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ORIGINAL RESEARCH

Several formulas and identities related to the Pochhammer k-symbol and application to a case of the Fuss–Catalan–Qi numbers

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Abstract We give some formulas and series identities involving the Pochhammer k-symbol. In particular, we give generalizations of formulas and identities about the Catalan numbers and the Catalan–Qi numbers given in [F. Qi and B.-N. Guo, *Integral representations of the Catalan numbers and their applications*, Mathematics **5** (2017), no. 3, Article 40, available online at https://doi.org/10.3390/math5030040.] and [W. Chammam, *Several formulas and identities related to Catalan–Qi and q-Catalan–Qi numbers*, Indian J. Pure Appl Math., **50**, (2019), 1039–1048; Available online at https://doi.org/10.1007/s13226-019-0372-1].

Keywords Formula · Pochhammer k-symbol · Catalan numbers · Catalan–Qi numbers · Fuss–Catalan–Qi numbers · Hypergeometric series

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1 Introduction

The factorial function is denoted and defined by

$$(x)_n = \begin{cases} x(x+1)(x+2)...(x+n-1) & , n \geq 1, \ x \neq 0, \\ 1 & , n = 0. \end{cases} \tag{1.1}$$

The function $(x)_n$ defined in relation (Eq. 1.1) is also known as the Pochhammer symbol. The properties of this symbol and different generalizations have been studied for a long time. These properties are important in many mathematical areas, such as combinatorial theory, special functions, probability and statistics. For example, several authors have focused on the approximation theory and computation using the identities satisfied by these symbols. Here, we can cite Ablinge [1], Bagdasaryan [2], Srivastava et al. [3], and others. Also, we can refer

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to recent work of Kouba [4] who has proven several identities involving the Pochhammer symbol using the combinatorial identity of Chaundy–Bullard.

In 2019, Qi, Shi and Cerone, in defining the Fuss–Catalan–Qi function, carried out important analytical studies of this subject (integral representations and their applications, convexity, formulas, identities, inequalities, and the like) [5], then, they introduced the numbers known as the Fuss–Catalan–Qi numbers as a generalization of the Catalan numbers, Fuss–Catalan numbers, and Catalan–Qi numbers, which are very important in combinatorics. The outline of this paper is as follows. In Sect. 2, we introduce the basic background and notation to be used throughout the paper and we present our main results. Indeed, we give a generalization of Theorems 22 and 23 of [6] and Theorem 3 of [7]. In Sect. 3, we give new identities for the case $p = q$ of the Fuss–Catalan–Qi numbers. Finally, we present some relations between the case $p = q$ of the Fuss–Catalan–Qi numbers and the generalized hypergeometric series.

2 New identities for series involving the Pochhammer k-symbol

Diaz and Pariguan [8, p. 180] introduced the Pochhammer k-symbol as

$$(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k), \quad n, k > 0. \tag{2.1}$$

It is clear that

$$(x)_{n,1} = (x)_n = \begin{cases} x(x+1)(x+2)...(x+n-1) & , n \geq 1, \\ 1 & , n = 0 \end{cases}$$

and

$$(-x)_n = (-1)^n (x-n+1)_n.$$

The relation between the Pochhammer symbol and the classical Euler gamma function is

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad n \geq 0, \tag{2.2}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathscr{R}(z) > 0.$$

The relation (Eq. 2.2) leads us to write the ratio between two Pochhammer symbol as

$$\frac{(x)_n}{(y)_n} = \frac{B(y, n+x)}{B(x, n+y)}, \ n \geq 0 \tag{2.3}$$

and

$$\frac{(x)_n}{(y)_n} = \frac{B(n+x, y-x)}{B(x, y-x)}, \ n \geq 0, \tag{2.4}$$

where $B(x, y)$ is the classical beta function defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt, \tag{2.5}$$

for $\mathscr{R}(x), \mathscr{R}(y) > 0$.

