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ORIGINAL RESEARCH

On some restricted overpartition functions

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Abstract We prove arithmetic properties of some restricted overpartition functions in which the parts are from certain residue classes of 8. For example, if $\overline{p}_{3,4,5}(n)$ and $\overline{p}_{1,4,7}(n)$ denote the number of overpartitions of a positive integer n into parts congruent to 3, 4, or 5 modulo 8 and congruent to 1, 4, or 7 modulo 8, respectively, then $\overline{p}_{3,4,5}(16n+1) \equiv 0 \pmod{16}$ and $\overline{p}_{1,4,7}(16n+9) \equiv 0 \pmod{16}$ for all non-negative integers n.

Keywords Overpartition · Pod function · Congruence · Theta function

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1 Introduction and results

An overpartition of n is a non-increasing sequence of positive integers whose sum is n in which the first occurrence of a number may be overlined. The overpartition function, denoted by $\overline{p}(n)$, is the number of overpartitions of n, with the convention that $\overline{p}(0) = 1$. For example, $\overline{p}(3) = 8$ because there are 8 overpartitions of 3, namely. $3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1$, and $\overline{1}+1+1$. Since the overlined parts of an overpartition form a partition into distinct parts and the non-overlined parts of an overpartition form an unrestricted partition, the generating function of $\overline{p}(n)$ is

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

where here and for the sequel, for any complex number a and q with $|q| < 1$, the standard q-product $(a;q)_\infty$ is defined by

$$(a;q)_\infty := \prod_{j=0}^{\infty} \left(1 - aq^j\right).$$

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After the publication of the paper [12] by Corteel and Lovejoy in 2004, overpartitions have been studied quite extensively from various points of view. For example, many arithmetic properties of overpartitions as well as overpartitions into odd parts have been proved by a number of mathematicians. For more information and references, see [8–11, 15, 17–19, 21–26, 28, 30–36].

In this paper, we prove arithmetic properties of some restricted overpartition functions in which the parts are from certain residue classes of 8. We consider the following restricted overpartition functions:

$$\begin{aligned}
\overline{p}_{3,5}(n) &= \text{number of overpartitions of } n \text{ into parts congruent to } \pm 3 \text{ modulo } 8,\\
\overline{p}_{3,4,5}(n) &= \text{number of overpartitions of } n \text{ into parts congruent to } \pm 3 \text{ or } 4 \text{ modulo } 8,\\
\overline{p}_{1,7}(n) &= \text{number of overpartitions of } n \text{ into parts congruent to } \pm 1 \text{ modulo } 8,\\
\overline{p}_{1,4,7}(n) &= \text{number of overpartitions of } n \text{ into parts congruent to } \pm 1 \text{ or } 4 \text{ modulo } 8.
\end{aligned}$$

By convention, we also set $\overline{p}_{3,5}(0) = \overline{p}_{3,4,5}(0) = \overline{p}_{1,7}(0) = \overline{p}_{1,4,7}(0) = 1$.

Example 1 $n = 8$

Then $\overline{p}_{3,5}(9) = 2$, $\overline{p}_{3,4,5}(9) = 6$, $\overline{p}_{1,7}(9) = 8$, $\overline{p}_{1,4,7}(9) = 16$, and the relevant partitions are given by

$$\begin{aligned}
&3+3+3 = \overline{3}+3+3,\\
&5+4 = \overline{5}+4 = 5+\overline{4} = \overline{5}+\overline{4} = 3+3+3 = \overline{3}+3+3,\\
&9 = \overline{9} = 7+1+1 = \overline{7}+1+1 = 7+\overline{1}+1 = \overline{7}+\overline{1}+1\\
&\quad = 1+1+1+1+1+1+1+1+1 = \overline{1}+1+1+1+1+1+1+1+1,\\
&9 = \overline{9} = 7+1+1 = \overline{7}+1+1 = 7+\overline{1}+1 = \overline{7}+\overline{1}+1 = 4+4+1\\
&\quad = \overline{4}+4+1 = 4+4+\overline{1} = \overline{4}+4+\overline{1} = 4+1+1+1+1+1\\
&\quad = \overline{4}+1+1+1+1+1 = 4+\overline{1}+1+1+1+1 = \overline{4}+\overline{1}+1+1+1+1\\
&\quad = 1+1+1+1+1+1+1+1+1 = \overline{1}+1+1+1+1+1+1+1+1.
\end{aligned}$$

Now, let pod(n) denote the number of partitions of n into non-repeating odd parts, that is, odd parts must be distinct (and even parts are unrestricted). This function has appeared in several earlier works. For example, see the papers by Andrews [3–5], Berkovich and Garvan [6], Alladi [1,2]. The function has also been studied by several authors from the arithmetical point of view. For example, see the papers by Hirschhorn and Sellers [20], Radu and Sellers [27], Lovejoy and Osburn [24], Cui, Gu and Ma [13], Wang [29], Fang, Xue and Yao[14], and some other papers cited thereon.

The functions $\overline{p}_{3,4,5}(n)$ and $\overline{p}_{1,4,7}(n)$ are connected to pod(n) by the following relations.

Theorem 11 *For any positive integer n, we have*

$$\overline{p}_{3,4,5}(n) = \sum_{k=-\infty}^{\infty} (-1)^{k(k+1)/2} pod\left(n - (2k^2 - k)\right), \tag{1.1}$$

$$\overline{p}_{1,4,7}(n) = \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} pod\left(n - (2k^2 - k)\right). \tag{1.2}$$

Example 2 $n = 9$

We have

$$\begin{aligned}
\overline{p}_{3,4,5}(9) &= \text{pod}(9) - \text{pod}(8) + \text{pod}(6) - \text{pod}(3)\\
&= 13 - 10 + 5 - 2 = 6,\\
\overline{p}_{1,4,7}(9) &= \text{pod}(9) + \text{pod}(8) - \text{pod}(6) - \text{pod}(3)\\
&= 13 + 10 - 10 - 5 - 2 = 16.
\end{aligned}$$

which are in agreement with Example 1.

In Section 2, we present a proof of Theorem 11.

Clearly, the generating functions of $\overline{p}_{3,4}$, $\overline{p}_{1,7}$, $\overline{p}_{3,4,5}$, and $\overline{p}_{1,4,7}(n)$ are given by

$$\sum_{n=0}^{\infty}\overline{p}_{3,5}(n)q^n=\frac{(-q^3;q^8)_\infty(-q^5;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty},\quad \sum_{n=0}^{\infty}\overline{p}_{1,7}(n)q^n=\frac{(-q;q^8)_\infty(-q^7;q^8)_\infty}{(q;q^8)_\infty(q^7;q^8)_\infty},$$

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(n)q^n=\frac{(-q^3;q^8)_\infty(-q^4;q^8)_\infty(-q^5;q^8)_\infty}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}, \tag{1.3}$$

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(n)q^n=\frac{(-q;q^8)_\infty(-q^4;q^8)_\infty(-q^7;q^8)_\infty}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}. \tag{1.4}$$

Since $(-q^\ell;q^8)_\infty\equiv(q^\ell;q^8)_\infty \pmod 2$, it is clear that for all $n>0$,

$$\overline{p}_{3,5}(n)\equiv\overline{p}_{3,4,5}(n)\equiv\overline{p}_{1,7}(n)\equiv\overline{p}_{1,4,7}(n)\equiv 0 \pmod 2.$$

We present the following congruences.

