



Hamilton and Souplet–Zhang type estimations on semilinear parabolic system along geometric flow

Sujit Bhattacharyya · Shahroud Azami · Shyamal Kumar Hui^{ID}

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Abstract In this article we derive both Hamilton type and Souplet–Zhang type gradient estimations for a system of semilinear equations along a geometric flow on a weighted Riemannian manifold.

Keywords Hamilton type estimation · Souplet Zhang type estimation · Gradient estimate · Semilinear equations · Weighted Laplacian · Geometric flow

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1 Introduction

The study of gradient estimation was started after the work of Li and Yau [11], where they derived a bound for the quantity $\frac{|\nabla u|}{u}$ and u satisfies

$$(\Delta - q(x) - \partial_t)u(x, t) = 0,$$

which is known as the Li–Yau type estimation. This field becomes much popular after introducing Ricci flow by Hamilton [5, 6]. Later Souplet and Zhang [14] developed this area. Gradient estimations were studied on many different system of equations. There is a vast literature for deriving gradient estimation for single differential equations, to get an idea see [7, 8] and the references therein. For system of equations we start by considering the work of Shen and Ding [13] where they considered the following system

$$\begin{cases} u_t = \Delta u^m + k_1(t)f_1(v), \\ v_t = \Delta v^n + k_2(t)f_2(u), \end{cases}$$

with nonlinear boundary conditions, where Δ is the Laplace Beltrami operator and proved that the solution of the above system blows up in finite time using differential inequality and Sobolev inequality. A parabolic system of the form

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S. Bhattacharyya · S. K. Hui (✉)
Department of Mathematics, The University of Burdwan, Golapbag, Burdwan, West Bengal 713104, India
E-mail: skhui@math.buruniv.ac.in

S. Bhattacharyya
E-mail: sujitbhattacharyya.1996@gmail.com

S. Azami
Department of Pure Mathematics, Faculty of Sciences, Imam Khomeini International University, Qazvin, Iran
E-mail: azami@sci.ikiu.ac.ir

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$

is called Keller-Segal system, which was studied by Winkler [16]. Global existence and finite time blow up of the solution for the following semilinear parabolic system

$$\begin{cases} u_t = \Delta u + e^{\alpha t} v^p, \\ v_t = \Delta v + e^{\beta t} u^q, \end{cases}$$

on hyperbolic space was showed by Wu and Yang [19]. System of nonlinear parabolic equations have numerous applications in quantum physics, fluid dynamics, laser-plasma interaction, study of Navier-Stokes equation and many other fields. For example, the system of Zakharov equations plays an important role in laser-plasma interaction. Various interesting results for this system have been studied by Zheng, Shang and Di [20]. Uniform stability for Memory-Type Elasticity System have been studied by Li and Bao [10]. In [4], Feng and Liu studied smooth solutions of compressible Navier-Stokes-Poisson system in $\mathbb{R}^3$, which represents the dynamics of electrons in semiconductors.

Motivated by the works of Wu [17], in this article, we acquire the Hamilton type and Souplet-Zhang type gradient estimations for the system of weighted semilinear equations

$$\begin{cases} (\Delta_\phi - \partial_t)f = -e^{\lambda_1 t} h^p, \\ (\Delta_\phi - \partial_t)h = -e^{\lambda_2 t} f^q, \end{cases} \quad (1.1)$$

on a complete weighted Riemannian manifold $(M^n, g, e^{-\phi} d\mu)$ along an abstract geometric flow

$$\frac{\partial}{\partial t} g_{ij} = 2S_{ij}, \quad (1.2)$$

where $g(t)$ is an one parameter family of the Riemannian metric evolving along (1.2), $e^{-\phi} d\mu$ is the weighted volume form, ϕ is a twice differentiable function on M , $p, q, \lambda_1, \lambda_2$ are positive constants and $\Delta_\phi := \Delta - \nabla \phi \nabla$ denotes the weighted Laplacian operator, $S_{ij}(t) := \mathcal{S}(e_i, e_j)$ is a smooth symmetric 2-tensor on $(M, g(t))$.

2 Preliminaries

This section is devoted to the basic results and evolution equations that will be used later.

Lemma 2.1 [Weighted Bochner formula][1] For any smooth function u , we have

$$\frac{1}{2} \Delta_\phi |\nabla u|^2 = |\text{Hess } u|^2 + \langle \nabla \Delta_\phi u, \nabla u \rangle + \text{Ric}_\phi(\nabla u, \nabla u),$$

where $\text{Ric}_\phi := \text{Ric} + \text{Hess } \phi$, is called the Bakry-Émery Ricci tensor [2], Hess is the Hessian operator and for any integer $m > n$, the $(m - n)$ -Bakry-Émery Ricci tensor is defined by

$$\text{Ric}_\phi^{m-n} := \text{Ric} + \text{Hess } \phi - \frac{\nabla \phi \otimes \nabla \phi}{m - n}.$$

Lemma 2.2 [1] If u is any smooth function on M then under the geometric flow (1.2), the expression $|\nabla u|^2$ evolves by

$$\frac{\partial}{\partial t} |\nabla u|^2 = -2\mathcal{S}(\nabla u, \nabla u) + 2\langle \nabla u, \nabla u_t \rangle. \quad (2.1)$$

Let $x_0 \in M$ and $R > 0, T > 0$ be any two real numbers. Define a compact subset of $M \times (-\infty, \infty)$ by

$$D_T(2R) = \{(x, t) : d_t(x, x_0) \leq 2R, 0 \leq t \leq T\},$$

where $d_t(x, x_0)$ is the geodesic distance between x and x_0 under the metric $g(t)$.



3 Hamilton type gradient estimate

In this section we derive both local and global Hamilton type estimation for positive solution of (1.1) along (1.2), for this we consider $(f, h) = (u^3, v^3)$ as a positive solution to the system (1.1). Substituting $f = u^3, h = v^3$ in (1.1) we get

$$\begin{cases} 3u^2(\Delta_\phi - \partial_t)u = -6u|\nabla u|^2 - e^{\lambda_1 t} v^{3p}, \\ 3v^2(\Delta_\phi - \partial_t)v = -6v|\nabla v|^2 - e^{\lambda_2 t} u^{3q}, \end{cases} \quad (3.1)$$

