Indian J Pure Appl Math (2026) 57:856–867
https://doi.org/10.1007/s13226-025-00752-2

ORIGINAL RESEARCH

Delta-shock for a class of non-strictly hyperbolic systems of conservation laws as self-similar viscosity limits

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Received: 18 February 2023 / Accepted: 14 January 2025 / Published online: 29 January 2025

Abstract This paper is focused on a class of non-strictly hyperbolic systems of conservation laws. With the use of the limiting viscosity method, the stability of the delta shock wave solution is established.

Keywords Conservation laws · Delta shock wave · Limiting viscosity method

1 Introduction

This paper is concerned with a class of non-strictly hyperbolic systems of conservation laws

$$\begin{cases} \rho_t + (\rho f(u))_x = 0, \\ (\rho g(u))_t + (\rho f(u) g(u))_x = 0, \end{cases} \tag{1.1}$$

where $t \in R^+$ represents the time variable, $x \in R$ stands for the space variable, the sign of $\rho = \rho(x,t)$ is assumed to be unchanging, $u = u(x,t)$ is given smooth function, $f(u)$ and $g(u)$ are respectively smooth and strictly monotone functions.

The system (1.1) coincides with one-dimensional zero-pressure gas dynamics when $f(u) = g(u) = u$, where $\rho \geq 0$ and $u \in R$ are viewed as the density and velocity of gas, respectively. When $g(u) = u$, Yang [1] studied the Riemann problem for the model (1.1) with initial data

$$(\rho, u)(t = 0, x) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0, \end{cases} \tag{1.2}$$

by the characteristic analysis and vanishing viscosity method. Yang and Sun [2] studied the Riemann problem with delta initial data for the system (1.1) when $g(u) = u$. Mitrovic and Nedeljkov [3] proved that the delta-shocks of the model (1.1) can be formed as the limits of shocks of the perturbation model (1.1) as the pressure vanishes. Recently, Li [4] solved the Riemann problem for the system (1.1) with characteristic analysis and obtained the stability of the Riemann solutions under certain perturbation of the initial data. A noteworthy feature is that delta shock waves appear in solutions to the (1.1). As for the delta shock waves, we refer the reader to the papers [5–11] for more details.

The purpose of this paper is to explore the stability of the solution containing delta shock wave of (1.1) and (1.2) by self-similar vanishing viscosity method introduced by Dafermos [12]. The solutions for the broad classes of 2×2 hyperbolic systems including the equations of isentropic, Lagrangian, gas dynamics were constructed

Communicated by V D Sharma.

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by Dafermos and Diperna [13]. This method was widely used in [14,15] for the admissibility of weak solutions to the Riemann problem for hyperbolic systems of conservation laws. By the method, Tan, Zhang and Zheng [10] initially established the stability of delta shock wave for a hyperbolic system of conservation laws. Sheng and Zhang [8] studied the Riemann problem for zero-pressure gas dynamics, and established the stability of the delta shock wave and vacuum solutions. Chhatria, Sen and Raja Sekhar [16] discussed the Riemann problem for a strictly hyperbolic system of conservation laws, and proved the existence, uniqueness and stability of the Riemann solution involving delta shock wave. Sen and Raja Sekhar [17] solved the Riemann problem for a strictly hyperbolic system of conservation laws, and proved the existence, uniqueness and stability of the delta-shock wave solution. Sun [18] considered the Riemann problem for the changed form of the chromatography system, and proved the existence and uniqueness of solutions involving the delta shock wave. Hu [19] studied the stability of the solutions including delta shock waves for a nonstrictly hyperbolic system of conservation laws by using a vanishing viscosity approach. Hu [20] also discussed the equations of constant pressure fluid dynamics. For more applications of this method, readers can see [21] for the multidimensional zero-pressure gas dynamics by Li and Yang, [22] for a class of nonstrictly hyperbolic systems of conservation laws by Yang and Zhang, and [23,24] for zero-pressure gas dynamics with energy conservation law and a class of systems of conservation laws of the Keyfitz-Kranzer type by Li, etc.

It can be seen from the above analysis that for a 2×2 hyperbolic system of conservation laws, one can add the self-similar viscosity terms to the right of the two equations simultaneously, or just the right of the one to analyse the stability of the Riemann solutions. In present study, we consider the following viscous perturbations

$$\begin{cases} \rho_t + (\rho f(u))_x = \varepsilon t \rho_{xx}, \\ (\rho g(u))_t + (\rho f(u) g(u))_x = \varepsilon t (\rho g(u))_{xx} \end{cases} \tag{1.3}$$

with (1.2). Throughout this paper, we assume $\rho \geq 0$, $f'(u) > 0$ and $g'(u) > 0$. The rest of the cases can be discussed similarly. First, the existence of a smooth self-similar solution of the (1.3) with (1.2) is established. Then we show that in the limit $\varepsilon \to 0$, the solution $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ $(\xi = x/t)$ of (1.3) and (1.2) converges weakly star to the delta shock wave solution of (1.1)-(1.2) when $u_- > u_+$. Sections 3-5 show that $u(x, t)$ is always a bounded monotone function of $\xi = x/t$. However, $\rho(x, t)$ may be unbounded simultaneously along the single line $x/t = \xi_\sigma$. Moreover, $\rho(x, t)$ is a sum of a step function plus a distributional weighted Dirac δ-function with the discontinuity $x = f(u(\sigma))t = \sigma t$ as its support when $u_- > u_+$. In the meantime, we also verify the generalized Rankine-Hugoniot relation proposed in Sect. 2. It is discovered that the delta shock wave solution to (1.1)-(1.2) has stability under the viscous perturbations.

This paper consists of five parts, and is structured as follows. Section 2 recalls the Riemann solutions to (1.1)-(1.2). Section 3 proves the existence of smooth solutions to (1.3) and (1.2). Sections 4-5 analyse the limiting behavior of the smooth solutions to (1.3) and (1.2) as $\varepsilon \to 0$.

2 Riemann solutions to (1.1) with (1.2)

This section recalls the Riemann solutions to (1.1) with (1.2), for which the detailed investigations can be found in [4]. Throughout this paper, we assume $\rho \geq 0$, $f'(u) > 0$ and $g'(u) > 0$. The initial data $\rho_\pm > 0$ and $u_\pm$ are all given constants. The other cases can be treated in a similar way. The double eigenvalue is $\lambda = f(u)$ with associated right eigenvector $\overrightarrow{r} = (1, 0)^T$ satisfying $\nabla\lambda \cdot \overrightarrow{r} = 0$. Hence the (1.1) is extremely non-strictly hyperbolic.

