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**ORIGINAL RESEARCH**

# String topology of quaternionic Stiefel manifolds

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**Abstract** The Gerstenhaber algebra structure on the rational loop homology $\mathbb{H}_*(LX, \mathbb{Q})$ of a simply connected compact manifold $X$ can be determined in terms of derivations of the minimal Sullivan model of $X$. Using this result we compute the Gerstenhaber algebra structure on the rational loop homology of quaternionic Stiefel manifolds $V_{n,k}(\mathbb{H})$. Further, we compute the $S^1$-equivariant cohomology of $LV_{n,k}(\mathbb{H})$.

**Keywords** Gerstenhaber algebra · Sullivan algebra · Sullivan model · loop homology · loop bracket · string homology

**Mathematics Subject Classification** 55P62 · 55P35

## 1 Introduction

Chas and Sullivan [1] showed that the loop homology $\mathbb{H}_*(LX)$ of an orientable compact manifold $X$ of dimension $n$ can be endowed with algebraic structures viz., a product called "loop product" and a degree $+1$ Lie bracket called "loop bracket", making $\mathbb{H}_*(LX)$ a Gerstenhaber algebra.

By definition, the *loop homology* $\mathbb{H}_*(LX)$ of an orientable compact manifold $X$ of dimension $n$ is the ordinary homology of the free loop space $LX = \mathrm{Map}(S^1, X)$ of $X$ with a shift of degrees by $n$, i.e., $\mathbb{H}_*(LX) = H_{*+n}(LX)$.

A *Gerstenhaber algebra* is a graded commutative algebra $A = \{A_i\}_{i\in\mathbb{Z}}$ equipped with a linear map $\{,\} : (A \otimes A)_i \to A_{i+1}$, called a *Lie bracket*, satisfying

(i) $\{a, b\} = -(-1)^{(|a|+1)(|b|+1)}\{b, a\}$ (anti-commutativity)
(ii) $\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)}\{b, \{a, c\}\}$ (Jacobi identity)
(iii) $\{a, bc\} = \{a, b\}c + (-1)^{|b|(|a|+1)}b\{a, c\}$. (Poisson identity)

If $d : A_i \to A_{i-1}$ is a differential on $A$ satisfying

(iv) $d(ab) = d(a)b + (-1)^{|a|}ad(b)$
(v) $d\{a, b\} = \{da, b\} + (-1)^{(|a|+1)}\{a, db\}$,

with the bracket defined by $\{[a], [b]\} := [\{a, b\}]$, $H_*(A)$ becomes a Gerstenhaber algebra.

For a simply connected compact manifold $X$, Gatsinzi [2] showed that the Gerstenhaber algebra structure on the rational loop homology $\mathbb{H}_*(LX)$ can be determined in terms of derivations of the minimal Sullivan model of $X$.

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This can be concisely described as follows. Let $(\wedge V, d)$ be the minimal Sullivan model of $X$, where $V = \{V^i\}_{i\geq 2}$ is a finite dimensional graded vector space. For $k \in \mathbb{Z}$, let $(\mathrm{Der} \wedge V)_k$ be the vector space of all linear maps $\delta : (\wedge V)^i \to (\wedge V)^{i-k}$ that satisfies $\delta(ab) = \delta(a)b + (-1)^{k|a|}a\delta(b)$, and let $\mathrm{Der} \wedge V = \{(\mathrm{Der} \wedge V)_k\}_{k\in\mathbb{Z}}$. For $i \geq 2$, let $V_i^\# = \mathrm{Hom}(V^i, \mathbb{Q})$ and $V^\# = \{V_i^\#\}_{i\geq 2}$. Let $s^{-1}V^\#$ be the desuspension of $V^\#$ i.e., $\left(s^{-1}V^\#\right)_i = V_{i+1}^\#$. Then with the grading convention $(\wedge V)_i = (\wedge V)^{-i}$, it was shown that $(\wedge V \otimes \wedge s^{-1}V^\#, \{,\})$ is a Gerstenhaber algebra, where the bracket $\{,\}$ is described as follows. For all $a, b \in \wedge V$, and $s^{-1}\theta, s^{-1}\eta \in s^{-1}V^\#$,

$$\{a \otimes 1, b \otimes 1\} = 0, \quad \{a \otimes s^{-1}\theta, b \otimes 1\} = a\widetilde{\theta}(b) \otimes 1,$$
$$\{a \otimes s^{-1}\theta, b \otimes s^{-1}\eta\} = a\widetilde{\theta}(b) \otimes s^{-1}\eta - (-1)^{|a\widetilde{\theta}||b\widetilde{\eta}|}b\widetilde{\eta}(a) \otimes s^{-1}\theta,$$

where $\widetilde{\theta}, \widetilde{\eta}$ are the derivations of $\wedge V$ extended from $\theta$ and $\eta$ respectively.

Let $\wedge_{\wedge V}s^{-1}\,\mathrm{Der} \wedge V$ be the exterior algebra obtained as the quotient of the $\wedge V$-tensor algebra $T_{\wedge V}s^{-1}\,\mathrm{Der} \wedge V = \oplus_{k\geq 0}T_{\wedge V}^k s^{-1}\,\mathrm{Der} \wedge V$, by the ideal generated by elements of the form $x \otimes y - (-1)^{|x||y|}y \otimes x$, where $T_{\wedge V}^k s^{-1}\,\mathrm{Der} \wedge V = s^{-1}\,\mathrm{Der} \wedge V \otimes_{\wedge V} \cdots \otimes_{\wedge V} s^{-1}\,\mathrm{Der} \wedge V$ ($k$-times), and

$T_{\wedge V}^0 s^{-1}\,\mathrm{Der} \wedge V = \wedge V$. Since $V$ is finite dimensional it is isomorphic to $V^\#$, and this gives rise to a $\wedge V$-algebra isomorphism

$$\varphi : \wedge_{\wedge V}s^{-1}\,\mathrm{Der} \wedge V \to \wedge V \otimes \wedge s^{-1}V^\#.$$

Using this isomorphism, Gatsinzi [2] defined a differential $D$ on $\wedge V \otimes \wedge s^{-1}V^\#$ by

$$D(x) = -\{\varphi(s^{-1}d), x\},$$

for $x \in \wedge V \otimes \wedge s^{-1}V^\#$, which satisfies condition (iv) and (v) above, and he established that $\mathbb{H}_*(LX)$ is isomorphic to $H_*(\wedge V \otimes s^{-1}V^\#)$ as Gerstenhaber algebras.

If $(\wedge V, d)$, $V = \{V^i\}_{i\geq 2}$, is a Sullivan model of a simply connected topological space $X$, with $V$ finite dimensional, we call $\left(\wedge V \otimes \wedge s^{-1}V^\#, \{,\}, D\right)$ a *differential Sullivan Gerstenhaber algebra model of X*.

Let $X$ be a simply connected compact manifold whose minimal Sullivan model is $(\wedge V, 0)$, where $V = \{V^i\}_{i\geq 2}$ is finite dimensional. Then interestingly it turns out that the differential $D$ on $\wedge V \otimes \wedge s^{-1}V^\#$ is also zero so that the Gerstenhaber algebra structure on $\mathbb{H}_*(LX, \mathbb{Q})$ is just the differential Sullivan Gerstenhaber algebra model of $X$ itself.