Lemma 3.1 *Let (u, v) be a positive solution to the system of equations (3.1). If there exist positive constants k_1, k_2 such that $\text{Ric}_\phi \geq -(n-1)k_1 g$ and $\mathcal{S} \geq -k_2 g$ then the function $\mathcal{H}_1 := u|\nabla u|^2$ satisfies*

$$\begin{aligned} (\Delta_\phi - \partial_t)\mathcal{H}_1 &\geq 4u^{-3}\mathcal{H}_1^2 - 4u^{-1}\langle \nabla u, \nabla \mathcal{H}_1 \rangle - 2pe^{\lambda_1 t} v^{3p-\frac{3}{2}} u^{-\frac{3}{2}} \sqrt{\mathcal{H}_1 \mathcal{H}_2} \\ &\quad + e^{\lambda_1 t} v^{3p} u^{-2} \mathcal{H}_1 - 2((n-1)k_1 + k_2) \mathcal{H}_1 \end{aligned} \quad (3.2)$$

and the function $\mathcal{H}_2 := v|\nabla v|^2$ satisfies

$$\begin{aligned} (\Delta_\phi - \partial_t)\mathcal{H}_2 &\geq 4v^{-3}\mathcal{H}_2^2 - 4v^{-1}\langle \nabla v, \nabla \mathcal{H}_2 \rangle - 2qe^{\lambda_2 t} u^{3q-\frac{3}{2}} v^{-\frac{3}{2}} \sqrt{\mathcal{H}_2 \mathcal{H}_1} \\ &\quad + e^{\lambda_2 t} u^{3q} v^{-2} \mathcal{H}_2 - 2((n-1)k_1 + k_2) \mathcal{H}_2. \end{aligned} \quad (3.3)$$

Proof Differentiating $\mathcal{H}_1 = u|\nabla u|^2$ with respect to t gives

$$\partial_t \mathcal{H}_1 = |\nabla u|^2 u_t + u \partial_t |\nabla u|^2. \quad (3.4)$$

In view of Lemma 2.2, (3.4) gives

$$\partial_t \mathcal{H}_1 = |\nabla u|^2 u_t - 2u\mathcal{S}(\nabla u, \nabla u) + 2u\langle \nabla u, \nabla u_t \rangle. \quad (3.5)$$

Using weighted Bochner formula (Lemma 2.1) we have

$$\begin{aligned} \Delta_\phi \mathcal{H}_1 &= |\nabla u|^2 \Delta_\phi u + 2u|\text{Hess } u|^2 + 2u\langle \nabla \Delta_\phi u, \nabla u \rangle + 2u\text{Ric}_\phi(\nabla u, \nabla u) \\ &\quad + 4\text{Hess } u(\nabla u, \nabla u). \end{aligned} \quad (3.6)$$

Subtracting (3.5) from (3.6) we have

$$\begin{aligned} (\Delta_\phi - \partial_t)\mathcal{H}_1 &= |\nabla u|^2(\Delta_\phi - \partial_t)u + 2u|\text{Hess } u|^2 + 2u\langle \nabla(\Delta_\phi - \partial_t)u, \nabla u \rangle \\ &\quad + 2u(\text{Ric}_\phi + \mathcal{S})(\nabla u, \nabla u) + 4\text{Hess } u(\nabla u, \nabla u). \end{aligned} \quad (3.7)$$

Applying the first equation of the system (3.1) in (3.7) we have

$$\begin{aligned} (\Delta_\phi - \partial_t)\mathcal{H}_1 &= -\frac{1}{3}|\nabla u|^2 e^{\lambda_1 t} v^{3p} u^{-2} + 2u|\text{Hess } u|^2 + 4\text{Hess } u(\nabla u, \nabla u) \\ &\quad + \frac{2}{u}|\nabla u|^4 - 8\text{Hess } u(\nabla u, \nabla u) - 2pe^{\lambda_1 t} v^{3p-1} u^{-1} \langle \nabla u, \nabla v \rangle \\ &\quad + \frac{4}{3}e^{\lambda_1 t} v^{3p} u^{-2} |\nabla u|^2 + 2u(\text{Ric}_\phi + \mathcal{S})(\nabla u, \nabla u). \end{aligned} \quad (3.8)$$

Note that

$$\begin{aligned} 2u|\text{Hess } u|^2 + 4\text{Hess } u(\nabla u, \nabla u) + \frac{2}{u}|\nabla u|^4 &= 2u \left| \text{Hess } u + \frac{\nabla u \otimes \nabla u}{u} \right|^2 \\ &\geq 0. \end{aligned} \quad (3.9)$$

Hence (3.8) reduces to

$$\begin{aligned} (\Delta_\phi - \partial_t)\mathcal{H}_1 &\geq e^{\lambda_1 t} v^{3p} u^{-2} \mathcal{H}_1 - 8\text{Hess } u(\nabla u, \nabla u) - 2pe^{\lambda_1 t} v^{3p-1} u^{-1} |\nabla u| |\nabla v| \\ &\quad + 2u(\text{Ric}_\phi + \mathcal{S})(\nabla u, \nabla u) \\ &= e^{\lambda_1 t} v^{3p} u^{-2} \mathcal{H}_1 - 8\text{Hess } u(\nabla u, \nabla u) - 2pe^{\lambda_1 t} v^{3p-\frac{3}{2}} u^{-\frac{3}{2}} \sqrt{\mathcal{H}_1 \mathcal{H}_2} \\ &\quad + 2u(\text{Ric}_\phi + \mathcal{S})(\nabla u, \nabla u). \end{aligned} \quad (3.10)$$



We see that

$$8\text{Hess } u(\nabla u, \nabla u) = 4u^{-1} \langle \nabla u, \nabla \mathcal{H}_1 \rangle - 4u^{-3} \mathcal{H}_1^2. \quad (3.11)$$

Applying (3.11) and the bounds of Ric_ϕ , $\mathcal{S}$ in (3.10) we get (3.2).

Due to symmetry we can easily obtain the result (3.3) from (3.2).

This completes the proof. $\square$

Let $\tilde{\kappa}_1, \kappa_1, \tilde{\kappa}_2, \kappa_2, k_1, k_2$ be positive constants. Let $T_1 \in (0, T]$ and $(x_1, t_1) \in D_{T_1}(2R)$ be any point. For any $x_1 \in M$ and $R > 0$ we may find a cut-off function $\psi : [0, \infty) \rightarrow [0, 1]$ defined by [11]

$$\psi(s) = \begin{cases} 1, & s \in [0, 1], \\ 0, & s \in [2, \infty), \end{cases} \quad (3.12)$$

belonging to $C^2(M)$ satisfying $-c_0 \leq \psi'(s) \leq 0$, $\psi''(s) \geq -c_1$ and $\frac{|\psi''(s)|^2}{\psi(s)} \leq c_1$, where c_0, c_1 are positive constants. Let $R > 1$ be a constant and define

$$\eta(x, t) = \psi\left(\frac{d_t(x, x_1)}{R}\right). \quad (3.13)$$

To assume everywhere smoothness of ψ so that we can use maximum principle, we apply Calabi's argument [3]. By generalized Laplacian comparison theorem [9, 12, 18] we get

- (1) $\Delta_\phi d_t(x, x_1) \leq (n-1)\sqrt{k_1} \coth(\sqrt{k_1}d_t(x, x_1))$,
- (2) $\Delta_\phi \eta \geq -\frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R}) - \frac{c_1}{R^2}$,
- (3) $\frac{|\nabla \eta|^2}{\eta} \leq \frac{c_1}{R^2}$.

