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ORIGINAL RESEARCH

Congruence relation between Stirling numbers of the first and second kinds

A. Lalchhuangliana · **S. S. Singh**

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Abstract This paper consists of certain congruence properties of Stirling numbers of the first and second kinds. Some congruence relations between $s(n,k)$ and $S(n,k)$ for different modulo are obtained through their generating functions. We also present some exact p-adic valuations of $s(n,k)$ and $S(n,k)$ for some cases, mainly when $n-k$ is divisible by $p-1$ for odd prime p. Some estimates of the p-adic valuation of these two numbers are also presented when $p-1$ does not divide $n-k$.

Keywords Congruence · Generating function · Primes · p-adic valuation · Stirling numbers

Mathematics Subject Classification 11A07 · 11B73 · 11E95.

1 Introduction

The p-adic valuation of an integer a denoted by $v_p(a)$ is the highest exponent of p that divides a, where p is a prime. It follows that $p^{v_p(a)} \mid a$ but $p^{1+v_p(a)} \nmid a$. Every integer has a unique p-adic expansion. If

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n,$$

where $0 \le a_i \le p-1$, this expression is unique, and a_i's are called the p-adic digits. The sum of a's p-adic digits is denoted by $s_p(a)$. Thus,

$$s_p(a) = \sum_{i=0}^{n} a_i.$$

This sum is related to the p-adic valuation of binomial coefficients [14] as

$$v_p\left(\binom{n}{k}\right) = \frac{s_p(k) + s_p(n-k) - s_p(n)}{p-1}.$$

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This author contributed equally to this work.

A. Lalchhuangliana
Department of Mathematics and Computer Science, Mizoram University, Tanhril, Aizawl, Mizoram 796004, India
E-mail: abchhak@gmail.com

S. S. Singh
Department of Mathematics, Government Champhai College, Vengthlang North, Champhai, Mizoram 796321, India
E-mail: saratcha32@yahoo.co.uk

Some important generating functions of Stirling numbers of the second kind $S(n,k)$ and the first kind $s(n,k)$ are

$$x^{\underline{n}} = \sum_{i=0}^{n} s(n,i)x^i, \quad x^{\overline{n}} = \sum_{i=0}^{n} (-1)^{n-i} s(n,i)x^i, \tag{1}$$

$$\prod_{i=1}^{n-1}(1-ix) = \sum_{i=0}^{n-1} s(n,n-i)x^i, \quad \prod_{i=1}^{n-1}(1+ix) = \sum_{i=0}^{n-1}(-1)^i s(n,n-i)x^i, \tag{2}$$

$$\prod_{i=1}^{k}\frac{1}{(1-ix)} = \sum_{n=0}^{\infty} S(n+k,k)x^n, \quad \prod_{i=1}^{k}\frac{1}{(1+ix)} = \sum_{n=0}^{\infty}(-1)^n S(n+k,k)x^n, \tag{3}$$

where the notations $x^{\underline{n}}$ and $x^{\bar{n}}$ stand for the falling factorial and the rising factorial of x, respectively.
Stirling numbers are related to several sequences of numbers, making them interesting to many researchers. In recent years, divisibility properties of Stirling numbers of the second kind for an even prime appeared in Adelberg [2], Hong *et al.* [7], Lengyel [11], Wannemacker [22] and Zhao *et al.* [24].
For positive integers a, k, and a prime p such that $a < k < p$, and for each positive integer $n \equiv a \pmod{p^{m_0-1}(p-1)}$, Miska [15] confirmed that

$$v_p(S(n,k)) = v_p(S(a+(p-1)p^{m_0-1},k)) + v_p(n-a) - m_0 + 1.$$

Singh and Lalchhuangliana [18] proved, using a combinatorial approach and a concept of minimum periods, that

$$v_p(S(p^n,kp)) \geq 2,$$

for any integer $n \geq 2$ and $2 \leq k \leq p-1$.
Chan and Manna [4] obtained a congruence relation for $S(n,kp^m)$ modulo p^m and found that p^m always divides $S(n,kp^m)$ whenever $n \not\equiv k \pmod{p-1}$. Adelberg [1] discussed the concept of Minimum Zero Case (MZC), and if $n = ap^h$, $1 \leq a \leq p-1$, and $(p-1) \mid (n-k)$, then $S(n,k)$ is MZC with exact p-adic valuation

$$v_p(S(n,k)) = \frac{s_p(n) - s_p(k)}{p-1}.$$

For more results about the divisibility properties of Stirling numbers of the second kind, we refer to Davis [5], Sagan [17], Singh *et al.* [19], Tsumura [21], Sun [20], and Young [23].
For any prime p, Lengyel [12] confirmed that

$$v_p(s(ap^n+b, ap^n+b-k)) = v_p(s(b,b-k)),$$

where a, k, b, and n are integers such that $a \geq 1$ with $(a,p) = 1$, $2 \leq k+1 \leq b$ and n is sufficiently large. Qiu and Hong [16] showed that $v_2(s(2^n,k)) = v_2(s(2^n+1,k+1))$ for $1 \leq k \leq 2^n$, and $v_2(s(2n,2n-k)) = 2n-2-v_2(k-1)$ if k is odd and $2 \leq k \leq 2^{n-1}+1$. Komatsu and Young [10] proved that

$$v_p(s(n+1,k+1)) = v_p(n!) - v_p(k!) - kr,$$

where n, k, m, and r are positive integers such that $n = kp^r + m$ and $m < p^r$. More results about the divisibility of Stirling numbers of the first kind can be seen in Cao and Pan [3], Hong and Qiu [8], Howard [9], and Leonetti and Sanna [13].
This paper present the congruences of $S(n,kp^m)$ and $s(kp^m,n)$ explicitly in terms of binomial coefficient when $n \equiv k \pmod{p-1}$. We further obtain congruences for $S(n,k)$ and $s(k,a)$ modulo p, p^m, and p^n, where $m = \lfloor log_p(k) \rfloor$ and $n \geq m$. We reduce the congruences to simpler results for some special cases. The exact values of $v_p(S(n,k))$ and $v_p(s(n,k))$ for some special cases of $n \equiv k \pmod{p-1}$ are obtained. We also discuss the case when $S(n,k)$ and $S(n-1,k-1)$ have the same congruence property.

