For any real number $\beta \neq 0$, let $S_\beta(G) = \sum_{i=1}^{n-1} \partial_i^\beta$ be the sum of β-th powers of the $(n-1)$ largest distance Laplacian eigenvalues of G. In this paper, we obtain various bounds for the graph invariant $S_\beta(G)$, which relates it with different parameters associated to the structure of the graph. We also obtain bounds for $S_\beta(G)$ in terms of the number of components of the complement graph $\overline{G}$.

Keywords Distance matrix · Distance Laplacian matrix · Distance Laplacian eigenvalues · Transmission · Diameter · Wiener index

Mathematics Subject Classification 05C50 · 05C12 · 15A18

1 Introduction

A graph $G = (V, E)$ consists of the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and the edge set $E(G)$. We assume that all the graphs under consideration are simple and connected. Further, $|V(G)| = n$ is the *order* and $|E(G)| = m$ is the *size* of G. The *degree* of a vertex v, denoted by $d_G(v)$ (we simply write d_v) is the number of edges incident on the vertex v. As usual, K_n is the complete graph with n vertices and $K_{1,n-1}$ is the star graph with n vertices. Also, $K_{a,b}$ is the complete bipartite graph with two partite sets V_1 and V_2 of cardinalities a and b, respectively.

In G, the *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv}, is defined as the length of a shortest path between u and v. The *diameter* of G is the maximum distance between any two vertices of G. The *distance matrix* of G is denoted by $D(G)$ and is defined as $D(G) = (d_{uv})_{u,v \in V(G)}$. The *vertex transmission* $Tr_G(v)$ of a vertex v is defined as the sum of the distances from v to all other vertices in G, that is, $Tr_G(v) = \sum\limits_{u \in V(G)} d_{uv}$. The *Wiener index* (also called *transmission*) of a graph G, denoted by $\sigma(G)$, is the sum of distances between

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all unordered pairs of vertices in G. Clearly, $\sigma(G) = \frac{1}{2} \sum_{v \in V(G)} Tr_G(v)$. For any vertex $v_i \in V(G)$, the vertex transmission $Tr_G(v_i)$ is also called the *transmission degree* of v_i.

Let $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [2] defined the Laplacian and signless Laplacian for the distance matrix of a connected graph as $D^L(G) = Tr(G) - D(G)$ (or simply D^L) and $D^Q(G) = Tr(G) + D(G)$ (or simply D^Q) respectively. Since $D^L(G)$ is a real symmetric positive semi-definite matrix, we denote its eigenvalues by ∂_i's and order them as $\partial_1 \geq \partial_2 \geq \ldots \partial_{n-1} > \partial_n = 0$. The largest distance Laplacian eigenvalue ∂_1 is called the distance Laplacian spectral radius of G. More work on distance Laplacian eigenvalues can be found in [3,6,7,10,14,16,18].

For a real number $\alpha \neq 0$, Zhou [19] studied the sum of αth powers of the Laplacian eigenvalues and Akbari et al. [1] investigated the sum of αth powers of signless Laplacian eigenvalues of G. Pirzada et al. [13] also investigated the sum of αth powers of distance signless Laplacian eigenvalues of G. Clearly, the smallest distance Laplacian eigenvalues ∂_n is always zero and $\partial_i \geq n$ for all $i = 1, 2 \ldots, (n-1)$. Now, for a real number $\beta \neq 0$, let $S_\beta(G) = \sum_{i=1}^{n-1} \partial_i^\beta$ be the sum of βth powers of the $(n-1)$ non zero distance Laplacian eigenvalues of G.

In Section 2, we obtain bounds for $S_\beta(G)$ in terms of order n, Wiener index $\sigma(G)$ and maximum transmission Tr_1. We characterize the extremal cases of these bounds. In Section 3, we obtain a lower and an upper bound for $S_\beta(G)$ in terms of number of the vertices n and the number of components C_k in its complement graph $\overline{G}$ and determine the conditions for extremal cases. As a consequence, we obtain an upper bound for $S_\beta(G)$ in terms of order n only, when $\beta > 0$ and a lower bound for $S_\beta(G)$ in terms of order, when $\beta < 0$ and characterize the extremal graphs in both cases.