Look for the solution depending on the single variable $\xi = x/t$. Then the Riemann problem (1.1) and (1.2) is reduced to

$$\begin{cases} -\xi \rho_\xi + (\rho f(u))_\xi = 0, \\ -\xi (\rho g(u))_\xi + (\rho f(u) g(u))_\xi = 0 \end{cases} \tag{2.1}$$

with

$$(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm). \tag{2.2}$$

Besides the constant state solution $(\rho, u)(\xi) =$ constant and the vacuum solution

$$\xi = f(u), \quad \rho = 0, \tag{2.3}$$

the elementary wave of (1.1) only contains contact discontinuity

$$\xi = f(u_-) = f(u_+). \tag{2.4}$$

Using the constants, vacuum state and contact discontinuity, two kinds of solutions to (1.1) and (1.2) can be constructed. When $u_- < u_+$, the solution is made up of two contact discontinuities and a vacuum state besides two constant states:

$$(\rho, u)(\xi) = \begin{cases} (\rho_-, u_-), \ -\infty < \xi < f(u_-), \\ (0, u(\xi)), \ f(u_-) \le \xi \le f(u_+), \\ (\rho_+, u_+), \ f(u_+) < \xi < +\infty, \end{cases} \tag{2.5}$$

where $u(\xi)$ satisfies $u(f(u_-)) = u_-$ and $u(f(u_+)) = u_+$.

When $u_- > u_+$, the solution contains a delta shock wave. In order to define the delta shock wave solution of (1.1) with (1.2) in the sense of distributions, a weighted δ-measure $\omega(s)\delta_L$ supported on a smooth curve L parameterized as $x = x(s)$, $t = t(s)$ $(c \le s \le d)$ is defined by

$$\langle \omega(s)\delta_L, \varphi(x,t)\rangle = \int_c^d \omega(s)\varphi(x(s), t(s))ds \tag{2.6}$$

for all test functions $\varphi(x,t) \in C_0^\infty(R \times R^+)$.

With this definition, the following delta shock wave solution

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_-), & x < \sigma t, \\ (\omega(t)\delta(x - \sigma t), u_\delta), & x = \sigma t, \\ (\rho_+, u_+), & x > \sigma t, \end{cases} \tag{2.7}$$

can be constructed, where the weight $\omega(t)$ and velocity σ, u_δ satisfy the generalized Rankine-Hugoniot relation

$$\begin{cases} \dfrac{dx(t)}{dt} = \sigma = f(u_\delta), \\ \dfrac{d\omega(t)}{dt} = -[\rho]\sigma + [\rho f(u)], \\ \dfrac{d(\omega(t)g(u_\delta))}{dt} = -[\rho g(u)]\sigma + [\rho f(u)g(u)] \end{cases} \tag{2.8}$$

and entropy condition

$$f(u_+) < \sigma < f(u_-), \tag{2.9}$$

with $[j] = j_- - j_+$ being the jump of j across the discontinuity. Under initial data $x(0) = 0$, $\omega(0) = 0$ and (2.9), the (2.8) has a unique solution $(\sigma, \omega(t), u_\delta)$.

Theorem 2.1 *Suppose that $\rho \ge 0$, $f'(u) > 0$, $g'(u) > 0$. Then Riemann problem (1.1) with (1.2) possesses a unique entropy solution, which is composed of two contact discontinuities and a vacuum state when $f(u_-) < f(u_+)$ and a delta shock wave when $f(u_-) > f(u_+)$.*

Remark 2.2 By the same discussion as in Sects. 2-3 of [4], the result similar to the Theorem 2.1 for the cases

(i) $\rho \ge 0$, $f'(u) > 0$, $g'(u) < 0$, (ii) $\rho \ge 0$, $f'(u) < 0$, $g'(u) > 0$, (iii) $\rho \ge 0$, $f'(u) < 0$, $g'(u) < 0$, (iv) $\rho \le 0$, $f'(u) > 0$, $g'(u) < 0$,

(v) $\rho \le 0$, $f'(u) > 0$, $g'(u) > 0$, (vi) $\rho \le 0$, $f'(u) < 0$, $g'(u) > 0$,

(vii) $\rho \le 0$, $f'(u) < 0$, $g'(u) < 0$

can be derived.

3 Existence of the solutions to (1.3) with (1.2)

In this section, we establish the existence of the solutions to (1.3) with (1.2). Performing self-similar transformation, the problem turns into the boundary value problem

$$\begin{cases} -\xi\rho_\xi + (\rho f(u))_\xi = \varepsilon\rho_{\xi\xi}, \\ -\xi(\rho g(u))_\xi + (\rho f(u)g(u))_\xi = \varepsilon(\rho g(u))_{\xi\xi} \end{cases} \tag{3.1}$$

with (2.2).

What we are interested is the delta shock wave, so we only discuss the case $u_+ < u_-$. In order to prove the existence of a smooth solution to (3.1) and (2.2) for any fixed $\varepsilon > 0$, we deal with the following the boundary value problem

$$\begin{cases} -\xi\rho_\xi + \mu(\rho f(u))_\xi = \varepsilon\rho_{\xi\xi}, \\ -\xi(\rho g(u))_\xi + \mu(\rho f(u)g(u))_\xi = \varepsilon(\rho g(u))_{\xi\xi} \end{cases} \tag{3.2}$$

and

$$\rho(\pm R) = \underline{\rho} + \mu(\rho_\pm - \underline{\rho}), \quad u(\pm R) = \mu u_\pm, \tag{3.3}$$

where $0 \le \mu \le 1$, $R \ge 1$, $\underline{\rho} = \min\{\rho_-, \rho_+\}$. When $\mu = 1$ and $R \to +\infty$, (3.2)-(3.3) is reduced to (3.1) with (2.2).

Inspired by in [14,19,20], we have the following theorem.

Theorem 3.1 *Assume that there exist two positive constants C and C_0 depending on ρ_+, ρ_-, u_+, u_-, ε (and thus independent of μ and R) such that any solution $(\rho(\xi), u(\xi))$ of (3.2)-(3.3) with $\rho(\xi) > 0$, corresponding to any $0 \le \mu \le 1$, $R \ge 1$, satisfies*

$$\sup_{\xi\in[-R,R]} \big(|\rho(\xi)| + |u(\xi)|\big) < C, \qquad \inf_{\xi\in[-R,R]} |\rho(\xi)| \ge C_0. \tag{3.4}$$

Then (3.1) with (2.2) admits a solution, denoted again by $(\rho(\xi), u(\xi))$ such that $\rho(\xi) > 0$ on $-\infty < \xi < +\infty$.

To get (3.4), the following some lemmas are needed.

Lemma 3.2 *Let $(\rho(\xi), u(\xi))$ with $\rho(\xi) > 0$ be a nonconstant solution of (3.2)-(3.3) in $[-R, R]$ for some $\mu \in (0, 1]$, $R \ge 1$. Then $u'(\xi) < 0$ in $[-R, R]$, and $\rho(\xi)$ satisfies one of the following*

(a) No critical point: $\rho(\xi)$ is a monotone function on $[-R, R]$;
(b) One maximum point: $\rho(\xi)$ has a maximum at τ in $[-R, R]$.