Let $\mathbb{H}$ denote the field of quaternions. For a positive integer $n$, let $Sp(n)$ denote the symplectic group which consists of all linear transformations of $\mathbb{R}^{4n}$ that respects the norm $\|\omega\| = \sqrt{\omega\overline{\omega}}$, where $\omega \in \mathbb{H}$ and $\overline{\omega}$ the conjugate of $\omega$. For $1 \leq k \leq n$, the quaternionic Stiefel manifold $V_{n,k}(\mathbb{H})$ of orthonormal $k$-frames in $\mathbb{H}^n$ is given by, $V_{n,k}(\mathbb{H}) = Sp(n)/Sp(n-k)$, if $1 \leq k < n$, and $V_{n,n}(\mathbb{H}) = Sp(n)$. Since $V_{n,k}(\mathbb{H})$ is simply connected and has minimal Sullivan model of the form $(\wedge V, 0)$ [3], from the preceding paragraph we have, as Gerstenhaber algebra $\mathbb{H}_*(LV_{n,k}(\mathbb{H}), \mathbb{Q})$ is nothing but the differential Sullivan Gerstenhaber algebra model of $V_{n,k}(\mathbb{H})$.

In [1], Chas and Sullivan also defined a BV operator on $\mathbb{H}_*(LX)$ and showed that the loop product with this operator makes $\mathbb{H}_*(LX)$ a Batalin-Vilkovisky (BV) algebra. Recall, a *Batalin-Vilkovisky algebra* is a graded commutative algebra $A = \{A_i\}_{i\in\mathbb{Z}}$ with an operator $\Delta : A_i \to A_{i+1}$, known as a BV operator, that satisfies $\Delta \circ \Delta = 0$ and

$$\begin{aligned}\Delta(abc) = \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac)\\ - \Delta(a)bc - (-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c).\end{aligned}$$

If we denote the loop product, the loop bracket and the BV operator on $\mathbb{H}_*(LX)$ by $\bullet$, $\{,\}$ and $\Delta$ respectively, then as shown in [1] the three are related as

$$\{a, b\} = (-1)^{|a|}\Delta(a \cdot b) - (-1)^{|a|}\Delta(a) \cdot b - a \cdot \Delta(b). \tag{1}$$

In [4], H. Tamanoi calculated the structure of a Batalin-Vilkovisky (BV) algebra on the integral loop homology of complex Stiefel manifolds, whence the Gerstenhaber algebra structure on the same can be determined using the above relation (1). In this paper we show explicitly the Gerstenhaber algebra structure on the loop homology of quaternionic Stiefel manifolds with rational coefficients.

Given an orientable compact manifold $X$, one can see that $S^1$ acts diagonally on $ES^1 \times LX$, where $ES^1$ is the total space of the universal $S^1$-bundle. Let $ES^1 \times_{S^1} LX$ be the quotient of the diagonal action of $S^1$ on $ES^1 \times LX$, i.e.,

$$ES^1 \times_{S^1} LX := (ES^1 \times LX)/S^1.$$

The $S^1$-*equivariant homology (cohomology)* of $LX$ is defined as the homology (cohomology) of $ES^1 \times_{S^1} LX$. It is called the *string homology (cohomology)* of $X$, denoted by $H_*^{S^1}(LX)$ $\left(H^*_{S^1}(LX)\right)$.

Let $X$ be a simply connected space with $(\wedge(x_1, x_2, \ldots), d)$ as its minimal Sullivan model, then it has been proved by Vigué-Poirrier and Sullivan in [5] that the minimal Sullivan model of the loop space $LX$ is $(\wedge(x_1, y_1, x_2, y_2, \ldots), \delta)$, where $|y_1| = |x_1| - 1$ and the differential $\delta$ is defined by $\delta(x_i) = d(x_i)$ and on $y_i$'s it is defined using $\delta \circ s + s \circ \delta = 0$, where $s : \wedge(x_1, y_1, x_2, y_2, \ldots) \to \wedge(x_1, y_1, x_2, y_2, \ldots)$ is the derivation defined by $s(x_i) = y_i$ and $s(y_i) = 0$. Furthermore in [6], Vigué-Poirrier and Burghelea proved that the minimal Sullivan model of $ES^1 \times_{S^1} LX$ is $(\wedge(x_1, y_1, x_2, y_2, \ldots, u), D)$, where $|u| = 2$, and the differential $D$ is given by $D(u) = 0$, $D(z) = \delta(z) + s(z)u$ for $z \in \wedge(x_1, y_1, x_2, y_2, \ldots)$.

Basu in his dissertation [7] computed the string cohomology of projective spaces, spheres and product of spheres over rational coefficients. Here we compute the string cohomology of quaternionic Stiefel manifolds over rational coefficient.

In what follows, the underlying field is the field $\mathbb{Q}$ of rational numbers.

## 2 Some results on the Gerstenhaber algebra structure on the loop homology of a simply connected manifold

First we state the following theorem which is due to Gatsinzi [2].

**Theorem 1** *Let $(\wedge V, d)$ be the minimal Sullivan model of a simply connected compact manifold $X$, where $V = \{V^i\}_{i \geq 2}$ is finite dimensional. Then $\mathbb{H}_*(LX) \cong H_*((\wedge V \otimes \wedge s^{-1}V^{\#}, D))$ as Gerstenhaber algebras.*

Now in the above theorem if we restrict to simply connected manifolds $X$, whose minimal Sullivan models are of the type $(\wedge V, 0)$, with $V = \{V^i\}_{i \geq 2}$ finite dimensional, we have the following corollary which serves well in computing the Gerstenhaber algebra structure of such type of spaces.

**Corollary 1** *Let $X$ be a simply connected compact manifold with minimal Sullivan model $(\wedge V, 0)$, where $V = \{V^i\}_{i \geq 2}$ is finite dimensional. Then $\mathbb{H}_*(LX) \cong \wedge V \otimes \wedge s^{-1}V^{\#}$ as Gerstenhaber algebras.*

*Proof* Let $x \in \wedge V \otimes \wedge s^{-1}V^{\#}$. By definition $D(x) = -\{\varphi(s^{-1}0), x\} = -\{0, x\} = 0$. Therefore $\mathbb{H}_*(LX) \cong H_*((\wedge V \otimes \wedge s^{-1}V^{\#}, 0)) \cong \wedge V \otimes \wedge s^{-1}V^{\#}$. □

We apply corollary 1 to compute the Gerstenhaber algebra structure on the rational loop homology of quaternionic Stiefel manifolds.

## 3 Gerstenhaber algebra structure on $\mathbb{H}_*(LV_{n,k}(\mathbb{H}), \mathbb{Q})$

Recall that the minimal Sullivan model of $V_{n,k}(\mathbb{H})$ is $(\wedge(y_1, \ldots, y_k), 0)$, where $|y_i| = 4(n-k+i) - 1$ [3]. As a direct consequence of Corollary 1 we have proved the following proposition.

**Proposition 2** *For $1 \leq k \leq n$, the Gerstenhaber algebra structure on the rational loop homology of $V_{n,k}(\mathbb{H})$ is given by*

$$\mathbb{H}_*(LV_{n,k}(\mathbb{H}), \mathbb{Q}) \cong \wedge(y_1, \ldots, y_k) \otimes \wedge(s^{-1}\eta_1, \ldots, s^{-1}\eta_k),$$

*where $\{\eta_1, \ldots, \eta_k\}$ is the dual basis corresponding to the basis $\{y_1, \ldots, y_k\}$.*

The next proposition gives an explicit expression of the bracket on $\wedge(y_1, \ldots, y_k) \otimes \wedge(s^{-1}\eta_1, \ldots, s^{-1}\eta_k)$ that corresponds to the loop bracket on $\mathbb{H}_*(LV_{n,k}(\mathbb{H}), \mathbb{Q})$.

