ORIGINAL RESEARCH

Generalized Hamming Weight for self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$

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Abstract
We consider the self-dual codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ to discuss the generalized Hamming weight with respect to rank over the ring $\mathbb{F}_2 + u\mathbb{F}_2$. We also discuss the bounds in terms of rank for generalized Lee weight.

Keywords Linear codes · Hamming distance · Generalized Hamming weight · Singleton bound

Mathematics Subject Classification MSC 06D50 · MSC 11F50 · MSC 94B05 · MSC 94A62 · MSC 94B65

1 Introduction

Generalized Hamming weights were introduced by Helleseth et al. [1] for a linear code over finite field while studying the weight distribution of irreducible cyclic codes and later Wei [2] rediscovered the idea and gave definition of generalized Hamming weights. Janwa and Lal [3] discussed the generalized Hamming weight of binary cyclic codes of different odd length and gave an algorithm of time complexity of the order $O(r \cdot n \cdot q^{r(k-1)})$ to calculate r^{th} generalized Hamming distance over the field $\mathbb{F}_q$. Helleseth et al. [4] further discussed some results on generalized Hamming weight for Goethals and Preparata codes over the ring $\mathbb{Z}_4$ and gave an upper bound on the rth generalized Hamming weights for the Goethals code and also discussed the rth generalized Hamming weight for the Preparata code. Horimoto and Shiromoto [5] introduced the generalized Hamming weights with respect to the rank for linear codes over finite chain rings.

A Bonnecaze and P. Udaya [6] introduced linear codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ and discussed that self-dual codes of odd length exist over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ as in the case of $\mathbb{Z}_4$-codes. They also found that some non-free codes give rise to optimal binary linear codes and extremal self-dual codes through a linear Gray map.

We got the motivation from [5, 7, 8] to work on this paper. We consider the ring $R = \mathbb{F}_2 + u\mathbb{F}_2$, with $u^2 = 0$. The ring R shares some interesting properties of the ring $\mathbb{Z}_4$ and field $\mathbb{F}_4$. The set of elements of the ring R is $\{0, 1, u, \overline{u} = 1 + u\}$, R is a local ring with the maximal ideal $\{0, u\}$. Also R can be expressed as $\frac{\mathbb{F}_2[u]}{\langle u^2 \rangle}$.

The correspondence is organised as follows. Section 2 is devoted to some basic definitions and assumptions which we use in this paper. Section 3 talks about the codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ and the Lee weight of a code C over the ring $\mathbb{F}_2 + u\mathbb{F}_2$. Singleton bound and bounds for GLWR over the ring is given in Section 4. Section 5 is devoted towards the determination of generalized weight.

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2 Preliminaries

A linear code C over the ring R is an additive submodule of R^n. We define an inner product over the ring R as

$$\langle x, y\rangle = \sum_i x_i y_i,$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

The dual code $C^\perp$ is defined as

$$C^\perp = \{y \mid \langle x, y\rangle = 0, \text{ for all } x \in C\}.$$

A code C is self-dual if $C = C^\perp$.

Two codes C_1 and C_2 are said to be permutation equivalent if all codewords of C_2 can be obtained from the codewords of C_1 by applying a permutation of coordinates $\{1, 2, \ldots, n\}$ on the codewords of C_1 and vice versa.

Let $x = (x_1, x_2, \ldots, x_n)$ be a codeword in a linear code C over the ring R and let $n_0(x)$ denotes the number of $x_i = 0$, $n_2(x)$ denotes the number of $x_i = u$ and $n_1(x) = n - n_0(x) - n_2(x)$, then the Lee weight w_L of $x = (x_1, x_2, \ldots, x_n)$ is given by

$$n_1(x) + 2n_2(x).$$

For a vector $x \in R^n$, we define the norm as

$$|x| = |\, supp(x)\,|,$$

where $supp(x) = \{i \mid x_i \neq 0\}$. By exchanging it to sub codes, let C be a code of length n and let D be any subcode of C, then

$$|D| = |\, supp(D)\,|,$$

where

$$supp(D) = \{\, i \mid x \in D \text{ with } x_i \neq 0\,\} = \bigcup_{x \in D} supp(x)$$

For a linear code C over a ring R and any g, $1 \leq g \leq \text{rank}(C)$, we define g^{th} generalized Hamming weight as

$$d_g^H(c) = \min\{|D| : D \text{ is an } R - \text{submodule of } C \text{ with rank}(D) = r\}.$$

The minimum Hamming weight of a code C is $d_1^H(C)$.

