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ORIGINAL RESEARCH

On a sum involving divisor function and the integral part function

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Abstract For any real number t, let $\lfloor t \rfloor$ be the largest integer not exceeding y. As usual, let $d(n)$ be the divisor function. Recently, Ma and Sun obtain the following

$$\sum_{n\leq x} d\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = \lambda x + O_{\varepsilon}\left(x^{11/23+\varepsilon}\right).$$

where $\lambda = \sum_{k=1}^{\infty} \frac{d(k)}{k(k+1)}$ is an absolute constant and ε is an arbitrarily small positive number. In this article, we give a slight generalization of their formula. Specifically, we prove that for any $c > 0$, the asymptotic formula

$$\sum_{n\leq x^{1/c}} d\left(\left\lfloor \frac{x}{n^c} \right\rfloor\right) = d_c x^{1/c} + O_{\varepsilon,c}\left(x^{\theta_c+\varepsilon}\right)$$

holds, where $\theta_c < \frac{1}{c}$ and $d_c = \sum_{k\geq 1} d(k)\left(\frac{1}{k^{1/c}} - \frac{1}{(k+1)^{1/c}}\right)$ is a constant.

Keywords Divisor function · Asymptotic formula · Exponent pair

Mathematics Subject Classification Primary 11N37 · Secondary 11L07

1 Introduction

Let $d(n)$ be the divisor function, and let $\lfloor t \rfloor$ be the integral part of real number t. The celebrate result of Dirichlet showed that

$$\sum_{n\leqslant x} d(n) = x\log x + (2\gamma - 1)x + O\left(\sqrt{x}\right),$$

where $\gamma = 0.577215664\ldots$ is the Euler constant. The explorations of the error term turn out to be a significant issue of number theory [5]. From [3], we know that

$$\sum_{n\leqslant x} d\left(\left\lfloor \frac{x}{n} \right\rfloor\right) = x\log x + (2\gamma - 1)x + O\left(x^{517/1648+o(1)}\right).$$

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In 2019, Bordellès–Dai–Heyman–Pan–Shparlinski [1] established an asymptotic formula of

$$H_f(x) = \sum_{n \le x} f\left(\left\lfloor \frac{x}{n} \right\rfloor\right)$$

under some assumptions of f. In [6], Liu, Wu and Yang established the following variant prime number theorem

$$\sum_{n \le x} \Lambda\left(\lfloor x/n \rfloor\right) = x \sum_{k \ge 1} \frac{\Lambda(k)}{k(k+1)} + O_\varepsilon(x^{9/19+\varepsilon}),$$

whose error term improved upon the former ones given by Wu [9], Zhai [10], Ma–Wu [8] and Bordellès [2].

Using the asymptotic formula obtained by Wu [9] to the divisor function, we can get

$$H_d(x) = x \sum_{k \ge 1} \frac{d(k)}{k(k+1)} + O_\varepsilon\left(x^{1/2+\varepsilon}\right).$$

In 2021, The error term $x^{1/2+\varepsilon}$ was improved by Ma–Sun [7] to $x^{11/23+\varepsilon}$ by using the method of exponent pairs.

Inspired by the above works, in this article we study the asymptotic formula of

$$S_{d,c}(x) = \sum_{n \le x^{1/c}} d\left(\left\lfloor \frac{x}{n^c} \right\rfloor\right),$$

where c is a positive real number. Now, we state our theorem in the following.

Theorem 1.1 *For any $\varepsilon > 0$ and $x \to \infty$, we have*

$$S_{d,c}(x) = x^{1/c} \sum_{k \ge 1} d(k) \left(\frac{1}{k^{1/c}} - \frac{1}{(k+1)^{1/c}}\right) + O_{\varepsilon,c}(x^{\theta_c+\varepsilon}),$$

where

$$\theta_c = \begin{cases} \frac{2}{3c+2} & \text{if } 0 < c < \frac{2}{11}, \\ \frac{11}{11c+12} & \text{if } c \ge \frac{2}{11}. \end{cases} \tag{1.1}$$

Taking $c = 1$ in our theorem, the results are reduced to those of Ma–Sun, so Theorem 1.1 extended Ma–Sun's theorem.

2 Auxiliary Lemmas

From now on, let x be a large positive number and n stand for the positive integers. Let $\varepsilon > 0$ be arbitrary small positive number which may not be the same throughout our paper. Put $e(y) = e^{2\pi i y}$. For two complex-valued functions $f(x)$ and $g(x)$, the notation $f(x) = O(g(x))$ or $f(x) \ll g(x)$ means there exists a positive constant M such that $|f(x)| \le M|g(x)|$. And $D < d \le 2D$ is abbreviated to $d \sim D$.

