Indian J Pure Appl Math (2026) 57:836–855
https://doi.org/10.1007/s13226-024-00599-z

ORIGINAL RESEARCH

On Padovan or Perrin numbers as products of three repdigits in base δ

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Received: 13 February 2023 / Accepted: 17 May 2024 / Published online: 28 May 2024

Abstract Let P_m and E_m be the m-th Padovan and Perrin numbers, respectively. In this paper, we prove that for a fixed integer δ with $\delta \geq 2$ there exists finitely many Padovan and Perrin numbers that can be represented as products of three repdigits in base δ. Moreover, we explicitly find these numbers for $2 \leq \delta \leq 10$ as an application.

Keywords Padovan and Perrin numbers · Linear forms in logarithms · Reduction method

Mathematics Subject Classification 11B39 · 11J86 · 11D61 · 11D72 · 11Y50

1 Introduction

A positive integer x is called a repdigit in base $\delta \geq 2$ if it has only one digit in its base δ expansion, i.e. a number of the form

$$x = \underbrace{\overline{a \dots a}}_{\ell \ times}{}_{(\delta)} = a\left(\frac{\delta^{\ell} - 1}{\delta - 1}\right),$$

where a and ℓ are positive integers with $1 \leq a \leq \delta - 1$.
The Padovan sequence $\{P_n\}_{n\geq 0}$ is defined by

$$P_{n+3} = P_{n+1} + P_n,$$

for $n \geq 0$ where $P_0 = P_1 = P_2 = 1$. This is the sequence A000931 in the OEIS (see [13]). The first few terms of this sequence are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, \dots$$

Communicated by B. Sury.

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Let $\{E_n\}_{n\geq 0}$ be the Perrin sequence given by

$$E_{n+3} = E_{n+1} + E_n,$$

for $n \geq 0$, where $E_0 = 3$, $E_1 = 0$ and $E_2 = 2$. Its first few terms are

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, \ldots$$

It is the sequence A001608 in the OEIS (see [13]).
Let us now recall some facts and properties of the Padovan and the Perrin sequences which will be used later. The characteristic equation

$$x^3 - x - 1 = 0$$

has roots $\alpha, \beta, \lambda = \overline{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12}$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \quad \text{and} \quad r_1 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Let

$$\begin{aligned} a &= \frac{(1-\beta)(1-\lambda)}{(\alpha-\beta)(\alpha-\lambda)} = \frac{1+\alpha}{-\alpha^2+3\alpha+1}, \\ b &= \frac{(1-\alpha)(1-\lambda)}{(\beta-\alpha)(\beta-\lambda)} = \frac{1+\beta}{-\beta^2+3\beta+1}, \\ c &= \frac{(1-\alpha)(1-\beta)}{(\lambda-\alpha)(\lambda-\beta)} = \frac{1+\lambda}{-\lambda^2+3\lambda+1} = \overline{b}. \end{aligned}$$

Binet's formula for P_n is

$$P_n = a\alpha^n + b\beta^n + c\lambda^n, \quad \text{for all } n \geq 0, \tag{1}$$

and Binet's formula for E_n is

$$E_n = \alpha^n + \beta^n + \lambda^n, \quad \text{for all } n \geq 0. \tag{2}$$

From [[2], formula (6)], we know that the minimal polynomial of a over the integers is given by

$$23x^3 - 23x^2 + 6x - 1, \tag{3}$$

and has zeros a, b, c with $|a|,\ |b|,\ |c| < 1$.
Numerically, the following estimates hold:

$$\begin{aligned} 1.32 < \alpha &< 1.33, \\ 0.86 < |\lambda| = |\beta| = \alpha^{-\frac{1}{2}} &< 0.87, \\ 0.72 < a &< 0.73, \\ 0.24 < |b| = |c| &< 0.25. \end{aligned}$$

Note that it is easy to see that the contribution of the complex conjugate roots β and λ, to the right-hand side of (1), is very small if n is large. In particular, setting

$$e_1(m) := P_m - a\alpha^m = b\beta^m + c\lambda^m, \tag{4}$$

we have $|e_1(m)| < \frac{1}{2\alpha^{m/2}}$, for all $m \geq 1$.
Furthermore, by induction, one can prove that

$$\alpha^{m-3} \leq P_m \leq \alpha^{m-1} \quad \text{for } m \geq 1. \tag{5}$$

Also, it follows that the difference between the right hand side of equation (2) and α^m becomes quite small as m increases. More specifically, let

$$e_2(m) := E_m - \alpha^m = \beta^m + \lambda^m, \tag{6}$$

we have $|e_2(m)| < \frac{2}{\alpha^{m/2}}$, for all $m \geq 1$. Similarly, one has

$$\alpha^{m-2} \leq E_m \leq \alpha^{m+1} \text{ for } m \geq 2. \tag{7}$$

In the literature, there are many papers that combine repdigits and recurrent sequences. For more details the reader can refer to ([3], [5], [7], [8], [9], [11]).
Recently, the second author, Filipin and Togbé found in [2] all Padovan and Perrin numbers which are products of two repdigits in base b with $2 \leq b \leq 10$. In fact the purpose of this work is to extend the work of the authors from [2]. This study deals with Padovan and Perrin numbers which are products of three repdigits in base δ with $\delta \geq 2$. In other words, we study the Diophantine equations

$$P_k = d_1 \frac{\delta^\ell - 1}{\delta - 1} \cdot d_2 \frac{\delta^m - 1}{\delta - 1} \cdot d_3 \frac{\delta^n - 1}{\delta - 1}, \tag{8}$$

and

$$E_k = d_1 \frac{\delta^\ell - 1}{\delta - 1} \cdot d_2 \frac{\delta^m - 1}{\delta - 1} \cdot d_3 \frac{\delta^n - 1}{\delta - 1}, \tag{9}$$

where $d_1, d_2, d_3, k, \ell, m$ and n are positive integers that satisfy $1 \leq d_1, d_2, d_3 \leq \delta - 1$ and $\delta \geq 2$ with $n \geq 1$, $\ell \leq m \leq n$. Our method for the proofs is similar to that of [1] where the authors study Fibonacci and Lucas numbers as products of three repdigits in base $g \geq 2$.

Here are the main results of this paper.

Theorem 1 *Let $\delta \geq 2$ be an integer. Then the Diophantine equation (8) has only finitely many solutions in integers $k, d_1, d_2, d_3, \delta, \ell, m, n$ such that $n \geq 1$. Furthermore, we have*

$$\ell \leq m \leq n < 7.4 \times 10^{49} \log^9 \delta \quad \textit{and} \quad k < 9.62 \times 10^{50} \log^{10} \delta.$$

Now, as an application, we solve the Diophantine equation (8) for $2 \leq \delta \leq 10$. In this range, the bound on k according to Theorem 1 becomes

$$k < 4.1 \times 10^{54}.$$

Thus, our main result in this case is the following.

Theorem 2 *The only Padovan numbers which are products of three repdigits in base δ with $2 \leq \delta \leq 10$ are follows*

$$1, 2, 3, 4, 5, 6, 7, 9, 12, 16, 21, 28, 49, 86, 114, 200, 465, 616 \textit{ and } 4410.$$

The following result is a consequence of Theorem 2.

Corollary 3 *The largest Padovan number which can be represented as products of three repdigits in base δ with $2 \leq \delta \leq 10$ is $P_{32} = 4410$. More precisely, we have*

$$\begin{aligned} P_{32} = 4410 &= 2\left(\frac{4^2-1}{3}\right) \cdot 1 \left(\frac{4^3-1}{3}\right) \cdot 1 \left(\frac{4^3-1}{3}\right), \\ &= \overline{22}_4 \cdot \overline{111}_4 \cdot \overline{111}_4. \end{aligned}$$

Theorem 4 *Let $\delta \geq 2$ be an integer. Then the Diophantine equation (9) has only finitely many solutions in integers $k, d_1, d_2, d_3, \delta, \ell, m, n$ such that $n \geq 1$. Namely, we have*

$$\ell \leq m \leq n < 4.8 \times 10^{48} \log^9 \delta \quad \textit{and} \quad k < 6 \times 10^{49} \log^{10} \delta.$$

Theorem 5 *The only Perrin numbers which are products of three repdigits in base δ with $2 \leq \delta \leq 10$ are* 2, 3, 5, 7, 10, 12, 90, 486 *and* 76725.

