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ORIGINAL RESEARCH

Generalized Bernstein polynomials

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Abstract This paper investigates the approximation properties of a generalized Bernstein operator introduced by Cao (J Math Anal Appl 209:140–146, 1997). Employing modulus of continuity and Peetre's K-functional, we establish asymptotic and quantitative Voronovskaya-type theorems. The non-multiplicativity of the operators is explored through a Grüss–Voronovskaya-type theorem. Additionally, we determine the rate of convergence for functions with derivatives of bounded variation. To validate our theoretical findings, we conduct numerical experiments using MATLAB. These experiments showcase the operators' convergence behavior and computational efficiency through detailed analysis of tables and graphs. Our results demonstrate the effectiveness of the proposed operators in approximating functions, highlighting their practical applicability.

Keywords Voronovskaya theorem · Rate of convergence · Chebyshev–Grüss inequality · Grüss Voronovskaya theorem

Mathematics Subject Classification 41A10 · 41A25 · 41A36 · 26A15

1 Introduction

In order to give a simple constructive proof of the fundamental Weierstrass approximation theorem, for $\mathfrak{z}(t) \in \mathcal{C}(J)$, $J = [0, 1]$, the space of all continuous functions on J endowed with the uniform norm denoted by $\|.\|$, Bernstein [1] defined a sequence of positive linear operators as:

$$B_n(\mathfrak{z}; \varkappa) = \sum_{k=0}^{n} p_{n,k}(\varkappa)\, \mathfrak{z}\left(\frac{k}{n}\right). \tag{1.1}$$

Subsequently, numerous generalizations of the operators (1.1) have been introduced by researchers ([2–4] and the references therein) to approximate functions in the various function spaces and their properties have been extensively investigated.

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For $\mathfrak{z}(t) \in \mathcal{C}(J)$, Cao [5] proposed a generalization of the Bernstein polynomials with the aid of a sequence (v_n) of natural numbers as:

$$\mathfrak{C}_n\left(\mathfrak{z}(t), v_n, \varkappa\right) := \frac{1}{v_n}\sum_{k=0}^{n}\left(\sum_{j=0}^{v_n-1}\mathfrak{z}\left(\frac{k+j}{n+v_n-1}\right)\right)p_{n,k}\left(\varkappa\right), \tag{1.2}$$

where $p_{n,k}\left(\varkappa\right) := \binom{n}{k}\varkappa^k(1-\varkappa)^{n-k}$ and $\varkappa \in J$.

It is evident that if $v_n = 1, \forall\ n \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers, then the operator (1.2) reduces to (1.1). The author [5] established the necessary and sufficient conditions for the uniform convergence of the operators (1.2) to $\mathfrak{z}$ for all $\mathfrak{z}(t) \in \mathcal{C}(J)$ and also determined the approximation degree of these operators with the aid of the usual modulus of continuity.

The purpose of the present article is to extend the study by Cao [5]. We investigate the degree of approximation for the operators (1.2) by means of the moduli of continuity and the Peetre's K- functional. We establish asymptotic and quantitative Voronovskaya type theorems along with a Grüss-Voronovskaya type theorem. We also discuss the convergence rate of these operators for functions wiht derivatives of bounded variation. Lastly, to validate the theoretical results obtained in the paper, we present numerical experiments using MATLAB.

2 Preliminaries

In this section, we present some estimates of the moments and the central moments for the operators (1.2) which will be required to establish the main results of this paper.

Let us assume $e_i(\varkappa) = \varkappa^i$, for all $i \in \mathbb{N}\cup\{0\}$ and $\varkappa \in J$.

Lemma 1 *For the operators $\mathfrak{C}_n\left(\mathfrak{z}, v_n, \varkappa\right)$, for all $\varkappa \in J$, we have*

(i) $\mathfrak{C}_n\left(e_0, v_n, \varkappa\right) = 1$;

(ii) $\mathfrak{C}_n\left(e_1, v_n, \varkappa\right) = \frac{n\varkappa}{n+v_n-1} + \frac{v_n-1}{2(n+v_n-1)}$;

(iii) $\mathfrak{C}_n\left(e_2, v_n, \varkappa\right) = \frac{n(n-1)\varkappa^2}{(n+v_n-1)^2} + \frac{nv_n\varkappa}{(n+v_n-1)^2} + \frac{(v_n-1)(2v_n-1)}{6(n+v_n-1)^2}$;

(iv) $\mathfrak{C}_n\left(e_3, v_n, \varkappa\right) = \frac{1}{(n+v_n-1)^3}\left\{n\left(n-1\right)\left(n-2\right)\varkappa^3 + \frac{3}{2}n\left(n-1\right)\left(v_n+1\right)\varkappa^2 + nv_n^2\varkappa + \frac{v_n(v_n-1)^2}{4}\right\}$;

(v) $\mathfrak{C}_n\left(e_4, v_n, \varkappa\right) = \frac{1}{(n+v_n-1)^4}\Big\{n\left(n-1\right)\left(n-2\right)\left(n-3\right)\varkappa^4 + 2n\left(n-1\right)\left(n-2\right)\left(v_n+2\right)\varkappa^3$
$+n\left(n-1\right)\left(2v_n^2+3v_n+2\right)\varkappa^2 + nv_n^3\varkappa + \frac{(v_n-1)(2v_n-1)\left(3v_n^2-3v_n-1\right)}{30}\Big\}$.

Let us denote $u_\varkappa^r(t) = (t-\varkappa)^r$, and $\mu_{n,r}\left(v_n, \varkappa\right) = \mathfrak{C}_n\left(u_\varkappa^r(t), v_n, \varkappa\right)$, $for\ all\ r \in \mathbb{N}\cup\{0\}$ and $t, \varkappa \in J$. Then as a consequence of Lemma 1, it follows that

Lemma 2 *For the operators $\mathfrak{C}_n\left(\mathfrak{z}, v_n, \varkappa\right)$, following equalities hold:*

(i) $\mu_{n,1}\left(v_n, \varkappa\right) = \frac{(v_n-1)(1-2\varkappa)}{2(n+v_n-1)}$;

(ii) $\mu_{n,2}\left(v_n, \varkappa\right) = \frac{1}{(n+v_n-1)^2}\left\{\left(n-\left(v_n-1\right)^2\right)\left(\varkappa-\varkappa^2\right) + \frac{(2v_n-1)(v_n-1)}{6}\right\}$;

(iii) $\mu_{n,4}\left(v_n, \varkappa\right) = \frac{1}{(n+v_n-1)^4}\Big\{\left(3n^2-4n+4nv_n-6nv_n^2+\left(v_n-1\right)^4\right)\varkappa^4 - (6n^2-8n+8nv_n$
$-12nv_n^2+2(v_n-1)^4)\varkappa^3 + \left(3n^2-4n+5nv_n-8nv_n^2+\left(v_n-1\right)^3\left(2v_n-1\right)\right)\varkappa^2$
$+\left(2nv_n^2-nv_n-v_n(v_n-1)^3\right)\varkappa + \frac{1}{30}\left(6v_n^4-15v_n^3+10v_n^2-1\right)\Big\}$.

Throughout our further discussion, let us suppose that $\lim\limits_{n\to\infty} v_n = c$. Then from Lemma 2, by a simple calculation we easily deduce:

Lemma 3 *For all $\varkappa \in J$, the operators (1.2) verify the following relations:*

(i) $\lim\limits_{n\to\infty} n\mu_{n,1}\left(v_n, \varkappa\right) = \frac{(c-1)(1-2\varkappa)}{2}$;

(ii) $\lim_{n\to\infty} n\mu_{n,2}(v_n,\varkappa) = \varkappa(1-\varkappa)$;

(iii) $\lim_{n\to\infty} n^2\mu_{n,4}(v_n,\varkappa) = 3\varkappa^2(1-\varkappa)^2$.