Lemma 2.1 ([4, Lemma 2.4]) *Suppose that*

$$L(H) = \sum_{i=1}^{m} A_i H^{a_i} + \sum_{j=1}^{n} B_j H^{-b_j},$$

where A_i, B_j, a_i and b_j are positive. Assume that $H_1 \le H_2$. Then there is some H with $H_1 < H < H_2$ and

$$L(H) \ll \sum_{i=1}^{m} \sum_{j=1}^{n} \left(A_i^{b_j} B_j^{a_i}\right)^{1/(a_i+b_j)} + \sum_{i=1}^{m} A_i H_1^{a_i} + \sum_{j=1}^{n} B_j H_2^{-b_j}.$$

The implied constants depend only on m and n.

The next one is a standard result in the Fourier analysis.

Lemma 2.2 ([4, Theorem A.6]) *Let t be any real number. For $x \geq 1$ and $H \geq 1$, we have*

$$\psi(x) = -\sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(hx)}{2\pi i h} + R_H(x),$$

where $\Phi(t) = \pi t(1-|t|)\cot(\pi t) + |t|$ and

$$|R_H(x)| \leq \frac{1}{2H+2} \sum_{0 \leq |h| \leq H} \left(1 - \frac{|h|}{H+1}\right) e(hx). \tag{2.1}$$

For $D \leq x$, let

$$\Sigma_{\delta,\lambda}(x, D) = \sum_{k \sim D} d(k) \psi\left(\frac{x^\lambda}{(k+\delta)^\lambda}\right). \tag{2.2}$$

Next lemma gives us a bound of $\Sigma_{\delta,\lambda}(x, D)$.

Lemma 2.3 *Let δ be a fixed nonnegative constant. For large x, we have*

$$\Sigma_{\delta,\lambda}(x, D) \ll_\varepsilon x^\varepsilon \left((x^{2\lambda} D^{11-2\lambda})^{1/14} + (x^{2\lambda} D^{9-2\lambda})^{1/12} + x^{-\lambda} D^{\lambda+1}\right)$$

for $1 \leq D \leq x$.

Proof Using the well known $d(k) = \sum_{mn=k} 1$ and the symmetry of factors, we get

$$\begin{aligned}
\Sigma_{\delta,\lambda}(x, D) &= \sum_{k \sim D} \sum_{mn=k} \psi\left(\frac{x^\lambda}{(k+\delta)^\lambda}\right) \\
&= \sum_{m \leq 2D} \sum_{n \sim D/m} \psi\left(\frac{x^\lambda}{(mn+\delta)^\lambda}\right) \\
&= \sum_{m \leq \sqrt{2D}} \sum_{n \sim D/m} \psi\left(\frac{x^\lambda}{(mn+\delta)^\lambda}\right) + \sum_{\sqrt{2D} < m \leq 2D} \sum_{n \sim D/m} \psi\left(\frac{x^\lambda}{(mn+\delta)^\lambda}\right) \\
&= 2 \sum_{m \leq \sqrt{2D}} \sum_{n \sim D/m} \psi\left(\frac{x^\lambda}{(mn+\delta)^\lambda}\right).
\end{aligned}$$

By Lemma 2.2, we have

$$\Sigma_{\delta,\lambda}(x, D) = -\frac{1}{\pi i}\left(\Sigma'_{\delta,\lambda} + \overline{\Sigma'_{\delta,\lambda}}\right) + \Sigma''_{\delta,\lambda} \tag{2.3}$$

where

$$\Sigma'_{\delta,\lambda} = \sum_{m \leq \sqrt{2D}} \sum_{n \sim D/m} \sum_{1 \leq h \leq H} \frac{1}{h} \Phi\left(\frac{h}{H+1}\right) e\left(\frac{hx^\lambda}{(mn+\delta)^\lambda}\right)$$

and

$$\Sigma''_{\delta,\lambda} = 2 \sum_{m \leq \sqrt{2D}} \sum_{n \sim D/m} R_H\left(\frac{x^\lambda}{(mn+\delta)^\lambda}\right).$$