2 Preliminaries

The generating functions of $S(n,k)$ and $s(n,k)$ play a significant role in obtaining their congruence properties. We will assume p as an odd prime unless stated otherwise. We use the congruence property of the polynomial Chan and Manna [4]

$$\prod_{i=1}^{kp^m}(1-ix) \equiv (1-x^{p-1})^{kp^{m-1}} \pmod{p^m}. \tag{4}$$

If we replace x with $1/y$, we get

$$\prod_{i=1}^{kp^m}(1-ix) = \frac{1}{y^{kp^m}}\prod_{i=1}^{kp^m}(y-i)$$

and

$$(1-x^{p-1})^{kp^{m-1}} = \frac{1}{y^{k(p-1)p^{m-1}}}(y^{p-1}-1)^{kp^{m-1}}.$$

It follows that

$$\prod_{i=1}^{kp^m}(y-i) \equiv y^{kp^{m-1}}(y^{p-1}-1)^{kp^{m-1}} \pmod{p^m}. \tag{5}$$

We can also write this result as

$$x^{\underline{kp^m}} \equiv x^{kp^{m-1}}(x^{p-1}-1)^{kp^{m-1}} \pmod{p^m}. \tag{6}$$

Replacing x by $-x$ in Equations (4) and (6), we obtain

$$\prod_{i=1}^{kp^m}(1+ix) \equiv (1-x^{p-1})^{kp^{m-1}} \pmod{p^m} \tag{7}$$

and

$$x^{\overline{kp^m}} \equiv (-1)^k x^{kp^{m-1}}(x^{p-1}-1)^{kp^{m-1}} \pmod{p^m}. \tag{8}$$

Davis and Webb [6] proved, for a prime $p > 3$, that

$$\binom{np}{kp} \equiv \binom{n}{k} \pmod{p^e}, \tag{9}$$

where $e = 3 + v_p(n) + v_p(k) + v_p(n-k) + v_p(\binom{n}{k})$. With the help of Equation (9), it is easy to confirm

$$(1-x)^{p^n} \equiv (1-x^{p^{n-m}})^m \pmod{p^{m+1}} \tag{10}$$

for any integers n and m such that $0 \le m \le n$.

3 Results

In this section, we prove our main results which we present in theorems and corollaries. The first theorem gives the congruence relations between Stirling numbers of the first kind and Binomial coefficients.

Theorem 1 *If p is an odd prime and k is a positive integer not divisible by p, then for any positive integer m, the following congruences hold;*

$$\begin{aligned}
&a)\ \ s(kp^m, kp^m - b) \equiv (-1)^k s(kp^m, kp^{m-1} + b) \pmod{p^m},\\
&b)\ \ s(kp^m, kp^m - b) \equiv \binom{kp^{m-1}}{\frac{b}{p-1}}(-1)^{\frac{b}{p-1}} \pmod{p^m},\\
&\qquad\qquad \text{if } b \equiv 0 \pmod{p-1} \text{ and } b \le k(p-1)p^{m-1}.\\
&c)\ \ s(kp^m, b) \equiv 0 \pmod{p^m},\\
&\qquad\qquad \text{if } b \le kp^{m-1} \text{ or } kp^{m-1} - b \not\equiv 0 \pmod{p-1}.\\
&d)\ \ s(kp^m, b) \equiv s(kp^m + 1, b + 1) \pmod{p^m},
\end{aligned}$$

for any integer b such that $1 \le b \le kp^m - 1$.

Proof We know that

$$\prod_{i=1}^{kp^m-1}(1 - ix) = \sum_{i=0}^{kp^m-1} s(kp^m, kp^m - i)x^i$$

$$x^{\underline{kp^m}} = \sum_{i=0}^{kp^m} s(kp^m, i)x^i.$$

Due to Equations (4) and (6), we get

$$\prod_{i=1}^{kp^m-1}(1 - ix) \equiv (1 - x^{p-1})^{kp^{m-1}} \pmod{p^m};$$

$$x^{\underline{kp^m}} \equiv x^{kp^{m-1}}(x^{p-1} - 1)^{kp^{m-1}} \pmod{p^m}.$$

It follows that

$$\sum_{i=0}^{kp^m-1} s(kp^m, kp^m - i)x^i \equiv \sum_{j=0}^{kp^{m-1}} \binom{kp^{m-1}}{j}(-1)^j x^{j(p-1)} \pmod{p^m};$$

$$\sum_{i=0}^{kp^m} s(kp^m, i)x^i \equiv x^{kp^{m-1}} \sum_{j=0}^{kp^{m-1}} \binom{kp^{m-1}}{j}(-1)^{k-j} x^{j(p-1)} \pmod{p^m}.$$

Comparing the coefficients of x^b, we get the first three results, and the last result is obtained from the congruence

$$\prod_{i=1}^{kp^m}(1 - ix) \equiv \prod_{i=1}^{kp^m-1}(1 - ix) \pmod{p^m}. \tag{11}$$

Hence, the theorem follows. □

Corollary 1 *Let $n = kp^m$, $m \ge 1$, be an integer with only one non-zero p-adic digit, and $p > 3$ be a prime. Then, $s(n, a)$ is divisible by p if and only if $n \not\equiv a \pmod{(p-1)p^{m-1}}$. Further, $s(n, a)$ is divisible by p^{t+1}, $0 \le t \le m - 1$, if and only if $n - a < kp^{m-1}$ or $n \not\equiv a \pmod{(p-1)p^{m-1-t}}$.*

Proof Follow the proof of Theorem 1 and use Equation (10). □

Corollary 2 *For an odd prime p and integers k and m such that $p \nmid k$ and $m \geq 1$, we have*

$$v_p(s(kp^m + 1, a + 1)) = v_p(s(kp^m, a)) = m - 1 - v_p(a), \tag{12}$$

whenever $m - 1 > v_p(a)$, $kp^{m-1} \leq a \leq kp^m$, and $k \equiv a \pmod{p-1}$.

Proof The proof is based on the equality $v_p(\binom{ap^n}{bp^m}) = n - m$ when $p \nmid a$, $p \nmid b$, and $n > m$. □

Remark 1 If $n \geq m$, the p-adic valuation of the binomial coefficient $\binom{ap^n}{bp^m}$ is equal to $v_p(\binom{a}{bp^{m-n}})$. It follows that if $m - 1 \leq v_p(a)$ and $v_p(\binom{k}{kp-ap^{1-m}}) \leq m - 1$, we have

$$v_p(s(kp^m + 1, a + 1)) = v_p(s(kp^m, a)) = v_p\left(\binom{k}{kp - ap^{1-m}}\right). \tag{13}$$

It is also trivial from Theorem 1 that

$$Min\{v_p(s(kp^m + 1, a + 1)),\ v_p(s(kp^m, a))\} \geq m \tag{14}$$

when $a < kp^{m-1}$ or $kp^{m-1} - a \not\equiv 0 \pmod{p-1}$.

The following theorem is a generalization of Theorem 1.

Theorem 2 *For an odd prime p and positive integers k, m, a, and b, the following congruences hold;*

$$s(kp^m + a, kp^{m-1} + b) \equiv \sum_i (-1)^{k-i} \binom{kp^{m-1}}{i} s(a, b - i(p-1)) \pmod{p^m} \tag{15}$$

if $b \leq a + k(p-1)p^{m-1}$, and

$$s(kp^m + a, b) \equiv 0 \pmod{p^m}. \tag{16}$$

if $b \leq kp^{m-1}$.