3 Codes over $\mathbb{F}_2 + u\mathbb{F}_2$

A non-zero linear code C over R has a generator matrix which, after a suitable permutation of co-ordinates can be written in the form

$$G = \begin{pmatrix} I_{k_1} & A & B \\ 0 & uI_{k_2} & uD \end{pmatrix},$$

where A and B are matrices over R and D is a matrix over $\mathbb{F}_2$. Also $|C| = 2^{2k_1+k_2}$. The residue code $C_{(1)}$ is defined as

$$C_{(1)} = \{x \in \mathbb{F}_2^n \mid x + uy \in C \text{ for } y \in \mathbb{F}_2^n\},$$

and torsion code $C_{(2)}$ is defined as

$$C_{(2)} = \{x \in \mathbb{F}_2^n \mid ux \in C\}.$$

The Gray map $\phi : R \to \mathbb{F}_2^2$ is defined as

$$\phi(x + uy) = (y, x + y),$$

where $x, y \in \mathbb{F}_2$.

Lemma 1 [6] *If a linear code C is self-dual, then $\phi(C)$ is also self-dual. The minimum Hamming weight of $\phi(C)$ is equal to the minimum Lee weight of C.*

If a linear code C is self-dual, so is $\phi(C)$. The minimum Lee weight of C is equal to the minimum Hamming weight of $\phi(C)$. So we calculate Lee weight for each element $v \in R$. Lee weight for each element of $R = \{0, 1, u, 1 + u\}$ is given by $\{0, 1, 2, 1\}$ respectively.

Let C be a code of length n over $\mathbb{F}_2 + u\mathbb{F}_2$ and $A(C)$ be the $|C| \times n$ array of all codewords in C. We use the notation $u - \dim(C)$ to denote u dimension of C. $A(C)$ will correspond to the following three types:

(i) Columns containing only 0,
(ii) Columns containing 0 and u equally often,
(iii) Columns containing all the elements of $\mathbb{F}_2 + u\mathbb{F}_2$ equally often.

We define Lee-support weight of these columns as 0, 2, 1, respectively. For a code C, Lee-support weights $wt_L(C)$ is defined as the sum of the (Lee weights - support weights) of all columns of $A(C)$. The r^{th} generalized Lee weight with respect to the ranks $d_r^L(C)$ of C for $1 \leq r \leq 2k_1 + k_2$, is defined as

$$d_r^L(C) = \min\{wt_L(D) : D \text{ is a submodule of } C \text{ with } \operatorname{rank}(D) = r\}.$$

We define the r^{th} generalized Lee weight with respect to u-dimension (GLWT) of C as follows:

$$u - d_r^L(C) = \min\{wt_L(D) : D \text{ is a submodule of } C \text{ with } u - dim(D) = r\}.$$

We define (k_1, k_2) generalized Lee weight with respect to type $[k_1, k_2]$ as follows:

$$d_{k_1,k_2}^L(C) = \min\{wt_L(D) : D \text{ is a submodule of } C \text{ of type } [k_1, k_2]\}.$$

It can be noted that, $u - d_r^L(C)$ does not always correspond to the minimum Lee weight of C. In each case, the set $\{d_r^L(C)\}$, $\{d_{k_1,k_2}^L(C)\}$, $\{u - d_r^L(C)\}$ is called the Lee weight hierarchy of C.