We can deduce the following result.

Corollary 6 *The largest Perrin number which can be represented as products of three repdigits in base δ with $2 \le \delta \le 10$ is $E_{40} = 76725$. More precisely, we have*

$$\begin{aligned} E_{40} = 76725 &= 3\left(\frac{4^2-1}{3}\right)\cdot 3\left(\frac{4^2-1}{3}\right)\cdot 1\left(\frac{4^5-1}{3}\right), \\ &= \overline{33}_4 \cdot \overline{33}_4 \cdot \overline{11111}_4. \end{aligned}$$

The organization of this paper is as follows. In Section 2 we recall Matveev's theorem on linear form in logarithms of algebraic numbers. In Sections 3 and 4 we prove our mains results of this paper.

2 Matveev's result

We begin this section with a few reminders about logarithmic height of an algebraic number. Let η be an algebraic number of degree d, let $a_0 > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\eta = \eta^{(1)}, \ldots, \eta^{(d)}$ denote its conjugates. The quantity defined by

$$h(\eta) = \frac{1}{d}\left(\log|a_0| + \sum_{j=1}^{d} \log\max\left(1, \left|\eta^{(j)}\right|\right)\right)$$

is called the logarithmic height of η. Some properties of height are as follows. For η_1, η_2 algebraic numbers and $m \in \mathbb{Z}$, we have

$$\begin{aligned} h(\eta_1 \pm \eta_2) &\le h(\eta_1) + h(\eta_2) + \log 2, \\ h(\eta_1\eta_2^{\pm 1}) &\le h(\eta_1) + h(\eta_2), \\ h(\eta_1^m) &= |m| h(\eta_1). \end{aligned}$$

If $\eta = \dfrac{p}{q} \in \mathbb{Q}$ is a rational number in reduced form with $q > 0$, then the above definition reduces to $h(\eta) = \log(\max\{|p|, q\})$. We can now present the famous result of Matveev used in this study. Let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\eta_1, \ldots, \eta_s \in \mathbb{L}$ and $b_1, \ldots, b_s \in \mathbb{Z} \setminus \{0\}$. Let $B \ge \max\{|b_1|, \ldots, |b_s|\}$ and

$$\Lambda = \eta_1^{b_1} \cdots \eta_s^{b_s} - 1.$$

Let $A_1, \ldots, A_s$ be real numbers with

$$A_i \ge \max\{d_{\mathbb{L}} h(\eta_i), |\log \eta_i|, 0.16\}, \quad i = 1, 2, \ldots, s.$$

With the above notation, Matveev proved the following result [12].

Theorem 7 *Assume that $\Lambda \ne 0$. Then*

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 \cdot (1 + \log d_{\mathbb{L}}) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

3 Padovan numbers as products of three repdigits in base δ

3.1 Proof of Theorem 1

Note that if $n = 1$, then $\ell = m = 1$ and subsequently the Diophantine equation (8) becomes $P_k = d_1 d_2 d_3$. Combining this with (5), we have $k \le 3\log(\delta - 1)/\log\alpha + 3$. So in this case, Theorem 1 holds. For the rest of the proof we consider $n \ge 2$. The following result gives a relation between n and k in the equation (8).

Lemma 8 *All solutions of the Diophantine equation (8) satisfy*

$$k < 3n\frac{\log\delta}{\log\alpha} + 3 < 13n\log\delta.$$

Proof From (5), we have

$$\alpha^{k-3} \leq P_k = d_1 \frac{\delta^\ell - 1}{\delta - 1} \cdot d_2 \frac{\delta^m - 1}{\delta - 1} \cdot d_3 \frac{\delta^n - 1}{\delta - 1} \leq (\delta^n - 1)^3 < \delta^{3n}.$$

Taking logarithm on both sides, we get $(k-3)\log\alpha < 3n\log\delta$. Since, $n \geq 2$, $1.32 < \alpha < 1.33$ and $\delta \geq 2$, we obtain

$$k < 3n\frac{\log\delta}{\log\alpha} + 3 = \left(\frac{3}{\log\alpha} + \frac{3}{n\log\delta}\right) \cdot n\log\delta < 12.969 n\log\delta,$$

which leads to the desired inequalities. □

To start the proof of Theorem 1, we find upper bounds for the variables n, ℓ, m of (8). Using (4) and (8), we get

$$P_k = a\alpha^k + e_1(k) = d_1 \frac{\delta^\ell - 1}{\delta - 1} \cdot d_2 \frac{\delta^m - 1}{\delta - 1} \cdot d_3 \frac{\delta^n - 1}{\delta - 1}$$

to obtain

$$\begin{aligned} a\alpha^k - \frac{d_1d_2d_3\delta^{\ell+m+n}}{(\delta-1)^3} &= -e_1(k) - \frac{d_1d_2d_3\delta^{\ell+m}}{(\delta-1)^3} - \frac{d_1d_2d_3\delta^{n+\ell}}{(\delta-1)^3} + \frac{d_1d_2d_3\delta^{\ell}}{(\delta-1)^3} \\ &\quad - \frac{d_1d_2d_3\delta^{n+m}}{(\delta-1)^3} + \frac{d_1d_2d_3\delta^{m}}{(\delta-1)^3} + \frac{d_1d_2d_3\delta^{n}}{(\delta-1)^3} - \frac{d_1d_2d_3}{(\delta-1)^3}. \end{aligned} \tag{10}$$

Taking the absolute values of both sides of (10), we get

$$\begin{aligned} \left| a\alpha^k - \frac{d_1d_2d_3\delta^{\ell+m+n}}{(\delta-1)^3} \right| &\leq \frac{1}{2\alpha^{k/2}} + \frac{d_1d_2d_3\delta^{\ell+m}}{(\delta-1)^3} + \frac{d_1d_2d_3\delta^{n+\ell}}{(\delta-1)^3} + \frac{d_1d_2d_3\delta^{\ell}}{(\delta-1)^3} \\ &\quad + \frac{d_1d_2d_3\delta^{n+m}}{(\delta-1)^3} + \frac{d_1d_2d_3\delta^{m}}{(\delta-1)^3} + \frac{d_1d_2d_3\delta^{n}}{(\delta-1)^3} + \frac{d_1d_2d_3}{(\delta-1)^3}. \end{aligned} \tag{11}$$

Dividing both sides of (11) by $\frac{d_1d_2d_3\delta^{\ell+n+m}}{(\delta-1)^3}$ and using the fact that $\ell \leq m \leq n$ gives us the following inequalities.

$$\begin{aligned} \left| \frac{(\delta-1)^3 \cdot a\alpha^k \cdot \delta^{-(\ell+n+m)}}{d_1d_2d_3} - 1 \right| &\leq \frac{(\delta-1)^3}{2\alpha^{k/2} d_1d_2d_3\delta^{\ell+n+m}} + \frac{1}{\delta^n} + \frac{1}{\delta^m} + \frac{1}{\delta^{n+m}} \\ &\quad + \frac{1}{\delta^\ell} + \frac{1}{\delta^{n+\ell}} + \frac{1}{\delta^{\ell+m}} + \frac{1}{\delta^{\ell+m+n}} \\ &< 8 \cdot \delta^{-\ell}. \end{aligned}$$

From this, it follows that

$$\left| \frac{a(\delta-1)^3}{d_1d_2d_3} \cdot \alpha^k \cdot \delta^{-(\ell+n+m)} - 1 \right| < \frac{8}{\delta^\ell}. \tag{12}$$