Proof We have

$$x^{\underline{kp^m + a}} = \sum_{i=0}^{kp^m + a} s(kp^m + a, i) x^i \tag{17}$$

and

$$\begin{aligned} x^{\underline{kp^m + a}} &= \prod_{i=0}^{kp^m} (x - i) \prod_{i=1}^{a-1} (x - (kp^m + i)) \\ &\equiv x^{kp^{m-1}} (x^{p-1} - 1)^{kp^{m-1}} x^{\underline{a}} \pmod{p^m} \\ &\equiv x^{kp^{m-1}} \sum_{i=0}^{kp^{m-1}} \binom{kp^{m-1}}{i} (-1)^{k-i} x^{i(p-1)} \sum_{j=0}^{a} s(a, j) x^j \pmod{p^m}. \end{aligned} \tag{18}$$

Comparing the coefficients of $x^{kp^{m-1}+b}$ in the RHS of Equations (17) and (18), we obtain

$$s(kp^m + a, kp^{m-1} + b) \equiv \sum_{i(p-1)+j=b} (-1)^{k-i} \binom{kp^{m-1}}{i} s(a, j) \pmod{p^m}. \tag{19}$$

Changing the index j to $b - i(p-1)$ confirms the first result of the theorem. The coefficient of x^n on the right-hand side of Equation (18) vanishes if $n \leq kp^{m-1}$; hence, the second result follows. □

The following corollaries are special cases of the preceding theorem.

Corollary 3 *For an odd prime p and positive integers k, m, a, and b;*

$$(i)\ \ \sigma_a \equiv (-1)^k s(a,b) \pmod{p^m} \quad \text{if } b \le p-1,$$
$$(ii)\ v_p(\sigma_{p-1}) = \begin{cases} m-1-v_p(\lfloor \frac{b}{p-1} \rfloor), & \text{if } (p-1) \nmid b \text{ and } \lfloor \frac{b}{p-1} \rfloor < p^{m-1}; \\ m-1-v_p(\frac{b}{p-1}-1), & \text{if } (p-1) \mid b \text{ and } \frac{b}{p-1} < p^{m-1}. \end{cases}$$

where $\sigma_a = s(kp^m + a, kp^{m-1} + b)$.

Proof On observation of Equation (19), we can see that;
(i) If $b \le p-1$, then the only solution of $i(p-1)+j = b$ for (i, j) is $(0, b)$, unless $b = p-1$, in which case there are two solutions, namely $(0, p-1)$ and $(1, 0)$. The corresponding term for the solution $(1, 0)$ vanishes as $s(a, 0) = 0$. Hence, (i) follows.
(ii) Let $b = q(p-1)+r$ such that $0 \le r < p-1$. The only solution of $i(p-1)+j = q(p-1)+r$ with $j \le p-1$ is $(i, j) = (q, r)$, if $r \ne 0$. Thus, we get

$$s(kp^m + p - 1, kp^{m-1} + b) \equiv (-1)^{k-q} \binom{kp^{m-1}}{q} s(p-1, r) \pmod{p^m}. \tag{20}$$

Now, we obtain the congruence for $s(p-1, b)$:
We have

$$x^{\underline{p-1}} = \frac{x^{\underline{p}}}{x-p+1} \equiv (x^p - x)(1 - x + x^2 - \cdots) \pmod{p}.$$

It follows that

$$\sum_{i=0}^{p-1} s(p-1, i) x^i \equiv -x + x^2 - x^3 + \cdots + x^{p-1} \pmod{p}$$

and

$$s(p-1, i) \equiv (-1)^i \pmod{p} \tag{21}$$

if $1 \le i \le p-1$.
Therefore, the valuation of the binomial coefficient $\binom{kp^{m-1}}{q}$ is $m-1-v_p(q)$ and $s(p-1, r)$ is not divisible by p. Thus, the first case of (ii) follows.
On the other hand, if $r = 0$ or $b = q(p-1)$, then there are two solutions of $i(p-1)+j = q(p-1)$, namely $(q, 0)$ and $(q-1, p-1)$. The corresponding term for the index $(q, 0)$ is zero since $s(p-1, 0) = 0$. Following the proof of the first result, we get the second case of (ii). □

Corollary 4 *For an odd prime p and positive integers k, m, a, and b;*

$$s(kp^m + a, kp^{m-1} + b) \equiv (-1)^q \binom{kp^{m-1}}{q} s(a, r) \pmod{p^m}$$

if $a < p-1$ and $b = q(p-1)+r$ with $0 \le r < p-1$.

Proof Given Equation (19), the only solution of $i(p-1)+j = q(p-1)+r$ is $(i, j) = (q, r)$. Hence the result follows. □

Remark 2 The p-adic valuations of large classes of Stirling numbers of the first kind can be obtained using Theorem 1, Corollaries 3, and 4. The first result of Corollary 4 yields the following exact p-adic valuation,

$$v_p(s(kp^m + a, kp^{m-1} + b)) = m - 1 - v_p\Big(\Big\lfloor \frac{b}{p-1} \Big\rfloor\Big), \tag{22}$$

if $b \equiv 1, a, a-1$ or $a-3 \pmod{p-1}$, assuming conditions of the corollary apply.

Theorem 3 *Let p be an odd prime and $k,\ a,\ b,\ m$, and t be positive integers such that $max\{k,\ a\} \leq p-1$, $t < m$, and $b \leq ap^t$. Then,*

$$\theta_{ap^t} \equiv \begin{cases} s(ap^t, b) \pmod{p^m}, & \text{if } a \not\equiv b \pmod{p-1} \text{ or } b < ap^{t-1}; \\ s(ap^t, b) \pmod{p^{m-v_p(b)-1}}, & \text{otherwise.} \end{cases} \tag{23}$$

where $\theta_q = s(kp^m + q, kp^m + b)$.

Proof Replace a and b in Equation (19) with ap^t and $k(p-1)p^{m-1} + b$, respectively; we obtain

$$\theta_{ap^t} \equiv \sum_{i(p-1)+j=k(p-1)p^{m-1}+b} (-1)^{k-i} \binom{kp^{m-1}}{i} s(ap^t, j) \pmod{p^m}.$$

If we replace the index j with $(kp^{m-1} - i)(p-1) + b$, we get

$$\theta_{ap^t} \equiv \sum_i (-1)^{k-i} \binom{kp^{m-1}}{i} s(ap^t, (kp^{m-1} - i)(p-1) + b) \pmod{p^m}.$$

By reversing the index, we get

$$\theta_{ap^t} \equiv \sum_i (-1)^i \binom{kp^{m-1}}{i} s(ap^t, b + i(p-1)) \pmod{p^m}. \tag{24}$$