Let

$$\mathcal{G}_1 = \eta \mathcal{H}_1, \quad \mathcal{G}_2 = \eta \mathcal{H}_2. \quad (3.14)$$

We denote

$$\begin{aligned} \Omega_1(x, t) &= \Omega u^3 - e^{\lambda_1 t} v^{3p} u \eta + 2((n-1)k_1 + k_2)\eta u^3, \\ \Omega_2(x, t) &= \frac{4\sqrt{c_1}}{R} u^{\frac{3}{2}}, \\ \Omega_3(x, t) &= 2pe^{\lambda_1 t} v^{3p-\frac{3}{2}} u^{\frac{3}{2}} \eta \sqrt{\mathcal{G}_2}, \\ \bar{\Omega}_1(x, t) &= \Omega v^3 - e^{\lambda_2 t} u^{3q} v \eta + 2((n-1)k_1 + k_2)\eta v^3, \\ \bar{\Omega}_2(x, t) &= \frac{4\sqrt{c_1}}{R} v^{\frac{3}{2}}, \\ \bar{\Omega}_3(x, t) &= 2qe^{\lambda_2 t} u^{3q-\frac{3}{2}} v^{\frac{3}{2}} \eta \sqrt{\mathcal{G}_1}, \end{aligned}$$

and

$$\begin{aligned} \Omega &= \frac{c_0}{R}(m-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2 k_2, \\ \Omega_1^* &= \Omega \kappa_1^3 + 2((n-1)k_1 + k_2)\kappa_1^3, \\ \Omega_2^* &= \frac{4\sqrt{c_1}}{R} \kappa_1^{\frac{3}{2}}, \\ \bar{\Omega}_1^* &= \Omega \kappa_2^3 + 2((n-1)k_1 + k_2)\kappa_2^3, \\ \bar{\Omega}_2^* &= \frac{4\sqrt{c_1}}{R} \kappa_2^{\frac{3}{2}}, \\ \hat{\Omega}_1^* &= c_2 k_2 \kappa_1^3 + 2\kappa_1^3((n-1)k_1 + k_2), \\ \hat{\Omega}_2^* &= c_2 k_2 \kappa_2^3 + 2\kappa_2^3((n-1)k_1 + k_2), \end{aligned}$$

where $\eta, \mathcal{G}_1, \mathcal{G}_2$ are defined in (3.13) and (3.14) respectively.

Theorem 3.1 *If (f, h) is a positive solution to the system (1.1) along the flow (1.2) satisfying $\tilde{\kappa}_1^3 \leq f \leq \kappa_1^3$ and $\tilde{\kappa}_2^3 \leq h \leq \kappa_2^3$ in $D_T(2R)$ and $\text{Ric}_\phi \geq -(n-1)k_1 g$, $\mathcal{S} \geq -k_2 g$ on $D_T(2R)$ with $t \in [0, T]$ then*

$$\begin{cases} \frac{|\nabla f|}{\sqrt{f}} \leq 3\sqrt{\frac{\kappa_1}{\tilde{\kappa}_1}} \left(\sqrt{\frac{l'}{3}} + \sqrt{\frac{2l}{3}} + \left(\frac{3}{2^9} e^{8\lambda_1 t} + \frac{1}{3 \cdot 2^7} e^{8\lambda_2 t} \right)^{\frac{1}{4}} \right), \\ \frac{|\nabla h|}{\sqrt{h}} \leq 3\sqrt{\frac{\kappa_2}{\tilde{\kappa}_2}} \left(\sqrt{\frac{l'}{3}} + \sqrt{\frac{2l'}{3}} + \left(\frac{1}{2^8} e^{8\lambda_1 t} + \frac{1}{3 \cdot 2^6} e^{8\lambda_2 t} \right)^{\frac{1}{4}} \right), \end{cases} \quad (3.15)$$

where $l = \frac{(\Omega_1^*)^2}{2} + \frac{27}{256}(\Omega_2^*)^4 + \frac{3}{2^3} p^{\frac{8}{3}} \kappa_2^{8p-4} \kappa_1^4$ and $l' = \frac{(\bar{\Omega}_1^*)^2}{2} + \frac{27}{256}(\bar{\Omega}_2^*)^4 + \frac{3}{2^3} q^{\frac{8}{3}} \kappa_1^{8q-4} \kappa_2^4$.



Proof Let $\mathcal{G}_1, \mathcal{G}_2$ achieve maximum at $(x_1, t_1) \in D_{T_1}(2R)$. We assume that at (x_1, t_1) , $\mathcal{G}_1 \geq 0, \mathcal{G}_2 \geq 0$, otherwise the proof will be trivial. Hence at that point

$$\nabla \mathcal{G}_1 = 0, \Delta \mathcal{G}_1 \leq 0, \partial_t \mathcal{G}_1 \geq 0 \quad (3.16)$$

and

$$\nabla \mathcal{G}_2 = 0, \Delta \mathcal{G}_2 \leq 0, \partial_t \mathcal{G}_2 \geq 0. \quad (3.17)$$

For ease of calculation we proceed with (3.16) and get

$$\nabla \mathcal{H}_1 = -\frac{\mathcal{H}_1}{\eta} \nabla \eta. \quad (3.18)$$

By [15], there is a constant c_2 such that

$$-\mathcal{H}_1 \eta_t \geq -c_2 k_2 \mathcal{H}_1.$$

Hence

$$\begin{aligned} 0 &\geq (\Delta \phi - \partial_t) \mathcal{G}_1 \\ &= \mathcal{H}_1 (\Delta \phi - \partial_t) \eta + \eta (\Delta \phi - \partial_t) \mathcal{H}_1 \\ &\geq -\Omega \mathcal{H}_1 + \eta (\Delta \phi - \partial_t) \mathcal{H}_1 \end{aligned} \quad (3.19)$$

Using (3.2) in (3.19) we have

$$\begin{aligned} 0 &\geq -\Omega \mathcal{H}_1 + 4\eta u^{-3} \mathcal{H}_1^2 - 4\eta u^{-1} \langle \nabla u, \nabla \mathcal{H}_1 \rangle - 2\eta p e^{\lambda_1 t} v^{3p-\frac{3}{2}} u^{-\frac{3}{2}} \sqrt{\mathcal{H}_1 \mathcal{H}_2} \\ &\quad + \eta e^{\lambda_1 t} v^{3p} u^{-2} \mathcal{H}_1 - 2\eta ((n-1)k_1 + k_2) \mathcal{H}_1. \end{aligned} \quad (3.20)$$