Theorem 12 *For any non-negative integer n, we have*

$$\overline{p}_{3,5}(8n+7)\equiv 0 \pmod{2^2}, \tag{1.5}$$
$$\overline{p}_{3,5}(16n+14)\equiv 0 \pmod{2^2}, \tag{1.6}$$
$$\overline{p}_{3,5}(32n+28)\equiv 0 \pmod{2^2}, \tag{1.7}$$
$$\overline{p}_{3,5}(16n+1)\equiv 0 \pmod{2^3}, \tag{1.8}$$
$$\overline{p}_{3,5}(32n+14)\equiv 0 \pmod{2^3}, \tag{1.9}$$
$$\overline{p}_{3,5}(16n+7)\equiv 0 \pmod{2^4}, \tag{1.10}$$
$$\overline{p}_{3,5}(32n+2)\equiv 0 \pmod{2^4}, \tag{1.11}$$
$$\overline{p}_{3,4,5}(8n+7)\equiv 0 \pmod{2^2}, \tag{1.12}$$
$$\overline{p}_{3,4,5}(16n+15)\equiv 0 \pmod{2^3}, \tag{1.13}$$
$$\overline{p}_{3,4,5}(16n+1)\equiv 0 \pmod{2^4}, \tag{1.14}$$
$$\overline{p}_{3,4,5}(32n+r)\equiv 0 \pmod{2^4},\quad where\ r\in\{2,14\}, \tag{1.15}$$
$$\overline{p}_{1,7}(8n+7)\equiv 0 \pmod{2^2}, \tag{1.16}$$
$$\overline{p}_{1,7}(16n+14)\equiv 0 \pmod{2^2}, \tag{1.17}$$
$$\overline{p}_{1,7}(32n+28)\equiv 0 \pmod{2^2}, \tag{1.18}$$
$$\overline{p}_{1,7}(16n+9)\equiv 0 \pmod{2^3}, \tag{1.19}$$
$$\overline{p}_{1,7}(32n+30)\equiv 0 \pmod{2^3}, \tag{1.20}$$
$$\overline{p}_{1,7}(16n+15)\equiv 0 \pmod{2^4}, \tag{1.21}$$
$$\overline{p}_{1,7}(32n+18)\equiv 0 \pmod{2^4}, \tag{1.22}$$
$$\overline{p}_{1,4,7}(8n+7)\equiv 0 \pmod{2^2}, \tag{1.23}$$
$$\overline{p}_{1,4,7}(16n+7)\equiv 0 \pmod{2^3}, \tag{1.24}$$
$$\overline{p}_{1,4,7}(16n+9)\equiv 0 \pmod{2^4}, \tag{1.25}$$
$$\overline{p}_{1,4,7}(32n+r)\equiv 0 \pmod{2^4},\quad where\ r\in\{18,30\}. \tag{1.26}$$

The power of 2 in the modulus of each of the above congruences is sharp. Furthermore, we have checked numerically that there is no congruence modulo 2^5 for arguments of the form $An+B$ where $A=4, 8, 16, 32$, or 64 and $B=0,1,\ldots,A-1$.

In fact, we readily deduce each of the above congruences by finding the explicit generating function as a multiple of the corresponding modulus. The generating functions are in terms of Ramanujan's theta functions. For complex numbers a and b with $|ab| < 1$, his general theta function $f(a, b)$ is defined by

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}.$$

Two useful special cases of $f(a, b)$ are defined by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{1.27}$$

$$\psi(q) := f(q,q^3) = \frac{1}{2} f(1,q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.28}$$

The q-product representations in (1.27) and (1.28) arise from Jacobi's triple product identity [7, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{1.29}$$

We state some internal congruences modulo 2^2 and 2^3 in the following theorem.

Theorem 13 *For all nonnegative integers n and j,*

$$\overline{p}_{3,5}(2^j n) \equiv \overline{p}_{3,5}(n) \pmod{2^2}, \tag{1.30}$$

$$\overline{p}_{1,7}(2^j n) \equiv \overline{p}_{1,7}(n) \pmod{2^2}. \tag{1.31}$$

For n being an integer not of the form $(4k+2)^2$, where k is any nonnegative integer, we have

$$\overline{p}_{3,5}(2n) \equiv \overline{p}_{3,5}(n) \pmod{2^3}, \tag{1.32}$$

$$\overline{p}_{1,7}(2n) \equiv \overline{p}_{1,7}(n) \pmod{2^3}. \tag{1.33}$$

For n not being the square of an odd integer and j being any nonnegative integer, we have

$$\overline{p}_{3,5}(2^j n) \equiv \overline{p}_{3,5}(n) \pmod{2^3}, \tag{1.34}$$

$$\overline{p}_{1,7}(2^j n) \equiv \overline{p}_{1,7}(n) \pmod{2^3}. \tag{1.35}$$

We prove Theorem 13 in Section 3. We derive some infinite families of congruences modulo 2^2 and 2^3 from the previous two theorems.

Corollary 14 *For all nonnegative integers n and j,*

$$\overline{p}_{3,5}\left(2^j(8n+7)\right) \equiv 0 \pmod{2^2}, \tag{1.36}$$

$$\overline{p}_{3,5}\left(2^j(32n+14)\right) \equiv 0 \pmod{2^3}, \tag{1.37}$$

$$\overline{p}_{1,7}\left(2^j(8n+7)\right) \equiv 0 \pmod{2^2}, \tag{1.38}$$

$$\overline{p}_{1,7}\left(2^j(32n+30)\right) \equiv 0 \pmod{2^3}. \tag{1.39}$$

For $2n$ not being a triangular number and j being any nonnegative integer,

$$\overline{p}_{3,5}\left(2^j(16n+1)\right) \equiv 0 \pmod{2^3}. \tag{1.40}$$

For $2n+1$ not being a triangular number and j being any nonnegative integer,

$$\overline{p}_{1,7}\left(2^j(16n+9)\right) \equiv 0 \pmod{2^3}. \tag{1.41}$$

Proof The congruences of (1.5) and (1.30) immediately imply (1.36). Similarly, (1.16) and (1.31) imply (1.38).
Since $32n+14$ is not the square of an odd number for any n, (1.9) and (1.32) imply (1.37).
Similarly, $32n+30$ is not the square of an odd number for any n. Hence (1.20) and (1.33) imply (1.39).

Now, for any nonnegative integer k, $16n+1 \neq (2k+1)^2$ implies that $2n \neq \frac{k(k+1)}{2}$. Hence, (1.8) and (1.32) imply (1.40).

Similarly, for any nonnegative integer k, $16n+9 \neq (2k+1)^2$ implies that $2n+1 \neq \frac{k(k+1)}{2}$. Therefore, (1.19) and (1.33) imply (1.41). □

Remark 1 The congruences of (1.6) and (1.7) are special cases of (1.36). Similarly (1.17) and (1.18) are special cases of (1.38).

In the following theorem, we present the explicit generating functions implying congruences in (1.5)–(1.26).