Using Theorem 1, $s(ap^t, b + i(p-1))$ is divisible by p^t if $a \not\equiv b \pmod{p-1}$ or $b < ap^{t-1}$. The valuation of $\binom{kp^{m-1}}{i}$ is $m - 1 - v_p(i)$ unless $i = 0$. Thus, the valuation of the $i-$th terms, $i \neq 0$, of the right-hand side of Equation (24) is greater than or equal to $m - 1 + t - v_p(i)$. The range of the index i is determined by the inequality $b \leq b + i(p-1) \leq ap^t$, which implies that $0 \leq i \leq a \sum_{r=0}^{t-1} p^r$. It follows that $v_p(i) \leq t - 1$ and consequently $m - 1 + t - v_p(i) \geq m$. Hence, p^m divides all the $i - th$ terms except the term with $i = 0$. Thus, the first case of the theorem follows.
Now, we assume that $a \equiv b \pmod{p-1}$ and $ap^{t-1} \leq b \leq ap^t$. Therefore, we can express b as $ap^t - q(p-1)$ with $0 \leq q < ap^{t-1}$. Equation (24) becomes

$$\theta_{ap^t} \equiv \sum_i (-1)^i \binom{kp^{m-1}}{i} s(ap^t, ap^t - (q-i)(p-1)) \pmod{p^m}. \tag{25}$$

Given Theorem 1, the valuation of the i-th term on the RHS of the preceding congruence is $m + t - 2 - v_p(i) - v_p(q-i)$ if $i > 0$. Therefore, two sub cases arise, namely $v_p(q) < v_p(i)$ and $v_p(i) \leq v_p(q)$;
If $v_p(q) < v_p(i)$, we have $v_p(q-i) = v_p(q)$ and

$$m + t - 2 - v_p(i) - v_p(q-i) = m - 1 - v_p(q) + (t - 1 - v_p(i)) \geq m - 1 - v_p(q),$$

since $v_p(i) \leq t - 1$.
If $v_p(i) \leq v_p(q)$, we get

$$m + t - 2 - v_p(i) - v_p(q-i) \geq m - 1 - v_p(q) + (t - 1 - v_p(q-i)) \geq m - 1 - v_p(q),$$

since $v_p(q-i) \leq t - 1$.
The equality $b = ap^t - q(p-1)$ also implies that $v_p(b) = v_p(q)$ for the given condition. It follows that all the terms except when $i = 0$ are divisible by $p^{m-1-v_p(b)}$. Hence, the second case of the theorem follows. □

3.1 Valuations of $S(n, kp^m)$

Using Equations (3) and (4), we have the following result [4];

$$\sum_{n=0}^{\infty} S(n+kp^m, kp^m)x^n \equiv \sum_{j=0}^{\infty} \binom{kp^{m-1}+j-1}{j} x^{j(p-1)} \pmod{p^m}, \tag{26}$$

which implies $S(n, kp^m) \equiv 0 \pmod{p^m}$ if $n \not\equiv k \pmod{p-1}$; otherwise,

$$S(kp^m + a, kp^m) \equiv \binom{kp^{m-1} + \frac{a}{p-1} - 1}{kp^{m-1} - 1} \pmod{p^m} \tag{27}$$

for any non-negative integer a with $a \equiv 0 \pmod{p-1}$.
Due to Equation (11), we also have the congruence

$$S(kp^m + a - 1, kp^m - 1) \equiv S(kp^m + a, kp^m) \pmod{p^m}. \tag{28}$$

The following theorem is a consequence of Equations (26), (27), and (28).

Theorem 4 *Let p be an odd prime and k be an integer not divisible by p. For any positive integer n such that $n \equiv k \pmod{p-1}$ and $n < kp^m + (p-1)p^{m-1}$ with $m > 1$;*

$$v_p(S(n-1, kp^m - 1)) = v_p(S(n, kp^m)) = m - 1 - v_p(n). \tag{29}$$

If $n \not\equiv k \pmod{p-1}$, then

$$min\{v_p(S(n, kp^m)),\ v_p(S(n-1, kp^m - 1))\} \geq m. \tag{30}$$

Proof The second result is trivial from Equations (26) and (28).
We assume $n \geq kp^m$ and let $n = kp^m + b(p-1)$ with $b < p^{m-1}$. From Equations (27) and (28), we have

$$\begin{aligned} S(kp^m + b(p-1) - 1, kp^m - 1) &\equiv S(kp^m + b(p-1), kp^m) \\ &\equiv \binom{kp^{m-1} + b - 1}{b} \pmod{p^m}. \end{aligned}$$

The p-adic valuation of the above binomial coefficient is given as

$$v_p\left(\binom{kp^{m-1} + b - 1}{b}\right) = \frac{s_p(b) + s_p(kp^{m-1} - 1) - s_p(kp^{m-1} + b - 1)}{p-1}, \tag{31}$$

and we have

$$\begin{aligned} s_p(kp^{m-1} - 1) &= s_p((k-1)p^{m-1} + p^{m-1} - 1) \\ &= s_p((k-1)p^{m-1}) + s_p(p^{m-1} - 1) \\ &= s_p(k-1) + (m-1)(p-1). \end{aligned} \tag{32}$$

Further, k is not divisible by p and then $s_p(k-1) = s_p(k) - 1$. Therefore, Equation (32) reduces to

$$s_p(kp^m - 1) = s_p(k) + (m-1)(p-1) - 1. \tag{33}$$

Since $m > 1$ and $b < p^{m-1}$, we have

$$\begin{aligned} s_p(kp^{m-1} + b - 1) &= s_p(kp^{m-1}) + s_p(b-1) \\ &= s_p(k) + s_p(b-1). \end{aligned} \tag{34}$$

Let $b = b'p^{v_p(b)}$, $p \nmid b'$, for some positive integer b'. Replacing kp^m with b in Equation (33); we get

$$s_p(b-1) = s_p(b) - 1 + v_p(b)(p-1) \tag{35}$$

since $s_p(b) = s_p(b')$. Therefore, combining Equations (31), (33), (34), and (35), we get

$$v_p\left(\binom{kp^{m-1} + b - 1}{b}\right) = m - 1 - v_p(b).$$

It is also trivial from our assumption of b that $v_p(b) = v_p(n)$. Hence the theorem follows. □

Definition 1 Let a be a positive integer whose p-adic expansion is given by

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_t p^t.$$

For a fixed integer k, $1 \leq k \leq p-1$, we define $\rho_{p,k,m}(a)$ as

$$\rho_{p,k,m}(a) = \begin{cases} 0, & (\text{if } k + a_m < p), \\ 1+n, & (\text{if} k + a_m \geq p, a_{m+1} = \cdots = a_{m+n} = p-1 \\ & \qquad \text{and } a_{m+n+1} < p-1). \end{cases} \tag{36}$$

Here, $\rho_{p,k,m}(a)$ is the number of carries when adding a and kp^m in base p. Using Kummer's theorem, we can see that $\rho_{p,k,m}(a)$ is, in fact, $v_p(\binom{a+kp^m}{a})$.
The preceding theorem restricts the value of n to less than some particular value. The following theorem gives an alternate result of Theorem 4 when there is no restriction on the values of n but restrict $k < p$.