It is easy to see that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(z+w)}, \quad \mathscr{R}(x), \mathscr{R}(y) > 0. \tag{2.6}$$

It is well known that the Wallis ratio is defined by

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)}{\Gamma(n+1)},\ n \geq 0.$$

Hence, it is easy to see that

$$W_n = \frac{\left(\frac{1}{2}\right)_n}{(1)_n},\ n \geq 0. \tag{2.7}$$

The Wallis ratio has been studied and applied by many mathematicians, see [9,10], for example, and plenty of literature therein.

Now we give an important recurrence relation for the Pochhammer p-symbol. In the following, we write $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Lemma 1 *If a and b are complex numbers such that $\mathscr{R}(a), \mathscr{R}(b) > 0$ and $p \in \mathbb{N}^*$, we have the following recurrence relation:*

$$\begin{cases} \prod_{k=0}^{p-1} \frac{(k+a)_{0,p}}{(k+b)_{0,p}} = 1, \\ (np+a)_p \prod_{k=0}^{p-1} \frac{(k+a)_{n,p}}{(k+b)_{n,p}} = (np+b)_p \prod_{k=0}^{p-1} \frac{(k+a)_{n+1,p}}{(k+p)_{n+1,p}},\ n \geq 0. \end{cases} \tag{2.8}$$

Proof If a and b are complex numbers such that $\mathscr{R}(a), \mathscr{R}(b) > 0$, $p \in \mathbb{N}^*$ and $n \geq 0$, using

$$(a)_{n+1,p} = (a+np)(a)_{n,p},$$

we have

$$\begin{aligned} (np+b)_p \prod_{k=0}^{p-1} \frac{(k+a)_{n+1,p}}{(k+b)_{n+1,p}} &= (np+b)_p \prod_{k=0}^{p-1} \frac{(k+np+a)(k+a)_{n,p}}{(k+np+b)(k+b)_{n,p}} \\ &= (np+b)_p \prod_{k=0}^{p-1} \frac{(k+np+a)}{(k+np+b)} \prod_{k=0}^{p-1} \frac{(k+a)_{n,p}}{(k+b)_{n,p}} \\ &= (np+b)_p \frac{(np+a)_p}{(np+b)_p} \prod_{k=0}^{p-1} \frac{(k+a)_{n,p}}{(k+b)_{n,p}} \\ &= (np+a)_p \prod_{k=0}^{p-1} \frac{(k+a)_{n,p}}{(k+b)_{n,p}}. \end{aligned}$$

The required proof is complete. □

Theorem 1 *If a and b are complex numbers such that $\mathscr{R}(a) > 0$, $\mathscr{R}(b) > p \in \mathbb{N}^*$, $\mathscr{R}(b) \neq \mathscr{R}(a+p)$ and $n \geq 0$, then*

$$\sum_{\nu=0}^{n} \prod_{k=0}^{p-1} D(a, b, p; \nu) \frac{(k+a)_{\nu,p}}{(k+b)_{\nu,p}} = (a)_p + D(a, b, p; 0) - (np+a)_p \prod_{k=0}^{p-1} \frac{(k+a)_{n,p}}{(k+b)_{n,p}} \tag{2.9}$$

and

$$\sum_{\nu=0}^{n} \prod_{k=0}^{p-1} D(a, b, p; \nu) \frac{(k+a)_{\nu,p}}{(k+b)_{\nu,p}} = (a)_p + D(a, b, p; 0) - (np+b)_p \prod_{k=0}^{p-1} \frac{(k+a)_{n+1,p}}{(k+b)_{n+1,p}}, \tag{2.10}$$

with

$$D(a, b, p; n) = (p[n-1]+b)_p - (np+a)_p. \tag{2.11}$$

Proof For $n = 0$, it is not difficult to verify Eqs. (2.9) and (2.10).