Using the relation $\eta \langle \nabla u, \nabla \mathcal{H}_1 \rangle = -\mathcal{H}_1 \langle \nabla u, \nabla \eta \rangle \leq \frac{\sqrt{c_1}}{R} \eta^{\frac{1}{2}} \mathcal{H}_1 |\nabla u|$ in (3.20) and multiplying the resultant equation with ηu^3 , we obtain

$$\begin{aligned} 0 &\geq -\Omega u^3 \mathcal{G}_1 + 4\mathcal{G}_1^2 - 4\frac{\sqrt{c_1}}{R} u^{\frac{3}{2}} \mathcal{G}_1^{\frac{3}{2}} - 2p e^{\lambda_1 t} v^{3p-\frac{3}{2}} u^{\frac{3}{2}} \eta \sqrt{\mathcal{G}_1 \mathcal{G}_2} + e^{\lambda_1 t} v^{3p} u \eta \mathcal{G}_1 \\ &\quad - 2((n-1)k_1 + k_2) \eta u^3 \mathcal{G}_1, \end{aligned} \quad (3.21)$$

or equivalently,

$$0 \geq 4\mathcal{G}_1^2 - \Omega_1 \mathcal{G}_1 - \Omega_2 \mathcal{G}_1^{\frac{3}{2}} - \Omega_3 \sqrt{\mathcal{G}_1}. \quad (3.22)$$

Applying Young's inequality to the terms in the right hand side of (3.22) we have

$$\Omega_1 \mathcal{G}_1 \leq \frac{(\Omega_1^*)^2}{2} + \frac{\mathcal{G}_1^2}{2}, \quad (3.23)$$

$$\Omega_2 \mathcal{G}_1^{\frac{3}{2}} \leq \frac{27}{256} (\Omega_2^*)^4 + \mathcal{G}_1^2, \quad (3.24)$$

$$\Omega_3 \sqrt{\mathcal{G}_1} \leq \frac{3}{2^{\frac{2}{3}}} p^{\frac{8}{3}} \kappa_2^{8p-4} \kappa_1^4 + \frac{1}{27} e^{8\lambda_1 t} + \mathcal{G}_2^2 + \frac{1}{2} \mathcal{G}_1^2. \quad (3.25)$$

Using (3.23), (3.24) and (3.25) in (3.22) we get

$$2\mathcal{G}_1^2 \leq \frac{(\Omega_1^*)^2}{2} + \frac{27}{256} (\Omega_2^*)^4 + \frac{3}{2^{\frac{2}{3}}} p^{\frac{8}{3}} \kappa_2^{8p-4} \kappa_1^4 + \frac{1}{27} e^{8\lambda_1 t} + \mathcal{G}_2^2. \quad (3.26)$$

Similarly, using (3.17) it can be showed that

$$2\mathcal{G}_2^2 \leq \frac{(\bar{\Omega}_1^*)^2}{2} + \frac{27}{256} (\bar{\Omega}_2^*)^4 + \frac{3}{2^{\frac{2}{3}}} q^{\frac{8}{3}} \kappa_1^{8q-4} \kappa_2^4 + \frac{1}{27} e^{8\lambda_2 t} + \mathcal{G}_1^2. \quad (3.27)$$



For convenience, we denote $l = \frac{(\Omega_1^*)^2}{2} + \frac{27}{256}(\Omega_2^*)^4 + \frac{3}{2^{\frac{8}{3}}}p^{\frac{8}{3}}\kappa_2^{8p-4}\kappa_1^4$ and $l' = \frac{(\bar{\Omega}_1^*)^2}{2} + \frac{27}{256}(\bar{\Omega}_2^*)^4 + \frac{3}{2^{\frac{8}{3}}}q^{\frac{8}{3}}\kappa_1^{8q-4}\kappa_2^4$, thus (3.26) and (3.27) becomes

$$2\mathcal{G}_1^2 \leq l + \frac{1}{2^7}e^{8\lambda_1 t} + \mathcal{G}_2^2, \quad (3.28)$$

$$2\mathcal{G}_2^2 \leq l' + \frac{1}{2^7}e^{8\lambda_2 t} + \mathcal{G}_1^2. \quad (3.29)$$

Applying (3.29) in (3.28) and using the result in (3.29) we deduce

$$\begin{cases} \mathcal{G}_1^2 \leq \frac{1}{3}l' + \frac{2}{3}l + \Psi_1(t), \\ \mathcal{G}_2^2 \leq \frac{1}{3}l + \frac{2}{3}l' + \Psi_2(t), \end{cases} \quad (3.30)$$

where $\Psi_1(t) = \frac{3}{2^9}e^{8\lambda_1 t} + \frac{1}{3 \cdot 2^7}e^{8\lambda_2 t}$ and $\Psi_2(t) = \frac{1}{2^8}e^{8\lambda_1 t} + \frac{1}{3 \cdot 2^6}e^{8\lambda_2 t}$. Following [1], $\tilde{\kappa}_1 \leq u$ and $\tilde{\kappa}_2 \leq v$ implies $\tilde{\kappa}_1^2 \mathcal{H}_1^2 \leq u^2 \mathcal{G}_1^2$ and $\tilde{\kappa}_2^2 \mathcal{H}_2^2 \leq v^2 \mathcal{G}_2^2$ respectively. Hence (3.30) reduces to

$$\begin{cases} \mathcal{H}_1^2 \leq \frac{\kappa_1^2}{\tilde{\kappa}_1^2} \left(\frac{1}{3}l' + \frac{2}{3}l + \Psi_1(t) \right), \\ \mathcal{H}_2^2 \leq \frac{\kappa_2^2}{\tilde{\kappa}_2^2} \left(\frac{1}{3}l + \frac{2}{3}l' + \Psi_2(t) \right). \end{cases} \quad (3.31)$$

Given that $\mathcal{H}_1 = u|\nabla u|^2$, $\mathcal{H}_2 = v|\nabla v|^2$ with $u = f^{\frac{1}{3}}$ and $v = h^{\frac{1}{3}}$, so (3.31) becomes

$$\begin{cases} \left(\frac{1}{9} \frac{|\nabla f|^2}{f} \right)^2 \leq \frac{\kappa_1^2}{\tilde{\kappa}_1^2} \left(\frac{1}{3}l' + \frac{2}{3}l + \Psi_1(t) \right), \\ \left(\frac{1}{9} \frac{|\nabla h|^2}{h} \right)^2 \leq \frac{\kappa_2^2}{\tilde{\kappa}_2^2} \left(\frac{1}{3}l + \frac{2}{3}l' + \Psi_2(t) \right). \end{cases} \quad (3.32)$$