Theorem 1 *Let C be a linear code of length n over R with type $2^{2k_1+k_2}$. Then we have*

$$wt_L(C) = \frac{1}{4^{k_1-1}2^{k_2}} \sum_{x \in C} wt_L(x) - wt(x)$$

Proof In an array $A(C)$, let n_0 be the number of columns in which 0 and 4 are balanced and let n_1 be the number of columns in which 0, 1, u, $\overline{u}$ occurs equally often". So we have

$$wt_L(C) = 2n_0 + n_1.$$

Now

$$\begin{aligned}\sum_{x\in C}(wt_L(x)-wt(x)) &= \left(\frac{n_0|C|}{2}\cdot 2\right)+n_1\left(\frac{|C|}{4}\cdot 1+\frac{|C|}{4}\cdot 2+\frac{|C|}{4}\cdot 1\right)\\ &\quad-\left(\frac{n_0|C|}{2}\cdot 1\right)-n_1\left(\frac{|C|}{4}\cdot 1+\frac{|C|}{4}\cdot 1+\frac{|C|}{4}\cdot 1\right)\\ &= \frac{|C|}{4}(4n_0+4n_1-4n_0-3n_1)\\ &= \frac{|C|}{4}(2n_0+n_1)\\ &= \frac{|C|}{4}(wt_L(C)).\end{aligned}$$

Thus

$$wt_L(C)=\frac{1}{4^{k_1-1}2^{k_2}}\sum_{x\in C}(wt_L(x)-wt(x)).$$

□

We now give a MacWilliams relation for a code C over the ring R.

Theorem 2 [9] *Let C be a code of length n over R. Then the relation between complete weight enumerator and its dual is given by*

$$\begin{aligned}W^u_{C^\perp}(z_{00}+z_{01}+z_{10}+z_{11}) = \frac{1}{|C|}W^u_C(&z_{00}+z_{01}+z_{10}+z_{11},\\ &z_{00}-z_{01}+z_{10}-z_{11},\\ &z_{00}+z_{01}-z_{10}-z_{11},\\ &z_{00}-z_{01}-z_{10}+z_{11}).\end{aligned}$$

4 Bounds

4.1 A Singleton Bound

The finite ring R which has the Jacobson radical $J(R)\neq 0$ is called a finite chain ring if its principal left ideals form a ring [10]. Any finite chain ring can be seen as a local ring with $J(R)=R\theta$ for any $\theta\in J(R)/J(R)^2$. The ring $\mathbb{F}_2+u\mathbb{F}_2$ with $u^2=0$ is a finite chain ring. For generalized Hamming weight with respect to rank (GHWR) over finite chain rings, the singleton type bound is given by Horimoto and Shiromoto [5].

Proposition 1 *Let C be a linear code of length n over a finite chain ring R. For any r, $1\leq r\leq$ rank (C), we have*

$$d_r^H(C)\leq n-\operatorname{rank}(C)+r.$$

Let p and q be integers with q dividing p, by application of Chinese remainder theorem a map from

$$\psi_q:\left(\frac{\mathbb{F}_p[u]}{\langle u^2\rangle}\right)^n\longrightarrow\left(\frac{\mathbb{F}_q[u]}{\langle u^2\rangle}\right)^n$$

is defined as follows:

$$\psi_q(\alpha_1,\ldots,\alpha_n)=(\alpha_1 \pmod q,\ldots,\alpha_n \pmod q),$$

where $v = (\alpha_1, \ldots, \alpha_n)$ and $v \in \left(\frac{\mathbb{F}_p[u]}{\langle u^2\rangle}\right)$.

Let k be a positive integer with $k = \prod_{i=1}^{s} q_i$ and gcd $(q_i, q_j) = 1$.