2 $S_\beta(G)$ in terms of Wiener index and maximum transmission

For non-increasing sequence $(x) = (x_1, x_2, \ldots, x_n)$ and $(y) = (y_1, y_2, \ldots, y_n)$ of length n, we say (x) is weakly majorized by (y), denoted by $(x) \prec_\omega (y)$ if $\sum_{i}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for all $k = 1, 2, \ldots, n-1$. If in addition to this, (x) and (y) satisfy the condition $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, we say (x) is majorized by (y) and is denoted by $(x) \preceq (y)$.

Lemma 2.1 [9] Let $(x) = (x_1, x_2, \ldots, x_n)$ and $(y) = (y_1, y_2, \ldots, y_n)$ be non-increasing sequence of real numbers of length n. If $(x) \preceq (y)$, Then for any convex function ψ, we have $\sum_{i=n}^{n} \psi(x_i) \leq \sum_{i=1}^{n} \psi(y_i)$. Furthermore, if $(x) \prec (y)$ and ψ is strictly convex, then $\sum_{i=1}^{n} \psi(x_i) < \sum_{i=1}^{n} \psi(y_i)$.

The proof of the next lemma follows by using Schur's theorem.

Lemma 2.2 *Let G be a connected of order $n \geq 3$ having distance Laplacian eigenvalues $\partial_1, \partial_2, \ldots, \partial_n$ and transmission degrees $Tr_1, Tr_2, \ldots, Tr_n$. Then*

$$(Tr_1, Tr_2, \ldots, Tr_n) \preceq (\partial_1, \partial_2, \ldots, \partial_n) \text{ and}$$
$$(Tr_1, Tr_2, \ldots, Tr_{n-1}) \prec_\omega (\partial_1, \partial_2, \ldots, \partial_{n-1}).$$

The following result gives bounds for $S_\beta(G)$ in terms of the transmission degrees of G.

Theorem 2.3 *Let G be a connected graph of order $n \geq 3$ having transmission degrees $Tr_1, Tr_2, \ldots Tr_n$.*
(i) If $\beta < 0$ or $\beta > 1$, then $S_\beta(G) > \sum_{i=1}^{n-1} (Tr_i)^\beta$.
(ii) If $0 < \beta < 1$, then $S_\beta(G) < \sum_{i=1}^{n-1} (Tr_i)^\beta$.

Proof (i) For $x > 0$, the function $g(x) = x^\beta$ is an increasing convex function if $\beta > 1$ or $\beta < 0$. Let $(X) = (Tr_1, Tr_2 \ldots, Tr_{n-1})$ and $(Y) = (\partial_1, \partial_2, \ldots, \partial_{n-1})$. By Lemma 2.2, we have $(X) <_\omega (Y)$. So, it follows that $S_\beta \geq \sum_{i=1}^{n-1} (Tr_i)^\beta$. By Lemma 2.1, equality occurs if and only if $(X) = (Y)$. That is, if and only if $\partial_i = Tr_i$, for all $i = 1, 2 \ldots, n-1$. But then, $trace(A) = \partial_1 + \partial_2 + \cdots + \partial_{n-1} = Tr_1 + Tr_2 + \cdots + Tr_{n-1} < Tr_1 + Tr_2 + \cdots + Tr_{n-1} + Tr_n = trace(A)$, which is a contradiction. Therefore, $S_\beta > \sum_{i=1}^{n-1}(Tr_i)^\beta$.

(ii) For $x > 0$, we can see that the function $g(x) = -x^\beta$ is a convex function if $0 < \beta < 1$. Let $(X') = (Tr_1, Tr_2, \ldots, Tr_{n-1}, Tr_n)$ and $(Y') = (\partial_1, \partial_2, \ldots, \partial_{n-1})$. Then by proceeding similarly as in part (i) and using Lemma 2.2, (ii) follows. □

The following lemma states that deleting an edge does not decrease the distance Laplacian eigenvalues.

Lemma 2.4 [2] Let G be a connected graph with n vertices and $m \geq n$ edges. Let G* be the connected graph obtained from G by deleting an edge. Let $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$ and $\partial_1^* \geq \partial_2^* \geq \cdots \geq \partial_n^*$ be the distance Laplacian spectrum of G and G^*, respectively. Then $\partial_i^* \geq \partial_i$ for all $i = 1, 2, \ldots, n$.