Applying the elementary inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for positive x, y , in (3.32) we get

$$\begin{cases} \frac{|\nabla f|}{\sqrt{f}} \leq 3\sqrt{\frac{\kappa_1}{\tilde{\kappa}_1}} \left(\sqrt[4]{\frac{l'}{3}} + \sqrt[4]{\frac{2l}{3}} + \sqrt[4]{\Psi_1(t)} \right), \\ \frac{|\nabla h|}{\sqrt{h}} \leq 3\sqrt{\frac{\kappa_2}{\tilde{\kappa}_2}} \left(\sqrt[4]{\frac{l}{3}} + \sqrt[4]{\frac{2l'}{3}} + \sqrt[4]{\Psi_2(t)} \right). \end{cases} \quad (3.33)$$

Using the definition of $\Psi_1(t)$, $\Psi_2(t)$ on the above relations completes the proof. $\square$

Now we derive global Hamilton type estimate using Theorem 3.1.

Corollary 3.1 [Global Hamilton type estimate] *If (f, h) is a positive solution to the system (1.1) along the flow (1.2) satisfying $\tilde{\kappa}_1^3 \leq f \leq \kappa_1^3$ and $\tilde{\kappa}_2^3 \leq h \leq \kappa_2^3$ in $M \times [0, T]$ and $\text{Ric}_\phi \geq -(n-1)k_1g$ and $\mathcal{S} \geq -k_2g$ on $M \times [0, T]$ then*

$$\begin{cases} \frac{|\nabla f|}{\sqrt{f}} \leq 3\sqrt{\frac{\kappa_1}{\tilde{\kappa}_1}} \left(\sqrt[4]{\frac{L'}{3}} + \sqrt[4]{\frac{2L}{3}} + \left(\frac{3}{2^9}e^{8\lambda_1 t} + \frac{1}{3 \cdot 2^7}e^{8\lambda_2 t} \right)^{\frac{1}{4}} \right), \\ \frac{|\nabla h|}{\sqrt{h}} \leq 3\sqrt{\frac{\kappa_2}{\tilde{\kappa}_2}} \left(\sqrt[4]{\frac{L}{3}} + \sqrt[4]{\frac{2L'}{3}} + \left(\frac{1}{2^8}e^{8\lambda_1 t} + \frac{1}{3 \cdot 2^6}e^{8\lambda_2 t} \right)^{\frac{1}{4}} \right), \end{cases} \quad (3.34)$$

where $t \in [0, T]$, $L = \frac{1}{2}(\hat{\Omega}_1^*)^2 + \frac{3}{2^{\frac{8}{3}}}p^{\frac{8}{3}}\kappa_2^{8p-4}\kappa_1^4$ and $l' = \frac{1}{2}(\hat{\Omega}_1^*)^2 + \frac{3}{2^{\frac{8}{3}}}q^{\frac{8}{3}}\kappa_1^{8q-4}\kappa_2^4$.

Proof Taking $R \rightarrow +\infty$ in (3.15) we obtain (3.34). $\square$



4 Souplet–Zhang type gradient estimation

In this section we derive Souplet–Zhang type estimation for positive solution of (1.1) along (1.2). Thus throughout this section we consider $(f, h) = (e^u, e^v)$ as a positive solution to the system of equations (1.1). Putting $f = e^u$, $h = e^v$ in (1.1) we have

$$\begin{cases} (\Delta_\phi - \partial_t)u = -|\nabla u|^2 - e^{\lambda_1 t + vp - u} \\ (\Delta_\phi - \partial_t)v = -|\nabla v|^2 - e^{\lambda_2 t + uq - v}. \end{cases} \quad (4.1)$$

Let $\bar{u} = -e^{\lambda_1 t + vp - u}$ and $\bar{v} = -e^{\lambda_2 t + uq - v}$ then system (4.1) reduces to

$$\begin{cases} (\Delta_\phi - \partial_t)u = -|\nabla u|^2 + \bar{u} \\ (\Delta_\phi - \partial_t)v = -|\nabla v|^2 + \bar{v}. \end{cases} \quad (4.2)$$

Let $\kappa_1, \tilde{\kappa}_1, \kappa_2, \tilde{\kappa}_2$ be positive constants. Define $\rho_1 = 1 + \kappa_1$ and $\rho_2 = 1 + \kappa_2$. In this section we assume that a positive solution (f, h) of the system (1.1) satisfies $\tilde{\kappa}_1 \leq f \leq \kappa_1$ and $\tilde{\kappa}_2 \leq h \leq \kappa_2$. Hence a solution (u, v) of (4.2) satisfies $\ln \tilde{\kappa}_1 \leq u \leq \ln \kappa_1$ and $\ln \tilde{\kappa}_2 \leq v \leq \ln \kappa_2$. Let $x_0 \in M$ and $R > 0$ be any real number.

Lemma 4.1 *Let (u, v) be a solution to the equation (4.2). If there exist positive constants k_1, k_2 such that*

$$\text{Ric}_\phi \geq -(n-1)k_1 g \text{ and } \mathcal{S} \geq -k_2 g$$

on $D_T(2R)$, then the function $\mathcal{W}_1 := \frac{|\nabla u|^2}{(\rho_1 - u)^2}$ satisfies

$$\begin{aligned} (\Delta_\phi - \partial_t)\mathcal{W}_1 &\geq 2 \left(\frac{u - \ln \kappa_1}{\rho_1 - u} \right) \langle \nabla \mathcal{W}_1, \nabla u \rangle + \frac{2|\nabla u|^4}{(\rho_1 - u)^3} - \frac{2|\nabla u|^2}{(\rho_1 - u)^3} \bar{u} \\ &\quad - \frac{2\bar{u}}{(\rho_1 - u)^2} \left(p \langle \nabla v, \nabla u \rangle - |\nabla u|^2 \right) - \frac{2}{(\rho_1 - u)^2} ((n-1)k_1 + k_2) |\nabla u|^2, \end{aligned} \quad (4.3)$$

and the function $\mathcal{W}_2 := \frac{|\nabla v|^2}{(\rho_2 - v)^2}$ satisfies

$$\begin{aligned} (\Delta_\phi - \partial_t)\mathcal{W}_2 &\geq 2 \left(\frac{v - \ln \kappa_2}{\rho_2 - v} \right) \langle \nabla \mathcal{W}_2, \nabla v \rangle + \frac{2|\nabla v|^4}{(\rho_2 - v)^3} - \frac{2|\nabla v|^2}{(\rho_2 - v)^3} \bar{v} \\ &\quad - \frac{2\bar{v}}{(\rho_2 - v)^2} \left(q \langle \nabla u, \nabla v \rangle - |\nabla v|^2 \right) - \frac{2}{(\rho_2 - v)^2} (k_1 + k_2) |\nabla v|^2. \end{aligned} \quad (4.4)$$