Theorem 5 *Let p be an odd prime and k be a positive integer less than p. For positive integers m and n such that $n \equiv k \pmod{p-1}$;*

$$v_p(S(n-1, kp^m - 1)) = v_p(S(n, kp^m)) = \rho_{p,k-1,m-1}\Big(\frac{n-kp^m}{p-1}\Big), \tag{37}$$

if $\rho_{p,k-1,m-1}(\frac{n-kp^m}{p-1}) \leq m-1 \leq v_p(n)$. However, if $\rho_{p,k,m-1}(\frac{n-kp^m}{p-1}) \leq v_p(n) < m-1$, then

$$\begin{aligned} v_p(S(n-1, kp^m - 1)) &= v_p(S(n, kp^m)) \\ &= m - 1 - v_p(n) + \rho_{p,k,m-1}\Big(\frac{n-kp^m}{p-1}\Big). \end{aligned} \tag{38}$$

Proof Since $n \equiv k \pmod{p-1}$, we can write $n = kp^m + a(p-1)$. Let $a = \sum_{i=0}^{q} a_i p^i$ be the p-adic expansion of $a = \frac{n-kp^m}{p-1}$ for some positive integer q. To prove the theorem, it is enough to obtain $v_p(\binom{kp^{m-1}+a-1}{a})$ for both cases. For the first case, we have

$$a = a' p^t$$

for some positive integers a' and t such that $p \nmid a'$ and $t \geq m-1$. Therefore,

$$s_p(kp^{m-1} + a' p^t - 1) = s_p(k + a' p^{t-m+1} - 1) + s_p(p^{m-1} - 1)$$

and

$$s_p(kp^{m-1} - 1) = k - 1 + s_p(p^{m-1} - 1).$$

Thus, the valuation of the binomial coefficient is

$$v_p\left(\binom{kp^{m-1} + a - 1}{a}\right) = \frac{s_p(a') - s_p(k - 1 + a' p^{t-m+1}) + k - 1}{p-1}. \tag{39}$$

Suppose $t > m-1$, the sum of the digits $s_p(k - 1 + a' p^{t-m+1})$ can be split into the sum $s_p(k-1) + s_p(a')$ since we assume $1 \leq k \leq p-1$. In this case, the valuation of the binomial coefficient becomes zero, which is also equal to $\rho_{p,k,m-1}(a)$, since the $(m-1)$-th p-adic digit of a is zero and $k + a_{m-1} = k < p$. Now, we assume that $v_p(a) = m-1$, which means that $t = m-1$ and $a' = a_{m-1} + a_m p + a_{m+1} p^2 + \cdots$. It follows that if $k - 1 + a_{m-1} < p$, then

$$s_p(kp^{m-1} + a - 1) = s_p(a) + s_p(kp^{m-1} - 1)$$

and $v_p(\binom{kp^{m-1}+a-1}{a}) = 0 = \rho_{p,k-1,m-1}(a)$.
If $k - 1 + a_{m-1} > p$, then

$$a = a_{m-1}p^{m-1} + (p-1)\sum_{i=m}^{m-2+\rho_{p,k-1,m-1}(a)} p^i + \sum_{i=m-1+\rho_{p,k-1,m-1}(a)}^{q} a_i p^i,$$

$$\begin{aligned} kp^{m-1} + a - 1 &= (a_{m-1} + k - 1 - p)p^{m-1} \\ &\quad + (a_{m-1+\rho_{p,k-1,m-1}(a)} + 1)p^{m-1+\rho_{p,k-1,m-1}(a)} \\ &\quad + \sum_{i=m+\rho_{p,k-1,m-1}(a)}^{q} a_i p^i, \end{aligned}$$

which implies

$$s_p(a) = a_{m-1} + (p-1)(\rho_{p,k-1,m-1}(a) - 1) + \sum_{i=m+\rho_{p,k-1,m-1}(a)}^{q} a_i,$$

$$\begin{aligned} s_p(kp^{m-1} + a - 1) &= k - p + a_{m-1} + \sum_{i=m+\rho_{p,k-1,m-1}(a)}^{q} a_i \\ &= s_p(a) - (p-1)\rho_{p,k-1,m-1}(a) + k - 1. \end{aligned}$$

Therefore, we get

$$\begin{aligned} v_p\left(\binom{kp^{m-1} + a - 1}{a}\right) &= \frac{s_p(a) - s_p(k-1+a) + k - 1}{p-1} \\ &= \rho_{p,k-1,m-1}(a). \end{aligned}$$

Using this valuation in Equation (27), the first result of the theorem follows. The second result of the theorem can be obtained through the same method. □

Remark 3 If there is no restriction on the value of k and $v_p(a) < m-1$, we can write a as $a = cp^{m-1} + b$, where $b < p^{m-1}$. Therefore,

$$v_p\left(\binom{kp^{m-1} + a - 1}{a}\right) = m - 1 - v_p(b) + v_p\left(\binom{k+c}{c}\right).$$

For $v_p(a) \geq m-1$, we have $a = cp^{m-1}$ for some integer c and hence

$$v_p\left(\binom{kp^{m-1} + a - 1}{a}\right) = v_p\left(\binom{k+c-1}{c}\right).$$

If the valuations obtained are less than $m-1$ for both cases, then they are the valuations of $S(kp^m + a(p-1), kp^m)$ for both cases.

Theorem 6 *Let p be a prime greater than 3 and n, k, and m be positive integers. Then, p divides $S(n, kp^m)$ if $n \not\equiv k \pmod{(p-1)p^{m-1}}$. More precisely, p^{t+1}, $0 \leq t \leq m-1$, divides $S(n, kp^m)$ if $n \not\equiv k \pmod{(p-1)p^{m-1-t}}$.*

Proof The theorem can be proved using Equations (3), (4), and (10). □

3.2 Congruence relation between S(n,k) and s(n,k)

Now, we establish congruence relations between Stirling numbers of the first and the second kind using their generating functions. The next theorem gives the results for $S(n, k)$ modulo p. We use the notation $[n]$ to denote the set $\{0,\ 1,\ 2,\ \cdots,\ n\}$ for simplicity.

Theorem 7 *Let p be an odd prime and n, k, a, and d are positive integers such that $a < p$ and $0 \leq d < p-1$, then*

$$S(n, kp+a) \equiv (-1)^d s(p-a,\ p-a-d)\binom{\frac{n-k-a-d}{p-1}}{k} \pmod{p}$$

if $n-k-a \equiv d \pmod{p-1}$ for some $d \in [p-1-a]$, and

$$S(n, kp+a) \equiv 0 \pmod{p}$$

if $n-k-a \not\equiv d \pmod{p-1}$ for any $d \in [p-1-a]$.