For convenience, throughout this article let C_0 denote a positive constant not necessarily the same at each occurence.

Remark 1 From ([5], Lemma 1), we have

$$\mathfrak{C}_n\left(u_\varkappa^2(t), v_n, \varkappa\right) \le \frac{4}{3}.\frac{(v_n-1)^2}{n^2} + \frac{13}{12}.\frac{1}{n} = \nu_{n,v_n}^2,\ \forall \varkappa \in J.$$

Further from Lemma 3, for sufficiently large n and for each $\varkappa \in J$, we have

$$\mathfrak{C}_n\left(u_\varkappa^2(t), v_n, \varkappa\right) \le C_0\frac{\varkappa(1-\varkappa)}{n} = \eta^2(\varkappa) \tag{2.1}$$

and

$$\mathfrak{C}_n\left(u_\varkappa^4(t), v_n, \varkappa\right) \le C_0\frac{\varkappa^2(1-\varkappa)^2}{n^2}. \tag{2.2}$$

Lemma 4 *For $\mathfrak{z} \in \mathcal{C}(J)$, there holds:*

$$\|\mathfrak{C}_n(\mathfrak{z}, v_n, .)\| \le \|\mathfrak{z}\|.$$

Proof From Lemma 1, we have $\mathfrak{C}_n(1, v_n, \varkappa) = 1$. So,

$$|\mathfrak{C}_n(\mathfrak{z}, v_n, \varkappa)| \le \mathfrak{C}_n(1, v_n, \varkappa)\|\mathfrak{z}\| = \|\mathfrak{z}\|.$$

□

3 Main results

Our first result of this section is the uniform convergence theorem for the generalized Bernstein operators (1.2).

Theorem 1 *For all $\mathfrak{z} \in \mathcal{C}(J)$, there follows*

$$\lim_{n\to\infty}\|\mathfrak{C}_n(\mathfrak{z}, v_n, .) - \mathfrak{z}\| = 0.$$

Proof From Lemma 1, it is evident that $\lim_{n\to\infty}\|\mathfrak{C}_n(e_m, v_n, .) - e_m\| = 0$, for all $m \in \{0, 1, 2\}$, hence applying Bohman–Korovkin theorem, we get the required result. □

Next, we estimate order of convergence of (1.2) with the aid of first order modulus of continuity. The first order modulus of continuity of $\mathfrak{z} \in \mathcal{C}(J)$ is given by

$$\omega(\mathfrak{z};\delta) = \sup_{0<|h|<\delta}\ \sup_{\varkappa,\varkappa+h\in J}|\mathfrak{z}(\varkappa+h) - \mathfrak{z}(\varkappa)|,\ \ \delta > 0.$$

Theorem 2 *For $\mathfrak{z} \in \mathcal{C}(J)$ and any $\varkappa \in J$, the operators (1.2) verify the inequality:*

$$|\mathfrak{C}_n(\mathfrak{z}, v_n, \varkappa) - \mathfrak{z}(\varkappa)| \le 2\omega\left(\mathfrak{z}; \sqrt{\mu_{n,2}(v_n,\varkappa)}\right).$$

Proof Using the well known property of the first order modulus of continuity, for any $\delta > 0$, we have

$$|\mathfrak{z}(t) - \mathfrak{z}(\varkappa)| \le \omega(\mathfrak{z};\delta)\left(\frac{(t-\varkappa)^2}{\delta^2} + 1\right).$$

Hence in view of Lemma 1,

$$\begin{aligned}|\mathfrak{C}_n(\mathfrak{z}, v_n, \varkappa) - \mathfrak{z}(\varkappa)| &\le \mathfrak{C}_n(|\mathfrak{z}(t) - \mathfrak{z}(\varkappa)|, v_n, \varkappa)\\ &\le \omega(\mathfrak{z};\delta)\left(1 + \frac{1}{\delta^2}\mu_{n,2}(v_n,\varkappa)\right).\end{aligned}$$

Choosing $\delta = \sqrt{\mu_{n,2}(v_n,\varkappa)}$, the proof is completed. □

Let $\mathcal{C}^r(J),\ r\in\mathbb{N}$, denote the class of r-times continuously differentiable functions in J.

Theorem 3 *If $\mathfrak{z}\in\mathcal{C}^1(J)$, then*

$$|\mathfrak{C}_n(\mathfrak{z},v_n,\varkappa)-\mathfrak{z}(\varkappa)|\leq\frac{3}{2}\sqrt{\mu_{n,2}(v_n,\varkappa)}\,\omega\left(\mathfrak{z}';\sqrt{\mu_{n,2}(v_n,\varkappa)}\right).$$

Proof By our hypothesis, for any $\varkappa,t\in J$, we have

$$\mathfrak{z}(t)-\mathfrak{z}(\varkappa)=\mathfrak{z}'(\varkappa)(t-\varkappa)+\int_{\varkappa}^{t}\left(\mathfrak{z}'(y)-\mathfrak{z}'(\varkappa)\right)dy$$

and therefore

$$\mathfrak{C}_n(\mathfrak{z}(t)-\mathfrak{z}(\varkappa),v_n,\varkappa)=\mathfrak{z}'(\varkappa)\,\mathfrak{C}_n(t-\varkappa,v_n,\varkappa)+\mathfrak{C}_n\left(\int_{\varkappa}^{t}\left(\mathfrak{z}'(y)-\mathfrak{z}'(\varkappa)\right)dy,v_n,\varkappa\right).$$

Using the property of modulus of continuity

$$|\mathfrak{z}(y)-\mathfrak{z}(\varkappa)|\leq\omega(\mathfrak{z};\delta)\left(\frac{|y-\varkappa|}{\delta}+1\right),\ \delta>0,$$

we have

$$\left|\int_{\varkappa}^{t}\left|\mathfrak{z}'(y)-\mathfrak{z}'(\varkappa)\right|dy\right|\leq\omega\left(\mathfrak{z}';\delta\right)\left[\frac{(t-\varkappa)^2}{2\delta}+|t-\varkappa|\right].$$

Therefore,

$$|\mathfrak{C}_n(\mathfrak{z},v_n,\varkappa)-\mathfrak{z}(\varkappa)|\leq\left|\mathfrak{z}'(\varkappa)\right||\mathfrak{C}_n(t-\varkappa,v_n,\varkappa)|+\omega\left(\mathfrak{z}';\delta\right)\left\{\frac{1}{2\delta}\mathfrak{C}_n\left((t-\varkappa)^2,v_n,\varkappa\right)+\mathfrak{C}_n(|t-\varkappa|,v_n,\varkappa)\right\}.$$

Hence applying Cauchy-Schwarz inequality and Lemma 1, we have

$$\begin{aligned}|\mathfrak{C}_n(\mathfrak{z},v_n,\varkappa)-\mathfrak{z}(\varkappa)|&\leq\left|\mathfrak{z}'(\varkappa)\right||\mathfrak{C}_n(t-\varkappa,v_n,\varkappa)|\\&\quad+\omega\left(\mathfrak{z}';\delta\right)\left\{\frac{1}{2\delta}\sqrt{\mathfrak{C}_n\left((t-\varkappa)^2,v_n,\varkappa\right)}+1\right\}\sqrt{\mathfrak{C}_n\left((t-\varkappa)^2,v_n,\varkappa\right)}\\&\leq\omega\left(\mathfrak{z}';\delta\right)\left\{\frac{1}{2\delta}\sqrt{\mu_{n,2}(v_n,\varkappa)}+1\right\}\sqrt{\mu_{n,2}(v_n,\varkappa)}.\end{aligned}\tag{3.1}$$

Choosing $\delta=\sqrt{\mu_{n,2}(v_n,\varkappa)}$ in (3.1), we reach the desired result. □

In the next result, we examine the convergence rate of the operators (1.2) with the help of Peetre's K-functional.