Proof Using weighted Bochner formula (Lemma 2.1) we have

$$\begin{aligned} \Delta_\phi \mathcal{W}_1 &= \frac{6|\nabla u|^4}{(\rho_1 - u)^4} + \frac{2|\nabla u|^2}{(\rho_1 - u)^3} \Delta_\phi u + \frac{4}{(\rho_1 - u)^3} \langle \nabla |\nabla u|^2, \nabla u \rangle \\ &\quad + \frac{2}{(\rho_1 - u)^2} \left(|\text{Hess } u|^2 + \langle \nabla \Delta_\phi u, \nabla u \rangle + \text{Ric}_\phi(\nabla u, \nabla u) \right). \end{aligned} \quad (4.5)$$

By Lemma 2.2 we find

$$\partial_t \mathcal{W}_1 = -\frac{2}{(\rho_1 - u)^2} \mathcal{S}(\nabla u, \nabla u) + \frac{2}{(\rho_1 - u)^2} \langle \nabla u_t, \nabla u \rangle + \frac{2|\nabla u|^2}{(\rho_1 - u)^3} u_t. \quad (4.6)$$

Subtracting (4.6) from (4.5) then applying (4.2) we get

$$\begin{aligned} (\Delta_\phi - \partial_t)\mathcal{W}_1 &= \frac{6|\nabla u|^4}{(\rho_1 - u)^4} - \frac{2|\nabla u|^4}{(\rho_1 - u)^3} + \frac{2|\nabla u|^2}{(\rho_1 - u)^3} \bar{u} + \frac{8\text{Hess } u(\nabla u, \nabla u)}{(\rho_1 - u)^3} \\ &\quad + \frac{2|\text{Hess } u|^2}{(\rho_1 - u)^2} + \frac{2}{(\rho_1 - u)^2} (\text{Ric}_\phi + \mathcal{S})(\nabla u, \nabla u) \\ &\quad - \frac{4\text{Hess } u(\nabla u, \nabla u)}{(\rho_1 - u)^2} + \frac{2}{(\rho_1 - u)^2} \langle \nabla \bar{u}, \nabla u \rangle. \end{aligned} \quad (4.7)$$



Note that

$$\frac{|\text{Hess } u|^2}{(\rho_1 - u)^2} + \frac{2\text{Hess } u(\nabla u, \nabla u)}{(\rho_1 - u)^3} + \frac{|\nabla u|^4}{(\rho_1 - u)^4} = \frac{1}{(\rho_1 - u)^2} \left| \text{Hess } u + \frac{\nabla u \otimes \nabla u}{\rho_1 - u} \right|^2 \geq 0$$

and

$$\frac{2\text{Hess } u(\nabla u, \nabla u)}{(\rho_1 - u)^2} + \frac{2|\nabla u|^4}{(\rho_1 - u)^3} = \langle \nabla \mathcal{W}_1, \nabla u \rangle.$$

Hence (4.7) reduces to

$$\begin{aligned} (\Delta_\phi - \partial_t) \mathcal{W}_1 &\geq \frac{2\langle \nabla \mathcal{W}_1, \nabla u \rangle}{\rho_1 - u} - 2\langle \nabla \mathcal{W}_1, \nabla u \rangle + \frac{2|\nabla u|^4}{(\rho_1 - u)^3} - \frac{2|\nabla u|^2 \bar{u}}{(\rho_1 - u)^3} \\ &\quad - \frac{2}{(\rho_1 - u)^2} \langle \nabla \bar{u}, \nabla u \rangle + \frac{2}{(\rho_1 - u)^2} (\text{Ric}_\phi + \mathcal{S})(\nabla u, \nabla u) \end{aligned} \quad (4.8)$$

or,

$$\begin{aligned} (\Delta_\phi - \partial_t) \mathcal{W}_1 &\geq 2\langle \nabla \mathcal{W}_1, \nabla u \rangle \left(\frac{u - \ln \kappa_1}{\rho_1 - u} \right) + \frac{2|\nabla u|^4}{(\rho_1 - u)^3} - \frac{2|\nabla u|^2 \bar{u}}{(\rho_1 - u)^3} \\ &\quad - \frac{2\bar{u}}{(\rho_1 - u)^2} (p\langle \nabla v, \nabla u \rangle - |\nabla u|^2) + \frac{2}{(\rho_1 - u)^2} (\text{Ric}_\phi + \mathcal{S})(\nabla u, \nabla u). \end{aligned} \quad (4.9)$$

We apply the bounds of Ric_ϕ and $\mathcal{S}$ in the above equation to find (4.3).

Using the symmetry in (4.2) and applying similar method in (4.3) we obtain (4.4). This completes the proof. $\square$

Theorem 4.1 Let $(f, h) = (e^u, e^v)$ be a solution to the equation (1.1). If there exist positive constants k_1, k_2 such that

$$\text{Ric}_\phi \geq -(n-1)k_1 g \text{ and } \mathcal{S} \geq -k_2 g$$

on $D_T(2R)$, then

$$\begin{cases} \frac{|\nabla f|}{f} \leq \left(1 + \ln\left(\frac{\kappa_1}{\kappa_1}\right)\right) \left(\frac{20^{\frac{1}{4}}}{\sqrt{3}}b + \frac{2}{\sqrt{3}}b'\right), \\ \frac{|\nabla h|}{h} \leq \left(1 + \ln\left(\frac{\kappa_2}{\kappa_2}\right)\right) \left(\frac{2}{\sqrt{3}}b + \frac{20^{\frac{1}{4}}}{\sqrt{2}}b'\right), \end{cases} \quad (4.10)$$

where $b = \sqrt{\Lambda_1} + \frac{27^{\frac{1}{4}}}{\sqrt{2}}\Lambda_2 + \sqrt{p\bar{u}^*}$ and $b' = \sqrt{\bar{\Lambda}_1} + \frac{27^{\frac{1}{4}}}{\sqrt{2}}\Lambda_2 + \sqrt{q\bar{v}^*}$ with

$$\begin{aligned} \Lambda_1 &= \frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R} + \frac{3c_1}{R^2} + c_2k_2) + 4\bar{u}^* + 2((n-1)k_1 + k_2), \\ \Lambda_2 &= 2\ln\left(\frac{\kappa_1}{\kappa_1}\right) \frac{\sqrt{c_1}}{R}, \\ \bar{\Lambda}_1 &= \frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R} + \frac{3c_1}{R^2} + c_2k_2) + 4\bar{v}^* + 2((n-1)k_1 + k_2), \\ \bar{u}^* &= e^{\lambda_1 t + p\bar{\kappa}_2 - \kappa_1}, \\ \bar{v}^* &= e^{\lambda_2 t + q\bar{\kappa}_1 - \kappa_2} \text{ and } t \in [0, T]. \end{aligned}$$

Proof Consider ψ and η as in (3.12), (3.13). Let $G_1 = \eta\mathcal{W}_1$ and $G_2 = \eta\mathcal{W}_2$. Fix $T_2 \in (0, T]$ and assume G_i achieves maximum at $(x_0, t_0) \in D_{T_2}(2R)$ with $G_i(x_0, t_0) \geq 0$ (if $G_i(x_0, t_0) \leq 0$ then the proof is trivial) for $i = 1, 2$.