Proof We have

$$\frac{1}{\prod_{i=1}^{kp+a}(1-ix)} = \frac{\prod_{i=kp+a+1}^{(k+1)p}(1-ix)}{\prod_{i=1}^{(k+1)p}(1-ix)},$$

and we get the following congruences;

$$\frac{1}{\prod_{i=1}^{kp+a}(1-ix)} \equiv \frac{\prod_{i=0}^{p-a-1}(1+ix)}{(1-x^{p-1})^{k+1}} \pmod{p},$$

$$\sum_{n=0}^{\infty} S(n+kp+a, kp+a)x^n \equiv \sum_{i=0}^{p-a-1} s(p-a,\ p-a-i)(-1)^i x^i \times \sum_{j=0}^{\infty}\binom{k+j}{j}x^{j(p-1)} \pmod{p}.$$

It follows that if $n \equiv d \pmod{p-1}$ for $d \in [p-a-1]$, then

$$S(n+kp+a, kp+a) \equiv (-1)^d\binom{k+\frac{n-d}{p-1}}{k}s(p-a,\ p-a-d) \pmod{p}, \tag{40}$$

and if $n \not\equiv d \pmod{p-1}$ for any $d \in [p-a-1]$, then

$$S(n+kp+a, kp+a) \equiv 0 \pmod{p}. \tag{41}$$

If we replace $n+kp+a$ with n in Equations (40) and (41), we get the required results. □

Remark 4 The following results are consequences of Theorem 7, specifically for the prime $p = 3$ and $p = 5$:
For p=3, we have two classes (since $p-1 = 2$) for each $a \in \{0,\ 1,\ 2\}$. We get the following congruences:

$$S(n, 3k) \equiv \begin{cases} \binom{\frac{n-k}{2}}{k} \pmod{3}, & \text{if } n-k \text{ is even;} \\ 0 \pmod{3}, & \text{if } n-k \text{ is odd,} \end{cases}$$

$$S(n, 3k+1) \equiv \begin{cases} \binom{\frac{n-k-1}{2}}{k} \pmod{3}, & \text{if } n-k \text{ is odd;} \\ \binom{\frac{n-k-2}{2}}{k} \pmod{3}, & \text{if } n-k \text{ is even,} \end{cases}$$

$$S(n, 3k+2) \equiv \begin{cases} \binom{\frac{n-k-2}{2}}{k} \pmod{3}, & \text{if } n-k \text{ is even;} \\ 0 \pmod{3}, & \text{if } n-k \text{ is odd.} \end{cases}$$

For $p = 5$, we have four classes (since $p - 1 = 4$) for each $a \in \{0,\ 1,\ 2,\ 3,\ 4\}$. We get the following congruences:

$$S(n, 5k) \equiv \begin{cases} \binom{\frac{n-k}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \text{if } n \not\equiv k \pmod 4, \end{cases}$$

$$S(n, 5k+1) \equiv \begin{cases} \binom{\frac{n-k-1}{4}}{k} \pmod 5, & \text{if } n \equiv k+1 \pmod 4; \\ \binom{\frac{n-k-2}{4}}{k} \pmod 5, & \text{if } n \equiv k+2 \pmod 4; \\ \binom{\frac{n-k-3}{4}}{k} \pmod 5, & \text{if } n \equiv k+3 \pmod 4; \\ \binom{\frac{n-k-4}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4, \end{cases}$$

$$S(n, 5k+2) \equiv \begin{cases} \binom{\frac{n-k-2}{4}}{k} \pmod 5, & \text{if } n \equiv k+2 \pmod 4; \\ 3\binom{\frac{n-k-3}{4}}{k} \pmod 5, & \text{if } n \equiv k+3 \pmod 4; \\ 2\binom{\frac{n-k-4}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \text{if } n \equiv k+1 \pmod 4, \end{cases}$$

$$S(n, 5k+3) \equiv \begin{cases} \binom{\frac{n-k-3}{4}}{k} \pmod 5, & \text{if } n \equiv k+3 \pmod 4; \\ \binom{\frac{n-k-4}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \text{if } n \equiv k+1 \text{ or } k+2 \pmod 4, \end{cases}$$

$$S(n, 5k+4) \equiv \begin{cases} \binom{\frac{n-k-4}{4}}{k} \pmod 5, & \text{if } n \equiv k \pmod 4; \\ 0 \pmod 5, & \text{if } n \equiv k+1,\ k+2, \text{ or } k+3 \pmod 4. \end{cases}$$

In the case of $S(n, 5k)$, the multiplier $s(5, 5-d)$ where $d \in \{0,\ 1,\ 2,\ 3\}$ is divisible by 5 except when $d = 0$. It is also easy to see that the binomial coefficients on the RHS of the above equations reduce to 1 if $k = 0$. This observation leads us to acquire the following exact p-adic valuations:
For a prime $p = 3$ and any positive integer n,

$a)$ $v_3(S(2n, 2)) = 0$,
$b)$ $v_3(S(6n+3, 3)) = v_3(S(6n+5, 3)) = 0$,
$c)$ $v_3(S(6n, 4)) = v_3(S(6n+1, 4)) = v_3(S(6n+4, 4)) = v_3(S(6n+5, 4)) = 0$,
$d)$ $v_3(S(6n+1, 5)) = v_3(S(6n+1, 5)) = 0$,
$e)$ $v_3(S(6n, 6)) = 0$,
$f)$ $v_3(S(6n+1, 7)) = v_3(S(6n+2, 7)) = 0$,
$g)$ $v_3(S(6n, 6)) = 0$.

For a prime $p = 5$, we have the following p-adic valuations:

$a)$ $v_5(S(4n, 2)) = v_5(S(4n+2, 2)) = v_5(S(4n+3, 2)) = 0$,
$b)$ $v_5(S(4n, 3)) = v_5(S(4n+3, 3)) = 0$,
$c)$ $v_5(S(4n, 4)) = 0$,
$d)$ $v_5(S(20n+r, 5)) = 0$, if $r \in \{5, 9, 13, 17\}$,
$e)$ $v_5(S(20n+r, 6)) = 0$, if $r \in [19] \setminus \{2, 3, 4, 5\}$,
$f)$ $v_5(S(20n+r, 7)) = 0$, if $r \in [19] \setminus \{2, 3, 4, 5, 6, 10, 14, 18\}$,
$g)$ $v_5(S(20n+r, 8)) = 0$, if $r \in \{0, 1, 8, 9, 12, 13, 16, 17\}$,
$h)$ $v_5(S(20n+r, 9)) = 0$, if $r \in \{1, 9, 13, 17\}$.

The following theorem gives a generalization of Theorem 7 from modulo p to modulo p^m.

Theorem 8 *For an odd prime p and positive integers k, b, m, and n such that $b < p^{m-1}$, the following congruences hold;*

$$S(n+kp^m+b, kp^m+b) \equiv \sum_j (-1)^n \binom{(k+1)p^{m-1}+j-1}{j} \times s(c, c-n+j(p-1)) \pmod{p^m} \tag{42}$$

and

$$s(kp^m+b, kp^m+b-n) \equiv \sum_j (-1)^{n+j} \binom{(k+1)p^{m-1}}{j} \times S(n-j(p-1)+c, c) \pmod{p^m}, \tag{43}$$

where $c = p^m - b$.