**Proposition 3** *The bracket on the rational loop homology*

$$\mathbb{H}_*(LV_{n,k}(\mathbb{H}), \mathbb{Q}) \cong \wedge(y_1, \ldots, y_k) \otimes \wedge(s^{-1}\eta_1, \ldots, s^{-1}\eta_k)$$

*of $V_{n,k}(\mathbb{H})$ is given by:*
*for $k = 1$,*

$$\left\{y_1^{i_1} \otimes s^{-1}\eta_1^{p_1}, y_1^{j_1} \otimes s^{-1}\eta_1^{q_1}\right\} = (j_1 p_1 - i_1 q_1)(y_1^{i_1+j_1-1} \otimes s^{-1}\eta_1^{p_1+q_1-1}), \tag{2}$$

*for* $k=2$,

$$\begin{aligned}&\left\{y_1^{i_1}y_2^{i_2}\otimes s^{-1}\eta_1^{p_1}s^{-1}\eta_2^{p_2},\, y_1^{j_1}y_2^{j_2}\otimes s^{-1}\eta_1^{q_1}s^{-1}\eta_2^{q_2}\right\}\\&\quad=(-1)^{i_2j_1}\Big[(-1)^{i_2}(j_1p_1-i_1q_1)(y_1^{i_1+j_1-1}y_2^{i_2+j_2}\otimes s^{-1}\eta_1^{p_1+q_1-1}s^{-1}\eta_2^{p_2+q_2})\\&\qquad+(-1)^{j_1}(j_2p_2-i_2q_2)(y_1^{i_1+j_1}y_2^{i_2+j_2-1}\otimes s^{-1}\eta_1^{p_1+q_1}s^{-1}\eta_2^{p_2+q_2-1})\Big],\end{aligned}\tag{3}$$

*for* $3\leq k\leq n$,

$$\begin{aligned}&\left\{y_1^{i_1}\cdots y_k^{i_k}\otimes s^{-1}\eta_1^{p_1}\cdots s^{-1}\eta_k^{p_k},\, y_1^{j_1}\cdots y_k^{j_k}\otimes s^{-1}\eta_1^{q_1}\cdots s^{-1}\eta_k^{q_k}\right\}\\&\quad=(-1)^{\sum_{1\leq m<l\leq k}i_lj_m}\Big[(-1)^{\sum_{s=2}^k i_s}(j_1p_1-i_1q_1)(y_1^{i_1+j_1-1}\cdots y_k^{i_k+j_k}\otimes\\&s^{-1}\eta_1^{p_1+q_1-1}\cdots s^{-1}\eta_k^{p_k+q_k})+\sum_{r=2}^{k-1}(-1)^{\sum_{s=r+1}^k i_s+\sum_{t=1}^{r-1}j_t}(j_rp_r-i_rq_r)\\&(y_1^{i_1+j_1}\cdots y_r^{i_r+j_r-1}\cdots y_k^{i_k+j_k}\otimes s^{-1}\eta_1^{p_1+q_1}\cdots s^{-1}\eta_r^{p_r+q_r-1}\cdots s^{-1}\eta_k^{p_k+q_k})+\\&(-1)^{\sum_{t=1}^{k-1}j_t}(j_kp_k-i_kq_k)(y_1^{i_1+j_1}\cdots y_k^{i_k+j_k-1}\otimes s^{-1}\eta_1^{p_1+q_1}\cdots s^{-1}\eta_k^{p_k+q_k-1})\Big],\end{aligned}\tag{4}$$

*where* $0\leq i_1,\ldots,i_k,j_1,\ldots,j_k\leq 1$, *and* $p_1,\ldots,p_k,q_1,\ldots,q_k\geq 0$.

*Proof* For $k=1$, we have

$$\begin{aligned}&\left\{y_1^{i_1}\otimes s^{-1}\eta_1^{p_1},\, y_1^{j_1}\otimes s^{-1}\eta_1^{q_1}\right\}\\&\quad=\left\{(y_1^{i_1}\otimes 1)(1\otimes s^{-1}\eta_1^{p_1}),\,(y_1^{j_1}\otimes 1)(1\otimes s^{-1}\eta_1^{q_1})\right\}\\&\quad=(y_1^{i_1}\otimes 1)\left\{1\otimes s^{-1}\eta_1^{p_1},\,y_1^{j_1}\otimes 1\right\}(1\otimes s^{-1}\eta_1^{q_1})+\\&\qquad(1\otimes s^{-1}\eta_1^{p_1})\left\{y_1^{i_1}\otimes 1,\,1\otimes s^{-1}\eta_1^{q_1}\right\}(y_1^{j_1}\otimes 1)\\&\quad=j_1p_1(y_1^{i_1}\otimes 1)(1\otimes s^{-1}\eta_1^{p_1-1})(y_1^{j_1-1}\otimes 1)(1\otimes s^{-1}\eta_1^{q_1})-\\&\qquad i_1q_1(1\otimes s^{-1}\eta_1^{p_1})(y_1^{i_1-1}\otimes 1)(1\otimes s^{-1}\eta_1^{q_1-1})(y_1^{j_1}\otimes 1)\\&\quad=j_1p_1(y_1^{i_1+j_1-1}\otimes s^{-1}\eta_1^{p_1+q_1-1})-i_1q_1(y_1^{i_1+j_1-1}\otimes s^{-1}\eta_1^{p_1+q_1-1})\\&\quad\Longrightarrow\ \left\{y_1^{i_1}\otimes s^{-1}\eta_1^{p_1},\,y_1^{j_1}\otimes s^{-1}\eta_1^{q_1}\right\}=(j_1p_1-i_1q_1)(y_1^{i_1+j_1-1}\otimes s^{-1}\eta_1^{p_1+q_1-1}).\end{aligned}$$

For $k=2$,

$$\begin{aligned}&\left\{y_1^{i_1}y_2^{i_2}\otimes s^{-1}\eta_1^{p_1}s^{-1}\eta_2^{p_2},\, y_1^{j_1}y_2^{j_2}\otimes s^{-1}\eta_1^{q_1}s^{-1}\eta_2^{q_2}\right\}\\&\quad=\left\{(y_1^{i_1}\otimes s^{-1}\eta_1^{p_1})(y_2^{i_2}\otimes s^{-1}\eta_2^{p_2}),\,(y_1^{j_1}\otimes s^{-1}\eta_1^{q_1})(y_2^{j_2}\otimes s^{-1}\eta_2^{q_2})\right\}\\&\quad=(-1)^{i_1i_2}(y_2^{i_2}\otimes s^{-1}\eta_2^{p_2})\left\{y_1^{i_1}\otimes s^{-1}\eta_1^{p_1},\,y_1^{j_1}\otimes s^{-1}\eta_1^{q_1}\right\}(y_2^{j_2}\otimes s^{-1}\eta_2^{q_2})+\\&\qquad(-1)^{j_1j_2}(y_1^{i_1}\otimes s^{-1}\eta_1^{p_1})\left\{y_2^{i_2}\otimes s^{-1}\eta_2^{p_2},\,y_2^{j_2}\otimes s^{-1}\eta_2^{q_2}\right\}(y_1^{j_1}\otimes s^{-1}\eta_1^{q_1}).\end{aligned}$$

Now by applying the result (2) of the bracket for $k=1$, we get

$$\begin{aligned}&\left\{y_1^{i_1}y_2^{i_2}\otimes s^{-1}\eta_1^{p_1}s^{-1}\eta_2^{p_2},\, y_1^{j_1}y_2^{j_2}\otimes s^{-1}\eta_1^{q_1}s^{-1}\eta_2^{q_2}\right\}\\&\quad=(-1)^{i_2j_1}\Big[(-1)^{i_2}(j_1p_1-i_1q_1)(y_1^{i_1+j_1-1}y_2^{i_2+j_2}\otimes s^{-1}\eta_1^{p_1+q_1-1}s^{-1}\eta_2^{p_2+q_2})\\&\qquad+(-1)^{j_1}(j_2p_2-i_2q_2)(y_1^{i_1+j_1}y_2^{i_2+j_2-1}\otimes s^{-1}\eta_1^{p_1+q_1}s^{-1}\eta_2^{p_2+q_2-1})\Big].\end{aligned}$$