Put

$$\Lambda_1 := \frac{a(\delta-1)^3}{d_1d_2d_3} \cdot \alpha^k \cdot \delta^{-(\ell+n+m)} - 1.$$

To apply Theorem 7 to Λ_1, we choose the following data

$$(\eta_1, b_1) := \left(\frac{a(\delta-1)^3}{d_1d_2d_3}, 1\right), \ (\eta_2, b_2) := (\alpha, k), \ (\eta_3, b_3) := (\delta, -(\ell+m+n))$$

and $s := 3$. Note that $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$ and $b_1, b_2, b_3 \in \mathbb{Z}$. The degree $d_{\mathbb{L}} = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ is 3, where $\mathbb{L}$ is $\mathbb{Q}(\alpha)$. We need to show $\Lambda_1 \neq 0$. Suppose $\Lambda_1 = 0$, then

$$a\alpha^k = \frac{d_1d_2d_3}{(\delta-1)^3} \cdot \delta^{(\ell+m+n)} \in \mathbb{Q}.$$

Taking the norm in $\mathbb{L}$ on both sides of the above equality, we get

$$1 = |\pm 1| = \left[\frac{d_1 d_2 d_3}{(\delta - 1)^3} \cdot \delta^{(\ell+m+n)}\right]^3 > 1,$$

which is false. Thus $\Lambda_1 \neq 0$. Using to the inequalities from Lemma 8, we can take $B := 13n \log \delta$ because $B \geq \max\{|b_1|, |b_2|, |b_3|\}$. To estimate the parameters A_1, A_2, A_3, we calculate the logarithmic heights of η_1, η_2, η_3 as follows:

$$\begin{aligned} h(\eta_1) = h\left(\frac{a(\delta-1)^3}{d_1 d_2 d_3}\right) &\leq h\left(\frac{(\delta-1)^3}{d_1 d_2 d_3}\right) + h(a) \\ &= \log\left(\max\{(\delta-1)^3, d_1 d_2 d_3\}\right) + \frac{1}{3}\log 23 \\ &= 3\log(\delta - 1) + \frac{1}{3}\log 23 < 4.6 \log \delta \end{aligned}$$

and

$$h(\alpha) = \frac{1}{3}\log \alpha \quad \text{and} \quad h(\eta_3) = \log \delta.$$

Thus, one can take

$$A_1 = 13.8 \log \delta, \quad A_2 = \log \alpha \quad \text{and} \quad A_3 = 3 \log \delta.$$

Then, we apply Theorem 7 and find

$$\begin{aligned} \log |\Lambda_1| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2 \cdot (1 + \log 3) \cdot (1 + \log(13n \log \delta)) \times \\ 13.8 \log \delta \cdot \log \alpha \cdot 3 \log \delta. \end{aligned} \tag{13}$$

Comparing inequality (13) with (12) gives

$$\ell \log \delta - \log 8 < 3.2 \times 10^{13} \cdot (1 + \log(13n \log \delta)) \cdot \log^2 \delta. \tag{14}$$

Because for $\delta \geq 2$ and $n \geq 2$,

$$1 + \log(13n \log \delta) < 10 \log n \cdot \log \delta, \tag{15}$$

we would get

$$\ell < 3.3 \times 10^{14} \cdot \log n \cdot \log^2 \delta. \tag{16}$$

To see inequality (15), note that for fixed integer $\delta \geq 2$ the function $f(x) := 1 + \log(13x \log \delta) - 10 \log x \cdot \log \delta$ is decreasing for $x \geq 2$.

Secondly, we rewrite (8) as

$$\frac{a\alpha^k(\delta-1)}{d_1(\delta^\ell - 1)} + \frac{e_1(k)(\delta-1)}{d_1(\delta^\ell - 1)} = \frac{d_2 d_3}{(\delta-1)^2}\left(\delta^{n+m} - \delta^m - \delta^n + 1\right),$$

which implies

$$\begin{aligned} \frac{a\alpha^k(\delta-1)}{d_1(\delta^\ell - 1)} - \frac{d_2 d_3 \delta^{n+m}}{(\delta-1)^2} = -\frac{e_1(k)(\delta-1)}{d_1(\delta^\ell - 1)} - \frac{d_2 d_3 \delta^m}{(\delta-1)^2} - \frac{d_2 d_3 \delta^n}{(\delta-1)^2} \\ + \frac{d_2 d_3}{(\delta-1)^2}. \end{aligned} \tag{17}$$

Taking the absolute values of both sides of (17), we have

$$\left|\frac{a\alpha^k(\delta-1)}{d_1(\delta^\ell-1)}-\frac{d_2d_3\delta^{n+m}}{(\delta-1)^2}\right| \le \frac{(\delta-1)}{2d_1(\delta^\ell-1)\alpha^{k/2}}+\frac{d_2d_3\delta^m}{(\delta-1)^2}+\frac{d_2d_3\delta^n}{(\delta-1)^2} + \frac{d_2d_3}{(\delta-1)^2}.$$

Dividing the both sides of the inequality above by $\frac{d_2d_3\delta^{n+m}}{(\delta-1)^2}$ and using the fact that $n \ge 2$, leads to

$$\left|\frac{a(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)}\cdot\alpha^k\cdot\delta^{-(n+m)}-1\right| \le \frac{(\delta-1)^3}{2d_1d_2d_3(\delta^\ell-1)\alpha^{k/2}\delta^{n+m}} + \frac{1}{\delta^n}+\frac{1}{\delta^m}+\frac{1}{\delta^{n+m}} < 4\cdot\delta^{-m}.$$

Therefore,

$$\left|\frac{a(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)}\cdot\alpha^k\cdot\delta^{-(n+m)}-1\right| < \frac{4}{\delta^m}. \tag{18}$$

Now, let us apply Theorem 7 with

$$(\eta_1, b_1) := \left(\frac{a(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)}, 1\right), (\eta_3, b_3) := (\delta, -n-m),$$

$(\eta_2, b_2) := (\alpha, k)$, $s := 3$ and $B := 13n\log\delta$. Note that the numbers η_1, η_2 and η_3 are positive real numbers and elements of the field $\mathbb{L} = \mathbb{Q}(\alpha)$. It is obvious that the degree of the field $\mathbb{L}$ is 3. So $d_{\mathbb{L}} = 3$. Let

$$\Lambda_2 := \frac{a(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)}\cdot\alpha^k\cdot\delta^{-(n+m)}-1.$$

If $\Lambda_2 = 0$, then we get

$$a\alpha^k = \frac{d_1d_2d_3(\delta^\ell-1)\delta^{(n+m)}}{(\delta-1)^3} \in \mathbb{Q}.$$

This is impossible as $a\alpha^k$ is irrational for $k \ge 1$. Therefore, Λ_2 is nonzero. Moreover, since

$$\begin{aligned} h(\eta_1) = h\left(\frac{a(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)}\right) &\le h\left(\frac{(\delta-1)^3}{d_1d_2d_3}\right)+h\left(\frac{1}{\delta^\ell-1}\right)+h(a) \\ &= \log\max\{(\delta-1)^3, d_1d_2d_3\}+\log(\delta^\ell-1)+\frac{1}{3}\log 23 \\ &= 3\log(\delta-1)+\log(\delta^\ell-1)+\frac{1}{3}\log 23 < (4.6+\ell)\log\delta, \end{aligned}$$

and

$$h(\eta_2) = \frac{1}{3}\log\alpha, \quad h(\eta_3) = \log\delta,$$

we can take $A_1 := 3(\ell+4.6)\log\delta$, $A_2 := \log\alpha$ and $A_3 := 3\log\delta$. Thus, taking into account the inequality (18) and using Theorem 7, we obtain

$$m\log\delta - \log 4 < 1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2\cdot(1+\log 3)\cdot(1+\log B)\times 9\cdot(4.6+\ell)\cdot\log\alpha\cdot\log^2\delta.$$