Theorem 2.5 *Let G be a connected graph on $n \geq 3$ vertices with $m \geq n$ edges and let $G - e$ be the connected graph obtained from G by deleting an edge e. Then*

$$S_\beta(G) \leq S_\beta(G-e), \quad \text{if } \beta > 0 \quad \text{and} \quad S_\beta(G) \geq S_\beta(G-e), \quad \text{if } \beta < 0.$$

Proof For $0 < a \leq b$, we have $a^t \leq b^t$ for all $t > 0$ and $a^t \geq p^t$ for all $t < 0$. Using this property and Lemma 2.4, the proof follows. □

The following is an immediate consequence of Theorem 2.5.

Corollary 2.6 *Let G be a connected graph of order $n \geq 3$ with $m \geq n$ edges. Then for any arbitrary spanning tree T of G, we have*

$$(n-1)n^\beta \leq S_\beta(G) < S_\beta(T), \text{ if } \beta > 0 \quad \text{and} \quad S_\beta(T) < S_\beta(G) \leq (n-1)n^\beta, \text{ if } \beta < 0$$

equality on both left and right occurs if and only if $G \cong K_n$.

Proof Every connected graph with n vertices contains a spanning tree, say T, and is itself a spanning subgraph of the complete graph K_n. By using Theorem 2.5 and the fact that the distance Laplacian spectrum of the complete graph K_n is $(0, n^{[n-1]})$, we get the required result. □

The following lemma gives a lower bound of distance Laplacian spectral radius in terms of the order n and maximum transmission Tr_1.

Lemma 2.7 [8] Let G be a connected graph of order n, then $\partial_1 \geq \frac{n}{n-1} Tr_1$ with equality if and only if G is complete graph K_n.

Fact 1. Let $C \subseteq \mathbb{R}^n$ be a convex set, and let $f : C \to \mathbb{R}$ be convex on C. If $\lambda_1, \lambda_2, \ldots, \lambda_k \in [0, 1]$, $\sum_{i=1}^k \lambda_i = 1$, and $x^{(1)}, x^{(2)}, \ldots, x^{(k)} \in C$, then

$$f\left(\sum_{i=i}^{k} \lambda_i x^{(i)}\right) \leq \sum_{i=1}^{k} \lambda_i f(x^{(i)}).$$

If $f(x)$ is strictly convex on C and $\lambda_i > 0$ for $i = 1, 2, \ldots, n$, then the above inequality holds, with equality if and only if all the $x^{(i)}$ are equal.

Theorem 2.8 *Let G be a connected graph of order $n \geq 3$ and let $Tr_1 = \max_{1 \leq i \leq n} Tr_i$.*

(i) If $\beta < 0$ or $\beta > 1$, then

$$S_\beta(G) \geq \left(\frac{n}{n-1} Tr_1\right)^\beta + \frac{((n-1)2\sigma(G) - nTr_1)^\beta}{(n-1)^\beta (n-2)^{\beta-1}} \qquad \text{and}$$

(ii) If $0 < \beta < 1$, then

$$S_\beta(G) \leq \left(\frac{n}{n-1} Tr_1\right)^\beta + \frac{((n-1)2\sigma(G) - nTr_1)^\beta}{(n-1)^\beta (n-2)^{\beta-1}}$$

with equality if and only if $G \cong K_n$.

Proof (i) For $\beta \neq 0, 1$ and $x > 0$. The function x^{β} is strictly a convex function if $\beta < 0$ or $\beta > 1$. Suppose that $\beta < 0$ or $\beta > 1$. Then

$$\left(\sum_{i=2}^{n-1} \frac{1}{n-2} \partial_i\right)^{\beta} \leq \sum_{i=2}^{n-1} \frac{1}{n-2} \partial_i^{\beta},$$

which implies that

$$\sum_{i=2}^{n-1} \partial_i^{\beta} \geq \frac{1}{(n-2)^{\beta-1}} \left(\sum_{i=2}^{n-1} \partial_i\right)^{\beta},$$

Since x^{β} is strictly convex function if $\beta < 0$ or $\beta > 1$. Therefore by Fact 1, equality is possible if and only if $\partial_2 = \partial_3 = \cdots = \partial_{n-1}$.