Theorem 15 *We have*

$$\sum_{n=0}^{\infty} \overline{p}_{3,5}(8n+7)q^n = 4q\,\frac{\psi(-q^2)\psi(-q^4)f(-q^2,-q^{14})}{\varphi^2(-q)\varphi(-q^2)}, \tag{1.42}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \overline{p}_{3,5}(16n+14)q^n &= 4\,\frac{\varphi^4(q)\psi(-q^2)}{\varphi^9(-q^2)}\Big(2f(-q,q^3)\psi^2(q^2)\varphi(-q^8)\\ &\quad + qf(-q^2,-q^{14})\varphi^2(q)\psi(-q^4)\Big),\end{aligned} \tag{1.43}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \overline{p}_{3,5}(32n+28)q^n &= 4\frac{\varphi^8(q)}{\varphi^{17}(-q^2)}\Big(2f(-q,q^3)\varphi(-q^4)\psi(q^2)\\ &\quad \times \Big(\varphi(q)\psi(-q^4)\left(2\varphi^4(q)-\varphi^4(-q)\right)\\ &\quad + \varphi(-q^8)\psi(q^4)\left(4\varphi^4(q)-\varphi^4(-q)\right)\Big)\\ &\quad + qf(-q^2,-q^{14})\varphi^2(q)\psi(-q^2)\Big(\psi(-q^4)\left(4\varphi^4(q)-3\varphi^4(-q)\right)\\ &\quad + 16\varphi(q)\varphi(q^2)\varphi(-q^8)\psi^2(q^2)\Big)\Big),\end{aligned} \tag{1.44}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,5}(16n+1)q^n = 8q\,\frac{\varphi^3(-q^4)\psi(q^4)f(-q,-q^7)}{\varphi^5(-q)}, \tag{1.45}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{p}_{3,5}(32n+14)q^n &= 8\,\frac{\psi(-q)}{\varphi^9(-q)}\Big(f(-q^3,-q^5)\psi^2(q)\varphi(-q^4)\left(2\varphi^4(q)-\varphi^4(-q)\right)\\ &\quad + 2qf(-q,-q^7)\psi^2(q^2)\Big(\psi(-q^2)\left(4\varphi^4(q)-\varphi^4(-q)\right)\\ &\quad - 4\varphi^2(q)\psi^2(q)\varphi(-q^4)\Big)\Big),\end{aligned} \tag{1.46}$$

$$\sum_{n=0}^{\infty} \overline{p}_{3,5}(16n+7)q^n = 16q\,\frac{\psi(-q)\psi(-q^2)\psi^2(q^2)f(-q,-q^7)}{\varphi^5(-q)}, \tag{1.47}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{p}_{3,5}(32n+2)q^n &= 16q\,\frac{\psi(q)\psi(-q)\psi(-q^2)}{\varphi^9(-q)}\Big(2f(-q,-q^7)\big(2\varphi^4(q)\varphi(q^2)\\ &\quad + \varphi(-q)\varphi^4(-q^4)\big) - f(-q^3,-q^5)\varphi^4(-q)\psi(q^4)\Big),\end{aligned} \tag{1.48}$$

$$\sum_{n=1}^{\infty} \overline{p}_{3,4,5}(8n+7)q^n = 4\,\frac{\psi(-q^2)\psi(-q^4)}{\varphi^3(-q)\varphi(-q^2)}\Big(\varphi(q^4)f(-q,q^3) + q\varphi(q^2)f(-q^2,-q^{14})\Big), \tag{1.49}$$

$$\sum_{n=1}^{\infty}\overline{p}_{3,4,5}(16n+15)q^n = 8\,\frac{\psi(-q)\psi(-q^2)\varphi(q^2)f(-q,-q^7)}{\varphi^7(-q)}\Big(2\varphi^3(q)+\varphi^2(-q)\varphi(q^2)\Big), \tag{1.50}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(16n+1)q^n &= 16q\,\frac{\varphi(-q^2)\varphi(-q^4)\psi(q^4)}{\varphi^7(-q)}\Big(\varphi^3(q)f(-q,-q^7)\\ &\quad+\varphi^2(-q)\psi(q^4)f(-q^3,-q^5)\Big),\end{aligned} \tag{1.51}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{p}_{3,4,5}(32n+2)q^n &= 16q\,\frac{\varphi(-q^2)\varphi(-q^4)\psi(q^4)}{\varphi^{13}(-q)}\\ &\quad\times\Bigg(f(-q^3,-q^5)\psi(q^4)\varphi^2(-q)\Big(32\varphi^6(q)+\varphi^6(-q)\Big)\\ &\quad+f(-q,-q^7)\Big(32\varphi^9(q)-24\varphi^5(q)\varphi^4(-q)-7\varphi^3(q)\varphi^6(-q)\\ &\quad+2\varphi(q)\varphi^8(-q)+4\varphi^4(q)\varphi^4(-q)\varphi(q^2)\\ &\quad-6\varphi^2(q)\varphi^6(-q)\varphi(q^2)-\varphi^8(-q)\varphi(q^2)\Big)\Bigg),\end{aligned} \tag{1.52}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{p}_{3,4,5}(32n+14)q^n &= 16\frac{\psi(-q)\psi(-q^2)f(-q^3,-q^5)}{\varphi^{13}(-q)}\Bigg(32\varphi^{10}(q)-32\varphi^9(q)\varphi(q^2)\\ &\quad+24\varphi^8(q)\varphi^2(-q)-16\varphi^7(q)\varphi^2(-q)\varphi(q^2)\\ &\quad-22\varphi^6(q)\varphi^4(-q)+20\varphi^5(q)\varphi^4(-q)\varphi(q^2)\\ &\quad-13\varphi^4(q)\varphi^6(-q)+7\varphi^3(q)\varphi^6(-q)\varphi(q^2)+2\varphi^2(q)\varphi^8(-q)\\ &\quad-\varphi(q)\varphi^8(-q)\varphi(q^2)-2q\varphi^8(-q)\psi^2(q^4),\Bigg),\end{aligned} \tag{1.53}$$

$$\sum_{n=0}^{\infty}\overline{p}_{1,7}(8n+7)q^n = 4\,\frac{\varphi^2(q)\psi(-q^2)\psi(-q^4)f(-q^6,-q^{10})}{\varphi^5(-q)}, \tag{1.54}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{p}_{1,7}(16n+14)q^n &= 4\,\frac{\phi^4(q)\psi(-q^2)}{\varphi^9(-q^2)}\Big(2\psi^2(q^2)\varphi(-q^8)f(q,-q^3)\\ &\quad+\varphi^2(q)\psi(-q^4)f(-q^6,-q^{10})\Big),\end{aligned} \tag{1.55}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{p}_{1,7}(32n+28)q^n &= 4\,\frac{\varphi^8(q)}{\varphi^{17}(-q^2)}\Bigg(2f(q,-q^3)\varphi(-q^4)\psi(q^2)\\ &\quad\times\Big(\varphi(-q^8)\psi(q^4)\left(4\varphi^4(q)-\varphi^4(-q)\right)\\ &\quad\;+\varphi(q)\psi(-q^4)\left(2\varphi^4(q)-\varphi^4(-q)\right)\Big)\\ &\quad+f(-q^6,-q^{10})\Big(16\varphi(-q^4)\varphi(-q^8)\psi^6(q)\\ &\quad+\varphi^2(q)\psi(-q^2)\psi(-q^4)\left(4\varphi^4(q)-3\varphi^4(-q)\right)\Big)\Bigg),\end{aligned} \tag{1.56}$$