Proof Assume that $(\rho(\xi), u(\xi))$ is a nonconstant solution of (3.2)-(3.3) with $\rho(\xi) > 0$. Then it yields from (3.2) that

$$\Big(-\xi + \mu f(u) - 2\varepsilon\frac{\rho'}{\rho} - \varepsilon\frac{g''(u)}{g'(u)}u'\Big)u' = \varepsilon u''. \tag{3.5}$$

Integrating (3.5) over $(-R, \xi)$ gives

$$u'(\xi) = \lambda \exp\Big(\int_{-R}^{\xi} \frac{-s + \mu f(u) - 2\varepsilon\frac{\rho'}{\rho} - \varepsilon\frac{g''(u)}{g'(u)}u'}{\varepsilon}ds\Big), \tag{3.6}$$

where λ is determined by

$$\mu(u_+ - u_-) = \lambda\int_{-R}^{R} \exp\Big(\int_{-R}^{t} \frac{-s + \mu f(u) - 2\varepsilon\frac{\rho'}{\rho} - \varepsilon\frac{g''(u)}{g'(u)}u'}{\varepsilon}ds\Big)dt. \tag{3.7}$$

In virtue of $u_- > u_+$ and $\mu > 0$, we have $\lambda < 0$. Then it follows from (3.6) that $u'(\xi) < 0$ in $[-R, R]$.

Assume that τ is a critical point of $\rho(\xi)$, then $\rho'(\tau) = 0$. From (3.2), we arrive at $\varepsilon\rho''(\tau) = \mu\rho(\tau)f'(u(\tau))u'(\tau) < 0$, which means that τ is a maximum point. This finishes the proof.

For the case (a), it is evident that $u(\xi)$ and $\rho(\xi)$ are uniformly bounded in $[-R, R]$ with respect to $0 < \mu \le 1$, $R \ge 1$. It remains to discuss the case (b) in the Lemma 3.2. □

Lemma 3.3 *For the case (b), there exists a positive constant M such that*

$$\underline{\rho} \leq \rho(\xi) \leq M, \tag{3.8}$$

where $\underline{\rho} = \min\{\rho_-, \rho_+\}$ *and* M *are independent of* $\mu \in (0, 1]$ *and* $R \geq 1$.

Proof The left half of (3.8) is obvious in virtue of Lemma 3.2 and (3.3). To prove the right half of (3.8), we just need to show

$$\int_\alpha^\beta \rho(\xi)d\xi \leq (\beta - \alpha)\overline{\rho} + N \quad \text{for any interval } (\alpha, \beta) \subset (-R, R), \tag{3.9}$$

where $\overline{\rho} = \max_{0\leq\mu\leq1}\{\rho(-R), \rho(R)\} = \max\{\rho_-, \rho_+\}$ and $N = \overline{\rho}\max_{0\leq\mu\leq1}|f(u(R)) - f(u(-R))|$. To do this, set $\eta_1 = \sup\{\xi \in [-R, \alpha)|\rho(\xi) \leq \overline{\rho}\}$ if $\rho(\alpha) > \overline{\rho}$, while $\eta_1 = \inf\{\xi \in (\alpha, \beta)|\rho(\xi) \geq \overline{\rho}\}$ if $\rho(\alpha) \leq \overline{\rho}$ (if it is empty, the (3.9) is valid). Analogously, set $\eta_2 = \inf\{\xi \in (\beta, R]|\rho(\xi) \leq \overline{\rho}\}$ if $\rho(\beta) > \overline{\rho}$, while $\eta_2 = \sup\{\xi \in (\alpha, \beta)|\rho(\xi) \geq \overline{\rho}\}$ if $\rho(\beta) \leq \overline{\rho}$ (if it is empty, the (3.9) is satisfied). By the choices of η_1 and η_2, we have $\rho(\eta_1) = \rho(\eta_2) = \overline{\rho}$ and

$$\int_\alpha^\beta (\rho(\xi) - \overline{\rho})d\xi \leq \int_{\eta_1}^{\eta_2} (\rho(\xi) - \overline{\rho})d\xi = -\int_{\eta_1}^{\eta_2} \xi\rho'(\xi)d\xi. \tag{3.10}$$

With the help of Lemma 3.2, one can observe that $\rho'(\eta_1) \geq 0$ and $\rho'(\eta_2) \leq 0$. Integrating the first equation in (3.2) yields

$$\begin{aligned} -\int_{\eta_1}^{\eta_2} \xi\rho'(\xi)d\xi &= \varepsilon\rho'(\eta_2) - \varepsilon\rho'(\eta_1) - \mu\overline{\rho}(f(u(\eta_2)) - f(u(\eta_1))) \\ &\leq |\overline{\rho}(f(u(\eta_2)) - f(u(\eta_1)))| \leq N. \end{aligned} \tag{3.11}$$

As a result, (3.9) holds.

Using (3.9), we deduce

$$\underline{\rho} \leq \rho(\xi) \leq \overline{\rho} + \frac{N}{|\tau - \xi|}, \qquad \text{for } \xi \in [-R, R]\backslash\{\tau\}. \tag{3.12}$$

Without loss of generality, we suppose $\rho(\tau) > \overline{\rho}$. We take $\beta \in [-R, \tau]$ satisfying $\rho(\beta) = \overline{\rho}$. For any $\xi_1 \in [\beta, \tau)$, there must be $\xi_2 \in (\tau, R]$ such that $\rho(\xi_1) = \rho(\xi_2)$. Integrating the first equation of (3.2) over (ξ_1, ξ_2) yields

$$\varepsilon\rho'(\xi_2) - \varepsilon\rho'(\xi_1) = -\int_{\xi_1}^{\xi_2} \xi\rho'(\xi)d\xi + \mu\rho(\xi_1)(f(u(\xi_2)) - f(u(\xi_1))). \tag{3.13}$$

Because of $\rho'(\xi_2) < 0$, $\rho'(\xi_1) > 0$ and $-\int_{\xi_1}^{\xi_2} \xi\rho'(\xi)d\xi = \int_{\xi_1}^{\xi_2}(\rho(\xi) - \rho(\xi_1))d\xi \geq 0$, we have

$$\varepsilon\rho'(\xi_1) \leq \mu\rho(\xi_1)(f(u(\xi_1)) - f(u(\xi_2))) \leq \rho(\xi_1)\frac{N}{\overline{\rho}}. \tag{3.14}$$

Integration of (3.14) leads to

$$\rho(\tau) \leq \rho(\xi_0)\exp^{\frac{N(\tau-\xi_0)}{\varepsilon\overline{\rho}}}, \quad \xi_0 \in [\beta, \tau]. \tag{3.15}$$

If $\tau - \beta < 1$, by choosing $\xi_0 = \beta$ in (3.15), we get $\rho(\tau) \leq \overline{\rho}\exp^{\frac{N}{\varepsilon\overline{\rho}}}$. If $\tau - \beta \geq 1$, by taking $\xi_0 = \tau - 1$ in (3.15) and considering (3.12), we arrive at

$$\rho(\tau) \leq \left(\overline{\rho} + \frac{N}{|\tau - \xi_0|}\right)\exp^{\frac{N}{\varepsilon\overline{\rho}}} = (\overline{\rho} + N)\exp^{\frac{N}{\varepsilon\overline{\rho}}}.$$

This finishes the proof. □

4 Existence of solutions to (1.1) with (1.2) when $\rho_\varepsilon(\xi)$ and $u_\varepsilon(\xi)$ are uniformly bounded

This section proves the existence of solutions to (1.1) with (1.2) when $\rho_\varepsilon(\xi)$ and $u_\varepsilon(\xi)$ are uniformly bounded. Similar to (3.12), one has

$$\underline{\rho} \le \rho_\varepsilon(\xi) \le \overline{\rho} + \frac{N_1}{|\tau_\varepsilon - \xi|}, \qquad \xi \in (-\infty, +\infty) \setminus \{\tau_\varepsilon\} \tag{4.1}$$

when $\rho_\varepsilon(\xi)$ possesses a maximum point τ_ε, where $N_1 = \bar{\rho}(f(u_-) - f(u_+))$. If $\{(\rho_\varepsilon(\xi), u_\varepsilon(\xi))|0 < \varepsilon < 1\}$ is uniformly bounded in ε, then the existence of a solution to Riemann problem (1.1) with (1.2) can be obtained by virtue of the Theorem 3.2 in [12].