Therefore, we have

$$S_{\beta}(G) \geq \partial_1^{\beta} + \frac{1}{(n-2)^{\beta-1}} \left(\sum_{i=2}^{n-1} \partial_i\right)^{\beta} = \partial_1^{\beta} + \frac{(2\sigma(G) - \partial_1)^{\beta}}{(n-2)^{\beta-1}}.$$

Consider the function

$$h(x) = x^{\beta} + \frac{(2\sigma(G) - x)^{\beta}}{(n-2)^{\beta-1}}.$$

The function $h(x)$ is increasing for $x \geq \frac{2\sigma(G)}{n-1}$. By Lemma 2.7, we have $\partial_1 \geq \frac{n}{n-1} Tr_1 \geq \frac{2\sigma(G)}{n-1}$. Thus,

$$\begin{aligned} S_{\beta}(G) \geq & h\left(\frac{n}{n-1} Tr_1\right) = \left(\frac{n}{n-1} Tr_1\right)^{\beta} + \frac{(2\sigma(G) - \frac{n}{n-1} Tr_1)^{\beta}}{(n-2)^{\beta-1}} \\ = & \left(\frac{n}{n-1} Tr_1\right)^{\beta} + \frac{(2\sigma(G)(n-1) - nTr_1)^{\beta}}{(n-1)^{\beta}(n-2)^{\beta-1}} \end{aligned}$$

with equality if and only if $\partial_2 = \partial_3 = \cdots = \partial_{n-1}$ and $\partial_1 = \frac{n}{n-1} Tr_1$. By Lemma 2.7, this is possible if and only if G is the complete graph K_n.

(ii) For $0 < \beta < 1$, we have

$$\left(\sum_{i=2}^{n-1} \partial_i\right)^{\beta} \geq \sum_{i=2}^{n-1} \frac{1}{n-1} \partial_i^{\beta},$$

with equality if and only if $\partial_2 = \partial_3 = \cdots = \partial_{n-1}$. Proceeding in the same way as above, (ii) can be proved. □

For a connected graph G of order $n \geq 3$, let $R = \prod_{i=1}^{n-1} \partial_i$, where $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_{n-1} > 0$ are the distance Laplacian eigenvalues. The following result gives a lower bound for $S_{\beta}(G)$ in terms of the number of vertices n, R and the maximum transmission Tr_1 of G.

Theorem 2.9 *Let $\beta \neq 0, 1$ be a real number and let G be a connected graph of order $n \geq 3$. Then*

$$S_{\beta}(G) \geq \left(\frac{n}{n-1} Tr_1\right)^{\beta} + (n-2) R^{\frac{\beta}{n-2}} \left(\frac{n}{n-1} Tr_1\right)^{\frac{-\beta}{n-2}}$$

with equality if and only if $G \cong K_n$.

Proof By the arithmetic-geometric mean inequality, we have

$$S_{\beta}(G) = (\partial_1)^{\beta} + \sum_{i=2}^{n-1} \partial_i^{\beta} \geq \partial_1^{\beta} + (n-2)(\prod_{i=2}^{n-1} \partial_i^{\beta})^{\frac{1}{n-2}} = \partial_1^{\beta} + (n-2)(\frac{R}{\partial_1})^{\frac{\beta}{n-2}}$$

with equality if and only if $\partial_2 = \partial_3 = \cdots = \partial_{n-1}$.

Consider the function $h(x) = x^{\beta} + (n-2)R^{\frac{\beta}{n-2}}x^{\frac{-\beta}{n-2}}$. Then the derivative of $h(x)$ is

$$h^{'}(x) = \beta x^{\frac{-\beta}{n-2}-1}(x^{\frac{(n-1)\beta}{n-2}} - R^{\frac{\beta}{n-2}}).$$

For $\beta \neq 0, 1,\ h(x)$ is increasing for $x \geq R^{\frac{1}{n-1}}$. By Lemma 2.7, we have

$$\partial_1 \geq \frac{n}{n-1}Tr_1 \geq \frac{2\sigma(G)}{n-1} > \sum_{i=1}^{n-1}\frac{Tr_i}{n-1} \geq (\prod_{i=n}^{n-1}\partial_i)^{\frac{1}{n-1}} = R^{\frac{1}{n-1}}.$$

Therefore,

$$S_{\beta}(G) \geq h\left(\frac{n}{n-1}Tr_1\right) = \left(\frac{n}{n-1}Tr_1\right)^{\beta} + (n-2)R^{\frac{\beta}{n-2}}\left(\frac{n}{n-1}Tr_1\right)^{\frac{-\beta}{n-2}}.$$

Equality occurs if and only if $\partial_2 = \partial_3 = \cdots = \partial_{n-1}$ and $\partial_1 = \frac{n}{n-1}Tr_1$. Clearly, by Lemma 2.7 $G \cong K_n$. □

Now, we obtain bounds for $S_{\beta}(G)$ in terms of order n, the sum of squares of transmissions and the Wiener index.