$$\sum_{n=0}^{\infty}\overline{p}_{1,7}(16n+9)q^n = 8\,\frac{\varphi^3(-q^4)\psi(q^4)f(-q^3,-q^5)}{\varphi^5(-q)}, \tag{1.57}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,7}(32n+30)q^n = 8\,\frac{\psi(-q)f(-q,-q^7)}{\varphi^9(-q)}\Big(\psi^2(q)\varphi(-q^4)\left(2\varphi^4(q)-\varphi^4(-q)\right)$$

$$+\Big(\varphi(q)+\varphi(q^2)\Big)\Big(\varphi(q^2)\psi(-q^2)\Big(4\varphi^4(q)-\varphi^4(-q)\Big)$$
$$+4\varphi^3(q)\varphi(q^2)\psi(q^2)\varphi(-q^4)\Big)\Big), \tag{1.58}$$

$$\sum_{n=0}^{\infty}\overline{p}_{1,7}(16n+15)q^n = 16\,\frac{\psi(-q)\psi(-q^2)\psi^2(q^2)f(-q^3,-q^5)}{\varphi^5(-q)}, \tag{1.59}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,7}(32n+18)q^n = 16\,\frac{\varphi(-q^2)\varphi(-q^4)\psi(q^4)}{\varphi^9(-q)}\Big(f(-q^3,-q^5)\Big(\varphi^5(q)$$
$$+4\varphi^4(q)\varphi(q^2)-\varphi^4(-q)\varphi(q^2)-2\varphi^3(q)\varphi^2(-q)\Big)$$
$$-qf(-q,-q^7)\varphi^4(-q)\psi(q^4)\Big), \tag{1.60}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,4,7}(8n+7)q^n = 4\,\frac{\psi(-q^2)\psi(-q^4)}{\varphi^3(-q)\varphi(-q^2)}\big(\varphi(q^2)f(-q^6,-q^{10})+\varphi(q^4)f(q,-q^3)\big), \tag{1.61}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,4,7}(16n+7)q^n = 8\,\frac{\psi(-q)\psi(-q^2)\varphi(q^2)f(-q^3,-q^5)}{\varphi^7(-q)}\big(2\varphi^3(q)-\varphi^2(-q)\varphi(q^2)\big), \tag{1.62}$$

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(16n+9)q^n = 16\,\frac{\varphi(-q^2)\psi(q^2)\psi(-q^2)}{\varphi^7(-q)}\Big(f(-q^3,-q^5)\varphi^3(q)$$
$$+qf(-q,-q^7)\varphi^2(-q)\psi(q^4)\Big), \tag{1.63}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,4,7}(32n+18)q^n = 16\frac{\varphi(-q^2)\varphi(-q^4)f(-q,-q^7)}{\varphi^{13}(-q)}\Big(16\varphi^{10}(q)$$
$$+16\varphi^9(q)\varphi(q^2)+4\varphi^8(q)\varphi^2(-q)-17\varphi^6(q)\varphi^4(-q)$$
$$-14\varphi^5(q)\varphi^4(-q)\varphi(q^2)-3\varphi^4(q)\varphi^6(-q)$$
$$-2\varphi^3(q)\varphi^6(-q)\varphi(q^2)+3\varphi^2(q)\varphi^8(-q)$$
$$+3\varphi(q)\varphi^6(-q)\varphi^3(q^2)-q\varphi^8(-q)\psi^2(q^4),\Big), \tag{1.64}$$

$$\sum_{n=1}^{\infty}\overline{p}_{1,4,7}(32n+30)q^n = 16\frac{\psi(-q)\psi(-q^2)}{\varphi^{13}(-q)}\Big(f(-q^3,-q^5)\psi(q^4)\varphi^8(-q)\Big(\varphi(q)$$
$$+\varphi(q^2)\Big)+f(-q,-q^7)\Big(32\varphi^{10}(q)+24\varphi^8(q)\varphi^2(-q)$$
$$-22\varphi^6(q)\varphi^4(-q)-13\varphi^4(q)\varphi^6(-q)+\varphi^2(q)\varphi^8(-q)$$
$$+32\varphi^9(q)\varphi(q^2)+16\varphi^7(q)\varphi^2(-q)\varphi(q^2)$$
$$-20\varphi^5(q)\varphi^4(-q)\varphi(q^2)-7\varphi^3(q)\varphi^6(-q)\varphi(q^2)\Big)\Big). \tag{1.65}$$

In Section 4, we prove Theorem 15. However, for brevity, we present only the proofs of (1.51) and (1.63). The other generating functions can be proved in a similar fashion.

We now end this section with a lemma containing some useful identities from [7, p. 40, p. 45, and p. 49].

Lemma 16 *We have*

$$\varphi(q)\varphi(-q)=\varphi^2(-q^2), \tag{1.66}$$
$$\psi(q)\psi(-q)=\psi(q^2)\varphi(-q^2), \tag{1.67}$$
$$\varphi(q)=\varphi(q^4)+2q\psi(q^8), \tag{1.68}$$

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4), \tag{1.69}$$
$$\psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}). \tag{1.70}$$

Furthermore, if $ab = cd$, then

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc), \tag{1.71}$$
$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{1.72}$$

2 Proof of Theorem 11

Proof of Theorem 11. Setting $a = -q$ and $b = q^3$ in (1.29), we have

$$f(-q, q^3) = \sum_{k=-\infty}^{\infty} (-1)^{k(k+1)/2} q^{2k^2-k} = (q; -q^4)_\infty (-q^3; -q^4)_\infty (-q^4; -q^4)_\infty. \tag{2.1}$$

Now,

$$(q; -q^4)_\infty = \prod_{j=0}^{\infty} \left(1 - q(-q^4)^j\right) = \prod_{j=0}^{\infty} \left(1 - q^{8j+1}\right)\left(1 + q^{8j+5}\right) = (q; q^8)_\infty (-q^5; q^8)_\infty,$$
$$(-q^3; -q^4)_\infty = \prod_{j=0}^{\infty} \left(1 + q^3(-q^4)^j\right) = \prod_{j=0}^{\infty} \left(1 + q^{8j+3}\right)\left(1 - q^{8j+7}\right) = (-q^3; q^8)_\infty (q^7; q^8)_\infty,$$
$$(-q^4; -q^4)_\infty = \prod_{j=0}^{\infty} \left(1 + q^4(-q^4)^j\right) = \prod_{j=0}^{\infty} \left(1 + q^{8j+4}\right)\left(1 - q^{8j+8}\right) = (-q^4; q^8)_\infty (q^8; q^8)_\infty.$$

Therefore, from (2.1), we find that

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} (-1)^{k(k+1)/2} q^{2k^2-k} &= (q; q^8)_\infty (-q^3; q^8)_\infty (-q^4; q^8)_\infty (-q^5; q^8)_\infty (q^7; q^8)_\infty (q^8; q^8)_\infty \\
&= \frac{(q; q^2)_\infty (-q^3; q^8)_\infty (-q^4; q^8)_\infty (-q^5; q^8)_\infty (q^8; q^8)_\infty}{(q^3; q^8)_\infty (q^5; q^8)_\infty} \\
&= \frac{(q; q^2)_\infty (q^4; q^4)_\infty (-q^3; q^8)_\infty (-q^4; q^8)_\infty (-q^5; q^8)_\infty}{(q^3; q^8)_\infty (q^4; q^8)_\infty (q^5; q^8)_\infty} \\
&= \frac{(q^2; q^2)_\infty (-q^3; q^8)_\infty (-q^4; q^8)_\infty (-q^5; q^8)_\infty}{(-q; q^2)_\infty (q^3; q^8)_\infty (q^4; q^8)_\infty (q^5; q^8)_\infty},
\end{aligned}$$

where we have used Euler's identity $(-q; q)_\infty = 1/(q; q^2)_\infty$ in the last equality. Applying (1.3) in the above identity, we have

$$\sum_{n=0}^{\infty} \overline{p}_{3,4,5}(n) q^n = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{k=-\infty}^{\infty} (-1)^{k(k+1)/2} q^{2k^2-k}. \tag{2.2}$$

