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ORIGINAL RESEARCH

Properties of block derangements

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Abstract A derangement is a permutation that has no fixed points. In this paper, we study a subclass of derangements on $r+n$ objects such that none of the first r objects returns to the first r objects location. We determine explicit formulas and recurrence relations for the number of this subclass of derangement. Also, the probability that a permutation is in this subclass of derangements is close to $1/e$. Finally, we will present the Stirling transformation formula of these numbers.

Keywords Derangement · Block derangement · Asymptotics · Stirling transform

1 Introduction

The introduction of derangement numbers goes back to as early as 1708 when Pierre Rémond de Montmo was counting the number of derangements on n letters (see [4,5]). Suppose that S_n is the symmetric group on the set $[n]=\{1,2,\ldots,n\}$. A derangement is a fixed point-free element of the symmetric group S_n. The set of these permutations on S_n is denoted by $\mathcal{D}_n$. In other words,

$$\mathcal{D}_n=\{\pi\in S_n : \pi(i)\neq i \text{ for all } 1\leq i\leq n\}.$$

Also, let $D(n)=|\mathcal{D}_n|$. The inclusion–exclusion principle gives

$$D(n)=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!},$$

and that

$$\lim_{n\to\infty}\frac{D(n)}{n!}=\frac{1}{e}.$$

Assume that $D(0)=1$ and $D(1)=0$. Then the number $D(n)$ is related to the numbers $D(n-1)$ and $D(n-2)$ (with $n\geq 2$) through a closed-form recursive relation

$$D(n)=(n-1)(D(n-1)+D(n-2)).$$

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In [6], authors started investigating a subclass of derangements on $r+n$ letters which restrict the first r of these to be in distinct cycles. (Such derangements are called r-derangement.) The number of r-derangements is denoted by $D_r(n)$.

Theorem 1 [6, Theorem 4] *Let $r \in \mathbb{N}^+$ and $s \in \{1, \ldots, r\}$. Then for each $n \geq s$ we have*

$$D_r(n) = \sum_{j=s}^{n} \binom{j-1}{s-1} \frac{n!}{(n-j)!} D_{r-s}(n-j).$$

In particular,

$$D_r(n) = \sum_{j=r}^{n} \binom{j-1}{r-1} \frac{n!}{(n-j)!} D(n-j), \ n \geq r.$$

Additionally, we have a closed formula for r-derangements numbers:

$$D_r(n) = \sum_{j=r}^{n} \binom{j}{r} \frac{n!}{(n-j)!} (-1)^{(n-j)}, \ n \geq r.$$

In this paper, we study a subclass of derangements on $r+n$ letters such that none of the first r letters return to the first r locations. More formally, let S_{r+n} be the set of permutations on the set $[r+n]$ and consider the following:

$$\mathcal{D}(r,n) = \{\pi \in \mathcal{D}_{r+n} : \pi(i) \neq i \text{ for all } 1 \leq i \leq n+r \text{ and } \pi(i) \notin [r] \text{ for all } i \in [r]\}.$$

We termed this set as *block derangements*. Also, let $D(r,n) = |\mathcal{D}(r,n)|$. In this paper, we obtain a relation between block derangements and r-derangements (which is defined in [6]) on S_{r+n}.

Remark 1 It follows from the definition of block derangements that n must be greater than or equal to r. For $n < r$, we have $D(r,n) = 0$. Now let $r = 0$. Then we have

$$D(0,n) = D(n) \text{ and } D(1,n) = D(n+1).$$

Thus the block derangement is a generalization of the derangement.

The Stirling number of the second kind $S(n,k)$ is the number of partitioning a set of n elements into k nonempty subsets. By [3], for given power series

$$f(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$$

we have the series transformation formula

$$f\left(\frac{\mu}{\lambda}(e^{\lambda t} - 1)\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left\{ \sum_{k=0}^{n} S(n,k) \lambda^{n-k} \mu^k a_k \right\} \tag{1}$$

where λ and μ are parameters. Let n be a nonnegative integer. The integers $b_n = \sum_{k=0}^{n} S(n,k)$ are known as Bell numbers. They give the number of ways a set of n elements can be partitioned into nonempty disjoint subsets. In this article, we prove the following results.