Theorem 2.10 *Let G be a graph of order $n \geq 3$ and $1 \leq k \leq n-1$, be a positive integer.*

(i) If $0 < \beta < 1$, then $S_{\beta}(G) \leq k^{1-\beta}\left(\frac{2k\sigma(G)}{n-1}\right)^{\beta} + (n-k-1)^{1-\beta}\left(2\sigma(G) - \frac{2k\sigma(G)}{n-1}\right)^{\beta}$.

(ii) If $\beta > 1$, then $S_{\beta}(G) \geq k^{1-\beta}\left(\frac{2k\sigma(G)}{n-1}\right)^{\beta} + (n-k-1)^{1-\beta}\left(2\sigma(G) - \frac{2k\sigma(G)}{n-1}\right)^{\beta}$.

(iii) If $\beta < 0$, then

$$S_{\beta}(G) \leq \min_{1\leq k\leq n-1}\left\{k^{1-\beta}\left[\frac{2k\sigma(G)+\sqrt{\psi}}{n-1}\right]^{\beta} + (n-k-1)^{1-\beta}\left[\frac{2\sigma(G)(n-k-1)-\sqrt{\psi}}{n-1}\right]^{\beta}\right\},$$

where $\psi = k(n-k-1)\left(2n\sum_{1\leq i<j\leq n}(d_{ij})^2 + n\sum_{i=1}^{n}Tr_i^2 - 4\sigma^2(G)\right)$.

Proof (i) Since $0 < \beta < 1$, by power mean inequality, we have

$$\left(\frac{\sum_{i=1}^{k}\partial_i^{\beta}}{k}\right)^{\frac{1}{\beta}} \leq \left(\frac{\sum_{i=1}^{k}\partial_i}{k}\right) \text{ or } \sum_{i=1}^{k}\partial_i^{\beta} \leq k^{1-\beta}\left(\sum_{i=1}^{k}\partial_i\right)^{\beta}$$

with equality if and only if $\partial_1 = \partial_2 = \partial_3 = \cdots = \partial_k$.

Similarly, we have

$$\sum_{i=k+1}^{n-1}\partial_i^{\beta} \leq (n-k-1)^{1-\beta}\left(2\sigma(G) - \sum_{i=1}^{k}\partial_i\right)^{\beta} \text{ as } \sum_{i=1}^{n-1}\partial_i = 2\sigma(G)$$

with equality if and only if $\partial_{k+1} = \partial_{k+2} = \partial_{k+3} = \cdots = \partial_{n-1}$.

As $\partial_1 \geq \partial_2 \geq \partial_3 \geq \cdots \geq \partial_{n-1}$, we have

$$\frac{1}{k}\sum_{i=1}^{k}\partial_i \geq \frac{1}{n-k-1}\sum_{i=k+1}^{n-1}\partial_i = \frac{1}{n-k-1}\left(2\sigma(G) - \sum_{i=1}^{k}\partial_i\right),$$

which implies that $\sum_{i=1}^{k}\partial_i \geq \dfrac{2k\sigma(G)}{n-1}$. Therefore, we have

$$\begin{aligned} S_{\beta}(G) &= \sum_{i=1}^{n-1}\partial_i^{\beta} = \sum_{i=1}^{k}\partial_i^{\beta} + \sum_{i=k+1}^{n-1}\partial_i^{\beta} \\ &\leq k^{1-\beta}\left(\sum_{i=1}^{k}\partial_i\right)^{\beta} + (n-k-1)^{1-\beta}\left(2\sigma(G) - \sum_{i=1}^{k}\partial_i\right)^{\beta} \end{aligned}$$

Now, consider the function

$$f(x) = k^{1-\beta}x^{\beta} + (n-k-1)^{1-\beta}\Big(2\sigma(G) - x\Big)^{\beta}.$$

Therefore, differentiating $f(x)$, we have

$$f'(x) = \beta\Bigg[(\frac{x}{k})^{\beta-1} - \Big(\frac{2\sigma(G)-x}{n-k-1}\Big)^{\beta-1}\Bigg] \leq 0 \ \ as \ \ 0<\beta<1, x \geq \frac{2k\sigma(G)}{n-1}.$$

Thus $f(x)$ is a decreasing function for $x \geq \frac{2k\sigma(G)}{n-1}$. Therefore, maximum of $f(x)$ occurs at $x = \frac{2k\sigma(G)}{n-1}$. So on substituting $x = \frac{2k\sigma(G)}{n-1}$ in $f(x)$ we get the required result.