Assume that Eqs. (2.9) and (2.10) are true for some $n > 0$. Then, by induction, using the recurrence relation (2.8), we have

$$\begin{aligned}\sum_{\nu=0}^{n+1}\prod_{k=0}^{p-1} D(a,b,p;\nu)\frac{(k+a)_{\nu,p}}{(k+b)_{\nu,p}} &= \sum_{\nu=0}^{n}\prod_{k=0}^{p-1} D(a,b,p;\nu)\frac{(k+a)_{\nu,p}}{(k+b)_{\nu,p}}\\ &+ D(a,b,p;n+1)\prod_{k=0}^{p-1}\frac{(k+a)_{n+1,p}}{(k+b)_{n+1,p}}\\ &= (a)_p + D(a,b,p;0) - (np+a)_p\prod_{k=0}^{p-1}\frac{(k+a)_{n,p}}{(k+b)_{n,p}}\\ &\quad +\Big[(np+b)_p - ((n+1)p+a)_p\Big]\prod_{k=0}^{p-1}\frac{(k+a)_{n+1,p}}{(k+b)_{n+1,p}}\\ &= (a)_p + D(a,b,p;0) - (np+a)_p\prod_{k=0}^{p-1}\frac{(k+a)_{n,p}}{(k+b)_{n,p}}\\ &\quad +(np+b)_p\prod_{k=0}^{p-1}\frac{(k+a)_{n+1,p}}{(k+b)_{n+1,p}}\\ &\quad -((n+1)p+a)_p\prod_{k=0}^{p-1}\frac{(k+a)_{n+1,p}}{(k+b)_{n+1,p}}\\ &= (a)_p + D(a,b,p;0) - ((n+1)p+a)_p\prod_{k=0}^{p-1}\frac{(k+a)_{n+1,p}}{(k+b)_{n+1,p}}.\end{aligned}$$

The proof of Eq. (2.10) is similar to the proof of Eq. (2.9). The proof of the theorem is complete. □

Corollary 1 *If a and b are complex numbers such that $\mathscr{R}(b) > \mathscr{R}(a+p) > p \in \mathbb{N}^*$ and $n \geq 0$, then*

$$\sum_{\nu=0}^{n}\prod_{k=0}^{p-1} D(a,b,p;\nu)\frac{(k+a)_{\nu,p}}{(k+b)_{\nu,p}} = (b-p)_p\left[1 - \frac{B(b-a-p,a+np+p)}{B(a,b-a-p)}\right]. \tag{2.12}$$

Consequently,

$$\sum_{n=0}^{\infty}\prod_{k=0}^{p-1} D(a,b,p;n)\frac{(k+a)_{n,p}}{(k+b)_{n,p}} = (b-p)_p. \tag{2.13}$$

Proof Using Theorem 1 and Eq. (2.6), we obtain the result. □

Example 1 For $a = \frac{1}{2}$, $b = 2$ and $p = 1$, we find the simple classical result for the Wallis ratio,

$$\sum_{\nu=0}^{n}\frac{W_\nu}{\nu+1} = 2 - \frac{(2n+1)}{n+1}W_n,\ \ n \geq 0. \tag{2.14}$$

Example 2 For $a = \frac{1}{2}$, $b = 2$ and $p = 1$, we find the classical results for the Catalan numbers [6],

$$\sum_{\nu=0}^{n}\frac{C_\nu}{4^\nu} = 2 - (2n+1)\frac{C_n}{4^n},\ \ n \geq 0 \tag{2.15}$$

and

$$\sum_{n=0}^{\infty}\frac{C_n}{4^n} = 2. \tag{2.16}$$

Example 3 For $a = p = 2$ and $b = 5$, we obtain

$$\sum_{\nu=0}^{n} \frac{1}{(\nu+1)(\nu+2)} = 1 - \frac{1}{n+2}, \quad n \geq 0. \tag{2.17}$$

Consequently,

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1. \tag{2.18}$$

Example 4 For $a = p = 3$ and $b = 4$, we have the classical result

$$\sum_{\nu=0}^{n} (\nu+1) = -\frac{1}{54}\left[3! - \frac{(3n+3)_3}{n+1}\right] = \frac{(n+1)(n+2)}{2}, \quad n \geq 0. \tag{2.19}$$