Since $\delta \ge 2$ and $1+\log B < 10\log n\log\delta$, it follows that

$$m < 7\times 10^{13}\cdot(4.6+\ell)\cdot\log n\cdot\log^2\delta+2. \tag{19}$$

By (16), we have

$$4.6 + \ell < 3.4 \times 10^{14} \cdot \log n \cdot \log^2 \delta. \tag{20}$$

Therefore from (19) and (20), we easily get

$$m < 2.4 \times 10^{28} \log^2 n \log^4 \delta. \tag{21}$$

Rearranging now equation (8) as

$$\frac{d_3\delta^n}{\delta-1} - \frac{a(\delta-1)^2\alpha^k}{d_1d_2(\delta^\ell-1)(\delta^m-1)} = \frac{d_3}{\delta-1} + \frac{e_1(k)(\delta-1)^2}{d_1d_2(\delta^\ell-1)(\delta^m-1)}$$

and taking absolute values of both sides of the equality above, we get

$$\left|\frac{d_3\delta^n}{\delta-1} - \frac{a(\delta-1)^2\alpha^k}{d_1d_2(\delta^\ell-1)(\delta^m-1)}\right| \le \frac{d_3}{\delta-1} + \frac{(\delta-1)^2}{2\alpha^{k/2}d_1d_2(\delta^\ell-1)(\delta^m-1)}. \tag{22}$$

Dividing both sides of (22) by $\dfrac{d_3\delta^n}{\delta-1}$ and using the fact that $n \ge 2$, we obtain

$$\left|1 - \frac{a(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)(\delta^m-1)} \cdot \delta^{-n} \cdot \alpha^k\right| < \frac{1}{\delta^n} + \frac{1}{\delta^{n-1}} < \frac{2}{\delta^{n-1}}. \tag{23}$$

Put

$$\Lambda_3 := \frac{a(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)(\delta^m-1)} \cdot \delta^{-n} \cdot \alpha^k - 1. \tag{24}$$

Next, we apply Theorem 7 on (24). First, we need to check that $\Lambda_3 \neq 0$. If it were, then we would get that

$$a\alpha^k = \frac{d_1d_2d_3(\delta^\ell-1)(\delta^m-1)\delta^n}{(\delta-1)^3} \in \mathbb{Q}$$

which is false. Thus, $\Lambda_3 \neq 0$. So, we apply Theorem 7 on (24) with the data:

$$s := 3,\ (\eta_1, b_1) := \left(\frac{a(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)(\delta^m-1)}, 1\right), (\eta_2, b_2) := (\delta, -n),$$

and $(\eta_3, b_3) := (\alpha, k)$. Because, $B \ge \max\{|b_1|, |b_2|, |b_3|\} = \max\{1, n, k\}$ and using the inequality from Lemma 8 we can take $B := 13n \log \delta$. Note that $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$. Observe that $\mathbb{L} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$, so $d_{\mathbb{L}} = 3$. Next,

$$\begin{aligned} h(\eta_1) &= h\left(\frac{a(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)(\delta^m-1)}\right) \\ &\le h\left(\frac{(\delta-1)^3}{d_1d_2d_3}\right) + h\left((\delta^\ell-1)(\delta^m-1)\right) + h(a) \\ &< (3+\ell+m)\log\delta + \frac{1}{3}\log 23 < (4.6+\ell+m)\log\delta. \end{aligned}$$

Thus, we can take

$$A_1 := 3(4.6+\ell+m)\log\delta, \quad A_2 := 3\log\delta \quad \text{and} \quad A_3 := \log\alpha.$$

Theorem 7 tells us that

$$\begin{aligned} \log|\Lambda_3| > &-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2 \cdot (1+\log 3) \cdot (1+\log B) \times \\ &9\log\delta \cdot \log\alpha \cdot (4.6+\ell+m)\log\delta \end{aligned} \tag{25}$$

and

$$1 + \log B < 10 \log n \log \delta. \tag{26}$$

Combining the above two inequalities with (23) gives

$$(n-1)\log\delta - \log 2 < 7 \times 10^{13}(4.6 + \ell + m)\log n \log^3 \delta,$$

which leads to

$$n < 7 \times 10^{13}(4.6 + \ell + m)\log n \log^2 \delta + 2. \tag{27}$$

Referring to the inequalities (16) and (21), we have

$$\begin{aligned} 4.6 + \ell + m &< 4.6 + 3.3 \times 10^{14} \cdot \log n \cdot \log^2 \delta + 2.4 \times 10^{28} \log^2 n \log^4 \delta \\ &< 4.8 \times 10^{28} \log^2 n \log^4 \delta. \end{aligned} \tag{28}$$

Inserting this in (27), we get that

$$n < 3.4 \times 10^{42} \cdot \log^3 n \cdot \log^6 \delta. \tag{29}$$

To get an upper bound of n in terms of δ we have to recall the following result.

Lemma 9 *(Lemma 7 of [10]) If* $l \geq 1$, $H > \left(4l^2\right)^l$ *and* $H > L/(\log L)^l$, *then*

$$L < 2^l H (\log H)^l.$$

Thus, we can apply Lemma 9 with

$$l = 3, \quad L = n \quad \text{and} \quad H := 3.4 \times 10^{42} \cdot \log^6 \delta.$$

It follows that

$$\begin{aligned} n &< 2^3 \cdot 3.4 \times 10^{42} \cdot \log^6 \delta \times \log^3 \left(3.4 \times 10^{42} \cdot \log^6 \delta\right) \\ &< 2^3 \cdot 3.4 \times 10^{42} \cdot \log^6 \delta \cdot (97.94 + 6 \log\log\delta)^3 \\ &< 7.4 \times 10^{49} \log^9 \delta, \end{aligned}$$

where we have used the fact that $97.94 + 6\log\log\delta < 139 \log\delta$, which holds for all $\delta \geq 2$. Hence, we summarize that all the solutions of (8) satisfy

$$n < 7.4 \times 10^{49} \log^9 \delta \quad \text{and} \quad k < 13 n \log \delta < 9.62 \times 10^{50} \log^{10} \delta.$$

This ends the proof of Theorem 1. □

3.2 Proof of Theorem 2

For the proof of Theorem 2 we need the following result due to Dujella and Pethő [6].

Lemma 10 *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon = ||\mu q|| - M \cdot ||\tau q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < |m\tau - n + \mu| < AB^{-w}$$

in positive integers m, n and k with

$$m \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Proof of Theorem 2 First, we have to reduce the bounds on ℓ, m, n and k. Put

$$\begin{aligned} z_1 &:= \log(\Lambda_1 + 1) \\ &= k \log \alpha - (\ell + m + n) \log \delta + \log\left(\frac{a(\delta-1)^3}{d_1 d_2 d_3}\right). \end{aligned}$$

Inequality (12) can be written as

$$\left|e^{z_1} - 1\right| < \frac{8}{\delta^\ell}.$$

For $\ell \ge 4$ and $2 \le \delta \le 10$, we have that $|e^{z_1} - 1| < \frac{8}{\delta^\ell} < \frac{1}{2}$ which implies that $\frac{2}{3} < e^{-z_1} < 2$. If $z_1 > 0$, then

$$0 < z_1 < e^{z_1} - 1 = \left|e^{z_1} - 1\right| < \frac{8}{\delta^\ell}.$$

If $z_1 < 0$, then

$$0 < |z_1| < e^{|z_1|} - 1 = e^{-z_1}(1 - e^{z_1}) < \frac{16}{\delta^\ell}.$$

Thus, in all cases we have $0 < |z_1| < \frac{16}{\delta^\ell}$, which implies that

$$0 < \left| k \frac{\log \alpha}{\log \delta} - (\ell + m + n) + \frac{\log\left(a(\delta-1)^3/d_1 d_2 d_3\right)}{\log \delta} \right| < (16/\log \delta) \cdot \delta^{-\ell}. \tag{30}$$

It is easy to see that $\frac{\log \alpha}{\log \delta}$ is irrational. In fact, if $\frac{\log \alpha}{\log \delta} = \frac{p}{q}$ ($p, q \in \mathbb{Z}$ and $p > 0, q > 0$, $\gcd(p, q) = 1$), then $\delta^p = \alpha^q \in \mathbb{Z}$ which is an absurdity. Now, we apply Lemma 10 with $w := \ell$,

$$\tau := \frac{\log \alpha}{\log \delta}, \quad \mu := \frac{\log\left(a(\delta-1)^3/d_1 d_2 d_3\right)}{\log \delta}, \quad A := 16/\log \delta, \quad B := \delta.$$

Since $k < 4.1 \times 10^{54}$, we can take $M := 4.1 \times 10^{54}$. Therefore, for the remaining proof, we use Mathematica to apply Lemma 10. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until we find the one that satisfies the conditions. Let q_t be the denominator of the t-th convergent of the continued fraction of τ. We find the following results.