(ii) Since, $\beta > 1$, using power mean inequality and proceeding as in (i), we have

$$S_{\beta}(G) \geq k^{1-\beta}\Big(\sum_{i=1}^{k}\partial_i\Big)^{\beta} + (n-k-1)^{1-\beta}\Big(2\sigma(G) - \sum_{i=1}^{k}\partial_i\Big)^{\beta}.$$

Also, $f(x)$ is an increasing function for $x \geq \frac{2k\sigma(G)}{n-1}$ as $\beta > 1$. Using the same technique as in (i), we get the required result.

(iii) Since $S_1(G) = \sum_{i=1}^{n-1} Tr_i = 2\sigma(G)$ and the sum of the squares of eigenvalues of a symmetric matrix is equal to the sum of the squares of all entries, we have,

$$S_2(G) = \textstyle\sum_{i=1}^{n-1}\partial_i^2 = 2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} T^2r_i.$$

Now, using this fact and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(2\sigma(G) - \sum_{i=1}^{k}\partial_i)^2 &= (\partial_{k+1} + \cdots + \partial_{n-1})^2\\
&= \Big(\sum_{i=k+1}^{n-1} 1.\partial_i\Big)^2\\
&\leq \sum_{i=k+1}^{n-1} 1^2. \sum_{i=k+1}^{n-1}\partial_i^2\\
&= (n-k-1)(\partial_{k+1}^2 + \cdots + \partial_{n-1}^2)\\
&= (n-k-1)\Big(2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n}Tr_i^2 - (\partial_1^2 + \cdots + \partial_k^2)\Big)\\
&\leq (n-k-1)\Big(2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n}Tr_i^2 - \frac{1}{k}(\sum_{i=1}^{k}\partial_i)^2\Big)
\end{aligned}$$

Thus, it follows that

$$\sum_{i=1}^{k}\partial_i \leq \frac{2k\sigma(G) + \sqrt{k(n-k-1)\Big(2n\sum_{1\leq i<j\leq n}(d_{ij})^2 + n\sum_{i=1}^{n}Tr_i^2 - 4\sigma^2(G)\Big)}}{n-1}.$$

For $\beta < 0$, from (i), we observe that $f(x)$ is an increasing function for

$$\frac{2k\sigma(G)}{n-1} \leq x \leq \frac{2k\sigma(G) + \sqrt{k(n-k-1)\Big(2n\sum_{1\leq i<j\leq n}(d_{ij})^2 + n\sum_{i=1}^{n}Tr_i^2 - 4\sigma^2(G)\Big)}}{n-1}.$$

Thus, for

$$\psi = k(n-k-1)\Big(2n \sum_{1\le i<j\le n} (d_{ij})^2 + n\sum_{i=1}^{n} Tr_i^2 - 4\sigma^2(G)\Big),$$

it follows that

$$f(x) \le k^{1-\beta}\left[\frac{2k\sigma(G)+\sqrt{\psi}}{n-1}\right]^{\beta} + (n-k-1)^{1-\beta}\left[\frac{2\sigma(G)(n-k-1)-\sqrt{\psi}}{n-1}\right]^{\beta}.$$

This proves the result. □

For a connected G of order $n \ge 3$, let $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_n > 0$ be the eigenvalues of the distance signless Laplacian matrix $D^Q(G)$ and $m_\alpha(G) = \sum_{i=1}^{n} \rho_i^\alpha$. The following result [13] gives a lower and an upper bound for $m_\alpha(G)$ in terms of the transmission degrees, sum of squares of distances and the transmission number of G.

Lemma 2.11 [13] Let G be a connected graph of order $n \ge 3$ having transmission number $\sigma(G)$. Then

$$m_\alpha(G) \ge \frac{(2\sigma(G))^{2-\alpha}}{\Big(2\sum_{1\le i<j\le n} (d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2\Big)^{1-\alpha}}, \qquad \text{if } \alpha<0,\ 0<\alpha<1$$

and

$$m_\alpha(G) \le \frac{(2\sigma(G))^{2-\alpha}}{\Big(2\sum_{1\le i<j\le n} (d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2\Big)^{1-\alpha}}, \qquad \text{if } 1<\alpha<2,\ \alpha>2.$$

The next result gives an upper and a lower bound for $S_\beta(G)$ in terms of the transmission degrees, sum of the squares of distances and the transmission number.