Finally, for the case $3 \leq k \leq n$, we prove by induction on $k$. Let $k = 3$. Now

$$\begin{aligned}
&\left\{y_1^{i_1} y_2^{i_2} y_3^{i_3} \otimes s^{-1}\eta_1^{p_1} s^{-1}\eta_2^{p_2} s^{-1}\eta_3^{p_3},\, y_1^{j_1} y_2^{j_2} y_3^{j_3} \otimes s^{-1}\eta_1^{q_1} s^{-1}\eta_2^{q_2} s^{-1}\eta_3^{q_3}\right\}\\
&= \left\{(y_1^{i_1} \otimes s^{-1}\eta_1^{p_1})(y_2^{i_2} y_3^{i_3} \otimes s^{-1}\eta_2^{p_2} s^{-1}\eta_3^{p_3}),\, (y_1^{j_1} \otimes s^{-1}\eta_1^{q_1})(y_2^{j_2} y_3^{j_3} \otimes s^{-1}\eta_2^{q_2} s^{-1}\eta_3^{q_3})\right\}\\
&= (-1)^{i_1(i_2+i_3)}(y_2^{i_2} y_3^{i_3} \otimes s^{-1}\eta_2^{p_2} s^{-1}\eta_3^{p_3})\left\{y_1^{i_1} \otimes s^{-1}\eta_1^{p_1},\, y_1^{j_1} \otimes s^{-1}\eta_1^{q_1}\right\}(y_2^{j_2} y_3^{j_3} \otimes s^{-1}\eta_2^{q_2} s^{-1}\eta_3^{q_3})\\
&\quad + (-1)^{j_1(j_2+j_3)}(y_1^{i_1} \otimes s^{-1}\eta_1^{p_1})\left\{y_2^{i_2} y_3^{i_3} \otimes s^{-1}\eta_2^{p_2} s^{-1}\eta_3^{p_3},\, y_2^{j_2} y_3^{j_3} \otimes s^{-1}\eta_2^{q_2} s^{-1}\eta_3^{q_3}\right\}(y_1^{j_1} \otimes s^{-1}\eta_1^{q_1}).
\end{aligned}$$

Applying the results (2) and (3) of the bracket for $k = 1$ and $k = 2$ respectively, we get

$$\begin{aligned}
&\left\{y_1^{i_1} y_2^{i_2} y_3^{i_3} \otimes s^{-1}\eta_1^{p_1} s^{-1}\eta_2^{p_2} s^{-1}\eta_3^{p_3},\, y_1^{j_1} y_2^{j_2} y_3^{j_3} \otimes s^{-1}\eta_1^{q_1} s^{-1}\eta_2^{q_2} s^{-1}\eta_3^{q_3}\right\}\\
&= (-1)^{\sum_{1\leq m<l\leq 3} i_l j_m}\Big[(-1)^{i_2+i_3}(j_1p_1 - i_1q_1)(y_1^{i_1+j_1-1} y_2^{i_2+j_2} y_3^{i_3+j_3} \otimes s^{-1}\eta_1^{p_1+q_1-1} s^{-1}\eta_2^{p_2+q_2} s^{-1}\eta_3^{p_3+q_3})\\
&\quad + (-1)^{i_3+j_1}(j_2p_2 - i_2q_2)(y_1^{i_1+j_1} y_2^{i_2+j_2-1} y_3^{i_3+j_3} \otimes s^{-1}\eta_1^{p_1+q_1} s^{-1}\eta_2^{p_2+q_2-1} s^{-1}\eta_3^{p_3+q_3})\\
&\quad + (-1)^{j_1+j_2}(j_3p_3 - i_3q_3)(y_1^{i_1+j_1} y_2^{i_2+j_2} y_3^{i_3+j_3-1} \otimes s^{-1}\eta_1^{p_1+q_1} s^{-1}\eta_2^{p_2+q_2} s^{-1}\eta_3^{p_3+q_3-1})\Big].
\end{aligned}$$

Therefore (4) holds for $k = 3$. Assume that the equation (4) holds. We will show that the same holds for $k + 1$. We have

$$\begin{aligned}
&\left\{y_1^{i_1} \cdots y_{k+1}^{i_{k+1}} \otimes s^{-1}\eta_1^{p_1} \cdots s^{-1}\eta_{k+1}^{p_{k+1}},\, y_1^{j_1} \cdots y_{k+1}^{j_{k+1}} \otimes s^{-1}\eta_1^{q_1} \cdots s^{-1}\eta_{k+1}^{q_{k+1}}\right\}\\
&\quad = \Big\{(y_1^{i_1} \otimes s^{-1}\eta_1^{p_1})(y_2^{i_2} \cdots y_{k+1}^{i_{k+1}} \otimes s^{-1}\eta_2^{p_2} \cdots s^{-1}\eta_{k+1}^{p_{k+1}}),\\
&\quad\quad (y_1^{j_1} \otimes s^{-1}\eta_1^{q_1})(y_2^{j_2} \cdots y_{k+1}^{j_{k+1}} \otimes s^{-1}\eta_2^{q_2} \cdots s^{-1}\eta_{k+1}^{q_{k+1}})\Big\}\\
&\quad = (-1)^{i_1(i_2+\cdots+i_{k+1})}(y_2^{i_2} \cdots y_{k+1}^{i_{k+1}} \otimes s^{-1}\eta_2^{p_2} \cdots s^{-1}\eta_{k+1}^{p_{k+1}})\\
&\quad\quad \left\{y_1^{i_1} \otimes s^{-1}\eta_1^{p_1},\, y_1^{j_1} \otimes s^{-1}\eta_1^{q_1}\right\}(y_2^{j_2} \cdots y_{k+1}^{j_{k+1}} \otimes s^{-1}\eta_1^{q_2} \cdots s^{-1}\eta_{k+1}^{q_{k+1}})\\
&\quad\quad + (-1)^{j_1(j_2+\cdots j_{k+1})}(y_1^{i_1} \otimes s^{-1}\eta_1^{p_1})\Big\{y_2^{i_2} \cdots y_{k+1}^{i_{k+1}} \otimes s^{-1}\eta_2^{p_2} \cdots s^{-1}\eta_{k+1}^{p_{k+1}},\\
&\quad\quad y_2^{j_2} \cdots y_{k+1}^{j_{k+1}} \otimes s^{-1}\eta_2^{q_2} \cdots s^{-1}\eta_{k+1}^{q_{k+1}}\Big\}(y_1^{j_1} \otimes s^{-1}\eta_1^{q_1}).
\end{aligned}$$