Example 5 In general, for $a = p$, $b = p + 1$ with $p \in \mathbb{N}\backslash\{0, 1\}$, we have

$$\sum_{\nu=0}^{n} \frac{D(p, p+1, p; \nu)}{\nu+1} = p! - \frac{(np+p)_p}{n+1} = p! - p(np+p+1)_{p-1}, \quad n \geq 0. \tag{2.20}$$

As $n \to \infty$, the sums of these series fail to have a limit.

Example 6 If a and b are complex numbers such that $\mathscr{R}(b) > \mathscr{R}(a+1) > 1$ and $p = 1$, then

$$\sum_{\nu=0}^{n} \frac{(a)_\nu}{(b)_\nu} = \frac{1}{b-a-1}\left[b - 1 - \frac{(a)_{n+1}}{(b)_n}\right], \quad n \geq 0 \tag{2.21}$$

and

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} = \frac{b-1}{b-a-1}. \tag{2.22}$$

3 Sums of finite and infinite series of certain cases of Fuss–Catalan–Qi numbers

In combinatorics, the Catalan numbers are a sequence of natural numbers that arise in various counting problems, often involving objects that are recursively defined. They are named after the Belgian mathematician Eug ne Charles Catalan. The nth Catalan number can be expressed directly in terms of binomial coefficients by

$$C_n = \frac{(2n)!}{n!(n+1)!} = \frac{4^n \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+2)}, \quad n \geq 0.$$

The first Catalan numbers for $0 \leq n \leq 11$ are

$$1, \quad 1, \quad 2, \quad 5, \quad 14, \quad 42, \quad 132, \quad 429, \quad 1430, \quad 4862, \quad 16796, \quad 58786.$$

Sequence A000108 in the OEIS. One finds in the literature many generalizations of the Catalan numbers, such as the q-Catalan numbers [11,12], super ballot numbers [13], Fuss–Catalan numbers [14–16]... .

Since 2014, Qi and his co-authors, in defining the Catalan–Qi numbers as a generalization of the Catalan numbers, have carried out important analytical studies of these numbers [6,10,16–27].

The nth Catalan–Qi number can be expressed directly in terms of the Pochhammer symbol by

$$C(a, b; n) = \left(\frac{b}{a}\right)^n \frac{(a)_n}{(b)_n}, \quad n \geq 0, \tag{3.1}$$

where a and b are complex numbers such that $\mathscr{R}(a), \mathscr{R}(b) > 0$.

In particular, we have

$$C\Big(\frac{1}{2}, 2; n\Big) = C_n, \quad n \geq 0.$$

In 2019 [5], Qi, Shi and Cerone introduced a unified generalization of Catalan numbers, Fuss numbers, Fuss–Catalan numbers and Catalan–Qi numbers, and discovered some analytical properties of the unified generalization, known as the Fuss–Catalan–Qi numbers, defined by

$$Q(a, b; p, q; n) = \frac{\Gamma(b)}{\Gamma(a)}\Big(\frac{b}{a}\Big)^{(q-p+1)n}[\Gamma(n+1)]^{q-p}\frac{\Gamma(pn+a)}{\Gamma(qn+b)}, \quad n \geq 0, \tag{3.2}$$

where $\mathscr{R}(a), \mathscr{R}(b) > 0$ and $p, q \in \mathbb{N}^*$. In particular,

$$Q\Big(\frac{1}{2}, 2; 1, 1; n\Big) = Q(1, 2; 2, 1; n) = C_n,$$

$$Q(a, b; 1, 1; n) = C(a, b; n).$$

Moreover,

$$Q(b, a; p, q; n) = \Big(\frac{a}{b}\Big)^{2(q-p)n}\frac{1}{Q(a, b; q, p; n)}$$

and, when $p = q$ or $a = b$,

$$Q(a, b; q, p; n)Q(b, a; p, q; n) = 1.$$

Now we will give some sums of finite and infinite series involving the case $p = q$ of the Fuss–Catalan–Qi numbers using the main results in Theorem 1.