For any $\delta>0$ and $\mathfrak{z}\in\mathcal{C}(J)$, the Peetre's K-functional is given by

$$K_2(\mathfrak{z},\delta)=\inf\left\{\|\mathfrak{z}-g\|+\delta\left\|g''\right\|:g\in\mathcal{C}^2(J)\right\}.$$

For $\mathfrak{z}\in\mathcal{C}(J)$ and any $\delta>0$, the second order modulus of continuity is defined as

$$\omega_2\left(\mathfrak{z},\sqrt{\delta}\right)=\sup_{0<|h|\leq\sqrt{\delta}}\ \sup_{\varkappa,\varkappa+2h\in J}|\mathfrak{z}(\varkappa+2h)-2\mathfrak{z}(\varkappa+h)+\mathfrak{z}(\varkappa)|.$$

From [6], there exists a positive constant $M>0$, such that

$$K_2(\mathfrak{z},\delta)\leq M\omega_2\left(\mathfrak{z},\sqrt{\delta}\right),\ \ \delta>0.\tag{3.2}$$

Theorem 4 *Let $\mathfrak{z}\in\mathcal{C}(J)$. Then for all $\varkappa\in J$, we have*

$$|\mathfrak{C}_n(\mathfrak{z},v_n,\varkappa)-\mathfrak{z}(\varkappa)|\leq M'\omega_2\left(\mathfrak{z},\frac{1}{2}\sqrt{\mu_{n,2}(v_n,\varkappa)+\left(\mu_{n,1}(v_n,\varkappa)\right)^2}\right)+\omega\left(\mathfrak{z},\left|\mu_{n,1}(v_n,\varkappa)\right|\right),$$

where M' is some positive constant.

Proof or any $\mathfrak{z} \in \mathcal{C}(J)$, let us define an auxiliary operator as follows:

$$\tilde{\mathfrak{C}}_n\left(\mathfrak{z}, \upsilon_n, \varkappa\right) = \mathfrak{C}_n\left(\mathfrak{z}, \upsilon_n, \varkappa\right) + \mathfrak{z}\left(\varkappa\right) - \mathfrak{z}\left(\frac{\left(\upsilon_n - 1\right)\left(1 - 2\varkappa\right)}{2\left(n + \upsilon_n - 1\right)}\right). \tag{3.3}$$

Using Lemma 2, from (3.3) we obtain

$$\tilde{\mathfrak{C}}_n\left(e_0, \upsilon_n, \varkappa\right) = 1, \quad \text{and} \quad \tilde{\mathfrak{C}}_n\left(e_1, \upsilon_n, \varkappa\right) = \varkappa.$$

For any $g \in C^2(J)$, by Taylor's formula we have

$$g(t) = g(\varkappa) + (t - \varkappa)g'(\varkappa) + \int_{\varkappa}^{t} (t - y)g''(y)dy.$$

Hence using Lemma 2 and (3.3), we obtain

$$\tilde{\mathfrak{C}}_n\left(g, \upsilon_n, \varkappa\right) = g\left(\varkappa\right) + \mathfrak{C}_n\left(\int_{\varkappa}^{t} \left(t - y\right) g''\left(y\right) dy, \upsilon_n, \varkappa\right) - \int_{\varkappa}^{\frac{(\upsilon_n-1)(1-2\varkappa)}{2(n+\upsilon_n-1)}} \left(\frac{\left(\upsilon_n - 1\right)\left(1 - 2\varkappa\right)}{2\left(n + \upsilon_n - 1\right)} - y\right) g''\left(y\right) dy.$$

Consequently,

$$\left|\tilde{\mathfrak{C}}_n\left(g, \upsilon_n, \varkappa\right) - g\left(\varkappa\right)\right| \le \left\|g''\right\| \left\{\mu_{n,2}\left(\upsilon_n, \varkappa\right) + \left(\mu_{n,1}\left(\upsilon_n, \varkappa\right)\right)^2\right\}. \tag{3.4}$$

From Lemma 4, for all $\varkappa \in J$ we have

$$\left|\tilde{\mathfrak{C}}_n\left(\mathfrak{z}, \upsilon_n, \varkappa\right)\right| \le 3\left\|\mathfrak{z}\right\|. \tag{3.5}$$

For $\mathfrak{z} \in \mathcal{C}(J)$ and any $g \in \mathcal{C}^2(J)$, using (3.3), (3.5) and (3.4) (in that order) we get

$$\begin{aligned}
\left|\mathfrak{C}_n\left(\mathfrak{z}, \upsilon_n, \varkappa\right) - \mathfrak{z}\left(\varkappa\right)\right| &\le \left|\tilde{\mathfrak{C}}_n\left(\mathfrak{z} - g, \upsilon_n, \varkappa\right)\right| + \left|\tilde{\mathfrak{C}}_n\left(g, \upsilon_n, \varkappa\right) - g\left(\varkappa\right)\right| + \left|g\left(\varkappa\right) - \mathfrak{z}\left(\varkappa\right)\right| \\
&\quad + \left|\mathfrak{z}\left(\frac{\left(\upsilon_n - 1\right)\left(1 - 2\varkappa\right)}{2\left(n + \upsilon_n - 1\right)}\right) - \mathfrak{z}\left(\varkappa\right)\right| \\
&\le 4\left\|\mathfrak{z} - g\right\| + \left\{\mu_{n,2}\left(\upsilon_n, \varkappa\right) + \left(\mu_{n,1}\left(\upsilon_n, \varkappa\right)\right)^2\right\} \left\|g''\right\| + \omega\left(\mathfrak{z}, \left|\mu_{n,1}\left(\upsilon_n, \varkappa\right)\right|\right).
\end{aligned}$$

Taking infimum on the right hand side of the above inequality over all $g \in \mathcal{C}^2(J)$, we have

$$\left|\mathfrak{C}_n\left(\mathfrak{z}, \upsilon_n, \varkappa\right) - \mathfrak{z}\left(\varkappa\right)\right| \le 4K_2\left(\mathfrak{z}, \frac{\mu_{n,2}\left(\upsilon_n, \varkappa\right) + \left(\mu_{n,1}\left(\upsilon_n, \varkappa\right)\right)^2}{4}\right) + \omega\left(\mathfrak{z}, \left|\mu_{n,1}\left(\upsilon_n, \varkappa\right)\right|\right).$$

Now, we use relation (3.2) to complete the proof.

In the following result, we establish the Voronovskaya type asymptotic theorem for the operators (1.2).

3.1 Asymptotic approximation by $\mathfrak{C}_n$

Theorem 5 *For all* $\mathfrak{z} \in \mathcal{C}^2(J)$,

$$\lim_{n\to\infty} n(\mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa) - \mathfrak{z}(\varkappa)) = \frac{(c - 1)(1 - 2\varkappa)}{2}\mathfrak{z}'(\varkappa) + \varkappa(1 - \varkappa)\frac{\mathfrak{z}''(\varkappa)}{2!}$$

holds uniformly in $\varkappa \in J$.