Now, the generating function of $\mathrm{pod}(n)$, the number of partitions of n into non-repeating odd parts, is given by

$$\sum_{n=0}^{\infty} \mathrm{pod}(n) q^n = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty}. \tag{2.3}$$

Thus, (2.2) reduces to

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(n)q^n=\left(\sum_{m=0}^{\infty}\mathrm{pod}(m)q^m\right)\left(\sum_{k=-\infty}^{\infty}(-1)^{k(k+1)/2}q^{2k^2-k}\right).$$

Equating the coefficients of q^n from both sides of the above, we arrive at (1.1).

Again, setting $a=q$ and $b=-q^3$ in (1.29), and proceeding as above, we find that

$$\sum_{k=-\infty}^{\infty}(-1)^{k(k-1)/2}q^{2k^2-k}=\frac{(q^2;q^2)_\infty(-q;q^8)_\infty(-q^4;q^8)_\infty(-q^7;q^8)_\infty}{(-q;q^2)_\infty(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty},$$

which, by (1.4) and (2.3), can be rewritten as

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(n)q^n=\left(\sum_{m=0}^{\infty}\mathrm{pod}(m)q^m\right)\left(\sum_{k=-\infty}^{\infty}(-1)^{k(k-1)/2}q^{2k^2-k}\right).$$

Equating the coefficients of q^n from both sides of the above, we arrive at (1.2) to finish the proof of Theorem 11. □

3 Proof of Theorem 13

Proof of (1.30). The generating function (1.3) of $\overline{p}_{3,4,5}(n)$ may be rewritten as

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,5}(n)q^n&=\frac{f(q^3,q^5)}{f(-q^3,-q^5)}=\frac{f(-q,-q^7)f(q^3,q^5)}{f(-q,-q^7)f(-q^3,-q^5)}\\&=\frac{\psi(q)f(-q,-q^7)f(q^3,q^5)}{(q^2;q^2)_\infty(q^8;q^8)_\infty^2}.\end{aligned}\tag{3.1}$$

Now, setting $a=q, b=q^7, c=-q^3, d=-q^5$ in (1.71) and (1.72), and then using (1.28), we find that

$$f(q,q^7)f(-q^3,-q^5)+f(-q,-q^7)f(q^3,q^5)=2\psi(-q^4)f(-q^6,-q^{10}),\tag{3.2}$$

$$f(q,q^7)f(-q^3,-q^5)-f(-q,-q^7)f(q^3,q^5)=2q\psi(-q^4)f(-q^2,-q^{14}).\tag{3.3}$$

Subtracting (3.3) from (3.2), we have

$$f(-q,-q^7)f(q^3,q^5)=\psi(-q^4)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right).\tag{3.4}$$

Invoking (3.4) and (1.70) in (3.1), and then manipulating the q-products, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,5}(n)q^n&=\frac{\psi(q)\psi(-q^4)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)}{(q^2;q^2)_\infty(q^8;q^8)_\infty^2}\\&=\frac{(q^4;q^4)_\infty(q^{16};q^{16})_\infty}{(q^2;q^2)_\infty(q^8;q^8)_\infty^3}\left(f(q^6,q^{10})+qf(q^2,q^{14})\right)\\&\quad\times\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right).\end{aligned}\tag{3.5}$$

Extracting the even terms from both sides of the above,

$$\sum_{n=0}^{\infty}\overline{p}_{3,5}(2n)q^n=\frac{(q^2;q^2)_\infty(q^8;q^8)_\infty}{(q;q)_\infty(q^4;q^4)_\infty^3}\left(f(q^3,q^5)f(-q^3,-q^5)-qf(q,q^7)f(-q,-q^7)\right)\tag{3.6}$$

Now, setting $a=q^3, b=q^5, c=-q^3, d=-q^5$ in (1.71), we have

$$f(q^3,q^5)f(-q^3,-q^5)=f(-q^6,-q^{10})f(-q^8,-q^8)=f(-q^6,-q^{10})\varphi(-q^8).\tag{3.7}$$

Setting $a=q, b=q^7, c=-q, d=-q^7$ in (1.71), we have

$$f(q,q^7)f(-q,-q^7)=f(-q^2,-q^{14})f(-q^8,-q^8)=f(-q^2,-q^{14})\varphi(-q^8). \tag{3.8}$$

Employing (1.27), (1.3), (3.7) and (3.8) in (3.6), and then manipulating the q-products, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{3,5}(2n)q^n &= \frac{(q^2;q^2)_\infty(q^8;q^8)_\infty}{(q;q)_\infty(q^4;q^4)_\infty^3}\varphi(-q^8)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)\\ &= \frac{(q^2;q^2)_\infty(q^8;q^8)_\infty^3}{(q;q)_\infty(q^4;q^4)_\infty^3(q^{16};q^{16})_\infty}\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)\\ &= \frac{(q^8;q^8)_\infty^6}{(q^4;q^4)_\infty^4(q^{16};q^{16})_\infty^2}\sum_{n=0}^{\infty}\overline{p}_{3,5}(n)q^n\\ &\equiv \sum_{n=0}^{\infty}\overline{p}_{3,5}(n)q^n \pmod 4. \end{aligned}\tag{3.9}$$

Hence, for any nonnegative integer n,

$$\overline{p}_{3,5}(2n)\equiv\overline{p}_{3,5}(n) \pmod{2^2}.$$

The result of (1.30) immediately follows from the above equation. □

Proof of (1.31). In a similar manner as that of the above, we can show that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{1,7}(2n)q^n &= \frac{(q^8;q^8)_\infty^6}{(q^4;q^4)_\infty^4(q^{16};q^{16})_\infty^2}\sum_{n=0}^{\infty}\overline{p}_{1,7}(n)q^n\\ &\equiv \sum_{n=0}^{\infty}\overline{p}_{1,7}(n)q^n \pmod 4. \end{aligned}\tag{3.10}$$

Hence, for any nonnegative integer n,

$$\overline{p}_{1,7}(2n)\equiv\overline{p}_{1,7}(n) \pmod{2^2}.$$

The result of (1.31) immediately follows from the above equation. □

Proof of (1.32). Subtracting (3.5) from (3.9), and employing (1.28) and (1.70), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{p}_{3,5}(2n)-\overline{p}_{3,5}(n))q^n\\ &\quad= \frac{\psi(q)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)}{(q^2;q^2)_\infty}\left(\frac{(q^8;q^8)_\infty^3}{(q^4;q^4)_\infty^3(q^{16};q^{16})_\infty}-\frac{(q^4;q^4)_\infty(q^{16};q^{16})_\infty}{(q^8;q^8)_\infty^3}\right)\\ &\quad= \frac{\psi(q)f(-q,q^3)}{(q^2;q^2)_\infty}\left(\frac{(q^8;q^8)_\infty^3}{(q^4;q^4)_\infty^3(q^{16};q^{16})_\infty}-\frac{(q^4;q^4)_\infty(q^{16};q^{16})_\infty}{(q^8;q^8)_\infty^3}\right).\end{aligned}$$