Theorem 2 *Suppose r and n are integers and $1 \leq r \leq n$. Then*

$$D(r,n) = r! \sum_{i=r}^{n} (-1)^{n-i} \binom{i}{r} \frac{n!}{(n-i)!}.$$

Theorem 3 *Suppose r and n are integers and $1 \leq r \leq n$. Then $D(r,n) = r! D_r(n)$.*

Theorem 4 *Suppose r and n are integers and $1 \leq r \leq n$. Then for each r the sequence $\left\{\frac{D(r,n)}{(r+n)!}\right\}_{n\geq 1}$ tends to $1/e$.*

Theorem 5 *Let r and n be integer numbers. Then we have*

$$\sum_{k=0}^{n}(-1)^k S(n,k)D(r,k) = \sum_{j=0}^{n}\sum_{i=0}^{r}(-1)^{n+i+r-j} r!\binom{r}{i}(1+i)^{n-j} b_j.$$

In [6, Theorem 6], it was shown that for each $r \in \mathbb{N}^+$, the sequence $\left\{\frac{D_r(n)}{(r+n)!}\right\}_{n\in\mathbb{N}}$ is convergent to $\frac{1}{r!e}$. Now, we can use Theorem 4, to establish asymptotic estimates for r-derangements in another way. Also, according to Theorem 3, it can be obtained, similar to the main results in [6], some conclusions about the number $D(r,n)$, so we avoid repeating them here. In Sect. 2, we obtain the recursion formula for $D(r,n)$. Then we prove the main results of this section. Boyadzhiev gave an interesting combinatorial Stirling transform of derangement numbers in [2]. Similarly, at the end of Sect. 2, we obtain the Stirling transformation formula of $D(r,n)$.

2 Properties of the number $D(r, n)$.

Throughout this section, the set $\{1, 2, \ldots, n\}$ and the set $\{i, i+1, \ldots, j\}$ are denoted by $[n]$ and $[i, j]$, respectively. Also, assume that a permutation $\pi \in S_n$ is represented as the word $\pi(1)\pi(2)\ldots\pi(n)$. Assume that $\Delta = [r]$ and denote by $\pi(\Delta)$, the action of a permutation $\pi \in S_{r+n}$ on the set Δ by defining $\pi(\Delta) = \{\pi(x), x \in \Delta\}$. Obviously, A derangement $\pi \in S_{r+n}$ is a block derangement if and only if $\pi(\Delta) \cap \Delta = \emptyset$.

Theorem 6 *If r and n are integers and $1 \leq r \leq n$, then*

$$D(r,n) = n(rD(r-1,n-1) + D(r,n-1)).$$

Proof Assume that $\Delta = [r]$, when $r, n \in \mathbb{N}$ and $r \leqslant n$. Let $\pi \in \mathcal{D}(r,n)$ be a block derangement, such that $\pi(1) = k$. Hence, $k \in [r+1, r+n]$. According to the possibilities of $\pi(k)$, there are two cases to consider.

In the first case, assume that $\pi(k) \in [r]$. We define $\pi' \in S_{r+n-2}$ as follows:

$$\pi'(i) = \begin{cases} \pi(i+1) - |\{j \in \{1,k\} : \pi(j) < \pi(i+1)\}| & \text{if } 1 \leq i \leq k-2 \\ \pi(i+2) - |\{j \in \{1,k\} : \pi(j) < \pi(i+2)\}| & \text{if } k-1 \leq i \leq n+r-2 \end{cases}$$

Let $i \in [r-1]$. We know that $\pi(1) = k$, $\pi(i+1) \neq k$ and $r < \pi(i+1)$. So, we have two sub-cases.

(1) Let us assume that $\pi(i+1) < k$. Therefore, $\pi(k) \leq r < \pi(i+1) < k$. Consequently,

$$\pi'(i) = \pi(i+1) - 1 > r - 1.$$

(2) Now, consider the case $\pi(i+1) > k$. Thus, $\pi'(i) = \pi(i+2) - 2$. But, we know that $\pi(i+1) > k$. Also

$$\pi(i+1) - r > k - r \geqslant (r+1) - r.$$

Then, $\pi(i+1) > r+1$ and $\pi'(i) > r-1$.