Proof We have

$$\frac{1}{\prod_{i=1}^{kp^m+b}(1-ix)} = \frac{\prod_{i=kp^m+b+1}^{(k+1)p^m}(1-ix)}{\prod_{i=1}^{(k+1)p^m}(1-ix)}$$

and

$$\prod_{i=1}^{kp^m+b-1}(1-ix) = \frac{\prod_{i=1}^{(k+1)p^m}(1-ix)}{\prod_{i=kp^m+b}^{(k+1)p^m}(1-ix)}.$$

We obtain the following two congruences

$$\frac{1}{\prod_{i=1}^{kp^m+b}(1-ix)} \equiv \frac{\prod_{i=1}^{p^m-b-1}(1+ix)}{(1-x^{p-1})^{(k+1)p^{m-1}}} \pmod{p^m}, \tag{44}$$

and

$$\prod_{i=1}^{kp^m+b-1}(1-ix) \equiv \frac{(1-x^{p-1})^{(k+1)p^{m-1}}}{\prod_{i=0}^{p^m-b}(1+ix)} \pmod{p^m}. \tag{45}$$

Equations (44) and (45) generate Equations (42) and (43), respectively. Hence the theorem follows. □

Theorem 9 *For an odd prime p and positive integers k, a, m, and n such that $kp^m < p^n$ and $m+1 \le n$, the following congruences hold;*

$$S(a+kp^m, kp^m) \equiv (-1)^a \sum_j \binom{p^{n-1}+j-1}{j} \times s(bp^m, bp^m-a+j(p-1)) \pmod{p^n} \tag{46}$$

and

$$s(kp^m, kp^m-a) \equiv \sum_j (-1)^{a+j} \binom{p^{n-1}}{j} \times S(a-j(p-1)+bp^m, bp^m) \pmod{p^n}, \tag{47}$$

where $b = p^{n-m} - k$.

Proof Using the same technique as in the proof of Theorem 8, we obtain the following two congruences;

$$\frac{1}{\prod_{i=1}^{kp^m}(1-ix)} \equiv \frac{\prod_{i=1}^{(p^{n-m}-k)p^m-1}(1+ix)}{(1-x^{p-1})^{p^{n-1}}} \pmod{p^n} \tag{48}$$

and

$$\prod_{i=1}^{kp^m-1}(1-ix) \equiv \frac{(1-x^{p-1})^{p^{n-1}}}{\prod_{i=0}^{(p^{n-m}-k)p^m}(1+ix)} \pmod{p^n}. \tag{49}$$

These two congruences generate the required results. Hence the theorem follows. □

Remark 5 In the second result of the preceding theorem, the generating function on the RHS of Equation (49) generates an infinite term. In contrast, the LHS generates $kp^m - 1$ terms only. It follows that the sum vanishes when $a \geq kp^m$, i.e.,

$$\sum_j (-1)^{a+j}\binom{p^{n-1}}{j} S(a-j(p-1)+bp^m, bp^m) \equiv 0 \pmod{p^n}, \tag{50}$$

if $a \geq kp^m$.
If we replace $n = m+1$ in Theorem 9, the second result of the theorem yields the following congruence;

$$\begin{aligned} s(kp^m, kp^m - a) \equiv (-1)^a [&S(a+bp^m, bp^m) \\ &- S(a-p^m(p-1)+bp^m, bp^m)] \pmod{p^{m+1}}, \end{aligned} \tag{51}$$

whenever $p-1 \nmid a$. Moreover, if $a < p^m(p-1)$ and $p-1 \nmid a$, we obtain

$$s(kp^m, kp^m - a) \equiv (-1)^a S(a+bp^m, bp^m) \pmod{p^{m+1}}. \tag{52}$$

We can utilize Theorem 9 to obtain some values of $v_p(S(n, kp^m))$, which are always greater than or equal to m. Although Theorem 5 deals with $v_p(S(n, kp^m))$, the theorem is restricted to valuations less than or equal to m since the key congruences used in Theorem 5 are in modulo p^m. The congruences obtained in Theorem 9 are in modulo p^n for arbitrary n, usually greater than or equal to m of $S(n, kp^m)$; the next theorem is an application of such congruence.

Theorem 10 *Let p be an odd prime and a, u, k, and m be positive integers such that $p \nmid a$ and $a \equiv 0 \pmod{p-1}$. If $p^{m-1} \leq u = \frac{a}{p-1} < p^m$ and $\frac{u}{p^{m-1}} + k \geq p$, then*

$$v_p(S(kp^m + a, kp^m)) = m. \tag{53}$$

Proof Replace n and a in the first result of Theorem 9 with $m+1$ and $u(p-1)$, respectively; we get

$$\begin{aligned} S(a+kp^m, kp^m) \equiv &\sum_j \binom{p^m + j - 1}{j} \\ &\times s(bp^m, bp^m - (u-j)(p-1)) \pmod{p^{m+1}} \end{aligned} \tag{54}$$

where $b = p - k$.
Let $u = \sum_{i=0}^{m-1} u_i p^i$ be the p-adic expansion of u. We know that $s(bp^m, bp^m - (u-j)(p-1))$ is divisible by p^m if $bp^m - (u-j)(p-1) < bp^{m-1}$ or $j < \sum_{i=0}^{m-1} u_i p^i - bp^{m-1} = \alpha$ (*say*). We can also confirm that p divides $\binom{p^m+j-1}{j}$ if $j < \alpha$, unless $j = 0$, in which case $s(bp^m, bp^m - (u-j)(p-1)) = 0$ since $bp^m - u(p-1) < 0$. Thus, all the j-th terms with $0 \leq j < \alpha$ are divisible by p^{m+1}, and we obtain

$$\begin{aligned} S(a+kp^m, kp^m) \equiv &\sum_{j=\alpha}^{u} \binom{p^m + j - 1}{j} \\ &\times s(bp^m, bp^m - (u-j)(p-1)) \pmod{p^{m+1}}. \end{aligned} \tag{55}$$

From Theorem 1(b), we have

$$s(bp^m, bp^m-(u-j)(p-1)) \equiv \binom{bp^{m-1}}{u-j}(-1)^{u-j} \pmod{p^m}. \tag{56}$$