δ	2	3	4	5	6	7	8	9	10
q_t	q_{123}	q_{105}	q_{113}	q_{100}	q_{111}	q_{102}	q_{118}	q_{109}	q_{118}
$\varepsilon >$	0.275	0.104	0.042	0.079	0.024	0.001	0.024	0.001	0.003
$\ell \le$	192	120	97	82	75	70	64	62	58

It follows from the above table that

$$\ell \le 192, \tag{31}$$

is valid in all cases. Substituting this upper bound for ℓ into (19) and combining the new bound obtained with (27), we get

$$n < 3 \times 10^{31} \cdot \log^2 n,$$

which implies $n < 6.4 \times 10^{35}$ using Lemma 9. Thus, by Lemma 8 we have $k < 2 \times 10^{37}$. Next, we need to reduce the bound on m. We return to (18) and put

$$
\begin{aligned}
z_2 &:= \log(\Lambda_2 + 1) \\
&= k \log\alpha - (n+m)\log\delta + \log\left(\frac{a(\delta-1)^3}{d_1 d_2 d_3(\delta^\ell - 1)}\right).
\end{aligned}
$$

Since $2 \le \delta \le 10$, we get from the inequality (18) and $m \ge 3$ that

$$
\left|e^{z_2} - 1\right| < \frac{4}{\delta^m} < \frac{1}{2},
$$

which implies $\frac{1}{2} < e^{z_2} < \frac{3}{2}$. If $z_2 > 0$, then $0 < z_2 < e^{z_2} - 1 < \frac{4}{\delta^m}$. If $z_2 < 0$, then

$$
0 < |z_2| < e^{|z_2|} - 1 = e^{-z_2} - 1 = e^{-z_2}(1 - e^{z_2}) < \frac{8}{\delta^m}.
$$

In any case, since $0 < |z_2| < \frac{8}{\delta^m}$, thus we have

$$
0 < \left| k\frac{\log\alpha}{\log\delta} - (n+m) + \frac{\log\left(a(\delta-1)^3/d_1d_2d_3(\delta^\ell - 1)\right)}{\log\delta} \right| < \frac{8/\log\delta}{\delta^m}. \tag{32}
$$

Again, we apply Lemma 10 with

$$
\tau := \frac{\log\alpha}{\log\delta},\ \ \mu := \frac{\log\left(a(\delta-1)^3/d_1d_2d_3(\delta^\ell - 1)\right)}{\log\delta},\ \ A := \frac{8}{\log\delta},\ \ B := \delta
$$

and $M := 2 \times 10^{37}$. With the help of Mathematica, we found the results recorded in the following table where we denote by q_t the denominator of the t-th convergent of the continued fraction of τ.

δ	2	3	4	5	6	7	8	9	10
q_t	q_{85}	q_{68}	q_{79}	q_{71}	q_{82}	q_{64}	q_{73}	q_{76}	q_{86}
$\varepsilon >$	0.002	0.002	0.001	0.0003	10^{-4}	0.0003	10^{-4}	0.0009	10^{-4}
$m \le$	139	88	69	62	56	50	47	44	44

So, we have

$$
m \le 139, \tag{33}
$$

which holds in all cases. Inserting the bounds from (31) and (33) in (27) while using the fact that $2 \le \delta \le 10$, we get

$$
n < 1.3 \times 10^{17} \log n + 2,
$$

which leads to $n < 5.6 \times 10^{18}$ and hence to $k < 1.7 \times 10^{20}$. Finally, we have to reduce the bound on n. From (23), we can put

$$
\begin{aligned}
z_3 &:= \log(\Lambda_3 + 1) \\
&= k \log\alpha - n\log\delta + \log\left(\frac{a(\delta-1)^3}{d_1 d_2 d_3(\delta^\ell - 1)(\delta^m - 1)}\right).
\end{aligned}
$$

By following what is done in previous cases, it is easy to see that for $n \ge 2$, we have

$$0 < \left| k\frac{\log\alpha}{\log\delta} - n + \frac{\log\left(a(\delta-1)^3/d_1d_2d_3(\delta^\ell-1)(\delta^m-1)\right)}{\log\delta} \right| < \frac{(2/\log\delta)}{\delta^{n-1}}. \tag{34}$$

Now, we apply Lemma 10 to (34) with $B := \delta$,

$$\tau := \frac{\log\alpha}{\log\delta},\ \mu := \frac{\log\left(a(\delta-1)^3/d_1d_2d_3(\delta^\ell-1)(\delta^m-1)\right)}{\log\delta},\ A := \frac{2}{\log\delta},$$

and $M := 1.7 \times 10^{20}$. With Mathematica, we get the results that we record in the following table.

δ	2	3	4	5	6	7	8	9	10
q_t	q_{50}	q_{39}	q_{46}	q_{47}	q_{50}	q_{39}	q_{38}	q_{40}	q_{46}
$\varepsilon >$	10^{-5}	10^{-6}	0.0002	10^{-6}	10^{-4}	10^{-4}	10^{-3}	0.007	0.007
$n-1 \leq$	86	56	41	37	32	28	26	25	24

Therefore, in all cases the inequality (34) holds for

$$n \leq 87. \tag{35}$$

Thus, it remains to check equation (8) for $2 \leq \delta \leq 10$, $1 \leq d_1, d_2, d_3 \leq \delta - 1$, $1 \leq n \leq 87$, $1 \leq k \leq 2604$, $1 \leq \ell \leq 192$ and $1 \leq m \leq 139$. Here, we use Maple and see that the Diophantine equation (8) has only the solutions listed in the statement of Theorem 2. This completes the proof of Theorem 2. □

4 Perrin numbers as products of three repdigits in base δ

In this section, we will follow the method from Section 3. For the sake of completeness, we will give most of details.

First, we have to recall the following result which will be useful in this part.

Lemma 11 *(Legendre [4]) Let κ be a real number and x, y integers such that*

$$\left|\kappa - \frac{x}{y}\right| < \frac{1}{2y^2}.$$

Then $x/y = p_k/q_k$ is a convergent of κ. Further, let M and N be a nonnegative integers such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality

$$\left|\kappa - \frac{x}{y}\right| \geq \frac{1}{(a(M)+2)y^2},$$

holds for all pairs (x, y) of positive integers with $0 < y < M$.

4.1 Proof of Theorem 4

Here too by taking $n = 1$, we easily verify that the bound of k from Theorem 4 is valid. Now let us see what happens for $n \geq 2$. The following result will be useful in proving our main result which gives a relation between n and k of equation (9).