For $\Sigma'_{\delta,\lambda}$, using exponential pair (k,l) on the sum over n and the fact that $0<\Phi(t)<1 (0<|t|<1)$, we obtain

$$\begin{aligned}
\Sigma'_{\delta,\lambda} &\ll \sum_{m\le\sqrt{2D}}\sum_{1\le h\le H}\frac{1}{h}\left\{\left(\frac{mhx^{\lambda}}{D^{\lambda+1}}\right)^{k}\left(\frac{D}{m}\right)^{l}+\frac{D^{\lambda+1}}{mhx^{\lambda}}\right\}\\
&\ll \sum_{m\le\sqrt{2D}}(x^{\lambda k}m^{k-l}D^{l-\lambda k-k}H^{k}+x^{-\lambda}D^{\lambda+1}m^{-1})\\
&\ll x^{\lambda k}D^{(l-2\lambda k-k+1)/2}H^{k}+x^{-\lambda}D^{\lambda+1}\log D
\end{aligned}\tag{2.4}$$

for all $H\ge 1$. For the theory of exponent pairs, we recommend readers refer to [4, Chapter 3].

For $\Sigma''_{\delta,\lambda}$. By using (2.1), we have

$$\begin{aligned}
\Sigma''_{\delta,\lambda} &\ll \sum_{m\le\sqrt{2D}}\sum_{n\sim D/m}\left|R_H\left(\frac{x^{\lambda}}{(mn+\delta)^{\lambda}}\right)\right|\\
&\ll \frac{1}{H}\sum_{m\le\sqrt{2D}}\sum_{n\sim D/m}\sum_{0\le|h|\le H}\left(1-\frac{|h|}{H+1}\right)e\left(\frac{hx^{\lambda}}{(mn+\delta)^{\lambda}}\right)\\
&\ll DH^{-1}\log D+|\widetilde{\Sigma''_{\delta,\lambda}}|,
\end{aligned}$$

where

$$\widetilde{\Sigma''_{\delta,\lambda}}=\frac{1}{H}\sum_{m\le\sqrt{2D}}\sum_{n\sim D/m}\sum_{1\le h\le H}\left(1-\frac{|h|}{H+1}\right)e\left(\frac{hx^{\lambda}}{(mn+\delta)^{\lambda}}\right).$$

Using the same method as $\Sigma'_{\delta,\lambda}$ on $\widetilde{\Sigma''_{\delta,\lambda}}$, we can get

$$\widetilde{\Sigma''_{\delta,\lambda}}\ll DH^{-1}\log D+x^{\lambda k}D^{(l-2\lambda k-k+1)/2}H^{k}+x^{-\lambda}D^{\lambda+1}\log D,\tag{2.5}$$

provided $H\ge 1$.

Combining (2.3), (2.4) and (2.5), we can obtain that

$$\Sigma_{\delta,\lambda}(x,D)\ll DH^{-1}\log D+x^{\lambda k}D^{(l-2\lambda k-k+1)/2}H^{k}+x^{-\lambda}D^{\lambda+1}\log D.$$

for $H\ge 1$. By taking

$$\begin{aligned}
&m=n=1,\quad a_1=k,\quad b_1=1,\\
&A_1=x^{\lambda k}D^{(l-2\lambda k-k+1)/2},\quad B_1=D\log D,\quad H_1=\frac{1}{2}\text{and}H_2\to+\infty
\end{aligned}$$

in Lemma 2.1, we get

$$\Sigma_{\delta,\lambda}(x,D)\ll\left(x^{2\lambda k}D^{l+k-2\lambda k+1}(\log D)^{2k}\right)^{1/(2k+2)}+x^{\lambda k}D^{(l-2\lambda k-k+1)/2}+x^{-\lambda}D^{\lambda+1}\log D.\tag{2.6}$$

Taking $(k,l)=\left(\frac{1}{6},\frac{2}{3}\right)$ we get that

$$\Sigma_{\delta,\lambda}(x,D)\ll_{\varepsilon}x^{\varepsilon}\left(x^{\lambda/7}D^{11/14-\lambda/7}+x^{\lambda/6}D^{3/4-\lambda/6}+x^{-\lambda}D^{\lambda+1}\right).$$

After a simple calculation, we can get the Lemma. □

3 Proof of Theorem 1.1

Let $\eta \in [1, x^{1/(c+1)}]$ be a parameter to be decided later. We split the sum $S_d(x)$ into the following two parts

$$S_d(x) = S_1 + S_2, \tag{3.1}$$

where

$$S_1 = \sum_{n \le \eta} d\left(\left\lfloor \frac{x}{n^c} \right\rfloor\right) \quad \text{and} \quad S_2 = \sum_{\eta < n \le x^{1/c}} d\left(\left\lfloor \frac{x}{n^c} \right\rfloor\right).$$