From the above discussion we know that $\pi'(\Delta') \cap \Delta' = \emptyset$ for $\Delta' = [r-1]$.

Now, we can show that $\pi'(i) \neq i$ for all $1 \leq i \leq r+n-2$. Let $k-1 \leq i \leq r+n-2$ and $\pi(i+2) > k$. On the contrary, assume that $\pi'(i) = i$. Then, $\pi(i+2) - 2 = i$. So, $\pi(i+2) = i+2$. This leads to a contradiction with $\pi \in \mathcal{D}(r,n)$. By the simple calculation in other cases, we can show that π' is a derangement on the set $[1, r+n-2]$ and so $\pi' \in \mathcal{D}(r-1, n-1)$.

According to the choice of $k \in [r+1, r+n]$ and $\pi(k) \in [r]$, in this case, we obtain that the number of all permutations is $nrD(r-1, n-1)$.

In the second case let $\pi(k) \in [r, n]$. Consider $\pi' \in S_{r+n}$ as follows:

$$\pi'(i) = \begin{cases} \pi(i) & i \in [r+n] - \{1,k\} \\ \pi(k) & i = 1 \\ k & i = k \end{cases}$$

Hence, we can define $\pi'' \in S_{r+n-1}$ by the rule:

$$\pi''(i) = \begin{cases} \pi'(i) - |\{j \in \{k\} : \pi'(j) < \pi'(i)\}| & \text{if } \ 1 \le i \le k-1 \\ \pi'(i+1) - |\{j \in \{k\} : \pi'(j) < \pi'(i+1)\}| & \text{if } \ k \le i \le r+n-1 \end{cases}$$

By a similar argument as in the previous case, we conclude that $\pi'' \in \mathcal{D}(r, n-1)$. Since $k \in [r+1, r+n]$, the number of permutations in this case is $nD(r, n-1)$. □

Remark 2 The recurrence in Theorem 6 generalizes the well-known recurrence for the derangements. Set $r = 1$ in Theorem 6 to get that $D(1, n) = n(D(0, n-1) + D(1, n-1))$ and hence $D(n+1) = n(D(n) + D(n-1))$.

Now we are ready to prove the main results of the paper, which state the properties of $D(r, n)$.

Proof of Theorem 2 We will apply the inclusion–exclusion principle. Consider that $\mathcal{A}_i = \{\pi : \pi \in S_{r+n},\ \pi(\Delta) \cap \Delta = \emptyset,\ \pi(i) = i\}$, where $\Delta = [r]$ and $r+1 \le i \le n+r$.

Let $\pi \in \mathcal{A}_i$. As $\pi(i) = i$ for some $r+1 \le i \le r+n$, we can choose the string $\pi(1)\pi(2)\ldots\pi(r)$ in $\binom{n-1}{r} r\,!$ ways and we can choose the string $\pi(r+1)\pi(r+2)\ldots\pi(r+n)$ in $(n-1)\,!$ ways. Hence, $|\mathcal{A}_i| = r!\binom{n-1}{r}(n-1)!$. By the same argument as above, we have

$$|\mathcal{A}_{i_1} \cap \mathcal{A}_{i_2} \cap \ldots \mathcal{A}_{i_k}| = \begin{cases} r!\binom{n-k}{r}(n-k)! & \text{if } r \le n-k; \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $i_j \in [r+1, r+n]$ and $j \in [k]$. Then, by the inclusion–exclusion principle, we have

$$\begin{aligned} \left|\bigcup_{i=1}^{n} \mathcal{A}_{r+i}\right| &= \sum_{\emptyset \ne I \subseteq [n]} (-1)^{|I|-1} \left|\bigcap_{i \in I} \mathcal{A}_{r+i}\right| \\ &= \sum_{k=1}^{n-r} (-1)^{k-1} r! \binom{n-k}{r}\binom{n}{k}(n-k)! \\ &= \sum_{j=r}^{n-1} (-1)^{n-j-1} r! \binom{j}{r} \frac{n!}{(n-j)!}. \end{aligned}$$