Theorem 2.12 *Let G be a connected graph of order $n \ge 3$ having transmission number $\sigma(G)$. Then*

$$S_\beta(G) \ge \frac{(2\sigma(G))^{2-\beta}}{\Big(2\sum_{1\le i<j\le n} (d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2\Big)^{1-\beta}}, \quad \text{if } \beta<0,\ 0<\beta<1$$

and

$$S_\beta(G) \le \frac{(2\sigma(G))^{2-\beta}}{\Big(2\sum_{1\le i<j\le n} (d_{ij})^2 + \sum_{i=1}^{n} Tr_i^2\Big)^{1-\beta}}, \quad \text{if } 1<\beta<2,\ \beta>2$$

with equality if and only if $G \cong K_n$.

Proof Since the distance Laplacian eigenvalues are non-negative and $\partial_n = 0$ is the only eigenvalue which is zero, the proof is simillar to that of Lemma 2.11.

The equality occurs if and only if $\partial_1 = \partial_2 = \partial_3 = \cdots = \partial_{n-1}$, which is possible if and only if $G \cong K_n$. □

The following observations will be used in the sequel.

Lemma 2.13 [15] Let G be a connected graph with $n \ge 2$ vertices and m edges having diameter d. Then

$$\sum_{1\le i<j\le n} d_{ij}^2 \le \frac{2m(1-d^2)+d^2 n(n-1)}{2}$$

with equality if and only if $d \le 2$.

Lemma 2.14 [15] Let G be a connected bipartite graph having $n \geq 3$ vertices, m edges and diameter d with bi-partition sets as V_1, V_2 with $|V_1| = a, |V_2| = b, a \geq b \geq 1$ and $a + b = n$. Then

$$\sum_i Tr_i^2 \leq mn(1-d)^2 + nd^2(n-1)^2 + 4md(1-d)(n-1)$$

with equality if and only if G is complete bipartite graph.

Corollary 2.15 *Let G be a connected bipartite graph having $n \geq 3$ vertices, m edges and diameter d. Then*

$$S_\beta(G) \geq \frac{(2\sigma(G))^{2-\beta}}{\left(2m(1-d^2) + d^2n(n-1) + mn(1-d)^2 + nd^2(n-1)^2 + 4md(1-d)(n-1)\right)^{1-\beta}},$$

if $\beta < 0$ or $0 < \beta < 1$
and

$$S_\beta(G) \leq (2\sigma(G))^{2-\beta}\Big(2m(1-d^2) + d^2n(n-1) + mn(1-d)^2 + nd^2(n-1)^2 + 4md(1-d)(n-1)\Big)^{\beta-1},$$

if $1 < \beta < 2$ or $\beta > 2$.

3 $S_\beta(G)$ in terms of complement of G

First we obtain a lower and an upper bound for $S_\beta(G)$ in terms of the number of vertices n and the number of components C_k in its complement $\overline{G}$.

Theorem 3.1 *Let G be a connected graph of order n having disconnected complement. Then*

$$S_\beta(G) \leq (n - C_k)(\partial_1^L(G))^\beta + (C_k - 1)n^\beta, \qquad if\ \beta > 0$$

and

$$S_\beta(G) \geq (n - C_k)(\partial_1^L(G))^\beta + (C_k - 1)n^\beta, \qquad if\ \beta < 0.$$

with equality if and only if $\partial_1^L(G)$ is of multiplicity $(n - C_k)$.

Proof Let C_k be the number of components in $\overline{G}$. Then n is an eigenvalue of $D^L(G)$ with multiplicity $C_k - 1$. Then, for any $\beta > 0$,

$$\begin{aligned} S_\beta(G) &= (\partial_1^L(G))^\beta + (\partial_2^L(G))^\beta + \cdots + (\partial_{n-C_k}^L(G))^\beta + (C_k - 1)n^\beta \\ &\leq (n - c_k)(\partial_1^L(G))^\beta + (C_k - 1)n^\beta. \end{aligned}$$

The equality holds in above if and only if $\partial_1^L(G) == \partial_2^L(G) = ... = \partial_{n-c_k}^L(G)$, that is, if and only if $\partial_1^L(G)$ is of multiplicity $n - C_k$.

Using similar arguments as in part first, part second follows. □

Lemma 3.2 [11] Let G be a connected graph on $n \geq 2$ vertices. Then $m(\partial_1^L) \leq n - 1$ with equality if and only if G is a complete graph K_n.