Using the results (2) and (4) we get

$$\begin{aligned}
&\left\{y_1^{i_1} \cdots y_{k+1}^{i_{k+1}} \otimes s^{-1}\eta_1^{p_1} \cdots s^{-1}\eta_{k+1}^{p_{k+1}},\, y_1^{j_1} \cdots y_{k+1}^{j_{k+1}} \otimes s^{-1}\eta_1^{q_1} \cdots s^{-1}\eta_{k+1}^{q_{k+1}}\right\}\\
&\quad = (-1)^{i_1(i_2+\cdots+i_{k+1})}(y_2^{i_2} \cdots y_{k+1}^{i_{k+1}} \otimes s^{-1}\eta_2^{p_2} \cdots s^{-1}\eta_{k+1}^{p_{k+1}})\\
&\quad\quad (j_1p_1 - i_1q_1)(y_1^{i_1+j_1-1} \otimes s^{-1}\eta_1^{p_1+q_1-1})(y_2^{j_2} \cdots y_{k+1}^{j_{k+1}} \otimes s^{-1}\eta_2^{q_2} \cdots s^{-1}\eta_{k+1}^{q_{k+1}})\\
&\quad\quad + (-1)^{j_1(j_2+\cdots j_{k+1})}(y_1^{i_1} \otimes s^{-1}\eta_1^{p_1})\Big[(-1)^{\sum_{2\leq m<l\leq k+1} i_l j_m}\Big((-1)^{\sum_{s=3}^{k+1} i_s}(j_2p_2 - i_2q_2)\\
&\quad\quad (y_2^{i_2+j_2-1} \cdots y_{k+1}^{i_{k+1}+j_{k+1}} \otimes s^{-1}\eta_2^{p_2+q_2-1} \cdots s^{-1}\eta_{k+1}^{p_{k+1}+q_{k+1}})\\
&\quad\quad + \sum_{r=3}^{k}(-1)^{\sum_{s=r+1}^{k+1} i_s + \sum_{t=2}^{r-1} j_t}(j_rp_r - i_rq_r)\\
&\quad\quad (y_2^{i_2+j_2} \cdots y_r^{i_r+j_r-1} \cdots y_{k+1}^{i_{k+1}+j_{k+1}} \otimes s^{-1}\eta_2^{p_2+q_2} \cdots s^{-1}\eta_r^{p_r+q_r-1} \cdots s^{-1}\eta_{k+1}^{p_{k+1}+q_{k+1}})\\
&\quad\quad + (-1)^{\sum_{t=2}^{k} j_t}(j_{k+1}p_{k+1} - i_{k+1}q_{k+1})\\
&\quad\quad (y_2^{i_2+j_2} \cdots y_{k+1}^{i_{k+1}+j_{k+1}-1} \otimes s^{-1}\eta_2^{p_2+q_2} \cdots s^{-1}\eta_{k+1}^{p_{k+1}+q_{k+1}-1})\Big)\Big](y_1^{j_1} \otimes s^{-1}\eta_1^{q_1})\\
&\quad = (-1)^{\sum_{1\leq m<l\leq k+1} i_l j_m}\Big[(-1)^{\sum_{s=2}^{k+1} i_s}(j_1p_1 - i_1q_1)\\
&\quad\quad (y_1^{i_1+j_1-1} \cdots y_{k+1}^{i_{k+1}+j_{k+1}} \otimes s^{-1}\eta_1^{p_1+q_1-1} \cdots s^{-1}\eta_{k+1}^{p_{k+1}+q_{k+1}})
\end{aligned}$$

$$+\sum_{r=2}^{k}(-1)^{\sum_{s=r+1}^{k+1} i_s+\sum_{t=1}^{r-1} j_t}(j_r p_r - i_r q_r)(y_1^{i_1+j_1}\cdots y_r^{i_r+j_r-1}\cdots y_{k+1}^{i_{k+1}+j_{k+1}}$$
$$\otimes s^{-1}\eta_1^{p_1+q_1}\cdots s^{-1}\eta_r^{p_r+q_r-1}\cdots s^{-1}\eta_{k+1}^{p_{k+1}+q_{k+1}}) + (-1)^{\sum_{t=1}^{k} j_t}(j_{k+1}p_{k+1} - i_{k+1}q_{k+1})$$
$$(y_1^{i_1+j_1}\cdots y_{k+1}^{i_{k+1}+j_{k+1}-1}\otimes s^{-1}\eta_1^{p_1+q_1}\cdots s^{-1}\eta_{k+1}^{p_{k+1}+q_{k+1}-1})\Big].$$

□

Since $V_{n,n}(\mathbb{H}) = Sp(n)$, one obtains the Gerstenhaber algebra structure on $\mathbb{H}_*(LSp(n),\mathbb{Q})$:

**Corollary 2** *The Gerstenhaber algebra structure on the rational loop homology of $Sp(n)$ is given by*

$$\mathbb{H}_*(LSp(n),\mathbb{Q}) \cong \wedge(y_1, y_2 \ldots, y_n)\otimes\wedge(s^{-1}\eta_1, s^{-1}\eta_2,\ldots, s^{-1}\eta_n),$$

*where $\{\eta_1,\eta_2\ldots,\eta_n\}$ is the dual basis corresponding to the basis $\{y_1, y_2,\ldots, y_n\}$, and the bracket on*

$$\mathbb{H}_*(LSp(n),\mathbb{Q}) \cong \wedge(y_1, y_2 \ldots, y_n)\otimes\wedge(s^{-1}\eta_1, s^{-1}\eta_2,\ldots, s^{-1}\eta_n)$$

*is:*
*for $n = 1$,*

$$\Big\{y_1^{i_1}\otimes s^{-1}\eta_1^{p_1}, y_1^{j_1}\otimes s^{-1}\eta_1^{q_1}\Big\} = (j_1p_1 - i_1q_1)(y_1^{i_1+j_1-1}\otimes s^{-1}\eta_1^{p_1+q_1-1}),$$

*for $n = 2$,*

$$\Big\{y_1^{i_1}y_2^{i_2}\otimes s^{-1}\eta_1^{p_1}s^{-1}\eta_2^{p_2}, y_1^{j_1}y_2^{j_2}\otimes s^{-1}\eta_1^{q_1}s^{-1}\eta_2^{q_2}\Big\}$$
$$= (-1)^{i_2 j_1}\Big[(-1)^{i_2}(j_1p_1 - i_1q_1)(y_1^{i_1+j_1-1}y_2^{i_2+j_2}\otimes s^{-1}\eta_1^{p_1+q_1-1}s^{-1}\eta_2^{p_2+q_2})$$
$$+ (-1)^{j_1}(j_2p_2 - i_2q_2)(y_1^{i_1+j_1}y_2^{i_2+j_2-1}\otimes s^{-1}\eta_1^{p_1+q_1}s^{-1}\eta_2^{p_2+q_2-1})\Big],$$

*for $n \geq 3$,*

$$\Big\{y_1^{i_1}\cdots y_n^{i_n}\otimes s^{-1}\eta_1^{p_1}\cdots s^{-1}\eta_n^{p_n},$$
$$y_1^{j_1}\cdots y_n^{j_n}\otimes s^{-1}\eta_1^{q_1}\cdots s^{-1}\eta_n^{q_n}\Big\}$$
$$= (-1)^{\sum_{1\leq m<l\leq n} i_l j_m}\Big[(-1)^{\sum_{s=2}^{n} i_s}(j_1p_1 - i_1q_1)(y_1^{i_1+j_1-1}\cdots y_n^{i_n+j_n}\otimes$$
$$s^{-1}\eta_1^{p_1+q_1-1}\cdots s^{-1}\eta_n^{p_n+q_n}) + \sum_{r=2}^{n-1}(-1)^{\sum_{s=r+1}^{n} i_s+\sum_{t=1}^{r-1} j_t}(j_r p_r - i_r q_r)$$
$$(y_1^{i_1+j_1}\cdots y_r^{i_r+j_r-1}\cdots y_n^{i_n+j_n}\otimes s^{-1}\eta_1^{p_1+q_1}\cdots s^{-1}\eta_r^{p_r+q_r-1}\cdots s^{-1}\eta_n^{p_n+q_n})$$
$$+ (-1)^{\sum_{t=1}^{n-1} j_t}(j_n p_n - i_n q_n)(y_1^{i_1+j_1}\cdots y_n^{i_n+j_n-1}\otimes s^{-1}\eta_1^{p_1+q_1}\cdots s^{-1}\eta_n^{p_n+q_n-1})\Big],$$