Theorem 4.1 *For every fixed $\varepsilon > 0$, let $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ be a solution of (3.1) with (2.2), and assume that $\{(\rho_\varepsilon(\xi), u_\varepsilon(\xi))|0 < \varepsilon < 1\}$ is uniformly bounded variation. Then $\{(\rho_\varepsilon(\xi), u_\varepsilon(\xi))|0 < \varepsilon < 1\}$ possesses a subsequence which converges a.e. on $(-\infty, +\infty)$ to a bounded variation function $(\rho(\xi), u(\xi))$. The $(\rho(\xi), u(\xi))$ is a weak solution to (1.1) with (1.2).*

Clearly, the case (a) holds. For the case (b), $\rho_\varepsilon(\xi)$ may tend to infinity as $\varepsilon \to 0$. Let τ_ε be a maximum point of $\rho_\varepsilon(\xi)$ on $(-\infty, +\infty)$ and $\tau_\varepsilon \to \tau_0$, $|\tau_0| \le \infty$ as $\varepsilon \to 0$ (pass to a further subsequence if necessary).

Theorem 4.2 *If $|\tau_0| = \infty$, then $\{\rho_\varepsilon(\xi)|0 < \varepsilon < 1\}$ is uniformly bounded.*

Proof If $\tau_0 = +\infty$, we infer from (4.1) that

$$\underline{\rho} \le \rho_\varepsilon(\xi) \le \overline{\rho} + N_1 \tag{4.2}$$

for any $\xi \in (-\infty, h]$ and ε small, where h is any fixed real number. Here we take $h = 2$. So we calculate

$$0 \le \rho_\varepsilon'(\xi_0) = \rho_\varepsilon(2) - \rho_\varepsilon(1) \le \overline{\rho} + N_1 - \underline{\rho}, \qquad \xi_0 \in [1, 2]. \tag{4.3}$$

Since $u_+ < u_\varepsilon(\tau_\varepsilon) < u_-$ and $\tau_\varepsilon \to +\infty$ as $\varepsilon \to 0$, we can choose ε be so small that $\tau_\varepsilon - f(u_\varepsilon(\tau_\varepsilon)) \ge 1$, $(\tau_\varepsilon - \xi_0)/(\tau_\varepsilon - f(u_\varepsilon(\tau_\varepsilon))) \le 2$. Integrating the first equation of (3.1) over $(\xi_0, \tau_\varepsilon)$, we arrive at

$$\begin{aligned}(\tau_\varepsilon - f(u_\varepsilon(\tau_\varepsilon)))\rho_\varepsilon(\tau_\varepsilon) &= \rho_\varepsilon(\xi_0)(\xi_0 - f(u_\varepsilon(\xi_0))) + \int_{\xi_0}^{\tau_\varepsilon} \rho_\varepsilon(\xi) d\xi + \varepsilon\rho_\varepsilon'(\xi_0)\\ &\le (\overline{\rho} + N_1)(\xi_0 - f(u_\varepsilon(\xi_0))) + (\tau_\varepsilon - \xi_0)\overline{\rho} + 2N_1 + \overline{\rho} - \underline{\rho},\end{aligned}$$

which gives

$$\rho_\varepsilon(\tau_\varepsilon) \le 2\overline{\rho} + 1.$$

If $\tau_0 = -\infty$, the process is similar. The proof is completed.

We next consider the case $|\tau_0| < \infty$. The singularity point of (3.1) is given by $\xi = f(u_\varepsilon(\xi))$ and denoted by ξ_σ^ε. Setting

$$\sigma = \xi_\sigma = \lim_{\varepsilon \to 0} \xi_\sigma^\varepsilon = \lim_{\varepsilon \to 0} f(u_\varepsilon(\xi_\sigma^\varepsilon)), \tag{4.4}$$

then we have $f(u_+) < \xi_\sigma < f(u_-)$ in virtue of $f(u_+) \le f(u_\varepsilon(\xi)) \le f(u_-)$.

□

Theorem 4.3 *ξ_σ is given as above. Then $\{\rho_\varepsilon(\xi)|0 < \varepsilon < 1\}$ is uniformly bounded if $\tau_0 \ne \xi_\sigma$.*

Proof We only discuss the case $\tau_0 > \xi_\sigma$, the case $\tau_0 < \xi_\sigma$ can be treated similarly. Integrating the first equation in (3.1) over $(\tau_\varepsilon, \tau_0 + 1)$ gives

$$\varepsilon\rho_\varepsilon'(\tau_0 + 1) = (\tau_\varepsilon - f(u_\varepsilon(\tau_\varepsilon)))\rho_\varepsilon(\tau_\varepsilon) - (\tau_0 + 1 - f(u_\varepsilon(\tau_0 + 1)))\rho_\varepsilon(\tau_0 + 1) + \int_{\tau_\varepsilon}^{\tau_0+1} \rho_\varepsilon(\xi) d\xi. \tag{4.5}$$

By virtue of $\rho_\varepsilon'(\tau_0 + 1) \le 0$,

$$\begin{aligned}\tau_\varepsilon - f(u_\varepsilon(\tau_\varepsilon)) &= \tau_\varepsilon - \xi_\sigma^\varepsilon + f(u_\varepsilon(\xi_\sigma^\varepsilon)) - f(u_\varepsilon(\tau_\varepsilon))\\ &= (1 - f'(u_\varepsilon(\eta_\varepsilon))u_\varepsilon'(\eta_\varepsilon))(\tau_\varepsilon - \xi_\sigma^\varepsilon) > \frac{1}{2}(\tau_0 - \xi_\sigma) > 0, \quad \eta_\varepsilon \in (\xi_\sigma^\varepsilon, \tau_\varepsilon)\end{aligned}$$

and (4.1), it follows from (4.5) that

$$\rho_\varepsilon(\tau_\varepsilon) \le \frac{2}{\tau_0 - \xi_\sigma}(\tau_0 + 1 - f(u_+))(\overline{\rho} + 2N_1). \tag{4.6}$$

Thus $\rho_\varepsilon(\xi)$ is uniformly bounded in ε. The proof is completed.