Theorem 2 *(Theorem 3.1, [5]) If $\mathscr{R}(a), \mathscr{R}(b) > 0$ and $p, q \in \mathbb{N}^*$, the Fuss–Catalan–Qi numbers $Q(a, b; p, q; n)$ can be expressed by*

$$Q(a, b; p, q; n) = \left[\Big(\frac{b}{a}\Big)^{(q-p+1)}\frac{\Gamma(b)\Gamma(p+a)}{\Gamma(a)\Gamma(q+b)}\right]^n \frac{\prod_{k=0}^{p-1} C\big(\frac{k+a}{p}, 1; n\big)}{\prod_{k=0}^{q-1} C\big(\frac{k+b}{q}, 1; n\big)}, \quad n \geq 0. \tag{3.3}$$

Corollary 2 *For $\mathscr{R}(a), \mathscr{R}(b) > 0$ and $p \in \mathbb{N}^*$, the case $p = q$ of the Fuss–Catalan–Qi numbers can be written as*

$$Q(a, b; p, p; n) = \Big(\frac{b}{a}\Big)^n \prod_{k=0}^{p-1}\frac{(k+a)_{n,p}}{(k+b)_{n,p}}, \quad n \geq 0. \tag{3.4}$$

Proof Using Eq. (3.3), we obtain

$$\begin{aligned}
Q(a,b;p,p;n) &= \left[\left(\frac{b}{a}\right)\frac{\Gamma(b)\Gamma(p+a)}{\Gamma(a)\Gamma(p+b)}\right]^n \prod_{k=0}^{p-1} \frac{C\left(\frac{k+a}{p},1;n\right)}{C\left(\frac{k+b}{p},1;n\right)} \\
&= \left[\left(\frac{b}{a}\right)\frac{\Gamma(b)\Gamma(p+a)}{\Gamma(a)\Gamma(p+b)}\right]^n \prod_{k=0}^{p-1} \frac{\left(\frac{p}{k+a}\right)^n \frac{1}{n!}\left(\frac{k+a}{p}\right)_n}{\left(\frac{p}{k+b}\right)^n \frac{1}{n!}\left(\frac{k+b}{p}\right)_n} \\
&= \left[\left(\frac{b}{a}\right)\frac{\Gamma(b)\Gamma(p+a)}{\Gamma(a)\Gamma(p+b)}\right]^n \prod_{k=0}^{p-1} \frac{\left(\frac{1}{k+a}\right)^n p^n\left(\frac{k+a}{p}\right)_n}{\left(\frac{1}{k+b}\right)^n p^n\left(\frac{k+b}{p}\right)_n} \\
&= \left[\left(\frac{b}{a}\right)\frac{\Gamma(b)\Gamma(p+a)}{\Gamma(a)\Gamma(p+b)}\right]^n \prod_{k=0}^{p-1} \frac{\left(\frac{1}{k+a}\right)^n}{\left(\frac{1}{k+b}\right)^n} \prod_{k=0}^{p-1} \frac{p^n\left(\frac{k+a}{p}\right)_n}{p^n\left(\frac{k+b}{p}\right)_n} \\
&= \left[\left(\frac{b}{a}\right)\frac{\Gamma(b)\Gamma(p+a)}{\Gamma(a)\Gamma(p+b)}\right]^n \left[\prod_{k=0}^{p-1} \frac{(k+b)}{(k+a)}\right]^n \prod_{k=0}^{p-1} \frac{(k+a)_{n,p}}{(k+b)_{n,p}} \\
&= \left(\frac{b}{a}\right)^n \left[\frac{(a)_p}{(b)_p}\right]^n \left[\frac{(b)_p}{(a)_p}\right]^n \prod_{k=0}^{p-1} \frac{(k+a)_{n,p}}{(k+b)_{n,p}} \\
&= \left(\frac{b}{a}\right)^n \prod_{k=0}^{p-1} \frac{(k+a)_{n,p}}{(k+b)_{n,p}}.
\end{aligned}$$