Using that $d(n) \ll_\varepsilon n^\varepsilon$, we have the trivial estimate

$$S_1 \ll_\varepsilon \eta x^\varepsilon. \tag{3.2}$$

For S_2, we denote $k = \lfloor \frac{x}{n^c} \rfloor$, then

$$k \le \frac{x}{n^c} < k + 1 \Leftrightarrow \left(\frac{x}{k+1}\right)^{\frac{1}{c}} < n \le \left(\frac{x}{k}\right)^{\frac{1}{c}}.$$

Thus we can write

$$\begin{aligned} S_2 &= \sum_{k \le x/\eta^c} d(k) \sum_{(x/k+1)^{1/c} < n \le (x/k)^{1/c}} 1 \\ &= \sum_{k \le x/\eta^c} d(k) \left\{ \frac{x^{1/c}}{k^{1/c}} - \psi\left(\frac{x^{1/c}}{k^{1/c}}\right) - \frac{x^{1/c}}{(k+1)^{1/c}} + \psi\left(\frac{x^{1/c}}{(k+1)^{1/c}}\right) \right\} \\ &= x^{1/c} \sum_{k \ge 1} d(k) \left(\frac{1}{k^{1/c}} - \frac{1}{(k+1)^{1/c}}\right) + R_0(x) - R_1(x) + O_\varepsilon(\eta x^\varepsilon). \end{aligned} \tag{3.3}$$

where we used the bounds

$$x^{1/c} \sum_{k > x/\eta^c} d(k) \left(\frac{1}{k^{1/c}} - \frac{1}{(k+1)^{1/c}}\right) \ll_\varepsilon \eta x^\varepsilon,$$

$$\sum_{k \le \eta} d(k) \left\{ \psi\left(\frac{x^{1/c}}{(k+1)^{1/c}}\right) - \psi\left(\frac{x^{1/c}}{k^{1/c}}\right) \right\} \ll \sum_{k \le \eta} d(k) \ll_\varepsilon \eta x^\varepsilon$$

and defined

$$R_\delta(x) = \sum_{\eta < k \le x/\eta^c} d(k) \psi\left(\frac{x^{1/c}}{(k+\delta)^{1/c}}\right)$$

with $\delta = 0, 1$.

From now on, let $D_i = x/(2^i \eta^c)$ with $0 \le i \le \log(x/\eta^{c+1})/\log 2$ such that $\eta \le D_i \le x/\eta^c$.

For $0 < c < \frac{2}{11}$, Lemma 2.3 allows us to deduce that

$$\begin{aligned} |R_\delta(x)| &\le \sum_{1 \le i \le \log(x/\eta^{c+1})/\log 2} \Sigma_{\delta, 1/c}(x, D_i) \\ &\ll_\varepsilon \sum_{1 \le i \le \log(x/\eta^{c+1})/\log 2} x^\varepsilon \left(\left(x^2 D_i^{11c-2}\right)^{1/(14c)} + \left(x^2 D_i^{9c-2}\right)^{1/(12c)} + \left(x^{-1} D_i^{1+c}\right)^{1/c} \right) \\ &\ll_\varepsilon x^\varepsilon \left(\left(x^2 \eta^{11c-2}\right)^{1/(14c)} + \left(x^2 \eta^{9c-2}\right)^{1/(12c)} + x\eta^{-(1+c)} \right). \end{aligned} \tag{3.4}$$

Taking $\eta^{2/(3c+2)}$, we obtain

$$R_\delta(x) \ll_\varepsilon x^{2/(3c+2)+\varepsilon}.$$

Similarly, we have

$$R_\delta(x) \ll_{\varepsilon,c} x^\varepsilon \left(\left(x^{11} \eta^{-11c+2} \right)^{1/(14c)} + \left(x^2 \eta^{9c-2} \right)^{1/(12c)} + x\eta^{-(1+c)} \right)$$

for $\frac{2}{11} \le c < \frac{2}{9}$, and

$$R_\delta(x) \ll_{\varepsilon,c} x^\varepsilon \left(\left(x^{11} \eta^{-11c+2} \right)^{1/(14c)} + \left(x^9 \eta^{-9c+2} \right)^{1/12} + x\eta^{-(1+c)} \right)$$

for $c \ge \frac{2}{9}$. Taking $\eta = x^{11/(11c+12)}$, we find

$$R_\delta(x) \ll_{\varepsilon,c} x^{11/(11c+12)+\varepsilon}.$$

Inserting these into (3.3), we obtain

$$S_{d,c}(x) = x^{1/c} \sum_{k \ge 1} d(k) \left(\frac{1}{k^{1/c}} - \frac{1}{(k+1)^{1/c}} \right) + O_{\varepsilon,c}(x^{\theta_c)+\varepsilon}).$$

This completes the proof.

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