So,

$$\begin{aligned} D(r, n) &= |\{\pi : \pi \in S_{r+n},\ \pi(i) \notin [r] \text{ for all } i \in [r]\}| - \left|\bigcup_{i=1}^{n} \mathcal{A}_{r+i}\right| \\ &= r! \sum_{j=r}^{n} (-1)^{n-j} \binom{j}{r} \frac{n!}{(n-j)!}. \end{aligned}$$

□

Proof of Theorem 3 Equality comes directly from the third formula in Theorems 1 and 2. Then, the proof of theorem is straightforward. □

Proof of Theorem 4 Since $D(r, n)$ is the size of a subclass of derangements on $r+n$ objects, thus $D(r, n) \le D(r+n)$. In order to construct a permutation in $D(r, n)$, since none of the first r objects returns to the first r object's location, we choose r elements from the set $[r+1, r+n]$. These elements can be ordered in $r!\binom{n}{r}$ ways. Then, n elements of the remaining $r+n$ elements of the permutation can not be fixed points. Hence, the elements in these remaining n positions can be ordered in more than $D(n)$ ways. Thus, $r!\binom{n}{r}D(n) \le D(r, n)$. So for any integer $n \ge 1$, we obtain the following inequality

$$r!\binom{n}{r}D(n) \le D(r, n) \le D(r+n).$$

In order to prove the theorem, it suffices to divide the previous inequality by $(r+n)!$ and let n tend to $+\infty$. □

In [6, Theorem 3], it was shown that for each $r \in \mathbb{N}^+$, the exponential generating function of the sequence $\{D_r(n)\}_{n\geq 1}$ is

$$F_r(x) := \sum_{n=0}^{\infty} \frac{D_r(n)}{n!} x^n = \frac{x^r}{(1-x)^{r+1}} e^{-x}.$$

Let $D_r(x)$ be defined as the exponential generating function of the sequence $\{D(r,n)\}_{n\geq 1}$. In other words,

$$D_r(x) = \sum_{n=0}^{\infty} D(r,n) \frac{x^n}{n!}. \tag{2}$$

Theorem 7 *Suppose r and n are integers and $0 \leq r \leq n$. Then*

$$D_r(x) = \frac{r! x^r}{(1-x)^{r+1}} e^{-x}.$$

Proof The formula for the exponential generating function of the sequence $(D(r,n))_{n\in\mathbb{N}}$ follows directly from the identity $D(r,n) = r! D_r(n)$ and the formula for the exponential generating function of the sequence $(D_r(n))_{n\in\mathbb{N}}$. Therefore,

$$D_r(x) = \sum_{n=0}^{\infty} \frac{D(r,n)}{n!} x^n = \sum_{n=0}^{\infty} \frac{r! D_r(n)}{(n-1)!} x^n = r! F_r(x).$$

This completes the proof. □

Proof of Theorem 5 We apply property (1) with $\lambda = 1$ and $\mu = -1$ to the generating function (2), replacing x with $e^t - 1$. This gives the representation

$$D_r(-(e^t-1)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left\{ \sum_{k=0}^{n} (-1)^k S(n,k) D(r,k) \right\}. \tag{3}$$

By [1], the exponential generating function for the Bell numbers is

$$e^{e^t-1} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}.$$

Therefore, by Theorem 7, we have

$$\begin{aligned} D_r(-(e^t-1)) &= (-1)^r r! e^{-t} (1-e^{-t})^r e^{e^t-1} \\ &= (-1)^r r! \left(\sum_{n=0}^{\infty} \sum_{i=0}^{r} (-1)^{n+i} \binom{r}{i} (1+i)^n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \right) \\ &= (-1)^r r! \sum_{n=0}^{\infty} \frac{t^n}{n!} \left\{ \sum_{j=0}^{n} \sum_{i=0}^{r} (-1)^{n+i-j} \binom{r}{i} (1+i)^{n-j} b_j \right\} \end{aligned} \tag{4}$$

The equality of the theorem follows immediately by comparing the representations (3) and (4). □

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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