*where $0 \leq i_1,\ldots,i_n, j_1,\ldots, j_n \leq 1$, and $p_1,\ldots,p_n, q_1,\ldots,q_n \geq 0$.*

## 4 Calculation of the $S^1$-equivariant cohomology of $LV_{n,k}(\mathbb{H})$

Now we compute the string cohomology of quaternionic Stiefel manifolds $V_{n,k}(\mathbb{H})$ i.e., the $S^1$-equivariant cohomology of $LV_{n,k}(\mathbb{H})$. As mentioned earlier, the minimal Sullivan model of $V_{n,k}(\mathbb{H})$ is $(\wedge(y_1,\ldots,y_k), 0)$, where $|y_i| = 4(n-k+i)-1$. Since $V_{n,k}(\mathbb{H})$ is simply connected, by the description given in the introduction one sees that the minimal Sullivan model of $LV_{n,k}(\mathbb{H})$ is $(\wedge(y_1, z_1,\ldots, y_k, z_k),\delta)$, where $|z_i| = 4(n-k+i)-2$. The differential $\delta$ is defined by $\delta(y_i) = 0$ and on $z_i's$ it is defined using $\delta\circ s + s\circ\delta = 0$, where $s : \wedge(y_1, z_1,\ldots, y_k, z_k)\to\wedge(y_1, z_1,\ldots, y_k, z_k)$ is the derivation defined by $s(y_i) = z_i$ and $s(z_i) = 0$. Thus we have $\delta(z_i) = \delta(s(y_i)) = -s(\delta(y_i)) = -s(0) = 0$. That is, $\delta$ is the zero differential.

The minimal Sullivan model of $ES^1\times_{S^1} LV_{n,k}(\mathbb{H})$ is given by $(\wedge(y_1, z_1,\ldots, y_k, z_k, u), D)$, where $|u| = 2$, and the differential $D$ is given by $D(u) = 0$, $D(x) = \delta(x) + s(x)u = s(x)u$ for $x\in\wedge(y_1, z_1,\ldots, y_k, z_k)$. Thus we have $D(y_i) = s(y_i)u = z_iu$, and $D(z_i) = s(z_i)u = 0$.

**Proposition 4** *The $S^1$-equivariant cohomology of $LV_{n,k}(\mathbb{H})$ $\left(H^*_{S^1}(LV_{n,k}(\mathbb{H}))\right)$ is given by:*

*(i) When k is even,*

$$H^{\text{even}}_{S^1}(LV_{n,k}(\mathbb{H})) = \mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} [u^{a_{k+1}}],\ [z_k^{a_k}\cdots z_{k-p}^{b_{k-p}}] & p=0,\ldots,k-1,\ 1\le s\le \frac{k-2}{2}, \\ & 1\le i_1<i_2<\cdots<i_{2s+1}\le k, \\ \left[\sum_{j=1}^{2s+1}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s+1}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}\right] & a_l\ge 0\ \forall\, l=1,\ldots,k+1,\ b_{k-p}\ge 1 \end{array}\right\}$$

$$H^{\text{odd}}_{S^1}(LV_{n,k}(\mathbb{H})) = \mathbb{Q}_{\text{span}}\left\{\left[\sum_{j=1}^{2s}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}\right]\ \middle|\ \begin{array}{c} 1\le s\le \frac{k}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ a_l\ge 0\ \forall\, l=1,\ldots,k \end{array}\right\}.$$

*(ii) When k is odd,*

$$H^{\text{even}}_{S^1}(LV_{n,k}(\mathbb{H})) = \mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} [u^{a_{k+1}}],\ [z_k^{a_k}\cdots z_{k-p}^{b_{k-p}}] & p=0,\ldots,k-1,\ 1\le s\le \frac{k-1}{2}, \\ & 1\le i_1<i_2<\cdots<i_{2s+1}\le k, \\ \left[\sum_{j=1}^{2s+1}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s+1}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}\right] & a_l\ge 0\ \forall\, l=1,\ldots,k+1,\ b_{k-p}\ge 1 \end{array}\right\}$$

$$H^{\text{odd}}_{S^1}(LV_{n,k}(\mathbb{H})) = \mathbb{Q}_{\text{span}}\left\{\left[\sum_{j=1}^{2s}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}\right]\ \middle|\ \begin{array}{c} 1\le s\le \frac{k-1}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ a_l\ge 0\ \forall\, l=1,\ldots,k \end{array}\right\}.$$

*Proof* Denote $C^*$ to be the equivariant cochain groups of $(\wedge(y_1, z_1, \ldots, y_k, z_k, u), D)$.

$(i)$ When $k$ is even. We have

$$C^{\text{even}} = \mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}u^{a_{k+1}}, & 1\le s\le \frac{k}{2},\ 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ y_{i_1}y_{i_2}\cdots y_{i_{2s}}z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}u^{a_{k+1}} & a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array}\right\}.$$

Now

$$D(z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}u^{a_{k+1}}) = 0,$$

$$D(y_{i_1}y_{i_2}\cdots y_{i_{2s}}z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}u^{a_{k+1}}) = \sum_{j=1}^{2s}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}u^{a_{k+1}+1},$$

$1\le s\le \frac{k}{2}$. Thus one obtains

$$\text{Image}^{\text{even}}\, D = \mathbb{Q}_{\text{span}}\left\{\sum_{j=1}^{2s}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}u^{a_{k+1}+1}\ \middle|\ \begin{array}{c} 1\le s\le \frac{k}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array}\right\},$$

$$\mathrm{Ker}^{\mathrm{even}}\, D = \mathbb{Q}_{\mathrm{span}}\left\{ \begin{array}{c} z_1^{a_1} z_2^{a_2}\cdots z_k^{a_k} u^{a_{k+1}}, \\ \sum_{j=1}^{2s+1}(-1)^{j-1} y_{i_1}\cdots \widehat{y_{i_j}}\cdots y_{i_{2s+1}} z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k} u^{a_{k+1}} \end{array} \middle| \begin{array}{c} 1\le s\le \frac{k-2}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s+1}\le k, \\ a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array} \right\}.$$

Also we have

$$C^{\mathrm{odd}} = \mathbb{Q}_{\mathrm{span}}\left\{ y_{i_1} y_{i_2}\cdots y_{i_{2s-1}} z_1^{a_1} z_2^{a_2}\cdots z_k^{a_k} u^{a_{k+1}} \middle| \begin{array}{c} 1\le s\le \frac{k}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s-1}\le k, \\ a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array} \right\}$$

$$D(y_{i_1} y_{i_2}\cdots y_{i_{2s-1}} z_1^{a_1} z_2^{a_2}\cdots z_k^{a_k} u^{a_{k+1}}) = \sum_{j=1}^{2s-1}(-1)^{j-1} y_{i_1}\cdots \widehat{y_{i_j}}\cdots y_{i_{2s-1}} z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k} u^{a_{k+1}+1},$$

$1\le s\le \frac{k}{2}$. Thus we have

$$\mathrm{Image}^{\mathrm{odd}}\, D = \mathbb{Q}_{\mathrm{span}}\left\{ \sum_{j=1}^{2s-1}(-1)^{j-1} y_{i_1}\cdots \widehat{y_{i_j}}\cdots y_{i_{2s-1}} z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k} u^{a_{k+1}+1} \middle| \begin{array}{c} 1\le s\le \frac{k}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s-1}\le k, \\ a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array} \right\},$$

$$\mathrm{Ker}^{\mathrm{odd}}\, D = \mathbb{Q}_{\mathrm{span}}\left\{ \sum_{j=1}^{2s}(-1)^{j-1} y_{i_1}\cdots \widehat{y_{i_j}}\cdots y_{i_{2s}} z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k} u^{a_{k+1}} \middle| \begin{array}{c} 1\le s\le \frac{k}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array} \right\}.$$