We define a map

$$\psi_q : \left(\frac{\mathbb{F}_p[u]}{\langle u^2\rangle}\right)^n \longrightarrow \left(\frac{\mathbb{F}_{q_1}[u]}{\langle u^2\rangle}\right)^n \times \cdots \times \left(\frac{\mathbb{F}_{q_s}[u]}{\langle u^2\rangle}\right)^n,$$

by

$$\psi(v) := (\psi_{q_1}(v), \ldots, \psi_{q_s}(v)).$$

If $C^{(q_1)}, \cdots, C^{(q_s)}$ be codes of length n, with $C^{(q_i)}$ a code over $\frac{\mathbb{F}_{q_i}[u]}{\langle u^2\rangle}$, define the product by

$$\text{CRT}(C^{(q_1)}, \cdots, C^{(q_s)}) = \{\psi^{-1}(v_1, v_2, \ldots, v_s) \mid v_i \in C^{(q_i)}\},$$

where $\psi^{-1}(v_1, v_2, \ldots, v_s)$ is the unique vector that is component-wise congruent to $v_i \pmod{q_i}$.

The generalized Chinese remainder theorem implies that CRT is the inverse image of the map ψ. We have the following fact:

$$\text{rank}(\text{CRT}(C^{(q_1)}, \cdots, C^{(q_s)}) = \max(\text{rank}(C^{(q_i)})).$$

Additionally, we can see that if $C = \text{CRT}(C^{(q_1)}, \cdots, C^{(q_s)})$ and D is a subcode of C of rank h, then

$$D = \text{CRT}(D^{(q_1)}, \cdots, D^{(q_s)}),$$

where $D^{(q_i)} \subseteq C^{(q_i)}$ and max (rank$D^{(q_i)}$) is h.

Lemma 2 *Let $C = CRT(C^{(q_1)}, \cdots, C^{(q_s)})$, then $d_g(C) = \min(d_g^H(C^{(q_i)}))$.*

Proof This follows from the fact that $D = \text{CRT}(0, \ldots, D^{(q_i)}, \ldots, 0)$ is a code of C of rank g for all i if $D^{(q_i)}$ has rank g. □

Theorem 3 *Let C be a linear code of length n over R of rank r. Then $d_g \leq n - r + g$, for any g, $1 \leq g \leq r$.*

Proof Proof follows from Proposition 1 and Lemma 2. □

The code which meet this bound is g^{th} maximum Hamming distance separable with respect to rank (g^{th} MHDR) codes.

The Hamming and Lee distance of the elements of the ring R is {0, 1, 1, 1} and {0, 1, 2, 1} respectively. Let n_0, n_1, n_2, n_3 be the number of 0, 1, u, $1+u$ respectively, then d^L and d^H can be calculated by $d^H = n_1 + n_2 + n_3$ and $d^L = n_1 + 2n_2 + n_3$.

$$\left\lceil \frac{d^L}{2} \right\rceil \left\lceil \frac{n_1 + 2n_2 + n_3}{2} \right\rceil \leq d^H. \tag{1}$$

4.2 Bounds For GLWR

By using the above identity we now discuss the bounds for GLWR.

Theorem 4 *If C is a linear code of length n over R and for any r, $1 \leq r \leq rank\ (C)$*

$$\left\lfloor \frac{d_r^L - 2r + 1}{2} \right\rfloor \leq n - \text{rank}(C).$$

Proof Let $d_r^L = d_r^L(C)$ and $k = \text{rank}(C)$. For contradiction let us assume that

$$\left\lfloor \frac{d_r^L - 2r + 1}{2} \right\rfloor > n - k. \tag{2}$$

Then

$$\left\lfloor \frac{d_r^L - 2r + 1}{2} \right\rfloor = \left\{ \begin{matrix} \frac{d_r^L - 2r}{2} & d_r^L : \text{even} \\ \frac{d_r^L - 2r + 1}{2} & d_r^L : \text{odd} \end{matrix} \right\}.$$

Case I: d_r^L is even, then bound in equation 2 becomes

$$d_r^L > 2n - 2k + 2r.$$

But from Theorem 3,

$$d_r^L \leq 2n - 2k + 2r, \tag{3}$$

which contradicts our assumption.
Case 2: If d_r^L is odd, then the bound in equation 2 is

$$d_r^L > 2n - 2k + 2r + 1.$$

Thus we have

$$d_r^L = 2n - 2k + 2r$$

from equation 3. This contradicts the fact that d_r^L is odd. Therefore the result follows. □

Remark For $r = 1$ we have $d_r^L = d^L$, so the bound becomes

$$\left\lfloor \frac{d^L - 1}{2} \right\rfloor \leq n - \text{rank}(C).$$

When the linear code of length n over the ring R meets the bound in Theorem 4 for r, i.e.,

$$\left\lfloor \frac{d^L - 1}{2} \right\rfloor \leq n - \text{rank}(C),$$

then the code C is called r-th maximum Lee distance separable with respect to Rank (r-th MLDR) code.