Hence at (x_0, t_0) we have

$$\nabla G_1 = 0, \Delta G_1 \leq 0, \partial_t G_1 \geq 0, \quad (4.11)$$

$$\nabla G_2 = 0, \Delta G_2 \leq 0, \partial_t G_2 \geq 0. \quad (4.12)$$



Therefore,

$$\nabla \mathcal{W}_1 = -\frac{\mathcal{W}_1}{\eta} \nabla \eta, \quad (4.13)$$

$$\nabla \mathcal{W}_2 = -\frac{\mathcal{W}_2}{\eta} \nabla \eta \quad (4.14)$$

and

$$0 \geq (\Delta_\phi - \partial_t)G_1 = \mathcal{W}_1(\Delta_\phi - \partial_t)\eta + \eta(\Delta_\phi - \partial_t)\mathcal{W}_1 + 2\langle \nabla \mathcal{W}_1, \nabla \eta \rangle, \quad (4.15)$$

$$0 \geq (\Delta_\phi - \partial_t)G_2 = \mathcal{W}_2(\Delta_\phi - \partial_t)\eta + \eta(\Delta_\phi - \partial_t)\mathcal{W}_2 + 2\langle \nabla \mathcal{W}_2, \nabla \eta \rangle. \quad (4.16)$$

By [15], there is a constant c_2 such that

$$-\mathcal{W}_1 \eta_t \geq -c_2 k_2 \mathcal{W}_1, \quad (4.17)$$

$$-\mathcal{W}_2 \eta_t \geq -c_2 k_2 \mathcal{W}_2. \quad (4.18)$$

For ease of calculation we now proceed with (4.11), (4.13), (4.15) and (4.17). Using generalized Laplacian comparison theorem, (4.13) and (4.17) in (4.15) we get

$$0 \geq -\Omega \mathcal{W}_1 + \eta(\Delta_\phi - \partial_t)\mathcal{W}_1, \quad (4.19)$$

where $\Omega = \frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R} + \frac{3c_1}{R^2} + c_2 k_2)$.

Using (4.3) from Lemma 4.1 in (4.19) we obtain

$$\begin{aligned} 0 \geq & -\Omega \mathcal{W}_1 + 2\eta \left(\frac{u - \ln \kappa_1}{\rho_1 - u} \right) \langle \nabla \mathcal{W}_1, \nabla u \rangle + \frac{2\eta |\nabla u|^4}{(\rho_1 - u)^3} - \frac{2\eta |\nabla u|^2}{(\rho_1 - u)^3} \bar{u} \\ & - \frac{2\eta \bar{u}}{(\rho_1 - u)^2} \left(p \langle \nabla v, \nabla u \rangle - |\nabla u|^2 \right) - \frac{2\eta}{(\rho_1 - u)^2} ((n-1)k_1 + k_2) |\nabla u|^2. \end{aligned} \quad (4.20)$$

Multiplying (4.20) with η we get

$$\begin{aligned} 0 \geq & -\Omega G_1 - 2 \ln \left(\frac{\kappa_1}{\bar{\kappa}_1} \right) \frac{\sqrt{c_1}}{R} G_1^{\frac{3}{2}} + 2(\rho_1 - u) G_1^2 - \frac{2G_1}{\rho_1 - u} \bar{u} \\ & - 2\bar{u} \eta^2 p \frac{|\nabla v| |\nabla u|}{(\rho_1 - u)^2} + 2\bar{u} \eta^2 \frac{|\nabla u|^2}{(\rho_1 - u)^2} - 2G_1 ((n-1)k_1 + k_2). \end{aligned} \quad (4.21)$$

We see that $\rho_1 - u \geq 1$, $\rho_2 - v \geq 1$, $\bar{u} \leq -e^{\lambda_1 t + p\bar{\kappa}_2 - \kappa_1}$ and $\eta = 1$ on $D_{T_2}(2R)$. Thus (4.21) becomes

$$0 \geq 2G_1^2 - \Lambda_1 G_1 - \Lambda_2 G_1^{\frac{3}{2}} - \Lambda_3 \sqrt{G_1}. \quad (4.22)$$

where $\Lambda_1 = \Omega + 4e^{\lambda_1 t + p\bar{\kappa}_2 - \kappa_1} + 2((n-1)k_1 + k_2)$, $\Lambda_2 = 2 \ln \left(\frac{\kappa_1}{\bar{\kappa}_1} \right) \frac{\sqrt{c_1}}{R}$, $\Lambda_3 = 2e^{\lambda_1 t + p\bar{\kappa}_2 - \kappa_1} \sqrt{G_2}$. By using Young's inequality in each terms of (4.22) we obtain the following results.

$$\Lambda_1 G_1 \leq \Lambda_1^2 + \frac{G_1^2}{4}, \quad (4.23)$$

$$\Lambda_2 G_1^{\frac{3}{2}} \leq \frac{27}{4} \Lambda_2^4 + \frac{G_1^2}{4}, \quad (4.24)$$

$$\Lambda_3 G_1^{\frac{1}{2}} \leq \frac{3}{4} \Lambda_3^4 + \frac{G_1^2}{4}. \quad (4.25)$$

Using (4.23), (4.24) and (4.25) in (4.22) we get

$$\frac{5}{4} G_1^2 \leq \Lambda_1^2 + \frac{27}{4} \Lambda_2^4 + \frac{3}{4} \Lambda_3^4. \quad (4.26)$$



Applying Young's inequality on $\frac{3}{4}\Lambda_3^{\frac{4}{3}}$ we have

$$\frac{5}{4}G_1^2 \leq \Lambda_1^2 + \frac{27}{4}\Lambda_2^4 + p^2(\bar{u}^*)^2 + G_2^2, \quad (4.27)$$

where $\bar{u}^* = e^{\lambda_1 t + p\tilde{\kappa}_2 - \kappa_1}$. Similarly from (4.12), (4.14), (4.16) and (4.18), we obtain

$$\frac{5}{4}G_2^2 \leq \bar{\Lambda}_1^2 + \frac{27}{4}\Lambda_2^4 + q^2(\bar{v}^*)^2 + G_1^2, \quad (4.28)$$

where $\bar{\Lambda}_1 = \Omega + 4e^{\lambda_2 t + q\tilde{\kappa}_1 - \kappa_2} + 2((n-1)k_1 + k_2)$ and $\bar{v}^* = e^{\lambda_2 t + q\tilde{\kappa}_1 - \kappa_2}$.