Corollary 3.3 *Let G be a connected graph of order n having n components of $\overline{G}$. Then*

$$S_\beta(G) = (n-1)n^\beta, \quad for\ all\ \beta \neq 0.$$

Proof By Lemma 3.2, K_n is the only graph having a distance Laplacian eigenvalue of multiplicity $n - 1$. Since distance Laplacian spectrum of K_n is $(n^{n-1}, 0)$ using the spectrum of K_n in 3.1 , we get the required result. □

Lemma 3.4 [11] Let G be a graph on $n \geq 3$ and $G \ncong K_n$. Then $m(\partial_1^L) \leq n - 2$ with equality if and only if G is the star S_n *or* $n = 2p$ for the complete bipartite graph $K_{p,p}$.

Corollary 3.5 *Let G be a connected graph with $m(\partial_1^L) = n-2$ and let the complement of G has two components. Then*

$$S_\beta(G) \leq (n-2)(2n-1)^\beta + n^\beta, \qquad if\ \beta > 0$$

and

$$S_\beta(G) \geq (n-2)(2n-1)^\beta + n^\beta, \qquad if\ \beta < 0$$

with equality if and only if $G \cong S_n$.

Proof Since $m(\partial_1^L) = n-2$ and the complement of G has two components, therefore, by Lemma 3.4, G is the star S_n or the complete bipartite graph $K_{p,p}$ when $n = 2p$. The D^L- spectrum of S_n is $(2n-1)^{(n-2)}, n, 0$ and the D^L-spectrum of $K_{p,p}\ is\ (3p)^{(n-2)}, n, 0$. Now, $2n-1 > 3p,\ for\ n \geq 4$, where $p + p = n$. Therefore, by Theorem 3.1, we have

$$S_\beta(G) \leq (n-2)(2n-1)^\beta + n^\beta, \qquad if\ \beta > 0$$

and

$$S_\beta(G) \geq (n-2)(2n-1)^\beta + n^\beta, \qquad if\ \beta < 0$$

with equality if and only if $G \cong S_n$. □

Lemma 3.6 [11] Let G be a connected graph of order n and multiplicity of $n \geq 4$ such that $m(\partial_1^L) = n-3$ and $\partial_{n-1}^L = n$ with multiplicity two. Then $G \cong \overline{K}_{n-2} \bigvee K_2$ or $G \cong K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$ or $G \cong K_{\frac{n-1}{2},\frac{n-1}{2}} \bigvee K_1$.

Corollary 3.7 *Let G be a connected graph with $m(\partial_1^L) = n-3$ and let the complement of G has three components. Then*

$$S_\beta(G) \leq (n-3)(2(n-1))^\beta + n^\beta, \qquad if\ \beta > 0$$

and

$$S_\beta(G) \geq (n-3)(2(n-1))^\beta + n^\beta, \qquad if\ \beta < 0$$

with equality if and only if $G \cong \overline{K}_{n-2} \bigvee K_2$.

Proof Since, $m(\partial_1^L) = n-3$ and the complement of G has three connected components, therefore, by Lemma 3.6, we have $G \cong \overline{K}_{n-2} \bigvee K_2$ or $G \cong K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$ or $G \cong K_{\frac{n-1}{2},\frac{n-1}{2}} \bigvee K_1$. As D^L-spectrum of $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}\ is\ (\frac{4n}{3})^{n-3}, n^2, 0$, D^L-spectrum of $K_{\frac{n-1}{2},\frac{n-1}{2}} \bigvee K_1\ is\ (\frac{3n-1}{2})^{n-3}, n^2, 0$ and D^L-spectrum of $\overline{K}_{n-2} \bigvee K_2\ is\ (2(n-1))^{n-3}, n^2, 0$. For $n \geq 4$, we have $2(n-1) > \frac{3n-1}{2} > \frac{4n}{3}$. Therefore, by Theorem 3.1 , we get the required result. □

4 Conclusion

We have obtained various upper and lower bounds of the graph invariant $S_\beta(G)$ in terms of several graph parameters like order n, Wiener index $\sigma(G)$, maximum transmission Tr_1, diameter d and number of components C_k in $\overline{G}$. In generality, we acknowledge that it is not easy to find the extremal graphs corresponding to these bounds. So it will be interesting to study and investigate the problem of finding the extremal cases of the bounds given in this paper. Also, it will be interesting to find the upper and lower bounds for $S_\beta(G)$ in terms of parameters not used in this paper.

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Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

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