Now

$$H_{S^1}^{\mathrm{even}}(LV_{n,k}(\mathbb{H})) = \frac{\mathrm{Ker}^{\mathrm{even}}\, D}{\mathrm{Image}^{\mathrm{odd}}\, D}$$

$$= \frac{\mathbb{Q}_{\mathrm{span}}\left\{ \begin{array}{c} z_1^{a_1} z_2^{a_2}\cdots z_k^{a_k} u^{a_{k+1}}, \\ \sum_{j=1}^{2s+1}(-1)^{j-1} y_{i_1}\cdots \widehat{y_{i_j}}\cdots y_{i_{2s+1}} z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k} u^{a_{k+1}} \end{array} \middle| \begin{array}{c} 1\le s\le \frac{k-2}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s+1}\le k, \\ a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array} \right\}}{\mathbb{Q}_{\mathrm{span}}\left\{ \sum_{j=1}^{2s-1}(-1)^{j-1} y_{i_1}\cdots \widehat{y_{i_j}}\cdots y_{i_{2s-1}} z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k} u^{a_{k+1}+1} \middle| \begin{array}{c} 1\le s\le \frac{k}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s-1}\le k, \\ a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array} \right\}}.$$

But then

$$\mathbb{Q}_{\mathrm{span}}\left\{ z_1^{a_1} z_2^{a_2}\cdots z_k^{a_k} u^{a_{k+1}} \middle| a_l\ge 0\ \forall\, l=1,\ldots,k+1 \right\}$$
$$= \mathbb{Q}_{\mathrm{span}}\left\{ \begin{array}{c} u^{a_{k+1}}, z_k^{a_k}\cdots z_{k-p}^{b_{k-p}}, \\ z_k^{a_k}\cdots z_{k-p}^{b_{k-p}} u^{c_{k-p}} \end{array} \middle| \begin{array}{c} p=0,\ldots,k-1 \\ a_l\ge 0\ \forall\, l=1,\ldots,k+1,\ b_{k-p}, c_{k-p}\ge 1 \end{array} \right\}.$$

Therefore

$$H^{\text{even}}_{S^1}(LV_{n,k}(\mathbb{H}))$$

$$= \frac{\mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} u^{a_{k+1}},\ z_k^{a_k}\cdots z_{k-p}^{b_{k-p}},\ z_k^{a_k}\cdots z_{k-p}^{b_{k-p}}u^{c_{k-p}} & p=0,\ldots,k-1,\ 1\le s\le\frac{k-2}{2}, \\ & 1\le i_1<i_2<\cdots<i_{2s+1}\le k, \\ \sum_{j=1}^{2s+1}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s+1}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}u^{a_{k+1}} & a_l\ge 0\ \forall\, l=1,\ldots,k+1, \\ & b_{k-p},c_{k-p}\ge 1 \end{array}\right\}}{\mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} & 1\le s\le\frac{k}{2}, \\ \sum_{j=1}^{2s-1}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s-1}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}u^{a_{k+1}+1} & 1\le i_1<i_2<\cdots<i_{2s-1}\le k, \\ & a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array}\right\}}$$

$$= \mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} [u^{a_{k+1}}],\ [z_k^{a_k}\cdots z_{k-p}^{b_{k-p}}] & p=0,\ldots,k-1,\ 1\le s\le\frac{k-2}{2}, \\ & 1\le i_1<i_2<\cdots<i_{2s+1}\le k, \\ \left[\sum_{j=1}^{2s+1}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s+1}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}\right] & a_l\ge 0\ \forall\, l=1,\ldots,k+1,\ b_{k-p}\ge 1 \end{array}\right\}.$$

Also

$$H^{\text{odd}}_{S^1}(LV_{n,k}(\mathbb{H})) = \frac{\text{Ker}^{\text{odd}}\,D}{\text{Image}^{\text{even}}\,D}$$

$$= \frac{\mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} & 1\le s\le\frac{k}{2}, \\ \sum_{j=1}^{2s}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}u^{a_{k+1}} & 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ & a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array}\right\}}{\mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} & 1\le s\le\frac{k}{2}, \\ \sum_{j=1}^{2s}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}u^{a_{k+1}+1} & 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ & a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array}\right\}}$$

$$= \mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} & 1\le s\le\frac{k}{2}, \\ \left[\sum_{j=1}^{2s}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}\right] & 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ & a_l\ge 0\ \forall\, l=1,\ldots,k \end{array}\right\}.$$

$(ii)$ When $k$ is odd.
We have

$$C^{\text{even}} = \mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}u^{a_{k+1}}, & 1\le s\le\frac{k-1}{2},\ 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ y_{i_1}y_{i_2}\cdots y_{i_{2s}}z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}u^{a_{k+1}} & a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array}\right\}.$$

Now

$$D(z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}u^{a_{k+1}})=0,$$

$$D(y_{i_1}y_{i_2}\cdots y_{i_{2s}}z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}u^{a_{k+1}}) = \sum_{j=1}^{2s}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}u^{a_{k+1}+1},$$

$1\le s\le\frac{k-1}{2}$. Thus one obtains

$$\text{Image}^{\text{even}}\,D$$

$$= \mathbb{Q}_{\text{span}}\left\{\begin{array}{c|c} & 1\le s\le\frac{k-1}{2}, \\ \sum_{j=1}^{2s}(-1)^{j-1}y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}}z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k}u^{a_{k+1}+1} & 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ & a_l\ge 0\ \forall\, l=1,\ldots,k+1 \end{array}\right\},$$

$$\mathrm{Ker}^{\mathrm{even}}\, D = \mathbb{Q}_{\mathrm{span}} \left\{ \begin{array}{c} z_1^{a_1} z_2^{a_2} \cdots z_k^{a_k} u^{a_{k+1}}, \\ \sum_{j=1}^{2s+1} (-1)^{j-1} y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_{2s+1}} z_1^{a_1} \cdots z_{i_j}^{a_{i_j}+1} \cdots z_k^{a_k} u^{a_{k+1}} \end{array} \left| \begin{array}{c} 1 \le s \le \frac{k-1}{2}, \\ 1 \le i_1 < i_2 < \cdots < i_{2s+1} \le k, \\ a_l \ge 0 \,\forall\, l = 1, \ldots, k+1 \end{array} \right. \right\}.$$

Also we have

$$C^{\mathrm{odd}} = \mathbb{Q}_{\mathrm{span}} \left\{ y_{i_1} y_{i_2} \cdots y_{i_{2s-1}} z_1^{a_1} z_2^{a_2} \cdots z_k^{a_k} u^{a_{k+1}} \left| \begin{array}{c} 1 \le s \le \frac{k+1}{2}, \\ 1 \le i_1 < i_2 < \cdots < i_{2s-1} \le k, \\ a_l \ge 0 \,\forall\, l = 1, \ldots, k+1 \end{array} \right. \right\}$$

$$D(y_{i_1} y_{i_2} \cdots y_{i_{2s-1}} z_1^{a_1} z_2^{a_2} \cdots z_k^{a_k} u^{a_{k+1}}) = \sum_{j=1}^{2s-1} (-1)^{j-1} y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_{2s-1}} z_1^{a_1} \cdots z_{i_j}^{a_{i_j}+1} \cdots z_k^{a_k} u^{a_{k+1}+1},$$