Proof By the Taylor's expansion of $\mathfrak{z}$ about $t=\varkappa$, we have

$$\mathfrak{z}(t)=\mathfrak{z}(\varkappa)+(t-\varkappa)\mathfrak{z}'(\varkappa)+\frac{(t-\varkappa)^2}{2!}(t-\varkappa)^2+\xi(t,\varkappa)(t-\varkappa)^2, \tag{3.6}$$

where $\xi(t,\varkappa)\in\mathcal{C}(J)$ and $\xi(t,\varkappa)\to 0$, as $t\to\varkappa$. Now applying operator $\mathfrak{C}_n(.,\upsilon_n,\varkappa)$ to both sides of the above equality (3.6), we get

$$\begin{aligned}\lim_{n\to\infty} n(\mathfrak{C}_n(\mathfrak{z},\upsilon_n,\varkappa)-\mathfrak{z}(\varkappa)) &= \lim_{n\to\infty} n(\mathfrak{C}_n((t-\varkappa),\upsilon_n,\varkappa))\mathfrak{z}'(\varkappa)\\ &\quad+\lim_{n\to\infty} n(\mathfrak{C}_n((t-\varkappa)^2,\upsilon_n,\varkappa))\frac{\mathfrak{z}''(\varkappa)}{2!}\\ &\quad+\lim_{n\to\infty} n(\mathfrak{C}_n(\xi(t,\varkappa)(t-\varkappa)^2,\upsilon_n,\varkappa)),\\ &=\frac{(c-1)(1-2\varkappa)}{2}\mathfrak{z}'(\varkappa)+\varkappa(1-\varkappa)\frac{\mathfrak{z}''(\varkappa)}{2!}\\ &\quad+\lim_{n\to\infty} n\mathfrak{C}_n(\xi(t,\varkappa)(t-\varkappa)^2,\upsilon_n,\varkappa),\end{aligned} \tag{3.7}$$

uniformly in $\varkappa\in J$.

Since $\xi(t,\varkappa)\to 0$ as $t\to\varkappa$, for a given $\epsilon>0$, $\exists$ a $\delta>0$ such that $|\xi(t,\varkappa)|<\delta$, whenever $|t-\varkappa|<\delta$, and for $|t-\varkappa|\geq\delta$, we have $|\xi(t,\varkappa)|\leq M\dfrac{(t-\varkappa)^2}{\delta^2}$, for some $M>0$.

Let $\chi_\delta(t)$ be the characteristic function of the interval $(\varkappa-\delta,\varkappa+\delta)$. Now using Lemma 3, we have

$$\begin{aligned}|\mathfrak{C}_n(\xi(t,\varkappa)(t-\varkappa)^2,\upsilon_n,\varkappa)| &\leq \mathfrak{C}_n(|\xi(t,\varkappa)|(t-\varkappa)^2\chi_\delta(t),\upsilon_n,\varkappa)+\mathfrak{C}_n(|\xi(t,\varkappa)|(t-\varkappa)^2(1-\chi_\delta(t)),\upsilon_n,\varkappa)\\ &<\epsilon\mathfrak{C}_n((t-\varkappa)^2,\upsilon_n,\varkappa)+\frac{M}{\delta^2}\mathfrak{C}_n((t-\varkappa)^4,\upsilon_n,\varkappa)\\ &=\epsilon O\left(\frac{1}{n}\right)+O\left(\frac{1}{n^2}\right),\ uniformly\ in\ \varkappa\in J,\ \ as\ n\to\infty.\end{aligned}$$

Hence due to the arbitrariness of $\epsilon>0$, we get

$$\lim_{n\to\infty} n\mathfrak{C}_n(\xi(t,\varkappa)(t-\varkappa)^2,\upsilon_n,\varkappa)=0,$$

uniformly in $\varkappa\in J$. Thus, the result follows from (3.7). □

4 Quantitative Voronovskaya type theorem

In this section, we derive a quantitative Voronovskaya type estimate for the operators $\mathfrak{C}_n$. In order to prove this result, we recall the definition of the Ditzian-Totik first order modulus of smoothness:

For any $r>0$,

$$\omega_1^\phi(\mathfrak{z};r)=\sup_{0<h\leq r}\left\{\left|\mathfrak{z}\left(x+\frac{h\phi(\varkappa)}{2}\right)-\mathfrak{z}\left(\varkappa-\frac{h\phi(\varkappa)}{2}\right)\right|,\varkappa\pm\frac{h\phi(\varkappa)}{2}\in J\right\},$$

where $\phi(\varkappa)=\sqrt{\varkappa(1-\varkappa)}$ and $\mathfrak{z}\in\mathcal{C}(J)$.

Further, the corresponding $K-$functional is given by

$$K_\phi(\mathfrak{z};s)=\inf_{g\in W_\phi(J)}\left\{\|\mathfrak{z}-g\|+s\left\|\phi g'\right\|\right\},\ \ (s>0),$$

where $W_\phi(J)=\left\{g:g\in AC_{loc}(J),\left\|\phi g'\right\|<\infty\right\}$ and $AC_{loc}(J)$ is the class of absolutely continuous functions on every interval $[a,b]\subset J$.

Between $K-$functional and the Ditizian–Totik first order modulus of smoothness, from [7] it is known that there holds the following relation:

$$K_\phi(\mathfrak{z};r)\leq C\omega_1^\phi(\mathfrak{z};r),$$

where $C>0$ is a constant.

Theorem 6 *For any $\mathfrak{z} \in \mathcal{C}^2(J)$ and sufficiently large n, the following inequality holds*

$$\left|\mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa) - \mathfrak{z}(\varkappa) - \mathfrak{z}'(\varkappa)\mathfrak{C}_n((t-\varkappa), \upsilon_n, \varkappa) - \frac{1}{2}\mathfrak{z}''(\varkappa)\mathfrak{C}_n\left((t-\varkappa)^2, \upsilon_n, \varkappa\right)\right| \leq \frac{C_1\phi^2(\varkappa)}{n}\left\{\|\mathfrak{z}'' - g\| + \frac{1}{\sqrt{n}}\|\phi g'\|\right\},$$

where $g \in W_\phi(J)$ and $\varkappa \in (0,1)$.

Proof Using Taylor's expansion, for $\mathfrak{z} \in \mathcal{C}^2(J)$ *and any* $t, \varkappa \in J$, we have

$$\mathfrak{z}(t) - \mathfrak{z}(\varkappa) = (t-\varkappa)\mathfrak{z}'(\varkappa) + \int_\varkappa^t (t-\mathfrak{u})\mathfrak{z}''(\mathfrak{u})\,d\mathfrak{u}.$$

Hence,

$$\begin{aligned}\mathfrak{z}(t) - \mathfrak{z}(\varkappa) - (t-\varkappa)\mathfrak{z}'(\varkappa) - \frac{1}{2}(t-\varkappa)^2\mathfrak{z}''(\varkappa) &= \int_\varkappa^t (t-\mathfrak{u})\mathfrak{z}''(\mathfrak{u})\,du - \int_\varkappa^t (t-\mathfrak{u})\mathfrak{z}''(\varkappa)\,d\mathfrak{u}\\ &= \int_\varkappa^t (t-\mathfrak{u})\left\{\mathfrak{z}''(\mathfrak{u}) - \mathfrak{z}''(\varkappa)\right\}d\mathfrak{u}.\end{aligned}$$

Applying $\mathfrak{C}_n(., \upsilon_n, \varkappa)$ to both sides in the above relation, we obtain

$$\begin{aligned}&\left|\mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa) - \mathfrak{z}(\varkappa) - \mathfrak{z}'(\varkappa)\mathfrak{C}_n((t-\varkappa), \upsilon_n, \varkappa) - \frac{1}{2}\mathfrak{z}''(\varkappa)\mathfrak{C}_n\left((t-\varkappa)^2, \upsilon_n, \varkappa\right)\right|\\ &\leq \mathfrak{C}_n\left(\left|\int_\varkappa^t |t-\mathfrak{u}|\left|\mathfrak{z}''(\mathfrak{u}) - \mathfrak{z}''(\varkappa)\right|d\mathfrak{u}\right|, \upsilon_n, \varkappa\right). \qquad (4.1)\end{aligned}$$