Lemma 3 *If C is an r-th MLDR code, then*

$$d_r^L(C) = d_r^H(C) \text{ or } d_r^H(C) - 1.$$

Proof Since C is an r-th MLDR code

$$\left\lfloor \frac{d_r^L - 2r + 1}{2} \right\rfloor = n - \text{rank}(C).$$

Let us assume that

$$d_r^L(C) < 2d_r^H(C) - 1.$$

If $d_r^L(C)$ is odd, then we have

$$d_r^L(C) = 2n - 2\text{rank}(C) + 2r - 1.$$

Since

$$d_r^L(C) < 2d_r^H(C) - 1,$$

which implies

$$2n - 2\,\text{rank}(C) + 2r - 1 < 2d_r^H(C) - 1 \Leftrightarrow n - \text{rank}(C) + r < d_r^H(C),$$

a contradiction.
Similarly we can proceed for even $d_r^L(C)$. Hence the result follows. □

Theorem 5 *Let C be a linear code of length n over the ring R. If C is an r-th MLDR code, then C is an r-th MHDR code.*

Proof From the above Lemma 3, we have

$$n - \text{rank}(C) = \left\lfloor \frac{d_r^L - 2r + 1}{2} \right\rfloor = d_r^H - r.$$

□

Theorem 6 *Let C be an r-th MHDR code of length n over R. Then C is an r-th MLDR code if and only if $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$.*

Proof By Theorem 5 C is an r-th MLDR code if and only if

$$\left\lfloor \frac{d_r^L - 2r + 1}{2} \right\rfloor = d_r^H(C) - r.$$

If $d_r^L(C)$ is odd, then

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = \frac{d_r^L(C) - 2r + 1}{2} = d_r^H(C) - r.$$

If $d_r^L(C)$ is even, then

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor = \frac{d_r^L(C) - 2r}{2} = d_r^H(C) - r.$$

□

Let C be a linear code of length n over R with $\text{rank}(C) = k$ and minimum Hamming weight d_H, then Griesmer type bound for the code C can be written as

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H}{2^i} \right\rceil.$$

Theorem 7 *For a linear code C of length n over the ring R and for any r, $1 \leq r \leq \text{rank}(C)$, then the generalized Griesmer type bound for GLWR can be written as*

$$d_r^L(C) \geq \sum_{i=0}^{r-1} \left\lceil \frac{\left\lceil \frac{d_1^L(C)}{2} \right\rceil}{2^i} \right\rceil.$$

Proof From the definition of Griesmer type bound and equation 1 we can write

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{\left\lceil \frac{d_1^L(C)}{2} \right\rceil}{2^i} \right\rceil,$$

where $d_1^L(C)$ is the minimum Lee weight and k is the rank(C).
Let D be a subcode of C with $wt_L(D) = d_r^L(C)$ and rank$(D) = r$ and D' be the code with generator matrix come from the generator matrix of D by deleting zero columns. As the length of D' is less than equal to $wt_L(D)$ and the minimum Lee weight of D' is greater than or equal to $d_l^L(C)$, the rest can be derived from above proposition. □

For an R-module M, the socle of M is the sum of all simple submodules of M and is denoted by Soc(M). For a linear code C over the ring R it is defined as Soc$(C) = \{x \in C \mid x\theta = 0\}$. It is known that if C is a linear code of length n over a finite ring with rank$(C) = k$ and minimum Hamming weight d_H, then soc(C) is isomorphic to a binary $[n, k, d_H]$ code [5].