The required proof is complete. □

According to Eq. (2.8), if $\mathscr{R}(a), \mathscr{R}(b) > 0$ and $p \in \mathbb{N}^*$, the Fuss–Catalan–Qi numbers with $p = q$ satisfy the following recurrence relation:

$$\begin{cases} Q(a,b;p,p;0) = 1, \\ (np+b)_p Q(a,b;p,p;n+1) = \left(\frac{b}{a}\right)(np+a)_p Q(a,b;p,p;n), \ n \geq 0. \end{cases} \tag{3.5}$$

Proposition 1 *If a and b are complex numbers such that $\mathscr{R}(a) > 0$, $\mathscr{R}(b) > p \in \mathbb{N}^*$, and $\mathscr{R}(b) \neq \mathscr{R}(a+p)$, the Fuss–Catalan–Qi numbers with $p = q$ satisfy*

$$\sum_{k=0}^{n} \left(\frac{a}{b}\right)^k D(a,b,p;k) Q(a,b;p,p;k) = (b-p)_p - (np+a)_p \left(\frac{a}{b}\right)^n Q(a,b;p,p;n) \tag{3.6}$$

and

$$\sum_{k=0}^{n} \left(\frac{a}{b}\right)^k D(a,b,p;k) Q(a,b;p,p;k) = (b-p)_p - (np+b)_p \left(\frac{a}{b}\right)^{n+1} Q(a,b;p,p;n+1) \tag{3.7}$$

for $n \geq 0$.

Proof Using Eq. (3.4) in Theorem 1, we obtain the results. □

Corollary 3 *If a and b are complex numbers such that $\mathscr{R}(b) > \mathscr{R}(a+p) > p \in \mathbb{N}^*$ and $n \geq 0$, then*

$$\sum_{k=0}^{n} \left(\frac{a}{b}\right)^k D(a,b,p;k) Q(a,b;p,p;k) = (b-p)_p \left[1 - \frac{B(b-a-p, a+np+p)}{B(a, b-a-p)}\right] \tag{3.8}$$

and

$$\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n D(a,b,p;n) Q(a,b;p,p;n) = (b-p)_p. \tag{3.9}$$

Proof This follows from using Eq. (3.4) in Corollary 1. □

Remark 1 It is clear that Proposition 1 is a generalizations of the formulas and identities for the Catalan numbers and Catalan–Qi numbers given by Qi and Guo [6] and by Chammam [7].

4 Some Relations Between the Fuss–Catalan–Qi Numbers and the Generalized Hypergeometric Series

The generalized hypergeometric series is defined for complex numbers $a_i \in \mathbb{C}$ and $b_i \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, for positive integers $p, q \in \mathbb{N}$, by

$$ {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{z^n}{n!}. $$

In [23], Qi et al. discovered that ${}_2F_1\left(a, 1; b; \frac{bt}{a}\right)$ is a generating function for the Catalan–Qi numbers $C(a, b; n)$.

Theorem 3 *([23, Theorem 10]) For $a, b > 0$ and $n \geq 0$, the Catalan–Qi numbers $C(a, b; n)$ can be generated by*

$$ {}_2F_1\left(a, 1; b; \frac{bt}{a}\right) = \sum_{n=0}^{\infty} C(a, b; n) t^n \tag{4.1} $$

and satisfy

$$ C(a, b; n) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} {}_2F_1\left(a, -k; b; -\frac{b}{a}\right). \tag{4.2} $$