Using (4.28) in (4.27) we conclude

$$G_1^2 \leq \frac{20}{9} \left(\Lambda_1^2 + \frac{27}{4}\Lambda_2^4 + p^2(\bar{u}^*)^2 \right) + \frac{16}{9} \left(\bar{\Lambda}_1^2 + \frac{27}{4}\Lambda_2^4 + q^2(\bar{v}^*)^2 \right) \quad (4.29)$$

and using (4.29) in (4.28) we deduce

$$G_2^2 \leq \frac{16}{9} \left(\Lambda_1^2 + \frac{27}{4}\Lambda_2^4 + p^2(\bar{u}^*)^2 \right) + \frac{20}{9} \left(\bar{\Lambda}_1^2 + \frac{27}{4}\Lambda_2^4 + q^2(\bar{v}^*)^2 \right). \quad (4.30)$$

Since $\eta = 1$ on $D_T(2R)$, so putting $u = \ln f$ in (4.29), $v = \ln h$ in (4.30) we get

$$\left(\frac{|\nabla f|^2}{f^2(\rho_1 - \ln f)} \right)^2 \leq \frac{20}{9} \left(\Lambda_1^2 + \frac{27}{4}\Lambda_2^4 + p^2(\bar{u}^*)^2 \right) + \frac{16}{9} \left(\bar{\Lambda}_1^2 + \frac{27}{4}\Lambda_2^4 + q^2(\bar{v}^*)^2 \right), \quad (4.31)$$

$$\left(\frac{|\nabla h|^2}{h^2(\rho_2 - \ln h)} \right)^2 \leq \frac{16}{9} \left(\Lambda_1^2 + \frac{27}{4}\Lambda_2^4 + p^2(\bar{u}^*)^2 \right) + \frac{20}{9} \left(\bar{\Lambda}_1^2 + \frac{27}{4}\Lambda_2^4 + q^2(\bar{v}^*)^2 \right). \quad (4.32)$$

Applying the inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$, for positive x, y twice in each of the above equations and using the fact that $\rho_1 - \ln f \leq 1 + \ln\left(\frac{\kappa_1}{\tilde{\kappa}_1}\right)$, $\rho_2 - \ln h \leq 1 + \ln\left(\frac{\kappa_2}{\tilde{\kappa}_2}\right)$ we obtain

$$\frac{|\nabla f|}{f} \leq \left(1 + \ln\left(\frac{\kappa_1}{\tilde{\kappa}_1}\right) \right) \left(\frac{20^{\frac{1}{4}}}{\sqrt{3}}b + \frac{2}{\sqrt{3}}b' \right), \quad (4.33)$$

$$\frac{|\nabla h|}{h} \leq \left(1 + \ln\left(\frac{\kappa_2}{\tilde{\kappa}_2}\right) \right) \left(\frac{2}{\sqrt{3}}b + \frac{20^{\frac{1}{4}}}{\sqrt{2}}b' \right), \quad (4.34)$$

where $b = \sqrt{\Lambda_1} + \frac{27^{\frac{1}{4}}}{\sqrt{2}}\Lambda_2 + \sqrt{p\bar{u}^*}$ and $b' = \sqrt{\bar{\Lambda}_1} + \frac{27^{\frac{1}{4}}}{\sqrt{2}}\Lambda_2 + \sqrt{q\bar{v}^*}$. This completes the proof. $\square$

As a result we have the global gradient estimate for (1.1) along (1.2) on M .

Corollary 4.1 [Global Souplet-Zhang type estimate] If (f, h) is a positive solution to the system (1.1) along the flow (1.2) satisfying $\tilde{\kappa}_1^3 \leq f \leq \kappa_1^3$ and $\tilde{\kappa}_2^3 \leq h \leq \kappa_2^3$ in $M \times [0, T]$ and $\text{Ric}_\phi \geq -(n-1)k_1g$ and $S \geq -k_2g$ on $M \times [0, T]$ then

$$\begin{cases} \frac{|\nabla f|}{f} \leq \left(1 + \ln\left(\frac{\kappa_1}{\tilde{\kappa}_1}\right) \right) \left(\frac{20^{\frac{1}{4}}}{\sqrt{3}}B + \frac{2}{\sqrt{3}}B' \right), \\ \frac{|\nabla h|}{h} \leq \left(1 + \ln\left(\frac{\kappa_2}{\tilde{\kappa}_2}\right) \right) \left(\frac{2}{\sqrt{3}}B + \frac{20^{\frac{1}{4}}}{\sqrt{2}}B' \right), \end{cases} \quad (4.35)$$

where $B = \sqrt{\hat{\Lambda}_1} + \sqrt{p\bar{u}^*}$ and $B' = \sqrt{\hat{\Lambda}_1} + \sqrt{q\bar{v}^*}$ with

$$\begin{aligned} \hat{\Lambda}_1 &= c_2k_2 + 4\bar{u}^* + 2((n-1)k_1 + k_2), \\ \hat{\Lambda}_1 &= c_2k_2 + 4\bar{v}^* + 2((n-1)k_1 + k_2), \\ \bar{u}^* &= e^{\lambda_1 t + p\tilde{\kappa}_2 - \kappa_1}, \\ \bar{v}^* &= e^{\lambda_2 t + q\tilde{\kappa}_1 - \kappa_2} \text{ and } t \in [0, T]. \end{aligned}$$

Proof From Theorem 4.1 we have the local gradient estimate for positive solutions of (1.1) along (1.2). Taking $R \rightarrow +\infty$ in (4.10) we get (4.35). $\square$



5 Concluding remark

To find an exact solution of a partial differential equation (PDE) in higher dimensions turns out to be impossible in most of the cases. Gradient estimation allows us to uncover possible nature of the solutions for certain PDE's, without calculating its exact solution. In this article we have studied the system (1.1) along (1.2) on a weighted Riemannian manifold and derived certain constants to bound the quantities $\frac{|\nabla f|}{\sqrt{f}}$, $\frac{|\nabla h|}{\sqrt{h}}$, $\frac{|\nabla f|}{f}$ and $\frac{|\nabla h|}{h}$ in a definite way, where (f, h) is a positive solution of (1.1). This allows us to get an idea about the possible upper and lower bound for the gradient of those solutions.

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