Using (1.27), (1.28) and (1.68), the above equation becomes

$$\begin{aligned}\sum_{n=0}^{\infty}(\overline{p}_{3,5}(2n)-\overline{p}_{3,5}(n))q^n &= \frac{(q^{16};q^{16})_\infty\psi(q)f(-q,q^3)}{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}\left(\varphi(q^4)-\varphi(-q^4)\right)\\ &= \frac{(q^{16};q^{16})_\infty\psi(q)f(-q,q^3)}{(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}\times 4q^4\psi(q^{32})\\ &= 4q^4\frac{(q^2;q^2)_\infty(q^{16};q^{16})_\infty(q^{64};q^{64})_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2(q^{32};q^{32})_\infty}f(-q,q^3).\end{aligned}\tag{3.11}$$

Since $(q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod 2$, the above equation leads to the following congruence.

$$\sum_{n=0}^{\infty}(\overline{p}_{3,5}(2n)-\overline{p}_{3,5}(n))q^n \equiv 4q^4\frac{(q^{32};q^{32})_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}f(-q,q^3) \pmod 8. \tag{3.12}$$

Now, (1.29), (1.28) and standard q-product manipulations imply,

$$\begin{aligned} f(-q,q^3) &= \prod_{n=1}^{\infty}\left(1-(-1)^n q^{4n+1}\right)\left(1+(-1)^n q^{4n+3}\right)\left(1-(-1)^n q^{4n+4}\right) \\ &\equiv \prod_{n=1}^{\infty}\left(1+q^{4n+1}\right)\left(1+q^{4n+3}\right)\left(1-q^{4n+4}\right) \pmod 2 \\ &\equiv f(q,q^3) \\ &\equiv \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \\ &\equiv (q;q)_\infty(q^2;q^2)_\infty \pmod 2. \end{aligned} \tag{3.13}$$

Utilizing (3.13) in (3.12) and employing Jacobi's identity[16, Eq. (1.7.1)], we have

$$\begin{aligned} \sum_{n=0}^{\infty}(\overline{p}_{3,5}(2n)-\overline{p}_{3,5}(n))q^n &\equiv 4q^4(q^{32};q^{32})_\infty^3 \pmod 8 \\ &\equiv 4q^4\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{16k(k+1)} \\ &\equiv 4\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{(4k+2)^2} \pmod 8. \end{aligned}$$

The above congruence immediately implies (1.32), which in turn implies the infinite family of congruences in (1.34). □

Proof of (1.33). In a similar manner as that of the above proof, we have

$$\sum_{n=0}^{\infty}(\overline{p}_{1,7}(2n)-\overline{p}_{1,7}(n))q^n \equiv 4\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{(4k+2)^2} \pmod 8. \tag{3.14}$$

Clearly, (1.33) follows from the above congruence, thereby implying the the infinite family of congruences in (1.35). □

4 Proof of Theorem 15

Proof of (1.51). With the help of (1.29), (1.27) and (1.28), the generating function (1.3) of $\overline{p}_{3,4,5}(n)$ may be rewritten as

$$\begin{aligned} \sum_{n=0}^{\infty}\overline{p}_{3,4,5}(n)q^n &= \frac{\varphi(-q^8)f(q^3,q^5)}{\varphi(-q^4)f(-q^3,-q^5)} = \frac{\varphi(-q^8)f(-q,-q^7)f(q^3,q^5)}{\varphi(-q^4)f(-q,-q^7)f(-q^3,-q^5)} \\ &= \frac{\varphi(-q^8)\psi(q)f(-q,-q^7)f(q^3,q^5)}{\varphi(-q^4)(q^2;q^2)_\infty(q^8;q^8)_\infty^2}. \end{aligned} \tag{4.1}$$

Invoking (3.4) and (1.70) in (4.1), and then manipulating the q-products, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{p}_{3,4,5}(n)q^n \\
&\quad = \frac{\varphi(-q^8)\psi(-q^4)}{\varphi(-q^4)(q^2;q^2)_\infty(q^8;q^8)_\infty^2}\left(f(q^6,q^{10})+qf(q^2,q^{14})\right)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right) \\
&\quad = \frac{\left(f(q^6,q^{10})+qf(q^2,q^{14})\right)\left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)}{(q^2;q^2)_\infty(q^4;q^4)_\infty}.
\end{aligned} \tag{4.2}$$

Extracting the odd terms from both sides of the above, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_{3,4,5}(2n+1)q^n = \frac{f(q,q^7)f(-q^3,-q^5)-f(-q,-q^7)f(q^3,q^5)}{(q;q)_\infty(q^2;q^2)_\infty}, \tag{4.3}$$

which, by (1.28), (1.70) and (3.3) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{p}_{3,4,5}(2n+1)q^n &= 2q\frac{\psi(q)\psi(-q^4)f(-q^2,-q^{14})}{(q^2;q^2)_\infty^3} \\
&= 2q\frac{\psi(-q^4)f(-q^2,-q^{14})}{(q^2;q^2)_\infty^3}\left(f(q^6,q^{10})+qf(q^2,q^{14})\right).
\end{aligned} \tag{4.4}$$

Extracting the even terms from both sides of the above, we find that

$$\sum_{n=0}^{\infty} \overline{p}_{3,4,5}(4n+1)q^n = 2q\frac{\psi(-q^2)f(q,q^7)f(-q,-q^7)}{(q;q)_\infty^3}. \tag{4.5}$$

From (3.8) and (4.5), we have

$$\sum_{n=0}^{\infty} \overline{p}_{3,4,5}(4n+1)q^n = 2q\frac{\psi(-q^2)\varphi(-q^8)f(-q^2,-q^{14})}{(q;q)_\infty^3}, \tag{4.6}$$

which, with the aid of (1.27) and (1.28), may be rewritten as

$$\sum_{n=0}^{\infty} \overline{p}_{3,4,5}(4n+1)q^n = 2q\frac{\varphi(-q^8)f(-q^2,-q^{14})\varphi(q)\psi(q)}{\varphi^3(-q^2)\varphi(-q^4)}. \tag{4.7}$$

Employing (1.70) and (1.68) in (4.7), extracting the even terms from both sides of the resulting identity, and then using (1.66), (3.4), (3.8), and (1.68), we find the following 2-dissection of $\overline{p}_{3,4,5}(8n+1)$:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{p}_{3,4,5}(8n+1)q^n \\
&\quad = 2q\frac{\varphi(q^2)\varphi(-q^4)f(q,q^7)f(-q,-q^7)}{\varphi(-q^2)\varphi^3(-q)} + 4q\frac{\psi(q^4)\varphi(-q^4)f(-q,-q^7)f(q^3,q^5)}{\varphi(-q^2)\varphi^3(-q)} \\
&\quad = 2q\frac{\varphi(q^2)\varphi(-q^4)f(q,q^7)f(-q,-q^7)\varphi^3(q)}{\varphi^7(-q^2)} + 4q\frac{\psi(q^4)\varphi(-q^4)f(-q,-q^7)f(q^3,q^5)\varphi^3(q)}{\varphi^7(-q^2)} \\
&\quad = 2q\frac{\varphi(q^2)\varphi(-q^4)\varphi(-q^8)f(-q^2,-q^{14})}{\varphi^7(-q^2)}\left(\varphi(q^4)+2q\psi(q^8)\right)^3 + 4q\frac{\psi(q^4)\psi(-q^4)\varphi(-q^4)}{\varphi^7(-q^2)} \\
&\qquad \times \left(f(-q^6,-q^{10})-qf(-q^2,-q^{14})\right)\left(\varphi(q^4)+2q\psi(q^8)\right)^3.
\end{aligned} \tag{4.8}$$