Lemma 2 *For $a, b > 0$ and $p \in \mathbb{N}^*$,*

$$ {}_{p+1}F_p\left(\frac{a}{p}, ..., \frac{a+p-1}{p}, 1; \frac{b}{p}, ..., \frac{b+p-1}{p}; t\right) = \sum_{n=0}^{\infty} \prod_{k=0}^{p-1} \frac{(k+a)_{n,p}}{(k+b)_{n,p}} t^n. \tag{4.3} $$

Proof For $a, b > 0$ and $p \in \mathbb{N}^*$, we have

$$ \begin{aligned} \sum_{n=0}^{\infty} \prod_{k=0}^{p-1} \frac{(k+a)_{n,p}}{(k+b)_{n,p}} t^n &= \sum_{n=0}^{\infty} \prod_{k=0}^{p-1} \frac{p^n \left(\frac{k+a}{p}\right)_n}{p^n \left(\frac{k+b}{p}\right)_n} t^n \\ &= \sum_{n=0}^{\infty} \prod_{k=0}^{p-1} \frac{\left(\frac{k+a}{p}\right)_n}{\left(\frac{k+b}{p}\right)_n} t^n \\ &= \sum_{n=0}^{\infty} \prod_{k=0}^{p-1} \frac{\left(\frac{k+a}{p}\right)_n}{\left(\frac{k+b}{p}\right)_n} \frac{n!}{n!} t^n \\ &= \sum_{n=0}^{\infty} \prod_{k=0}^{p-1} \frac{\left(\frac{a+k}{p}\right)_n (1)_n}{\left(\frac{b+k}{p}\right)_n} \frac{t^n}{n!} \\ &= {}_{p+1}F_p\left(\frac{a}{p}, ..., \frac{a+p-1}{p}, 1; \frac{b}{p}, ..., \frac{b+p-1}{p}; t\right). \end{aligned} $$

The required proof is complete. □

Theorem 4 *For $a, b > 0$, $p \in \mathbb{N}^*$ and $n \geq 0$, the Fuss–Catalan–Qi numbers $Q(a, b; p, p; n)$ can be generated by*

$$ {}_{p+1}F_p\left(\frac{a}{p}, ..., \frac{a+p-1}{p}, 1; \frac{b}{p}, ..., \frac{b+p-1}{p}; \frac{bt}{a}\right) = \sum_{n=0}^{\infty} Q(a, b; p, p; n) t^n \tag{4.4} $$

and satisfy

$$ (-1)^{np} \left(\frac{a}{b}\right)^{n(1-p)} Q(a, b; p, p; n) = \sum_{k=0}^{np} (-1)^k \binom{np}{k} {}_2F_1\left(a, -k; b; -\frac{b}{a}\right). \tag{4.5} $$

Proof Using Eq. (3.4) and Lemma 2 we can obtain Eq. (4.4), and using Eq. (4.2) we obtain Eq. (4.5). □

Definition 4.1 [28] Given two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, we say that the sequence b_n is the binomial transform of the sequence a_n if

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

Corollary 4 *For $a, b > 0$, $p \in \mathbb{N}^*$ and for all nonnegative integers m, n, s, the Fuss–Catalan–Qi numbers $Q(a, b; p, p; n)$ satisfy*

$$\begin{aligned} &\sum_{k=0}^{m} \binom{m}{kp} \binom{(n+k)p}{sp} \left(\frac{a}{b}\right)^{(k+n-s)(1-p)} Q(a, b; p, p; k+n-s) \\ &= \sum_{k=0}^{np} (-1)^{(n-k)p} \binom{np}{kp} \binom{m+kp}{sp} {}_2F_1\Big(a, m-(k+s)p; b; -\frac{b}{a}\Big). \end{aligned} \tag{4.6}$$

Proof Just note that the sequence ${}_2F_1\Big(a, -n; b; -\frac{b}{a}\Big)$ is the binomial transform of the sequence $(-1)^{np}$ $\left(\frac{a}{b}\right)^{n(1-p)} Q(a, b; p, p; n)$. Using theorem 3.2 in the paper [28] completes the required proof [29]. □

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