Given $\eta > 0$, using (4.1), we have $\rho_\varepsilon(\xi) \le \overline{\rho} + \frac{2N_1}{\eta}$ for $\xi \in (-\infty, \xi_\sigma - \eta) \cup (\xi_\sigma + \eta, +\infty)$ if $\tau_\varepsilon \to \tau_0 = \xi_\sigma$ as $\varepsilon \to 0$. Together with the Theorems 4.2-4.3, we come to the following conclusion. □

Theorem 4.4 *When $u_- > u_+$, $\{(\rho_\varepsilon(\xi), u_\varepsilon(\xi)) | 0 < \varepsilon < 1\}$ is uniformly bounded and uniformly bounded variation over the interval $(-\infty, \xi_\sigma - \eta) \cup (\xi_\sigma + \eta, +\infty)$ for any fixed $\eta > 0$.*

Theorem 4.5 *ξ_σ is defined as above. Then for any $\eta > 0$,*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(g(u_\varepsilon(\xi)))' = 0, \quad \textit{for } |\xi - \xi_\sigma| \ge \eta,\\
&\lim_{\varepsilon\to 0} u_\varepsilon(\xi) = u_-, \qquad \textit{for } \xi \le \xi_\sigma - \eta,\\
&\lim_{\varepsilon\to 0} u_\varepsilon(\xi) = u_+, \qquad \textit{for } \xi \ge \xi_\sigma + \eta
\end{aligned} \tag{4.7}$$

uniformly in the above intervals.

Proof According to the Theorem 4.4, there exists a positive number M_1 (independent of ε) such that

$$\underline{\rho} \le \rho_\varepsilon(\xi) \le M_1, \quad \xi \in \Big(-\infty, \xi_\sigma - \frac{\eta}{2}\Big) \cup \Big(\xi_\sigma + \frac{\eta}{2}, +\infty\Big). \tag{4.8}$$

Moreover, we have

$$g\Big(u_\varepsilon\Big(\xi_\sigma - \frac{\eta}{2}\Big)\Big) - g\Big(u_\varepsilon\Big(\xi_\sigma - \frac{3\eta}{4}\Big)\Big) = \frac{\eta}{4} g'(u_\varepsilon(\xi_1))u_\varepsilon'(\xi_1), \quad \xi_1 \in \Big(\xi_\sigma - \frac{3\eta}{4}, \xi_\sigma - \frac{\eta}{2}\Big), \tag{4.9}$$

giving

$$\frac{4}{\eta}(g(u_+) - g(u_-)) \le g'(u_\varepsilon(\xi_1))u_\varepsilon'(\xi_1) < 0. \tag{4.10}$$

It follows from (3.1) that

$$\Big(-\xi + f(u) - 2\varepsilon\frac{\rho'}{\rho} - \varepsilon\frac{g''(u)}{g'(u)}u'\Big)u' = \varepsilon u''. \tag{4.11}$$

Integrating (4.11) over (ξ, ξ_1) yields

$$g'(u_\varepsilon(\xi))u_\varepsilon'(\xi) = g'(u_\varepsilon(\xi_1))u_\varepsilon'(\xi_1)\frac{\rho_\varepsilon^2(\xi_1)}{\rho_\varepsilon^2(\xi)}\exp\Big(\int_{\xi_1}^{\xi}\frac{f(u_\varepsilon(s)) - s}{\varepsilon}ds\Big). \tag{4.12}$$

Furthermore, because of $s - \xi_\sigma^\varepsilon \to s - \xi_\sigma \le \xi_1 - \xi_\sigma \le -\eta/2$ as $\varepsilon \to 0$, we conclude

$$f(u_\varepsilon(s)) - s = f(u_\varepsilon(s)) - f(u_\varepsilon(\xi_\sigma^\varepsilon)) + \xi_\sigma^\varepsilon - s = (f'(u_\varepsilon(\zeta_\varepsilon))u_\varepsilon'(\zeta_\varepsilon) - 1)(s - \xi_\sigma^\varepsilon) \ge \frac{\eta}{4},\ \zeta_\varepsilon \in (s, \xi_\sigma^\varepsilon).$$

Hence we get

$$\big|(g(u_\varepsilon(\xi)))'\big| \le \frac{4}{\eta}(g(u_-) - g(u_+))\Big(\frac{M_1}{\underline{\rho}}\Big)^2\exp\Big(\frac{\eta}{4\varepsilon}(\xi - \xi_1)\Big),\ \ \xi \le \xi_1, \tag{4.13}$$

which means that $(g(u_\varepsilon(\xi)))' \to 0$ uniformly on $\xi \le \xi_\sigma - \eta$ as $\varepsilon \to 0$.

Noticing that $\xi \le \xi_\sigma - \eta$,

$$g(u_\varepsilon(\xi)) - g(u_-) = \int_{-\infty}^{\xi}(g(u_\varepsilon(s)))'ds$$

and (4.13), we obtain $\lim\limits_{\varepsilon\to 0} g(u_\varepsilon(\xi)) = g(u_-)$ uniformly. According to the monotonicity of $g(u)$, we arrive at

$$\lim_{\varepsilon\to 0} u_\varepsilon(\xi) = u_- \tag{4.14}$$

uniformly. In a similar manner, we have $\lim\limits_{\varepsilon\to 0} u_\varepsilon(\xi) = u_+$ uniformly for $\xi \ge \xi_\sigma + \eta$. The proof is completed. □

5 Existence of solutions to (1.1) with (1.2) when $\rho_\varepsilon(\xi)$ may tend to infinity as $\tau_0 = \xi_\sigma$

In this section, we discuss the existence of solutions to (1.1) and (1.2) when $\rho_\varepsilon(\xi)$ may tend to infinity as $\tau_0 = \xi_\sigma$. It is shown that the family $\{(\rho_\varepsilon(\xi), u_\varepsilon(\xi))|0 < \varepsilon < 1\}$ of (3.1) with (2.2) could generate the Riemann solutions to (1.1) and (1.2).

Theorem 5.1 *Assume that $(\rho_\varepsilon(\xi), u_\varepsilon(\xi))$ is a solution to (3.1) with (2.2), and $\tau_\varepsilon \to \tau_0 = \xi_\sigma$ as $\varepsilon \to 0$. Then the sequence $\{(\rho_\varepsilon(\xi), u_\varepsilon(\xi))|0 < \varepsilon < 1\}$ possesses a subsequence that converges a.e. on $(-\infty, +\infty)$ to a pair of functions $(\rho(\xi), u(\xi))$.*

Proof On the basis of the Theorem 4.4, $\rho_\varepsilon(\xi)$ is uniformly bounded in ε on the finite domain $I_2 = [-2, \tau_0 - \frac{1}{2}] \cup [\tau_0 + \frac{1}{2}, 2]$ (without loss of generality, assume that $|\tau_0| < 1$). By the Helly's theorem, there exists a convergent subsequence of $\{\rho_\varepsilon(\xi)\}$ (still denoted by original one). Analogously, we can pick out a convergent subsequence of $\{\rho_\varepsilon(\xi)\}$ on $I_3 = [-3, \tau_0 - \frac{1}{3}] \cup [\tau_0 + \frac{1}{3}, 3]$. Repeat this process on each $I_n = [-n, \tau_0 - \frac{1}{n}] \cup [\tau_0 + \frac{1}{n}, n]$, $n = 4, 5, \ldots\ldots$ Finally, take out the diagonal element at each enumerated sequence. The sequence of diagonal elements is convergent at each $\xi \neq \tau_0$ to a function $\rho(\xi)$ defined on $(-\infty, \tau_0) \cup (\tau_0, +\infty)$. For $u_\varepsilon(\xi)$, the result is obvious because of $u'_\varepsilon(\xi) < 0$. The proof is completed.