Lemma 12 *All solutions of the Diophantine equation (9) satisfy*

$$k < 3n\frac{\log\delta}{\log\alpha} + 2 < 12.3n\log\delta.$$

Proof From (7), we have

$$\alpha^{k-2} \le E_k = d_1 \frac{\delta^\ell - 1}{\delta - 1} \cdot d_2 \frac{\delta^m - 1}{\delta - 1} \cdot d_3 \frac{\delta^n - 1}{\delta - 1} \le (\delta^n - 1)^3 < \delta^{3n}.$$

Taking logarithm on both sides, we get $(k-2)\log\alpha < 3n\log\delta$. Since, $n \ge 2$ and $\delta \ge 2$, we obtain the desired inequalities. This ends the proof. □

First, we find the upper bounds for the variables n, ℓ, m of equation (9). From (6) and (9), we get

$$E_k = \alpha^k + e_2(k) = d_1 \frac{\delta^\ell - 1}{\delta - 1} \cdot d_2 \frac{\delta^m - 1}{\delta - 1} \cdot d_3 \frac{\delta^n - 1}{\delta - 1}$$

to obtain

$$\begin{aligned}\alpha^k - \frac{d_1 d_2 d_3 \delta^{\ell+m+n}}{(\delta-1)^3} &= -e_2(k) - \frac{d_1 d_2 d_3 \delta^{\ell+m}}{(\delta-1)^3} - \frac{d_1 d_2 d_3 \delta^{n+\ell}}{(\delta-1)^3} + \frac{d_1 d_2 d_3 \delta^{\ell}}{(\delta-1)^3} \\ &\quad - \frac{d_1 d_2 d_3 \delta^{n+m}}{(\delta-1)^3} + \frac{d_1 d_2 d_3 \delta^{m}}{(\delta-1)^3} + \frac{d_1 d_2 d_3 \delta^{n}}{(\delta-1)^3} - \frac{d_1 d_2 d_3}{(\delta-1)^3}. \end{aligned} \tag{36}$$

Taking the absolute values of both sides of (36), we get

$$\left| \frac{(\delta-1)^3}{d_1 d_2 d_3} \cdot \alpha^k \cdot \delta^{-(\ell+n+m)} - 1 \right| < \frac{9}{\delta^\ell}. \tag{37}$$

Let

$$\Lambda_4 := \frac{(\delta-1)^3}{d_1 d_2 d_3} \cdot \alpha^k \cdot \delta^{-(\ell+n+m)} - 1.$$

We need to show $\Lambda_4 \neq 0$. Suppose $\Lambda_4 = 0$, then

$$\alpha^n = \frac{d_1 d_2 d_3}{(\delta-1)^3} \cdot \delta^{\ell+m+n} \in \mathbb{Q}$$

which is false. Thus $\Lambda_4 \neq 0$. We apply Theorem 7 to Λ_4 and we that

$$\begin{aligned}\log|\Lambda_4| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2 \cdot (1 + \log 3) \cdot (1 + \log(12.3 n \log\delta)) \times \\ 27 \cdot \log\alpha \cdot \log^2\delta. \end{aligned} \tag{38}$$

Comparing inequality (38) with (37) gives

$$\ell \log\delta - \log 9 < 2.1 \times 10^{13} \cdot (1 + \log(12.3 n \log\delta)) \cdot \log^2\delta. \tag{39}$$

Note that for $\delta \ge 2$ and $n \ge 2$,

$$1 + \log(12.3 n \log\delta) < 10 \log n \cdot \log\delta,$$

we would get

$$\ell < 2.2 \times 10^{14} \cdot \log n \cdot \log^2\delta. \tag{40}$$

Now, we rewrite (9) as

$$\frac{\alpha^k(\delta-1)}{d_1(\delta^\ell - 1)} + \frac{e_2(k)(\delta-1)}{d_1(\delta^\ell - 1)} = \frac{d_2 d_3}{(\delta-1)^2}\left(\delta^{n+m} - \delta^m - \delta^n + 1\right),$$

which implies

$$\frac{\alpha^k(\delta-1)}{d_1(\delta^\ell-1)} - \frac{d_2d_3\delta^{n+m}}{(\delta-1)^2} = -\frac{e_2(k)(\delta-1)}{d_1(\delta^\ell-1)} - \frac{d_2d_3g^m}{(\delta-1)^2} - \frac{d_2d_3\delta^n}{(\delta-1)^2} + \frac{d_2d_3}{(\delta-1)^2}. \tag{41}$$

Taking the absolute values of both sides of (41), we have

$$\left|\frac{(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)}\cdot\alpha^k\cdot\delta^{-(n+m)}-1\right| < \frac{5}{\delta^m}. \tag{42}$$

Applying Theorem 7 with

$$\Lambda_5 := \frac{(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)}\cdot\alpha^k\cdot\delta^{-(n+m)}-1,$$

we obtain

$$m\log\delta-\log 5 < 1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2\cdot(1+\log 3)\cdot(1+\log B)\times 9\cdot(3+\ell)\cdot\log\alpha\cdot\log^2\delta.$$

Since $\delta\geq 2$ and $1+\log B < 10\log n\log\delta$, we obtain that

$$m < 7\times 10^{12}\cdot(3+\ell)\cdot\log n\cdot\log^2\delta + 2.4. \tag{43}$$

By (40), we have

$$3+\ell < 2.3\times 10^{14}\cdot\log n\cdot\log^2\delta. \tag{44}$$

Therefore from (43) and (44), we easily get

$$m < 1.7\times 10^{27}\log^2 n\log^4\delta. \tag{45}$$

Rearranging now equation (9) as

$$\frac{d_3\delta^n}{\delta-1} - \frac{(\delta-1)^2\alpha^k}{d_1d_2(\delta^\ell-1)(\delta^m-1)} = \frac{d_3}{\delta-1} + \frac{e_2(k)(\delta-1)^2}{d_1d_2(\delta^\ell-1)(\delta^m-1)}$$

and taking absolute values of both sides of the equality above, we get

$$\left|\frac{d_3\delta^n}{\delta-1} - \frac{(\delta-1)^2\alpha^k}{d_1d_2(\delta^\ell-1)(\delta^m-1)}\right| \leq \frac{d_3}{\delta-1} + \frac{2(\delta-1)^2}{\alpha^{k/2}d_1d_2(\delta^\ell-1)(\delta^m-1)}. \tag{46}$$

Dividing both sides of (46) by $\dfrac{d_3\delta^n}{\delta-1}$ and using the fact that $n\geq 2$, we obtain

$$\left|1-\frac{(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)(\delta^m-1)}\cdot\delta^{-n}\cdot\alpha^k\right| < \frac{1}{\delta^n}+\frac{2}{\delta^{n-1}} < \frac{3}{\delta^{n-1}}. \tag{47}$$

Put

$$\Lambda_6 := \frac{(\delta-1)^3}{d_1d_2d_3(\delta^\ell-1)(\delta^m-1)}\cdot\delta^{-n}\cdot\alpha^k - 1. \tag{48}$$

Note that $\Lambda_6\neq 0$. So, Theorem 7 tells us that

$$\log|\Lambda_6| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2\cdot(1+\log 3)\cdot(1+\log B)\times 3\log\delta\cdot\log\alpha\cdot 3(3+\ell+m)\log\delta \tag{49}$$

with $1+\log B < 10\log n\log\delta$. Thus, we get from (47) that

$$(n-1)\log\delta-\log 3 < 7\times 10^{13}(3+\ell+m)\log n\log^3\delta,$$

which leads to

$$n < 7 \times 10^{13}(3+\ell+m)\log n \log^2 \delta + 2.6. \tag{50}$$

Referring to the inequalities (40) and (45), we have

$$\begin{aligned} 3+\ell+m &< 3 + 2.2 \times 10^{14} \cdot \log n \cdot \log^2 \delta + 1.7 \times 10^{27} \log^2 n \log^4 \delta \\ &< 3.4 \times 10^{27} \log^2 n \log^4 \delta. \end{aligned} \tag{51}$$

Inserting this in (50) leads to

$$n < 2.4 \times 10^{41} \cdot \log^3 n \cdot \log^6 \delta. \tag{52}$$

We have everything ready to apply Lemma 9 with the data

$$l = 3, \quad L = n \quad \text{and} \quad H := 2.4 \times 10^{41} \cdot \log^6 \delta.$$

So, we get

$$\begin{aligned} n &< 2^3 \cdot 2.4 \times 10^{41} \cdot \log^6 \delta \times \log^3 \left(2.4 \times 10^{41} \cdot \log^6 \delta\right) \\ &< 2^3 \cdot 2.4 \times 10^{41} \cdot \log^6 \delta \cdot (95.3 + 6 \log\log \delta)^3 \\ &< 4.8 \times 10^{48} \log^9 \delta. \end{aligned}$$