It follows that the p-adic valuation of each term on the RHS of Equation (55) is $m-v_p(j)+m-1-v_p(u-j)$. If $p \nmid j$ and $j \neq u$, then the valuation is $2m-1-v_p(u-j)$, which is greater than or equal to $m+1$ unless $v_p(u-j)=m-1$. On the other hand, if $p \mid j$, then $p \nmid (u-j)$, and the valuation becomes $2m-1-v_p(j)$, which is greater than or equal to $m+1$ unless $v_p(j)=m-1$. If $v_p(u-j)=m-1$, then $u-j=rp^{m-1}$ for some r, $p \nmid r$. If $v_p(j)=m-1$, then $j=tp^{m-1}$ for some t, $p \nmid t$. The only remaining term whose p-adic valuation is less than $m+1$ is the term with $j=u$. Thus, Equation (55) reduces to

$$\begin{aligned} S(a+kp^m, kp^m) &\equiv \sum_{r=1}^{b} \binom{p^m+u-rp^{m-1}-1}{p^m-1} \varphi_{r(p-1)p^{m-1}} \\ &+ \binom{p^m+u-rp^{m-1}-1}{p^m-1} \varphi_{r(p-1)p^{m-1}} \\ &+ \sum_{t=u_{m-1}-b+1}^{u_{m-1}} \binom{p^m+tp^{m-1}-1}{p^m-1} \varphi_{(u-tp^{m-1})(p-1)} \pmod{p^{m+1}}, \end{aligned} \tag{57}$$

where $\varphi_q = s(bp^m, bp^m-q)$, and the preceding equation can be written as

$$\begin{aligned} S(a+kp^m, kp^m) &\equiv \sum_{r=0}^{b} \binom{p^m+u-rp^{m-1}-1}{p^m-1} \varphi_{r(p-1)p^{m-1}} \\ &+ \sum_{t=u_{m-1}-b+1}^{u_{m-1}} \binom{p^m+tp^{m-1}-1}{p^m-1} \varphi_{(u-tp^{m-1})(p-1)} \pmod{p^{m+1}}. \end{aligned} \tag{58}$$

Now, we have the following congruences

$$\begin{aligned} \binom{p^m+u-rp^{m-1}-1}{p^m-1} &= \frac{p^m}{p^m+u-rp^{m-1}} \binom{p^m+u-rp^{m-1}}{p^m} \\ &\equiv \frac{p^m}{u} \pmod{p^{m+1}}, \end{aligned} \tag{59}$$

$$\varphi_{rp^{m-1}(p-1)} \equiv \binom{bp^{m-1}}{rp^{m-1}}(-1)^r \equiv \binom{b}{r}(-1)^r \pmod{p}, \tag{60}$$

$$\begin{aligned} \binom{p^m+tp^{m-1}-1}{p^m-1} &= \frac{p^m}{p^m+tp^{m-1}} \binom{p^m+tp^{m-1}}{p^m} \\ &\equiv \frac{p}{t} \pmod{p^2}, \end{aligned} \tag{61}$$

and

$$\begin{aligned} \varphi_{(u-tp^{m-1})(p-1)} &\equiv \binom{bp^{m-1}}{u-tp^{m-1}}(-1)^{u-t} \pmod{p^m} \\ &\equiv \frac{bp^{m-1}}{u-tp^{m-1}} \binom{bp^{m-1}-1}{u-tp^{m-1}-1}(-1)^{u-t} \pmod{p^m} \\ &\equiv \frac{bp^{m-1}}{u} \binom{b-1}{u_{m-1}-t}(-1)^{s_p(u)-u_{m-1}-1}(-1)^{u-t} \pmod{p^m} \\ &\equiv \frac{bp^{m-1}}{u} \binom{b-1}{u_{m-1}-t}(-1)^{t-u_{m-1}-1} \pmod{p^m}. \end{aligned} \tag{62}$$

Let

$$X = \sum_{r=0}^{b} \binom{p^m + u - rp^{m-1} - 1}{p^m - 1} s(bp^m, bp^m - r(p-1)p^{m-1}) \tag{63}$$

and

$$Y = \sum_{t=u_{m-1}-b+1}^{u_{m-1}} \binom{p^m + tp^{m-1} - 1}{p^m - 1} s(bp^m, bp^m - (u - tp^{m-1})(p-1)). \tag{64}$$

From Equations (59), (60), and (63), we get

$$\begin{aligned} X &\equiv \frac{p^m}{u} \sum_{r=1}^{b} \binom{b}{r} (-1)^r \pmod{p^{m+1}} \\ &\equiv 0 \pmod{p^{m+1}}. \end{aligned} \tag{65}$$

From Equations (61), (62), and (64), we get

$$\begin{aligned} Y &\equiv \frac{p^m}{u} \sum_{t=u_{m-1}-b+1}^{u_{m-1}} \frac{b}{t} \binom{b-1}{u_{m-1}-t} (-1)^{t-u_{m-1}-1} \pmod{p^{m+1}} \\ &\equiv \frac{p^m}{u} \sum_{t=0}^{b-1} \frac{b}{u_{m-1}-t} \binom{b-1}{t} (-1)^{t-1} \pmod{p^{m+1}} \\ &\equiv \frac{p^m}{u} \sum_{t=0}^{b-1} \frac{b-t}{u_{m-1}-t} \binom{b}{t} (-1)^{t-1} \pmod{p^{m+1}}. \end{aligned} \tag{66}$$

Using partial fraction decomposition, we get

$$\sum_{t=0}^{b-1} \frac{b-t}{u_{m-1}-t} \binom{b}{t} (-1)^{b-t-1} = \frac{1}{\binom{u_{m-1}}{b}}. \tag{67}$$

From Equations (58) and (63) – (67), we obtain

$$S(a + kp^m, kp^m) \equiv \frac{(-1)^b p^m}{u\binom{u_{m-1}}{b}} \pmod{p^{m+1}}. \tag{68}$$

Since $p \nmid u\binom{u_{m-1}}{b}$,

$$v_p(S(a + kp^m, kp^m)) = m. \tag{69}$$

Hence, the theorem holds. □

The following theorem gives a generalization of Theorem 8 to congruence modulo p^n for any positive integer n greater than m.

Theorem 11 *For an odd prime p and positive integers a, u, and n such that $a \leq p^n$, the following two congruences holds;*

$$\begin{aligned} S(u + a, a) \equiv (-1)^u \sum_j &\binom{p^{n-1} + j - 1}{j} \\ &\times s(p^n - a, p^n - a - u + j(p-1)) \pmod{p^n} \end{aligned} \tag{70}$$

and

$$\begin{aligned} s(a, a - u) \equiv \sum_j &(-1)^{u+j} \binom{p^{n-1}}{j} \\ &\times S(u - j(p-1) + p^n - a, p^n - a) \pmod{p^n}. \end{aligned} \tag{71}$$

Proof The proof is similar to the proof of Theorems 8 and 9. □

Remark 6 It follows from Theorem 11 that if $0 \leq u < p-1$, then the index j has only one possible value, which is $j = 0$. Therefore,

$$S(u+a, a) \equiv (-1)^u s(p^n - a, p^n - a - u) \pmod{p^n} \tag{72}$$

and

$$s(a, a-u) \equiv (-1)^u S(u + p^n - a, p^n - a) \pmod{p^n}. \tag{73}$$

Declarations

Conflicts of interest The authors have no conflicts of interest to declare. All co-authors have seen and agree with the contents of the manuscript and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

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