□

Theorem 5.2 *Assume that $\tau_0 = \xi_\sigma$ and $u_- > u_+$. Then for any $k > 0$,*

$$\rho(\xi) = H_\rho(\xi - \sigma) + \omega_0 \delta(\xi - \sigma), \tag{5.1}$$

where

$$H_\rho(x) = \begin{cases} \rho_-, \ x < 0, \\ \rho_+, \ x > 0, \end{cases} \tag{5.2}$$

$\delta(x)$ is Dirac function supported at $x = 0$, $\omega_0 = -\sigma[\rho] + [\rho f(u)]$ denotes the strength of $\delta(x)$, where σ is given by (4.4).

Proof For $-\infty < \xi_2 < \xi_3 < \sigma - k$, we deduce from the first equation of (3.1) that

$$\varepsilon \int_{\xi_2}^{\xi_3} \rho_\varepsilon \psi'' d\xi = \int_{\xi_2}^{\xi_3} \rho_\varepsilon(\psi + \xi\psi' - f(u_\varepsilon)\psi') d\xi \tag{5.3}$$

holds for each $\psi \in C_0^\infty(\xi_2, \xi_3)$. It follows from (4.1) that

$$\underline{\rho} \le \rho_\varepsilon(\xi) \le \overline{\rho} + \frac{N_1}{k}, \qquad \xi \in [\xi_2, \xi_3]. \tag{5.4}$$

According to (4.7), (5.4) and Lebesgue dominated convergence theorem, we take $\varepsilon \to 0$ in (5.3) to get

$$\int_{\xi_2}^{\xi_3} \rho(\xi)(\psi(\xi)(\xi - f(u_-)))' d\xi = 0, \quad \text{for each } \psi(\xi) \in C_0^\infty(\xi_2, \xi_3), \tag{5.5}$$

which means that $\rho(\xi) = \rho_-$ in virtue of $\xi - f(u_-) \neq 0$ and $\rho(-\infty) = \rho_-$. Analogously, we have $\rho(\xi) = \rho_+$ for $\xi > \sigma + k$. Theorem 4.5 tells us that $\rho(\xi)$ and $u(\xi)$ own the same discontinuity point $\xi = \xi_\sigma$ on $(-\infty, +\infty)$.

Nevertheless, since $\rho_-\rho_+(f(u_-) - f(u_+))(g(u_-) - g(u_+)) \neq 0$ at $\xi = \xi_\sigma$, the Rankine-Hugoniot condition is no longer valid. For this reason, the limiting behavior of $\rho_\varepsilon(\xi)$ as $\varepsilon \to 0$ in the neighborhood of $\xi = \sigma$ will be studied.

We take $\psi(\xi) \in C_0^\infty(\xi_2, \xi_3)$ $(\xi_2 < \sigma < \xi_3)$ with the property $\psi(\xi) = \psi(\sigma)$ for ξ in a neighborhood $\widetilde{\Omega}$ of σ (ψ is called as a sloping test function). It follows from (5.3) that

$$\lim_{\varepsilon \to 0} \int_{\xi_2}^{\xi_3} \rho_\varepsilon(\psi + \xi\psi' - f(u_\varepsilon)\psi') d\xi = 0. \tag{5.6}$$

For any α_1 and α_2 near σ with property $\xi_2 < \alpha_1 < \sigma < \alpha_2 < \xi_3$, according to the Theorem 4.5, we conclude

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\xi_2}^{\xi_3} \rho_\varepsilon(\xi - f(u_\varepsilon))\psi' d\xi \\
&= \lim_{\varepsilon\to 0}\int_{\xi_2}^{\alpha_1} \rho_\varepsilon(\xi - f(u_\varepsilon))\psi' d\xi + \lim_{\varepsilon\to 0}\int_{\alpha_2}^{\xi_3} \rho_\varepsilon(\xi - f(u_\varepsilon))\psi' d\xi \\
&= \int_{\xi_2}^{\alpha_1} \rho_-(\xi - f(u_-))\psi' d\xi + \int_{\alpha_2}^{\xi_3} \rho_+(\xi - f(u_+))\psi' d\xi \\
&= (\rho_+ f(u_+) - \rho_- f(u_-) - \rho_+\alpha_2 + \rho_-\alpha_1)\psi(\sigma) - \int_{\xi_2}^{\alpha_1} \rho_-\psi(\xi)d\xi - \int_{\alpha_2}^{\xi_3} \rho_+\psi(\xi)d\xi.
\end{aligned}$$

Sending $\alpha_1 \to \sigma^-$ and $\alpha_2 \to \sigma^+$ gives

$$\lim_{\varepsilon\to 0}\int_{\xi_2}^{\xi_3} \rho_\varepsilon(\xi - f(u_\varepsilon))\psi' d\xi = \big(\sigma[\rho] - [\rho f(u)]\big)\psi(\sigma) - \int_{\xi_2}^{\xi_3} H_\rho(\xi - \sigma)\psi(\xi)d\xi. \tag{5.7}$$

Together with (5.6)-(5.7), we deduce

$$\lim_{\varepsilon\to 0}\int_{\xi_2}^{\xi_3} (\rho_\varepsilon - H_\rho(\xi - \sigma))\psi(\xi)d\xi = \big(-\sigma[\rho] + [\rho f(u)]\big)\psi(\sigma). \tag{5.8}$$

With the approximation process as in [10], one can easily check that (5.8) holds for any $\psi \in C_0^\infty(\xi_2, \xi_3)$. As a result, ρ_ε converges weakly star to the function $H_\rho(\xi - \sigma)$ plus a Dirac delta function with strength $\omega_0 = -\sigma[\rho] + [\rho f(u)]$. □

Theorem 5.3 *Suppose that $\tau_0 = \xi_\sigma$ and $u_- > u_+$. Then for any $k > 0$,*

$$\rho(\xi)g(u(\xi)) = H_1(\xi - \sigma) + \omega_0 g(u_\delta)\delta(\xi - \sigma), \tag{5.9}$$

where

$$H_1(x) = \begin{cases} \rho_- g(u_-), \ x < 0, \\ \rho_+ g(u_+), \ x > 0, \end{cases} \tag{5.10}$$

$\omega_0 g(u_\delta) = -\sigma[\rho g(u)] + [\rho f(u) g(u)]$, $u_\delta = \lim_{\varepsilon\to 0} u_\varepsilon(\xi_\sigma^\varepsilon) = u(\sigma)$.