Extracting the even terms from both sides of the above and then using (1.69), we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(16n+1)q^n\\
&= 4q\frac{\varphi(q)\varphi(-q^2)\varphi(-q^4)\psi(q^4)f(-q,-q^7)}{\varphi^7(-q)}\left(3\varphi^2(q^2)+4q\psi^2(q^4)\right)\\
&\quad + 8q\frac{\psi(q^2)\psi(-q^2)\varphi(-q^2)\psi(q^4)f(-q^3,-q^5)}{\varphi^7(-q^2)}\left(3\varphi^2(q^2)+4q\psi^2(q^4)\right)\\
&\quad - 4q\frac{\psi(q^2)\psi(-q^2)\varphi(q^2)\varphi(-q^2)f(-q,-q^7)}{\varphi^7(-q^2)}\left(\varphi^2(q^2)+12q\psi^2(q^4)\right)\\
&= 4q\frac{\varphi(q)\varphi(-q^2)\varphi(-q^4)\psi(q^4)f(-q,-q^7)}{\varphi^7(-q)}\left(\varphi^2(q)+2\varphi^2(q^2)\right)\\
&\quad + 8q\frac{\psi(q^2)\psi(-q^2)\varphi(-q^2)\psi(q^4)f(-q^3,-q^5)}{\varphi^7(-q^2)}\left(\varphi^2(q)+2\varphi^2(q^2)\right)\\
&\quad - 4q\frac{\psi(q^2)\psi(-q^2)\varphi(q^2)\varphi(-q^2)f(-q,-q^7)}{\varphi^7(-q^2)}\left(3\varphi^2(q)-2\varphi^2(q^2)\right).
\end{aligned}
\tag{4.9}
$$

To simplify the above identity further, we now derive an expression for $f(-q^3,-q^5)$ in terms of $f(-q,-q^7)$. To that end, we set $a=q$, $b=q^3$, $c=d=-q^2$ in (1.71) and (1.72) to arrive at

$$
\begin{aligned}
\psi(q)\varphi(-q^2)+\psi(-q)\varphi(q^2) &= 2f^2(-q^3,-q^5),\\
\psi(q)\varphi(-q^2)-\psi(-q)\varphi(q^2) &= 2qf^2(-q,-q^7).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
f^2(-q^3,-q^5)+qf^2(-q,-q^7) &= \psi(q)\varphi(-q^2),\\
f^2(-q^3,-q^5)-qf^2(-q,-q^7) &= \psi(-q)\varphi(q^2).
\end{aligned}
$$

Dividing both sides of each of the above identities by $f(-q^3,-q^5)f(-q,-q^7)$, that is, by $(q;q^2)_\infty(q^8;q^8)_\infty^2$, and then using (1.27) and (1.28), we obtain

$$
\frac{f(-q^3,-q^5)}{f(-q,-q^7)}+q\frac{f(-q,-q^7)}{f(-q^3,-q^5)}=\frac{\psi(q)\varphi(-q^2)}{(q;q^2)_\infty(q^8;q^8)_\infty^2}=\frac{\varphi(q)}{\psi(q^4)} \tag{4.10}
$$

and

$$
\frac{f(-q^3,-q^5)}{f(-q,-q^7)}-q\frac{f(-q,-q^7)}{f(-q^3,-q^5)}=\frac{\psi(-q)\varphi(q^2)}{(q;q^2)_\infty(q^8;q^8)_\infty^2}=\frac{\varphi(q^2)}{\psi(q^4)}. \tag{4.11}
$$

Adding the above two identities, we find that

$$
2\psi(q^4)f(-q^3,-q^5)=\left(\phi(q)+\phi(q^2)\right)f(-q,-q^7). \tag{4.12}
$$

Employing (4.12) in (4.9), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(16n+1)q^n\\
&= 4q\frac{\varphi(q)\varphi(-q^2)\varphi(-q^4)\psi(q^4)f(-q,-q^7)}{\varphi^7(-q)}\left(\varphi^2(q)+2\varphi^2(q^2)\right)\\
&\quad + 4q\frac{\psi(q^2)\psi(-q^2)\varphi(-q^2)f(-q,-q^7)}{\varphi^7(-q^2)}\left(\varphi(q)+\varphi(q^2)\right)\left(\varphi^2(q)+2\varphi^2(q^2)\right)\\
&\quad - 4q\frac{\psi(q^2)\psi(-q^2)\varphi(q^2)\varphi(-q^2)f(-q,-q^7)}{\varphi^7(-q^2)}\left(3\varphi^2(q)-2\varphi^2(q^2)\right).
\end{aligned}
$$

With the aid of (1.67) and (1.69), the above further reduces to

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(16n+1)q^n$$
$$= 8q\frac{\varphi(-q^2)\varphi(-q^4)\psi(q^4)f(-q,-q^7)}{\varphi^7(-q)}\left(2\varphi^3(q)+\varphi^2(-q)\left(\varphi(q)+\varphi(q^2)\right)\right). \tag{4.13}$$

Employing the expression of $\varphi(q)+\varphi(q^2)$ from (4.12) in (4.13), we have

$$\sum_{n=0}^{\infty}\overline{p}_{3,4,5}(16n+1)q^n$$
$$= 16q\frac{\varphi(-q^2)\varphi(-q^4)\psi(q^4)}{\varphi^7(-q)}\left(\varphi^3(q)f(-q,-q^7)+\varphi^2(-q)\psi(q^4)f(-q^3,-q^5)\right),$$

which is (1.51). □

Proof of (1.63). Applying (1.29), (1.27) and (1.28) in (1.4), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(n)q^n = \frac{\varphi(-q^8)f(q,q^7)}{\varphi(-q^4)f(-q,-q^7)} = \frac{\varphi(-q^8)f(q,q^7)f(-q^3,-q^5)}{\varphi(-q^4)(q;q^2)_\infty(q^8;q^8)^2_\infty}$$
$$= \frac{\varphi(-q^8)\psi(q)f(q,q^7)f(-q^3,-q^5)}{\varphi(-q^4)(q^2;q^2)_\infty(q^8;q^8)^2_\infty}. \tag{4.14}$$

Using (3.4), with q replaced by $-q$, and (1.70) in (4.14), we have

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(n)q^n = \frac{\varphi(-q^8)\psi(-q^4)}{\varphi(-q^4)(q^2;q^2)_\infty(q^8;q^8)^2_\infty}\left(f(q^6,q^{10})+qf(q^2,q^{14})\right)$$
$$\times\left(f(-q^6,-q^{10})+qf(-q^2,-q^{14})\right).$$