From [8, p. 337], we know that

$$\left|\int_\varkappa^t |t-\mathfrak{u}|\left|\mathfrak{z}''(\mathfrak{u}) - \mathfrak{z}''(\varkappa)\right|d\mathfrak{u}\right| \leq 2\|\mathfrak{z}'' - g\|(t-\varkappa)^2 + 2\|\phi g'\|\phi^{-1}(\varkappa)|t-\varkappa|^3, \qquad (4.2)$$

where $\varkappa \in (0,1)$, $g \in W_\phi(J)$. Utilizing Cauchy-Schwarz inequality, Eqs. (2.1), (2.2), (4.1) and (4.2) we have

$$\begin{aligned}&\left|\mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa) - \mathfrak{z}(\varkappa) - \mathfrak{z}'(\varkappa)\mathfrak{C}_n(t-\varkappa, \upsilon_n, \varkappa) - \frac{1}{2}\mathfrak{z}''(\varkappa)\mathfrak{C}_n\left((t-\varkappa)^2, \upsilon_n, \varkappa\right)\right|\\ &\leq 2\|\mathfrak{z}'' - g\|\mathfrak{C}_n\left((t-\varkappa)^2, \upsilon_n, \varkappa\right) + 2\|\phi g'\|\phi^{-1}(\varkappa)\mathfrak{C}_n\left(|t-\varkappa|^3, \upsilon_n, \varkappa\right)\\ &\leq 2\|\mathfrak{z}'' - g\|\mathfrak{C}_n\left((t-\varkappa)^2, \upsilon_n, \varkappa\right) + 2\|\phi g'\|\phi^{-1}(\varkappa)\left\{\mathfrak{C}_n\left((t-\varkappa)^2, \upsilon_n, \varkappa\right)\right\}^{\frac{1}{2}}\left\{\mathfrak{C}_n\left((t-\varkappa)^4\right), \upsilon_n, \varkappa\right\}^{\frac{1}{2}}\\ &\leq \frac{C_1\phi^2(\varkappa)}{n}\left\{\|\mathfrak{z}'' - g\| + \frac{1}{\sqrt{n}}\|\phi g'\|\right\}.\end{aligned}$$

□

5 Grüss Voronovskaja type theorem

Grüss [9] obtained an estimate of the difference

$$T(\mathfrak{z}, g) = \frac{1}{(b-a)}\int_a^b \mathfrak{z}(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b \mathfrak{z}(t)dt\int_a^b g(t)dt.$$

After publication, this inequality attracted a great deal of interest. Acu et al. [10] applied this inequality for the first time on the spaces of continuous functions defined on a metric space. Gal and Gonska [11] obtained the Grüss–Voronovskaja type estimate for Bernstein polynomials and a class of one parameter family of operators of Bernstein-Durrmeyer type introduced by Paltanea [12]. Subsequently, many researchers contributed in this direction, see for instance [13–15] etc. In this section, we establish the Grüss Voronovskaja type theorem for the operator $\mathfrak{C}_n$ to show its non-multiplicative character.

Theorem 7 *Let $\mathfrak{z}''$, $g'' \in \mathcal{C}(J)$. Then there holds:*

$$\lim_{n\to\infty} n\{\mathfrak{C}_n(\mathfrak{z}g, \upsilon_n, \varkappa) - \mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa)\mathfrak{C}_n(g, \upsilon_n, \varkappa)\} = \varkappa(1-\varkappa)\mathfrak{z}'(\varkappa)g'(\varkappa),$$

uniformly in $\varkappa \in J$.

Proof Using the identity

$$(\mathfrak{z}g)''(\varkappa) = \mathfrak{z}''(\varkappa)g(\varkappa) + 2g'(\varkappa)\mathfrak{z}'(\varkappa) + g''(\varkappa)\mathfrak{z}(\varkappa), \ \forall \varkappa \in J,$$

by simple calculations, we may write

$$n\{\mathfrak{C}_n(\mathfrak{z}g, \upsilon_n, \varkappa) - \mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa)\mathfrak{C}_n(g, \upsilon_n, \varkappa)\} = n\bigg(\mathfrak{C}_n(\mathfrak{z}g, \upsilon_n, \varkappa) - \mathfrak{z}(\varkappa)g(\varkappa) - (\mathfrak{z}g)'(\varkappa)\mathfrak{C}_n((t-\varkappa), \upsilon_n, \varkappa)$$
$$-\frac{\mathfrak{C}_n((t-\varkappa)^2, \upsilon_n, \varkappa)(\mathfrak{z}g)''(\varkappa)}{2} - g(\varkappa)\bigg(\mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa) - \mathfrak{z}(\varkappa) - \mathfrak{z}'(\varkappa)\mathfrak{C}_n((t-\varkappa), \upsilon_n, \varkappa) - \frac{\mathfrak{C}_n((t-\varkappa)^2, \upsilon_n, \varkappa)}{2}$$
$$\mathfrak{z}''(\varkappa)\bigg) - \mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa)\bigg(\mathfrak{C}_n(g, \upsilon_n, \varkappa) - g(\varkappa) - g'(\varkappa)\mathfrak{C}_n((t-\varkappa), \upsilon_n, \varkappa) - \frac{\mathfrak{C}_n((t-\varkappa)^2, \upsilon_n, \varkappa)}{2}g''(\varkappa)\bigg)$$
$$+\mathfrak{C}_n((t-\varkappa)^2, \upsilon_n, \varkappa)\mathfrak{z}'(\varkappa)g'(\varkappa) + g''(\varkappa)\frac{\mathfrak{C}_n((t-\varkappa)^2, \upsilon_n, \varkappa)}{2!}(\mathfrak{z}(\varkappa) - \mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa)) + g'(\varkappa)\mathfrak{C}_n((t-\varkappa), \upsilon_n, \varkappa)$$
$$(\mathfrak{z}(\varkappa) - \mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa))\bigg).$$

In view of Theorem 1, for any $h \in \mathcal{C}(J)$, it follows that $\mathfrak{C}_n(h, \upsilon_n, \varkappa) \to h(\varkappa)$, as $n \to \infty$, uniformly in $\varkappa \in J$. Also, following the proof of Theorem 5, for $h \in \mathcal{C}^2(J)$ we have $n\bigg(\mathfrak{C}_n(h;\varkappa) - h(\varkappa) - h'(\varkappa)\mathfrak{C}_n((t-\varkappa), \upsilon_n, \varkappa) - \dfrac{\mathfrak{C}_n((t-\varkappa)^2, \upsilon_n, \varkappa)h''(\varkappa)}{2}\bigg) \to 0$, as $n \to \infty$, uniformly in $\varkappa \in J$. Since $\mathfrak{z}''$, $g'' \in \mathcal{C}(J)$, taking into account these results and considering Lemma 3, we have $\lim_{n\to\infty} n\{\mathfrak{C}_n(\mathfrak{z}g, \upsilon_n, \varkappa) - \mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa)\mathfrak{C}_n(g, \upsilon_n, \varkappa)\} = \varkappa(1-\varkappa)\mathfrak{z}'(\varkappa)g'(\varkappa)$, uniformly in $\varkappa \in J$, which completes the proof. □

6 Approximation of functions with derivatives of bounded variation

Determination of the convergence rate for functions with derivatives of bounded variation by means of positive linear operators has been an interesting area of research in approximation theory. Bojanic and Cheng (see [16, 17]) pioneered the study in this direction by estimatiing the rate of approximation for such functions by using Bernstein-Fejér polynomials. Recent research by Lian and Cai [18, 19] has focused on the approximation properties of generalized Bernstein-type operators for absolutely continuous functions with derivatives of bounded variation. In their work, they investigated the order of convergence for both Bézier variants and Bernstein-Chlodovsky-Kantorovich operators. Kajla et al. [20] introduced Bézier variant of Baskakov-Beta type operators and studied their convergence rates for functions with derivatives of bounded variation. Several other researchers have explored similar approximation problems for various sequences of positive linear operators. For instance, the works of [21–27] provide valuable insights into related topics.