Proof It follows from Theorem 4.5 and Theorem 5.2 that

$$\rho g(u) = \rho_- g(u_-), \ \text{ for } \ \xi < \sigma - k, \qquad \rho g(u) = \rho_+ g(u_+), \ \text{ for } \ \xi > \sigma + k. \tag{5.11}$$

We take a function $\psi(\xi) \in C_0^\infty(\xi_2, \xi_3)$ ($\xi_2 < \sigma < \xi_3$) satisfying $\psi(\xi) = \psi(\sigma)$ for ξ in a neighborhood $\widetilde{\Omega}$ of σ. From the second equation of (3.1), we conclude

$$\int_{\xi_2}^{\xi_3} \rho_\varepsilon g(u_\varepsilon)(\psi + \xi\psi' - f(u_\varepsilon)\psi')d\xi = \varepsilon \int_{\xi_2}^{\xi_3} \rho_\varepsilon g(u_\varepsilon)\psi'' d\xi, \tag{5.12}$$

which gives

$$\lim_{\varepsilon\to 0}\int_{\xi_2}^{\xi_3} \rho_\varepsilon g(u_\varepsilon)(\psi + \xi\psi' - f(u_\varepsilon)\psi')d\xi = 0. \tag{5.13}$$

For any α_1 and α_2 near σ satisfying $\xi_2 < \alpha_1 < \sigma < \alpha_2 < \xi_3$, on the basis of Theorem 4.5, we deduce

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\xi_2}^{\xi_3} \rho_\varepsilon g(u_\varepsilon)(\xi\psi' - f(u_\varepsilon)\psi')d\xi \\
&= \lim_{\varepsilon\to 0}\int_{\xi_2}^{\alpha_1} \rho_\varepsilon g(u_\varepsilon)(\xi\psi' - f(u_\varepsilon)\psi')d\xi + \lim_{\varepsilon\to 0}\int_{\alpha_2}^{\xi_3} \rho_\varepsilon g(u_\varepsilon)(\xi\psi' - f(u_\varepsilon)\psi')d\xi \\
&= \int_{\xi_2}^{\alpha_1} \rho_- g(u_-)(\xi\psi' - f(u_-)\psi')d\xi + \int_{\alpha_2}^{\xi_3} \rho_+ g(u_+)(\xi\psi' - f(u_+)\psi')d\xi \\
&= (\rho_+ g(u_+) f(u_+) - \rho_- g(u_-) f(u_-) - \rho_+ g(u_+)\alpha_2 + \rho_- g(u_-)\alpha_1)\psi(\sigma) - \int_{\xi_2}^{\alpha_1} \rho_- g(u_-)\psi(\xi)d\xi \\
&- \int_{\alpha_2}^{\xi_3} \rho_+ g(u_+)\psi(\xi)d\xi.
\end{aligned}$$

Letting $\alpha_1 \to \sigma^-$, $\alpha_2 \to \sigma^+$ and considering (5.13), we have

$$\lim_{\varepsilon\to 0}\int_{\xi_2}^{\xi_3}(\rho_\varepsilon g(u_\varepsilon) - H_1(\xi-\sigma))\psi(\xi)d\xi = \big(-\sigma[\rho g(u)] + [\rho f(u)g(u)]\big)\psi(\sigma). \tag{5.14}$$

Further, one could conclude that (5.14) holds for all $\psi \in C_0^\infty(\xi_2, \xi_3)$ by the approximation process. Hence $\rho_\varepsilon g(u_\varepsilon)$ converges weakly star to the function $H_1(\xi-\sigma)$ given by (5.10) plus a Dirac delta function with strength $\big(-\sigma[\rho g(u)] + [\rho f(u)g(u)]\big)$.

Besides, if we take the test function as $\psi/(\tilde{g}(u_\varepsilon)+\mu)$ in (5.12), where $\tilde{g}(u_\varepsilon)$ is a modified function satisfying $g(u_\varepsilon(\sigma))$ inside $\widetilde{\Omega}$ and $g(u_\varepsilon)$ outside $\widetilde{\Omega}$, then we send $\mu \to 0$ to obtain

$$\lim_{\varepsilon\to 0}\int_{\xi_2}^{\xi_3}(\rho_\varepsilon - H_\rho(\xi-\sigma))\psi(\xi)d\xi \cdot g(u_\delta) = \big(-\sigma[\rho g(u)] + [\rho f(u)g(u)]\big)\psi(\sigma). \tag{5.15}$$

Hence (5.9) can be defined. The proof is finished. □

Theorem 5.4 *Suppose that $\tau_0 = \xi_\sigma$ and $u_- > u_+$. Then $u = H_u(\xi-\sigma)$ and $\sigma = f(u_\delta)$ is required such that (1.1) holds in the distributional sense, where*

$$H_u(x) = \begin{cases} u_-, \ x < 0, \\ u_+, \ x > 0. \end{cases} \tag{5.16}$$

Proof The Theorem 4.5 gives $u = H_u(\xi-\sigma)$. Let $\xi_2 < \sigma < \xi_3$, $\psi \in C_0^\infty(\xi_2, \xi_3)$. Then it follows from the first equation of (2.1) that

$$\int_{\xi_2}^{\xi_3}\rho(\psi + \xi\psi' - f(u)\psi')d\xi = 0,$$

i.e.,

$$\int_{\xi_2}^{\xi_3}\big(H_\rho(\xi-\sigma) + \big(-\sigma[\rho] + [\rho f(u)]\big)\delta(\xi-\sigma)\big)(\psi + \xi\psi' - f(u)\psi')d\xi = 0. \tag{5.17}$$

The left side of (5.17) equals to

$$\begin{aligned}
&\int_{\xi_2}^{\sigma} H_\rho(\xi-\sigma)(\psi+\xi\psi'-f(u)\psi')d\xi + \int_{\sigma}^{\xi_3} H_\rho(\xi-\sigma)(\psi+\xi\psi'-f(u)\psi')d\xi \\
&+\int_{\xi_2}^{\xi_3}\big(\big(-\sigma[\rho]+[\rho f(u)]\big)\delta(\xi-\sigma)\big)\psi d\xi + \int_{\xi_2}^{\xi_3}\big(\big(-\sigma[\rho]+[\rho f(u)]\big)\delta(\xi-\sigma)\big)(\xi\psi'-f(u)\psi')d\xi \\
&= \rho_-\psi(\xi)(\xi - f(u_-)\big|_{\xi_2}^{\sigma} + \rho_+\psi(\xi)(\xi-f(u_+)\big|_{\sigma}^{\xi_3} + \big(-\sigma[\rho]+[\rho f(u)]\big)\psi(\sigma) \\
&+\big(-\sigma[\rho]+[\rho f(u)]\big)(\sigma - f(u_\delta))\psi'(\sigma) \\
&= \big(-\sigma[\rho]+[\rho f(u)]\big)(\sigma - f(u_\delta))\psi'(\sigma).
\end{aligned}$$

Combining with (5.17), we get

$$\big(-\sigma[\rho]+[\rho f(u)]\big)(\sigma - f(u_\delta))\psi'(\sigma) = 0. \tag{5.18}$$

Noticing that $u_+ < u_\delta < u_-$, it follows from (4.4) that

$$f(u_+) < \sigma < f(u_-). \tag{5.19}$$

Because of the arbitrariness of ψ and $-\sigma[\rho] + [\rho f(u)] \neq 0$, so we have $\sigma = f(u_\delta)$. The proof is finished.