Extracting the odd terms and then using (3.2) and (1.70), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(2n+1)q^n$$
$$= \frac{\varphi(-q^4)\psi(-q^2)}{\varphi(-q^2)(q;q)_\infty(q^4;q^4)^2_\infty}\left(f(q,q^7)f(-q^3,-q^5)+f(-q,-q^7)f(q^3,q^5)\right),$$
$$= 2\frac{\psi(-q^2)\psi(-q^4)\varphi(-q^4)f(-q^6,-q^{10})}{(q^2;q^2)^2_\infty(q^4;q^4)^2_\infty\varphi(-q^2)}\left(f(q^6,q^{10})+qf(q^2,q^{14})\right). \tag{4.15}$$

Extracting the even terms, we find that

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(4n+1)q^n = 2\frac{\psi(-q)\psi(-q^2)\varphi(-q^2)f(q^3,q^5)f(-q^3,-q^5)}{(q;q)^2_\infty(q^2;q^2)^2_\infty\varphi(-q)}. \tag{4.16}$$

Employing (1.27), (1.28), (1.66) and (3.7) in (4.16), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(4n+1)q^n = 2\frac{\varphi(q)\psi(q)\psi(-q^2)\varphi(-q^8)f(-q^6,-q^{10})}{\varphi^2(-q^2)(q^2;q^2)^3_\infty},$$

from which, with the aid of (1.68) and (1.70), we have the following 2-dissection of the generating function of $\overline{p}_{1,4,7}(4n+1)$:

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(4n+1)q^n$$
$$=2\frac{\psi(-q^2)\varphi(-q^8)f(-q^6,-q^{10})}{\varphi^2(-q^2)(q^2;q^2)_\infty^3}\left(\varphi(q^4)+2q\psi(q^8)\right)\left(f(q^6,q^{10})+qf(q^2,q^{14})\right). \tag{4.17}$$

Extracting the even terms and then using (3.7), (3.4), and (1.66), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(8n+1)q^n &= 2\frac{\psi(-q)\varphi(-q^4)f(-q^3,-q^5)}{\varphi^2(-q)(q;q)_\infty^3}\left(f(q^3,q^5)\varphi(q^2)+2qf(q,q^7)\psi(q^4)\right)\\ &= 2\frac{\varphi(-q^4)\varphi^3(q)}{\varphi^7(-q^2)}\Big(\varphi(q^2)\varphi(-q^8)f(-q^6,-q^{10})\\ &\quad+2q\psi(q^4)\psi(-q^4)\left(f(-q^6,-q^{10})+qf(-q^2,-q^{14})\right)\Big).\end{aligned}$$

Employing (1.68) in the above, we arrive at the following 2-dissection of the generating function of $\overline{p}_{1,4,7}(8n+1)$:

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(8n+1)q^n &= 2\frac{\varphi(-q^4)}{\varphi^7(-q^2)}\left(\varphi(q^4)+2q\psi(q^8)\right)^3\Big(\varphi(q^2)\varphi(-q^8)f(-q^6,-q^{10})\\ &\quad+2q^2\psi(q^4)\psi(-q^4)f(-q^2,-q^{14})+2q\psi(q^4)\psi(-q^4)f(-q^6,-q^{10})\Big)\\ &= 2\frac{\varphi(-q^4)}{\varphi^7(-q^2)}\left(A(q^2)+2qB(q^2)\right)\left(C(q^2)+2qD(q^2)\right),\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}A(q)&=\varphi^3(q^2)+12q\varphi(q^2)\psi^2(q^4),\\ B(q)&=\psi(q^4)\left(3\varphi^2(q^2)+4q\psi^2(q^4)\right),\\ C(q)&=\varphi(q)\varphi(-q^4)f(-q^3,-q^5)+2q\psi(q^2)\psi(-q^2)f(-q,-q^7),\\ D(q)&=\psi(q^2)\psi(-q^2)f(-q^3,-q^5).\end{aligned}$$

Extracting the odd terms from both sides of (4.18), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(16n+9)q^n=4\frac{\varphi(-q^2)}{\varphi^7(-q)}\left(A(q)D(q)+B(q)C(q)\right). \tag{4.19}$$

Now, we simplify $B(q)C(q)$. Subtracting (4.11) from (4.10), we have

$$2q\psi(q^4)f(-q,-q^7)=\left(\varphi(q)-\varphi(q^2)\right)f(-q^3,-q^5). \tag{4.20}$$

Using (1.67) in the expression for $C(q)$ and then using (4.20), we have

$$\begin{aligned}B(q)C(q)&=\left(\psi(q^4)\left(3\varphi^2(q^2)+4q\psi^2(q^4)\right)\right)\\ &\quad\times\left(\varphi(q)\varphi(-q^4)f(-q^3,-q^5)+2q\psi(q^4)\varphi(-q^4)f(-q,-q^7)\right)\\ &=\psi(q^4)\varphi(-q^4)f(-q^3,-q^5)\left(3\varphi^2(q^2)+4q\psi^2(q^4)\right)\left(2\varphi(q)-\varphi(q^2)\right),\end{aligned}$$

which, by an application of (1.67) again, gives

$$B(q)C(q)=\psi(q^2)\psi(-q^2)f(-q^3,-q^5)\left(3\varphi^2(q^2)+4q\psi^2(q^4)\right)\left(2\varphi(q)-\varphi(q^2)\right). \tag{4.21}$$

On the other hand,

$$A(q)D(q) = \psi(q^2)\psi(-q^2)f(-q^3,-q^5)\left(\varphi^3(q^2) + 12q\varphi(q^2)\psi^2(q^4)\right). \tag{4.22}$$

Using (4.21) and (4.22) in (4.19), we find that

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(16n+9)q^n = 8\frac{\varphi(-q^2)\psi(q^2)\psi(-q^2)f(-q^3,-q^5)}{\varphi^7(-q)}\Big(\varphi^2(q^2)\left(\varphi(q)-\varphi(q^2)\right)$$
$$+ 2\varphi(q)\varphi^2(q^2) + 4q\varphi(q)\psi^2(q^4) + 4q\varphi(q^2)\psi^2(q^4)\Big).$$

Employing (4.20) in the above, we have

$$\sum_{n=0}^{\infty}\overline{p}_{1,4,7}(16n+9)q^n = 16\frac{\varphi(-q^2)\psi(q^2)\psi(-q^2)f(-q^3,-q^5)}{\varphi^7(-q)}\Big(q\psi(q^4)\varphi^2(q^2)\frac{f(-q,-q^7)}{f(-q^3,-q^5)}$$
$$+ \varphi(q)\varphi^2(q^2) + 2q\varphi(q)\psi^2(q^4) + 2q\varphi(q^2)\psi^2(q^4)\Big).$$

Employing (1.69) and (4.20) again in the above, we arrive at (1.63). □

5 Concluding remark

In this paper, we consider some restricted overpartition functions in which the parts are from certain residue classes of 8. We deduce a number of congruences modulo 2, 4, 8, and 16 by finding the explicit generating function for each of the congruence as a multiple of the corresponding modulus. As mentioned in the introduction, there is no congruence modulo 32 for these overpartition functions with arguments of the form $64n + B$ when $B = 0, 1, \ldots, 63$. It would be interesting to find congruences modulo 2^k for $k \geq 5$ as well as modulo primes other than 2.

Conflict of interest The authors declare that they have no competing interests.

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