Motivated by the above research, in this section we shall discuss the rate of approximation of the operators $\mathfrak{C}_n$ for functions with derivatives of bounded variation.

Let $DBV(J)$ denote the class of all functions in $\mathcal{C}(J)$ with derivatives of bounded variation on interval J and let $\mathfrak{z} \in DBV(J)$, then $\mathfrak{z}$ can be represented as:

$$\mathfrak{z}(\varkappa) = \int_0^{\varkappa} \varsigma(t)dt + \mathfrak{z}(0),$$

where $\varsigma \in BV(J)$, i.e. ς is a function of bounded variation on J. The operators $\mathfrak{C}_n$ can be expressed in an integral form as follows:

$$\mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa) = \int_0^1 \mathfrak{R}_n(\varkappa, \upsilon_n, \tau)\mathfrak{z}(\tau)d\tau,$$

where $\Re_n(\varkappa, \upsilon_n, \tau) = \frac{1}{\upsilon_n}\sum_{k=0}^{n}\sum_{j=0}^{\upsilon_n-1} p_{n,k}(\varkappa)\,\delta_{\frac{k+j}{n+\upsilon_n-1}}(\tau)$ and Dirac- delta function $\delta_\varkappa(\tau)$ is given by

$$\delta_\varkappa(\tau) = \begin{cases} 1, & \text{for } \tau = \varkappa \\ 0, & \text{for } \tau \neq \varkappa. \end{cases}$$

Lemma 5 *For a fixed $\varkappa \in (0, 1)$ and sufficiently large n, we have*

(i) $\nu_n(\varkappa, \upsilon_n, t) = \int_0^t \Re_n(\varkappa, \upsilon_n, \tau)du \leq \frac{\eta^2(\varkappa)}{(\varkappa - t)^2}$, $0 \leq t < \varkappa$,

(ii) $1 - \nu_n(\varkappa, \upsilon_n, \mathfrak{z}) = \int_{\mathfrak{z}}^1 \Re_n(\varkappa, \upsilon_n, \tau)du \leq \frac{\eta^2(\varkappa)}{(\mathfrak{z} - \varkappa)^2}$, $\varkappa < \mathfrak{z} \leq 1$,

where, $\eta^2(\varkappa)$ is as defined in Remark 1.

Proof Using (2.1), the proof of the lemma is straightforward. □

Theorem 8 *Let $\mathfrak{z} \in DBV(J)$ then for any $\varkappa \in (0, 1)$ and sufficiently large n, we have*

$$\begin{aligned} |\mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa) - \mathfrak{z}(\varkappa)| \leq{}& \frac{1}{2}|\mathfrak{z}'(\varkappa+) + \mathfrak{z}'(\varkappa-)|\left|\mu_{n,1}(\upsilon_n, \varkappa)\right| + \frac{1}{2}\eta(\varkappa|\mathfrak{z}'(\varkappa+) - \mathfrak{z}'(\varkappa-)| \\ & + \frac{\eta^2(\varkappa)}{\varkappa}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{\varkappa-\frac{\varkappa}{k}}^{\varkappa+\frac{\varkappa}{k}}(\mathfrak{z}'_\varkappa) + \\ & \frac{\varkappa}{\sqrt{n}}\left(\bigvee_{\varkappa-\frac{\varkappa}{\sqrt{n}}}^{\varkappa+\frac{\varkappa}{\sqrt{n}}}\mathfrak{z}'_\varkappa\right) + \frac{\eta^2(\varkappa)}{\varkappa^2}|\mathfrak{z}(2\varkappa) - \mathfrak{z}(\varkappa) - \varkappa\mathfrak{z}'(\varkappa+)| \\ & + \left(\frac{||\mathfrak{z}|| + |\mathfrak{z}(\varkappa)|}{\varkappa^2}\right)\eta^2(\varkappa) + |\mathfrak{z}'(\varkappa+)|\eta(\varkappa. \end{aligned}$$

where $\bigvee_a^b(\mathfrak{z})$ denotes the total variation of $\mathfrak{z}$ on $[a, b]$, $\mathfrak{z}'_\varkappa$ is defined by

$$\mathfrak{z}'_\varkappa(\tau) = \begin{cases} \mathfrak{z}'(\tau) - \mathfrak{z}(\varkappa-), & 0 \leq \tau < \varkappa \\ 0, & \textit{for } \tau = \varkappa \\ \mathfrak{z}'(\tau) - \mathfrak{z}'(\varkappa+), & \varkappa < \tau < 1, \end{cases} \tag{6.1}$$

and $\eta^2(\varkappa)$ is defined as in Remark 1.

Proof As, $\mathfrak{C}_n(1, \upsilon_n, \varkappa) = 1$, for each $\varkappa \in (0, 1)$, we have

$$\begin{aligned} \mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa) - \mathfrak{z}(\varkappa) &= \int_0^1 \Re_n(\varkappa, \upsilon_n, \tau)(\mathfrak{z}(\tau) - \mathfrak{z}(\varkappa))d\tau \\ &= \int_0^1 \Re_n(\varkappa, \upsilon_n, \tau)\int_\varkappa^\tau \mathfrak{z}'(u)du d\tau. \end{aligned}$$

For each $\mathfrak{z} \in DBV(J)$, we may write

$$\begin{aligned} \mathfrak{z}'(u) = {}& \mathfrak{z}'_\varkappa(u) + \frac{1}{2}(\mathfrak{z}'(\varkappa+) + \mathfrak{z}'(\varkappa-)) + \frac{1}{2}(\mathfrak{z}'(\varkappa+) - \mathfrak{z}'(\varkappa-))sgn(u - \varkappa) + \\ & \delta_\varkappa(u)\{\mathfrak{z}'(u) - \frac{1}{2}(\mathfrak{z}'(\varkappa+) + \mathfrak{z}'(\varkappa-))\}. \end{aligned}$$

Clearly,

$$\int_0^1\left(\int_\varkappa^\tau\left(\mathfrak{z}'(u) - \frac{1}{2}(\mathfrak{z}'(\varkappa+) + \mathfrak{z}'(\varkappa-))\right)\delta_\varkappa(u)du\right)\Re_n(\varkappa, \upsilon_n, \tau)d\tau = 0,$$

and applying the Cauchy-Schwarz inequality

$$\left|\int_0^1 \Re_n(\varkappa, \upsilon_n, \tau)\left(\int_\varkappa^\tau \tfrac{1}{2}(\mathfrak{z}'(\varkappa+)-\mathfrak{z}'(\varkappa-))sgn(u-\varkappa)du\right)d\tau\right|$$

$$\leq \frac{1}{2}|\mathfrak{z}'(\varkappa+)-\mathfrak{z}'(\varkappa-)|\int_0^1 \tau-\varkappa|\Re_n(\varkappa, \upsilon_n, \tau)d\tau$$
$$\leq \frac{1}{2}|\mathfrak{z}'(\varkappa+)-\mathfrak{z}'(\varkappa-)|\mathfrak{C}_n(|\tau-\varkappa|, \upsilon_n, \varkappa)$$
$$\leq \frac{1}{2}|\mathfrak{z}'(\varkappa+)-\mathfrak{z}'(\varkappa-)|(\mathfrak{C}_n((\tau-\varkappa)^2, \upsilon_n, \varkappa))^{\frac{1}{2}}.$$