It follows from Theorems 5.2-5.4 that

$$\begin{cases} \sigma = f(u_\delta), \\ \omega_0 = -\sigma[\rho] + [\rho f(u)], \\ \omega_0 g(u_\delta) = -\sigma[\rho g(u)] + [\rho f(u)g(u)]. \end{cases} \tag{5.20}$$

Under the entropy condition (5.19), it has been proved that there exists a unique solution $(\sigma, \omega_0, u_\delta)$.

Thus the following theorem holds. □

Theorem 5.5 *Assume $\rho \geq 0$, $f'(u) > 0$ and $g'(u) > 0$. When $u_- > u_+$, let $(\rho_\varepsilon, u_\varepsilon)(x,t)$ be the solution of (1.3) with (1.2). Then the limit functions $(\rho, u)(x,t)$ of $(\rho_\varepsilon, u_\varepsilon)(x,t)$ exist in the distributional sense, and $(\rho, u)(x,t)$ solves (1.1) with (1.2). The solution $(\rho, u)(x,t)$ can be given as*

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_-), & x < \sigma t, \\ (\omega(t)\delta(x - \sigma t), u_\delta), & x = \sigma t, \\ (\rho_+, u_+), & x > \sigma t, \end{cases} \tag{5.21}$$

where $\omega(t) = \omega_0 t$, ω_0, σ, u_δ can be uniquely determined by (5.20) and entropy condition (5.19).

Remark 5.6 For the case (i) $\rho \geq 0$, $f'(u) > 0$, $g'(u) < 0$, we only need to change (4.10) and (4.13), the conclusion similar to the Theorem 5.5 can be obtained. For the case (ii) $\rho \geq 0$, $f'(u) < 0$, $g'(u) > 0$ and the case (iii) $\rho \geq 0$, $f'(u) < 0$, $g'(u) < 0$, the $u(\xi)$ in the Lemma 3.2 satisfies $u'(\xi) > 0$ in $[-R, R]$, and $\rho(\xi)$ in the Lemma 3.2 satisfies (a) or (b). Then the conclusion similar to the Theorem 5.5 can be obtained by making several minor amendments to Sects. 4-5.

For the cases (iv)-(vii) (see Remark 2.2), the initial data $\rho_\pm < 0$ and $u_\pm$ are all given constants. For these cases, (3.3) should be $\rho(\pm R) = \overline{\rho} + \mu(\rho_\pm - \overline{\rho})$, $u(\pm R) = \mu u_\pm$, where $\bar{\rho} = \max\{\rho_-, \rho_+\}$. Following the method as in [20], by the same arguments as in Sects. 3-5 with only few modifications, the conclusion similar to the Theorem 5.5 for the cases (iv)-(vii) can be derived.

Acknowledgements The author would like to thank the anonymous referees for his or her useful comments on the original manuscript. This paper is supported by the Ph.D. Foundation of Henan University of Engineering (no. D2022028) and the Science and Technology Research Program of Education Department of Henan Province (no. 24A110002).

Data availability No data were used to support this study.

Declarations

Conflict of interest The author has no conflicts to disclose.

References

1. H. Yang, Riemann problems for a class of coupled hyperbolic systems of conservation laws, J. Differ. Equ., 159, (1999), 447-484.
2. H. Yang, W. Sun, The Riemann problem with delta initial data for a class of coupled hyperbolic systems of conservation laws, Nonlinear Anal. TMA., 67, (2007), 3041-3049.
3. D. Mitrovic, M. Nedeljkov, Delta-shock waves as a limit of shock waves, J. Hyperbolic Diff. Eq., 4, (2007), 629-653.
4. Li, S, Stability of Riemann solutions to a class of non-strictly hyperbolic systems of conservation laws, Quaest. Math., (2023), 1-17.
5. V. G. Danilov, V. M. Shelkovich, Dynamics of propagation and interaction of δ-shock waves in conservation law systems, J. Differ. Equ., 211, (2005), 333-381.
6. V. G. Danilov, D. Mitrovic, Delta shock wave formation in the case of triangular hyperbolic system of conservation laws, J. Differ. Equ., 245, (2008), 3704-3734.
7. Le Floch, An existence and uniqueness result for two nonstrictly hyperbolic systems, Nonlinear Evolution Equations that Change Type, IMA Vol. Math. Appl., vol. 27, Springer-Verlag, New York/Berlin (1990) 126-138.
8. W. Sheng, T. Zhang, The Riemann problem for transportation equation in gas dynamics, Mem. Amer. Math. Soc., 137, (1999), 1-77.
9. D. Tan, T. Zhang, Two-dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws I. Four-J cases, II. Initial data involving some rarefaction waves, J. Differ. Equ. 111, (1994), 203-282.
10. D. Tan, T. Zhang, Y. Zheng, Delta shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws, J. Differ. Equ., 112, (1994), 1-32.
11. S. Li, Riemann problem for a class of non-strictly hyperbolic systems of conservation laws, Acta Appl. Math., 177, (2022), 1-19.
12. M. Dafermos, Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method, Arch. Ration. Mech. Anal., 52, (1973), 1-9.
13. M. Dafermos, R. J. Diperna, The Riemann problem for certain classes of hyperbolic systems of conservation laws, J. Differ. Equ., 20, (1976), 90-114.
14. M. Slemrod, A. E. Tzavaras, A limiting viscosity approach for the Riemann problem in isentropic gas dynamics, Indiana Univ. Math. J., 38, (1989), 1047-1074.
15. A. E. Tzavaras, Wave interactions and variation estimates for self-similar zero-viscosity limits in systems of conservation laws, Arch. Ration. Mech. Anal., 135, (1996), 1-60.

16. B. Chhatria, A. Sen, T. Raja Sekhar, (2022) Self-similar viscosity approach to the Riemann problem for a strictly hyperbolic system of conservation laws, Math. Method. Appl. Sci., 46, , 7265-7284,
17. A. Sen, T. Raja Sekhar, (2019) Delta shock wave as self-similar viscosity limit for a strictly hyperbolic system of conservation laws, J. Math. Phys., 60, , 051510.
18. M. Sun, Delta shock waves for the chromatography equations as self-similar viscosity limits, Q. Appl. Math., 69, (2011), 425-443.
19. J. Hu, One-dimensional Riemann problem for equations of constant pressure fluid dynamics with measure solutions by the viscosity method, Acta Appl. Math., 55, (1999), 209-229.
20. J. Hu, A limiting viscosity approach to Riemann solutions containing delta-shock waves for nonstrictly hyperbolic conservation laws, Q. Appl. Math., 55, (1997), 361-373.
21. J. Li, H. Yang, Delta-shocks as limits of vanishing viscosity for multidimensional zero-pressure gas dynamics, Quart. Appl. Math., 59, (2001), 315-342.
22. H. Yang, Y. Zhang, Delta shock waves with Dirac delta function in both components for systems of conservation laws, J. Differ. Equ., 257, (2014), 4369-4402.
23. S. Li, Delta shock wave as limits of vanishing viscosity for zero-pressure gas dynamics with energy conservation law, ZAMM-J. Appl. Math. Mech., 103, (2022), e202100377.
24. S. Li, Delta-shock for a class of systems of conservation laws of the Keyfitz-Kranzer type, Math. Nachr., 297, (2024), 1042-1061.