Note that, the above inequality holds because $95.3 + 6\log\log\delta < 135\log\delta$ for all $\delta \geq 2$. Hence, we summarize that all the solutions of (9) satisfy

$$n < 4.8 \times 10^{48} \log^9 \delta \quad \text{and} \quad k < 12.3 n \log \delta < 6 \times 10^{49} \log^{10} \delta.$$

Hence, the proof of Theorem 4 is finished. □

4.2 Proof of Theorem 5

In this subsection, we search for all solutions of the equation (9) in the range $2 \leq \delta \leq 10$. So, in these cases according to Theorem 4, the bound on k becomes

$$k < 2.6 \times 10^{53}.$$

Proof First, we reduce the bounds on ℓ, m, n and k by putting

$$\begin{aligned} z_4 &:= \log(\Lambda_4 + 1) \\ &= k \log\alpha - (\ell+m+n)\log\delta + \log\left(\frac{(\delta-1)^3}{d_1 d_2 d_3}\right). \end{aligned}$$

Thus, inequality (37) can be written as

$$\left|e^{z_4} - 1\right| < \frac{9}{\delta^\ell}.$$

For $\ell \geq 5$ and $2 \leq \delta \leq 10$, we have $|e^{z_4} - 1| < \frac{9}{\delta^\ell} < \frac{1}{2}$ which implies that $\frac{2}{3} < e^{-z_4} < 2$. If $z_4 > 0$, then

$$0 < z_4 < e^{z_4} - 1 = \left|e^{z_4} - 1\right| < \frac{9}{\delta^\ell}.$$

If $z_4 < 0$, then

$$0 < |z_4| < e^{|z_4|} - 1 = e^{-z_4}(1 - e^{z_4}) < \frac{18}{\delta^\ell}.$$

In any case, it is always holds true $0 < |z_4| < \dfrac{18}{\delta^{\ell}}$, which implies

$$0 < \left| k\frac{\log\alpha}{\log\delta} - (\ell+m+n) + \frac{\log\left((\delta-1)^3/d_1d_2d_3\right)}{\log\delta} \right| < (18/\log\delta)\cdot\delta^{-\ell}. \tag{53}$$

Now, we have to study the following two cases.

Case 1: $(d_1, d_2, d_3) \neq (\delta-1, \delta-1, \delta-1)$. We apply Lemma 10 with $w := \ell$,

$$\tau := \frac{\log\alpha}{\log\delta}, \quad \mu := \frac{\log\left((\delta-1)^3/d_1d_2d_3\right)}{\log\delta}, \quad A := 18/\log\delta, \quad B := \delta.$$

Using the fact that $k < 2.6\times 10^{53}$, we can take $M := 2.6\times 10^{53}$. We use Mathematica to apply Lemma 10 and find the following results where q_t denotes the denominator of the t-th convergent of the continued fraction of τ. Note that in this case the base $\delta = 2$ cannot be consider because this leads to $\mu = 0$.

δ	3	4	5	6	7	8	9	10
q_t	q_{103}	q_{109}	q_{98}	q_{111}	q_{98}	q_{115}	q_{107}	q_{115}
$\varepsilon >$	0.132	0.144	0.006	0.004	0.024	0.027	0.001	0.005
$\ell \leq$	117	92	82	76	67	63	60	57

Thus, we can say from the above table that the solutions of inequality (53) satisfy

$$\ell = w \leq \frac{\log(Aq_{103}/0.132)}{\log\delta} \leq 117. \tag{54}$$

Case 2: $(d_1, d_2, d_3) = (\delta-1, \delta-1, \delta-1)$. In this case from (53), we have

$$0 < \left| k\frac{\log\alpha}{\log\delta} - (\ell+m+n) \right| < (18/\log\delta)\cdot\delta^{-\ell}.$$

Dividing both sides of this inequalities by k and using the fact that $2 \leq \delta \leq 10$, we get

$$0 < \left| \frac{\log\alpha}{\log\delta} - \frac{\ell+m+n}{k} \right| < \frac{26}{k\cdot\delta^{\ell}}. \tag{55}$$

Assume that $\ell > 200$. Then it can be seen that

$$\frac{\delta^{\ell}}{2(26)} \geq \frac{2^{\ell}}{2(26)} > 3\times 10^{58} > 2.6\times 10^{53} > k,$$

and so we have

$$\left| \frac{\log\alpha}{\log\delta} - \frac{\ell+m+n}{k} \right| < \frac{26}{k\cdot\delta^{\ell}} < \frac{1}{2k^2}.$$

From the known properties of continued fraction (Lemma 11), it is seen that the rational $\dfrac{\ell+m+n}{k}$ is a convergent to $\kappa := \dfrac{\log\alpha}{\log\delta}$ for $2 \leq \delta \leq 10$. So $\dfrac{\ell+m+n}{k}$ is of the form p_t/q_t for some t. According to Lemma 11, we take $M := 2.6\times 10^{53}$, since $k < 2.6\times 10^{53}$. For $2 \leq \delta \leq 10$, we use Mathematica to find the first convergent q_N such that $q_N > M$ and then we get $a(M) := \max\{a_i : i = 0, \ldots, N\}$. Thus, Lemma 11 tells us that

$$\frac{1}{(a(M)+2)\cdot k^2} \leq \left|\frac{\log\alpha}{\log\delta} - \frac{\ell+m+n}{k}\right| < \frac{26}{k\cdot\delta^{\ell}}$$

which implies

$$\ell < \frac{\log\left(26\cdot(a(M)+2)\cdot 2.6\times 10^{53}\right)}{\log\delta}.$$

We thus arrive at the results which are in the following table.

δ	2	3	4	5	6	7	8	9	10
$q_N > M$	q_{118}	q_{101}	q_{106}	q_{96}	q_{108}	q_{95}	q_{112}	q_{105}	q_{112}
$a(M)$	303	151	160	433	65	627	241	278	564
$\ell \leq$	190	119	94	82	72	68	63	60	57

It follows from the above table that $\ell \leq 190$, which contradicts the fact that $\ell > 200$. Thus $\ell \leq 200$. Therefore by referring to (54) we can conclude that the bound

$$\ell \leq 200, \tag{56}$$

holds for Case 1 and Case 2. Substituting the upper bound from (56) into (43) and combining the new bound obtained with (50), we get

$$n < 3\times 10^{30}\cdot\log^2 n,$$

which implies $n < 6\times 10^{34}$ using Lemma 9. Thus, by Lemma 12 we have $k < 1.7\times 10^{36}$. Next, to reduce the bound on m we return to (42) and put

$$\begin{aligned} z_5 &:= \log(\Lambda_5+1) \\ &= k\log\alpha - (n+m)\log\delta + \log\left(\frac{(\delta-1)^3}{d_1d_2d_3(\delta^{\ell}-1)}\right). \end{aligned}$$

From the inequality (42) and for $m \geq 4$, we conclude that

$$\left|e^{z_5}-1\right| < \frac{5}{\delta^m} < \frac{1}{2},$$

which implies that $\frac{1}{2} < e^{z_5} < \frac{3}{2}$. If $z_5 > 0$, then $0 < z_5 < e^{z_5} - 1 < \frac{5}{\delta^m}$. If $z_5 < 0$, then

$$0 < |z_5| < e^{|z_5|} - 1 = e^{-z_5} - 1 = e^{-z_5}(1-e^{z_5}) < \frac{10}{\delta^m}.$$

In any case, since $0 < |z_5| < \frac{10}{\delta^m}$, thus we have

$$0 < \left|k\frac{\log\alpha}{\log\delta} - (n+m) + \frac{\log\left((\delta-1)^3/d_1d_2d_3(\delta^{\ell}-1)\right)}{\log\delta}\right| < \frac{10/\log\delta}{\delta^m}. \tag{57}$$

The case $(\delta-1)^3 \neq d_1d_2d_3(\delta^{\ell}-1)$.