Further,

$$|\mathfrak{C}_n(\mathfrak{z}, \upsilon_n, \varkappa)-\mathfrak{z}(\varkappa)| \leq \frac{1}{2}|\mathfrak{z}'(\varkappa+)+\mathfrak{z}'(\varkappa-)|\left|\mu_{n,1}(\upsilon_n, \varkappa)\right| + \frac{1}{2}\eta(\varkappa)|\mathfrak{z}'(\varkappa+)-\mathfrak{z}'(\varkappa-)|$$
$$+\left|\int_0^\varkappa \left(\int_\varkappa^\tau \mathfrak{z}'_\varkappa(u)du\right)\Re_n(\varkappa, \upsilon_n, \tau)d\tau + \int_\varkappa^1\int_\varkappa^\tau\left(\mathfrak{z}'_\varkappa(u)du\right)\Re_n(\varkappa, \upsilon_n, \tau)d\tau\right|. \quad (6.2)$$

Assume,

$$I_1 = \int_0^\varkappa \left(\int_\varkappa^\tau \mathfrak{z}'_\varkappa(u)du\right)\Re_n(\varkappa, \upsilon_n, \tau)d\tau$$

and

$$I_2 = \int_\varkappa^1 \left(\int_\varkappa^\tau \mathfrak{z}'_\varkappa(u)du\right)\Re_n(\varkappa, \upsilon_n, \tau)d\tau.$$

In view of Lemma 5 and integration by parts, we have

$$I_1 = \int_0^\varkappa \left(\int_\varkappa^\tau \mathfrak{z}'_\varkappa(u)du\right)\Re_n(\varkappa, \upsilon_n, \tau)d\tau = -\int_0^\varkappa \mathfrak{z}'_\varkappa(\tau)\nu_n(\varkappa, \upsilon_n, \tau)d\tau.$$

Hence, we may write

$$|I_1| \leq \int_0^{\varkappa-\frac{\varkappa}{\sqrt{n}}} |\mathfrak{z}'_\varkappa(\tau)|\nu_n(\varkappa, \upsilon_n, \tau)d\tau + \int_{\varkappa-\frac{\varkappa}{\sqrt{n}}}^\varkappa |\mathfrak{z}'_\varkappa(\tau)|\nu_n(\varkappa, \upsilon_n, \tau)d\tau.$$

Since $\mathfrak{z}'_\varkappa(\varkappa) = 0$ in view of (6.1) and by Lemma 5, $\nu_n(\varkappa, \upsilon_n, \tau) \leq 1$, we obtain

$$\int_{\varkappa-\frac{\varkappa}{\sqrt{n}}}^\varkappa |\mathfrak{z}'_\varkappa(\tau)|\nu_n(\varkappa, \upsilon_n, \tau)d\tau = \int_{\varkappa-\frac{\varkappa}{\sqrt{n}}}^\varkappa |\mathfrak{z}'_\varkappa(\tau)-\mathfrak{z}'_\varkappa(\varkappa)|\nu_n(\varkappa, \upsilon_n, \tau)d\tau$$
$$\leq \frac{\varkappa}{\sqrt{n}} \bigvee_{\varkappa-\frac{\varkappa}{\sqrt{n}}}^{\varkappa} \mathfrak{z}'_\varkappa.$$

Now, taking $\tau = \varkappa - \frac{\varkappa}{u}$, we get

$$\int_0^{\varkappa-\frac{\varkappa}{\sqrt{n}}} |\mathfrak{z}'_\varkappa(\tau)|\nu_n(\varkappa, \upsilon_n, \tau)d\tau \leq \eta^2(\varkappa)\int_0^{\varkappa-\frac{\varkappa}{\sqrt{n}}} |\mathfrak{z}'_\varkappa(\tau)-\mathfrak{z}'_\varkappa(\varkappa)|\frac{d\tau}{(\varkappa-\tau)^2}$$
$$\leq \frac{\eta^2(\varkappa)}{\varkappa}\int_1^{\sqrt{n}} \bigvee_{\varkappa-\frac{\varkappa}{u}}^{\varkappa} \mathfrak{z}'_\varkappa du$$
$$\leq \frac{\eta^2(\varkappa)}{\varkappa}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{\varkappa-\frac{\varkappa}{k}}^{\varkappa} (\mathfrak{z}'_\varkappa).$$

Hence,

$$|I_1| \le \frac{\varkappa}{\sqrt{n}} \bigvee_{\varkappa-\frac{\varkappa}{\sqrt{n}}}^{\varkappa} (\mathfrak{z}'_\varkappa) + \frac{\eta^2(\varkappa)}{\varkappa} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{\varkappa-\frac{\varkappa}{k}}^{\varkappa} (\mathfrak{z}'_\varkappa). \tag{6.3}$$

Again using integration by parts and Lemma 5,

$$\begin{aligned}
|I_2| &= \left| \int_\varkappa^1 \left(\int_\varkappa^\tau \mathfrak{z}'_\varkappa(u)du \right) \Re_n(\varkappa, \upsilon_n, \tau) d\tau \right| \\
&= \left| \int_\varkappa^{2\varkappa} \left(\int_\varkappa^\tau \mathfrak{z}'_\varkappa(u)du \right) \frac{\partial}{\partial\tau}(1-\nu_n(\varkappa, \upsilon_n, \tau))d\tau + \int_{2\varkappa}^1 \left(\int_\varkappa^\tau \mathfrak{z}'_\varkappa(u)du \right) \frac{\partial}{\partial\tau}(1-\nu_n(\varkappa, \upsilon_n, \tau))d\tau \right| \\
&= \left| \left(\int_\varkappa^{2\varkappa} \mathfrak{z}'_\varkappa(u)du \right)(1-\nu_n(\varkappa, \upsilon_n, 2\varkappa) - \int_\varkappa^{2\varkappa} (\mathfrak{z}'_\varkappa(\tau)(1-\nu_n(\varkappa, \upsilon_n, \tau))d\tau \right. \\
&\quad \left. - \int_{2\varkappa}^1 (\mathfrak{z}(\tau)-\mathfrak{z}(\varkappa))\Re_n(\varkappa, \upsilon_n, \tau)d\tau + \mathfrak{z}'(\varkappa+) \int_{2\varkappa}^1 (\tau-\varkappa)\Re_n(\varkappa, \upsilon_n, \tau)d\tau \right| \\
&\le \left| \left(\int_\varkappa^{2\varkappa} \mathfrak{z}'_\varkappa(u)du \right)(1-\nu_n(\varkappa, \upsilon_n, 2\varkappa) \right| + \left| \int_\varkappa^{2\varkappa} (\mathfrak{z}'_\varkappa(\tau)(1-\nu_n(\varkappa, \upsilon_n, \tau))d\tau \right| \\
&\quad + \left| \int_{2\varkappa}^1 (\mathfrak{z}(\tau)-\mathfrak{z}(\varkappa))\Re_n(\varkappa, \upsilon_n, \tau)d\tau \right| + |\mathfrak{z}'(\varkappa+)| \left| \int_{2\varkappa}^1 (\tau-\varkappa)\Re_n(\varkappa, \upsilon_n, \tau)d\tau \right| \\
&= \sum_1 + \sum_2 + \sum_3 + \sum_4 .
\end{aligned} \tag{6.4}$$

Now using (6.1), we have

$$\sum_1 \le \frac{\eta^2(\varkappa)}{\varkappa^2} \left| \int_\varkappa^{2\varkappa} \left(\mathfrak{z}'(u) - \mathfrak{z}'(\varkappa+) \right) du \right| = \frac{\eta^2(\varkappa)}{\varkappa^2} \left| \mathfrak{z}(2\varkappa) - \mathfrak{z}(\varkappa) - \varkappa \mathfrak{z}'(\varkappa+) \right| . \tag{6.5}$$

Next, we estimate $\sum_2$.