Lemma 4 *For any r, $1 \leq r \leq$ rank(C), $d_r^H(C) = d_r^H(soc(C))$.*

5 Determination of Generalized Weight

Let C be a linear code over the ring R of length n and u-dimension k. For $x \in C$ we define

$$\omega_u(x) = \mid \{i \mid x_i = u\},$$

so from Theorem 1 for $1 \leq r \leq k$,

$$d_r^L(C) = \frac{1}{2^{r-2}} \min \left\{ \sum_{x \in D} \omega_u(x) \mid D \text{ is } [n, r] \text{ subcode of } C \right\},$$

but finding a generalized Lee weight is difficult as ω_u is not a metric function.

Lemma 5 *Let C be a linear code over $\mathbb{F}_2 + u\mathbb{F}_2$ with generator matrix in the form $[ug_1, ug_2, \ldots, ug_k]$, then for $1 \leq r \leq k$, we have $d_r^L(C) = 2d_r^H(C)$.*

5.1 First order Reed-Muller Code

The first order Reed-Muller code $R^{1,m}$ over $\mathbb{F}_2 + u\mathbb{F}_2$ is a code of length $n = 2^{m-1}$, rank $= m$ and u-dimension $= m + 1$ with minimum Hamming weight 2^{m-2} and minimum Lee weight 2^{m-1}.
We define first order Reed-Muller code, $R^{1,m}$ as any primitive polynomial $h(x)$ of order m dividing $x^{2^m-1} - 1$ and α as one of its roots. Then $\alpha, \alpha^2, \ldots, \alpha^{2^m-1}$ be also the roots of $h(x)$, then the generator matrix of $R^{1,m}$ is given in [11] as:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & \alpha & \alpha^2 & \ldots & \alpha^{n-1} \end{pmatrix}.$$

The second row of the above matrix is consists of α^i which corresponds to the columns $[a_{1i}, a_{2i}, \ldots, a_{mi}]$, where

$$\alpha^i = a_{1i} + a_{2i}\alpha + \ldots + a_{mi}\alpha^{m-1}.$$

By our definition above we can generate u-basis from

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ u & u & u & \dots & u \\ 0 & 1 & \alpha & \dots & \alpha^n \end{pmatrix}.$$

For $c \in R^{1,m}$, c can be written as a u-linear combination of the above matrix. So it can be expressed in the required format for lemma 5, so by using that lemma

$$d_r^L(C) = 2d_r^H(C).$$

Let $1 \leq r \leq m$ and let D be an r-dimensional subcode of $R^{1,m}$ generated by any r rows chosen from above matrix. Then the chosen r rows share 2^{m-r} common zero bit positions. Hence the support size of D is $\sum_{i=0}^{r-1} 2^{m-i+1}$. From this and the result of Lemma 5, it can be concluded that the Lee weight hierarchy of $R^{1,m}$ with respect to u-dimension is given by

$$u - d_r^L(C) = 2^{m-r}(2^r - 1) \text{ and } u - d_m^L(C) = 2^m.$$

5.2 Simplex Codes

Unlike binary simplex code, code over the ring R is divided in two types, i.e., α and β type. The generator matrix for both type of code can be derived inductively [12].
For the α type simplex code, the generator matrix is

$$G_k^\alpha = \begin{pmatrix} 0000 & 1111 & uuuu & \overline{uuuu} \\ G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha \end{pmatrix}$$

and $G_1^\alpha = [01u\overline{u}]$, this generator matrix has dimension $k \times 2^{2k}$.
For β type simplex code, the generator matrix can be written as follows:

$$G_k^\beta = \begin{pmatrix} 11\dots1 & 00\dots0 & uu\dots u \\ G_{k-1} & G_{k-1}^\beta & G_{k-1}^\beta \end{pmatrix},$$

for $k \geq 2$.
The generator matrix of β type simplex code is of dimension $k \times 2^{k-1}(2^k - 1)$. By using Lemma 4.4 and Lemma 5 from [8] we can prove that $d_r^H(s_k^\beta) = 2^{2k-r-1}(2^r - 1)$ and $d_r^H(s_k^\alpha) = 2^{2k-r}(2^r - 1)$.

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