So, we apply Lemma 10 with the following data

$$\tau := \frac{\log\alpha}{\log\delta},\ \mu := \frac{\log\left((\delta-1)^3/d_1d_2d_3(\delta^{\ell}-1)\right)}{\log\delta},\ A := 10/\log\delta,\ B := \delta$$

and $M := 1.7\times 10^{36}$. With Mathematica, we got the results mentioned in the following table.

δ	2	3	4	5	6	7	8	9	10
q_t	q_{107}	q_{129}	q_{159}	q_{163}	q_{186}	q_{189}	q_{222}	q_{235}	q_{232}
$\epsilon >$	10^{-14}	10^{-30}	10^{-43}	10^{-54}	10^{-61}	10^{-67}	10^{-73}	10^{-78}	10^{-83}
$m \leq$	205	202	202	203	201	201	201	201	201

It follows from the above table that inequalities (57) hold in all cases for $m \leq 205$.

The case $(\delta - 1)^3 = d_1 d_2 d_3(\delta^\ell - 1)$.

In this case from (57), we have

$$0 < \left| k \frac{\log \alpha}{\log \delta} - (m+n) \right| < (10/\log \delta) \cdot \delta^{-m}.$$

Because $2 \leq \delta \leq 10$, we get

$$0 < \left| \frac{\log \alpha}{\log \delta} - \frac{m+n}{k} \right| < \frac{14.5}{k \cdot \delta^m}. \tag{58}$$

Assume that $m > 140$. Then it can be seen that

$$\frac{\delta^m}{2(14.5)} > 4.8 \times 10^{40} > 1.7 \times 10^{37} > k,$$

and so we have

$$\left| \frac{\log \alpha}{\log \delta} - \frac{m+n}{k} \right| < \frac{14.5}{k \cdot \delta^m} < \frac{1}{2k^2}.$$

From Lemma 11, it is seen that the rational $\frac{m+n}{k}$ is a convergent to $\kappa = \frac{\log \alpha}{\log \delta}$. Since $k < 1.7 \times 10^{36}$, we can apply Lemma 11 with $M := 1.7 \times 10^{36}$. For $2 \leq \delta \leq 10$, we use Mathematica to find the first convergent q_N such that $q_N > M$ and then we get $a(M) := \max\{a_i : i = 0, \ldots, N\}$. Therefore, we deduce from Lemma 11 that

$$\frac{1}{(a(M)+2) \cdot k^2} \leq \left| \frac{\log \alpha}{\log \delta} - \frac{m+n}{k} \right| < \frac{14.5}{k \cdot \delta^m}$$

which implies

$$m < \frac{\log\left(14.5 \cdot (a(M)+2) \cdot 1.7 \times 10^{36}\right)}{\log \delta}.$$

So, in the range $2 \leq \delta \leq 10$, we obtain the results presented in the following table.

δ	2	3	4	5	6	7	8	9	10
$q_N > M$	q_{84}	q_{64}	q_{78}	q_{67}	q_{75}	q_{60}	q_{71}	q_{70}	q_{81}
$a(M)$	80	151	160	433	65	217	241	107	49
$m \leq$	130	82	65	57	50	47	44	41	39

It follows that the bound $m \leq 130$ is valid for each δ such that $2 \leq \delta \leq 10$, which contradicts the fact that $m > 140$. Therefore $m \leq 140$. Thus, we conclude that, whether $(\delta - 1)^3 \neq d_1 d_2 d_3(\delta^\ell - 1)$ or not the following bound holds

$$m \leq 205. \tag{59}$$

Using inequalities (56) and (59) together and substituting these upper bounds into (50), we get

$$n < 1.6 \times 10^{17} \log n,$$

which leads to $n < 1.3 \times 10^{19}$ and hence to $k < 3.7 \times 10^{20}$. To reduce the bound on n we must return to inequality (47) and put

$$\begin{aligned} z_6 &:= \log(\Lambda_6 + 1) \\ &= k \log\alpha - n\log\delta + \log\left(\frac{(\delta-1)^3}{d_1d_2d_3(\delta^\ell - 1)(\delta^m - 1)}\right). \end{aligned}$$

For $n \geq 2$, we can easily see that

$$0 < \left| k\frac{\log\alpha}{\log\delta} - n + \frac{\log\left((\delta-1)^3/d_1d_2d_3(\delta^\ell-1)(\delta^m-1)\right)}{\log\delta}\right| < \frac{6/\log\delta}{\delta^{n-1}}. \tag{60}$$

The case $(\delta-1)^3 \neq d_1d_2d_3(\delta^\ell-1)(\delta^m-1)$.

Therefore, we can apply Lemma 10 to inequalities (60) with $B = \delta$,

$$\tau = \frac{\log\alpha}{\log\delta},\ \mu = \frac{\log\left((\delta-1)^3/d_1d_2d_3(\delta^\ell-1)(\delta^m-1)\right)}{\log\delta},\ A = 6/\log\delta,$$

and $M = 3.7 \times 10^{20}$. With the help of Mathematica and taking into account the fact that $\ell \leq 200$ and $m \leq 205$, we obtain the bound

$$n \leq 210,$$

which holds for $2 \leq \delta \leq 10$.

The case $(\delta-1)^3 = d_1d_2d_3(\delta^\ell-1)(\delta^m-1)$.

We get from inequalities (60) that

$$0 < \left| k\frac{\log\alpha}{\log\delta} - n\right| < \frac{6/\log\delta}{\delta^{n-1}}.$$

Because $2 \leq \delta \leq 10$, we have $6/\log\delta < 6/\log 2$ and then

$$0 < \left| \frac{\log\alpha}{\log\delta} - \frac{n}{k}\right| < \frac{8.7}{k\cdot\delta^{n-1}}.$$

Now, we assume that $n > 80$. Then

$$\frac{\delta^{n-1}}{2(8.7)} > 3.4 \times 10^{22} > 3.7 \times 10^{20} > k,$$

and so we have

$$\left| \frac{\log\alpha}{\log\delta} - \frac{n}{k}\right| < \frac{8.7}{k\cdot\delta^{n-1}} < \frac{1}{2k^2}.$$

From Lemma 11, it is seen that the rational number $\frac{n}{k}$ is a convergent to $\kappa = \frac{\log\alpha}{\log\delta}$. Since $k < 3.7 \times 10^{20}$, we can take $M = 3.7 \times 10^{20}$ in order to apply Lemma 11. Now let $\frac{p_N}{q_N}$ be N-th convergent of the continued fraction of κ such that $q_N > M$. Then

$$\frac{1}{(a(M)+2)\cdot k^2} \leq \left| \frac{\log\alpha}{\log\delta} - \frac{n}{k}\right| < \frac{8.7}{k\cdot\delta^{n-1}}.$$

In all cases we get with Mathematica that $n \leq 77$ which is a contradiction because $n > 80$. Therefore $n \leq 80$. So, whether $(\delta-1)^3 \neq d_1d_2d_3(\delta^\ell-1)(\delta^m-1)$ or not we only have to consider

$$n \leq 210.$$

Finally, it remains to check the Diophantine equation (9) for $2 \leq \delta \leq 10$, $1 \leq d_1, d_2, d_3 \leq \delta - 1$, $1 \leq n \leq 210$, $1 \leq k \leq 5947$, $1 \leq \ell \leq 200$ and $1 \leq m \leq 205$. A quick inspection using Maple reveals the only solutions listed in the statement of Theorem 5. This marks the end of the proof of Theorem 5. □

Acknowledgements The first author is partially supported by Université de Kara (Togo) and the second author is supported by IMSP, Institut de Mathématiques et de Sciences Physiques de l'Université d'Abomey-Calavi.

Data availability Not applicable.

Declarations

Conflict of interest There is no conflict of interest related to this paper or this submission. The authors have freely chosen this journal for publication without any consideration.

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