$1 \le s \le \frac{k+1}{2}$. Thus

$$\mathrm{Image}^{\mathrm{odd}}\, D = \mathbb{Q}_{\mathrm{span}} \left\{ \sum_{j=1}^{2s-1} (-1)^{j-1} y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_{2s-1}} z_1^{a_1} \cdots z_{i_j}^{a_{i_j}+1} \cdots z_k^{a_k} u^{a_{k+1}+1} \left| \begin{array}{c} 1 \le s \le \frac{k+1}{2}, \\ 1 \le i_1 < i_2 < \cdots < i_{2s-1} \le k, \\ a_l \ge 0 \,\forall\, l = 1, \ldots, k+1 \end{array} \right. \right\},$$

$$\mathrm{Ker}^{\mathrm{odd}}\, D = \mathbb{Q}_{\mathrm{span}} \left\{ \sum_{j=1}^{2s} (-1)^{j-1} y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_{2s}} z_1^{a_1} \cdots z_{i_j}^{a_{i_j}+1} \cdots z_k^{a_k} u^{a_{k+1}} \left| \begin{array}{c} 1 \le s \le \frac{k-1}{2}, \\ 1 \le i_1 < i_2 < \cdots < i_{2s} \le k, \\ a_l \ge 0 \,\forall\, l = 1, \ldots, k+1 \end{array} \right. \right\}.$$

Now

$$H_{S^1}^{\mathrm{even}}(LV_{n,k}(\mathbb{H})) = \frac{\mathrm{Ker}^{\mathrm{even}}\, D}{\mathrm{Image}^{\mathrm{odd}}\, D}$$

$$= \frac{\mathbb{Q}_{\mathrm{span}} \left\{ \begin{array}{c} z_1^{a_1} z_2^{a_2} \cdots z_k^{a_k} u^{a_{k+1}}, \\ \sum_{j=1}^{2s+1} (-1)^{j-1} y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_{2s+1}} z_1^{a_1} \cdots z_{i_j}^{a_{i_j}+1} \cdots z_k^{a_k} u^{a_{k+1}} \end{array} \left| \begin{array}{c} 1 \le s \le \frac{k-1}{2}, \\ 1 \le i_1 < i_2 < \cdots < i_{2s+1} \le k, \\ a_l \ge 0 \,\forall\, l = 1, \ldots, k+1 \end{array} \right. \right\}}{\mathbb{Q}_{\mathrm{span}} \left\{ \sum_{j=1}^{2s-1} (-1)^{j-1} y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_{2s-1}} z_1^{a_1} \cdots z_{i_j}^{a_{i_j}+1} \cdots z_k^{a_k} u^{a_{k+1}+1} \left| \begin{array}{c} 1 \le s \le \frac{k+1}{2}, \\ 1 \le i_1 < i_2 < \cdots < i_{2s-1} \le k, \\ a_l \ge 0 \,\forall\, l = 1, \ldots, k+1 \end{array} \right. \right\}}$$

$$= \frac{\mathbb{Q}_{\mathrm{span}} \left\{ \begin{array}{c} u^{a_{k+1}},\ z_k^{a_k} \cdots z_{k-p}^{b_{k-p}},\ z_k^{a_k} \cdots z_{k-p}^{b_{k-p}} u^{c_{k-p}} \\ \sum_{j=1}^{2s+1} (-1)^{j-1} y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_{2s+1}} z_1^{a_1} \cdots z_{i_j}^{a_{i_j}+1} \cdots z_k^{a_k} u^{a_{k+1}} \end{array} \left| \begin{array}{c} p = 0, \ldots, k-1,\ 1 \le s \le \frac{k-1}{2}, \\ 1 \le i_1 < i_2 < \cdots < i_{2s+1} \le k, \\ a_l \ge 0 \,\forall\, l = 1, \ldots, k+1, \\ b_{k-p}, c_{k-p} \ge 1 \end{array} \right. \right\}}{\mathbb{Q}_{\mathrm{span}} \left\{ \sum_{j=1}^{2s-1} (-1)^{j-1} y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_{2s-1}} z_1^{a_1} \cdots z_{i_j}^{a_{i_j}+1} \cdots z_k^{a_k} u^{a_{k+1}+1} \left| \begin{array}{c} 1 \le s \le \frac{k+1}{2}, \\ 1 \le i_1 < i_2 < \cdots < i_{2s-1} \le k, \\ a_l \ge 0 \,\forall\, l = 1, \ldots, k+1 \end{array} \right. \right\}}$$

$$= \mathbb{Q}_{\mathrm{span}} \left\{ \begin{array}{c} [u^{a_{k+1}}],\ [z_k^{a_k} \cdots z_{k-p}^{b_{k-p}}] \\ \left[ \sum_{j=1}^{2s+1} (-1)^{j-1} y_{i_1} \cdots \widehat{y_{i_j}} \cdots y_{i_{2s+1}} z_1^{a_1} \cdots z_{i_j}^{a_{i_j}+1} \cdots z_k^{a_k} \right] \end{array} \left| \begin{array}{c} p = 0, \ldots, k-1,\ 1 \le s \le \frac{k-1}{2}, \\ 1 \le i_1 < i_2 < \cdots < i_{2s+1} \le k, \\ a_l \ge 0 \,\forall\, l = 1, \ldots, k+1,\ b_{k-p} \ge 1 \end{array} \right. \right\}.$$

Also

$$
\begin{aligned}
H_{S^1}^{\text{odd}}(LV_{n,k}(\mathbb{H})) &= \frac{\text{Ker}^{\text{odd}}\, D}{\text{Image}^{\text{even}}\, D} \\
&= \frac{\mathbb{Q}\text{span}\left\{ \sum_{j=1}^{2s}(-1)^{j-1} y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}} z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k} u^{a_{k+1}} \;\middle|\; \begin{array}{c} 1\le s\le \frac{k-1}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ a_l\ge 0\,\forall\, l=1,\ldots,k+1 \end{array} \right\}}{\mathbb{Q}\text{span}\left\{ \sum_{j=1}^{2s}(-1)^{j-1} y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}} z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k} u^{a_{k+1}+1} \;\middle|\; \begin{array}{c} 1\le s\le \frac{k-1}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ a_l\ge 0\,\forall\, l=1,\ldots,k+1 \end{array} \right\}} \\
&= \mathbb{Q}\text{span}\left\{ \left[ \sum_{j=1}^{2s}(-1)^{j-1} y_{i_1}\cdots\widehat{y_{i_j}}\cdots y_{i_{2s}} z_1^{a_1}\cdots z_{i_j}^{a_{i_j}+1}\cdots z_k^{a_k} \right] \;\middle|\; \begin{array}{c} 1\le s\le \frac{k-1}{2}, \\ 1\le i_1<i_2<\cdots<i_{2s}\le k, \\ a_l\ge 0\,\forall\, l=1,\ldots,k \end{array} \right\}.
\end{aligned}
$$

□

## References


1. M. Chas and D. Sullivan, String topology, arXiv preprint math/9911159, (1999).
2. J. B. Gatsinzi, Hochschild cohomology of a Sullivan algebra, Mediterranean Journal of Mathematics, 13.6 (2016): 3765-3776.
3. Félix, Yves, John Oprea, and Daniel Tanré, Algebraic models in geometry. OUP Oxford, (2008).
4. H. Tamanoi, Batalin-Vilkovisky Lie algebra structure on the loop homology of complex Stiefel manifolds, International Mathematics Research Notices, 9 (2006): 97193-97193.
5. M. Vigué-Poirrier and D. Sullivan, The homology theory of the closed geodesic problem, Journal of Differential Geometry 11.4 (1976): 633-644.
6. M. Vigué-Poirrier and Dan Burghelea, A model for cyclic homology and algebraic $K$-theory of 1-connected topological spaces, Journal of Differential Geometry 22.2 (1985): 243-253.
7. Somnath Basu, Transversal String Topology & Invariants of Manifolds, Doctoral dissertation, State University of New York at Stony Brook, (2011).