$$\begin{aligned}
\sum_2 &\le \int_\varkappa^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \left| \mathfrak{z}'_\varkappa(\tau) \right| (1-\nu_n(\varkappa, \upsilon_n, \tau))d\tau + \int_{\varkappa+\frac{\varkappa}{\sqrt{n}}}^{\varkappa} \left| \mathfrak{z}'_\varkappa(\tau) \right| (1-\nu_n(\varkappa, \upsilon_n, \tau))d\tau \\
&= J_1 + J_2.
\end{aligned}$$

Using Lemma 5, and (6.1 we have

$$\begin{aligned}
J_1 &= \int_\varkappa^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \left| \mathfrak{z}'_\varkappa(\tau) - \mathfrak{z}'_\varkappa(\varkappa) \right| (1-\nu_n(\varkappa, \upsilon_n, \tau))d\tau \\
&\le \int_\varkappa^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \left(\bigvee_\varkappa^\tau \mathfrak{z}'_\varkappa \right) d\tau \\
&\le \frac{\varkappa}{\sqrt{n}} \left(\bigvee_\varkappa^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \mathfrak{z}'_\varkappa \right).
\end{aligned}$$

Similarly, applying Lemma 5 and putting $\tau = \varkappa + \frac{\varkappa}{u}$, one has

$$
\begin{aligned}
J_2 &\leq \eta^2(\varkappa) \int_{\varkappa+\frac{\varkappa}{\sqrt{n}}}^{2\varkappa} \left| \mathfrak{z}'_\varkappa(\tau) - \mathfrak{z}'_\varkappa(\varkappa) \right| \frac{d\tau}{(\tau-\varkappa)^2} \\
&\leq \eta^2(\varkappa) \int_1^{\sqrt{n}} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{u}} \mathfrak{z}'_\varkappa \right) \frac{du}{\varkappa} \\
&\leq \frac{\eta^2(\varkappa)}{\varkappa} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{k}} \mathfrak{z}'_\varkappa \right)
\end{aligned}
$$

Hence,

$$
\sum_2 \leq \frac{\varkappa}{\sqrt{n}} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \mathfrak{z}'_\varkappa \right) + \frac{\eta^2(\varkappa)}{\varkappa} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{k}} \mathfrak{z}'_\varkappa \right). \tag{6.6}
$$

In order to estimate $\sum_3$, we note that $\tau \geq 2\varkappa$, therefore $(\tau - \varkappa) \geq \varkappa$, and hence

$$
\begin{aligned}
\sum_3 &\leq \int_{2\varkappa}^1 |\mathfrak{z}(\tau)| \mathfrak{R}_n(\varkappa, \upsilon_n, \tau) d\tau + \int_{2\varkappa}^1 |\mathfrak{z}(\varkappa)| \mathfrak{R}_n(\varkappa, \upsilon_n, \tau) d\tau \\
&\leq (\|\mathfrak{z}\| + |\mathfrak{z}(\varkappa)|) \int_{2\varkappa}^1 \mathfrak{R}_n(\varkappa, \upsilon_n, \tau) d\tau \leq (\|\mathfrak{z}\| + |\mathfrak{z}(\varkappa)|) \int_{2\varkappa}^1 \frac{(\tau-\varkappa)^2}{\varkappa^2} \mathfrak{R}_n(\varkappa, \upsilon_n, \tau) d\tau \\
&\leq (\|\mathfrak{z}\| + |\mathfrak{z}(\varkappa)|) \frac{\eta^2(\varkappa)}{\varkappa^2}.
\end{aligned} \tag{6.7}
$$

Lastly, applying Cauchy-Schwarz inequality,

$$
\sum_4 \leq |\mathfrak{z}'(\varkappa+)| \int_0^1 |\tau - \varkappa| \mathfrak{R}_n(\varkappa, \upsilon_n, \tau) d\tau \leq |\mathfrak{z}'(\varkappa+)| \eta(\varkappa). \tag{6.8}
$$

Combining (6.2)–(6.8), we get the desired result. □

7 Numerical simulation and performance evaluation

To demonstrate the practical performance of the operators $\mathfrak{C}_n(\mathfrak{z}(t), \upsilon_n, \varkappa)$, we conducted numerical experiments using MATLAB. These experiments assessed the convergence behavior and computational efficiency of these operators through detailed analysis of tables and graphs.

Example 1 To illustrate the practical performance of the operators defined by (1.2), we consider their application to the function $\mathfrak{z}(\varkappa) = \cos(\varkappa^2)$. Using $n = 10, 50, 100$ and the sequence $\upsilon_n = 2$ for all $n \in \mathbb{N}$, we conducted numerical experiments. The resulting graph, presented in Fig. 1, visually demonstrates the convergence of the operators (1.2) to the function $\mathfrak{z}(\varkappa) = \cos(\varkappa^2)$.

To further explore the impact of the parameter n, we conducted additional experiments with $\upsilon_n = 2, \ \forall\, n \in \mathbb{N}$ and varying n. As shown in Fig. 2 and Table 1, increasing the value of n consistently leads to improved approximation, with the maximum absolute error generally decreasing for all values of $\varkappa$. Overall, the data suggests that the approximation $\mathfrak{C}_n(\mathfrak{z}(t), \upsilon_n, \varkappa)$ converges to the exact solution $\mathfrak{z}(\varkappa)$ as the number of terms n increases. The rate of convergence appears to be dependent on the value of $\varkappa$, with faster convergence for smaller values.

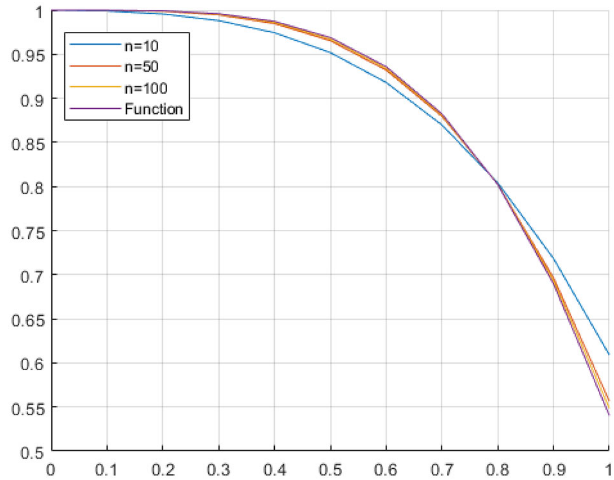

Fig. 1 Approximation process by $\mathfrak{C}_n\,(\mathfrak{z}(t), \upsilon_n, \varkappa)$ for $n = 10, 50, 100$ and $\upsilon_n = 2,\ \forall\, n$.

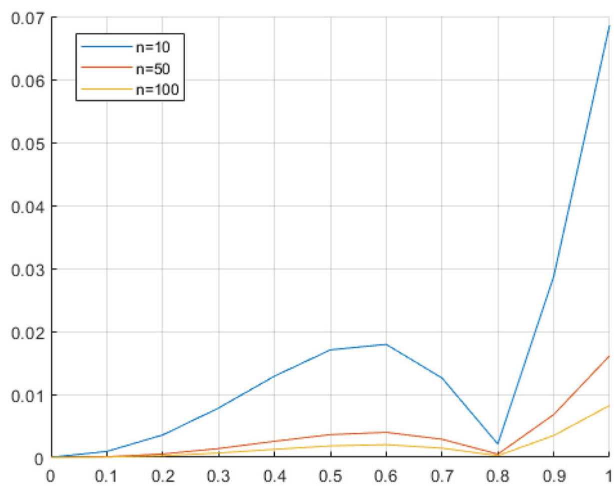

Fig. 2 Error of approximation by $\mathfrak{C}_n\,(\mathfrak{z}(t), \upsilon_n, \varkappa)$ for $n = 10, 50, 100$ and $\upsilon_n = 2,\ \forall\, n$.

Table 1 Maximum absolute errors $|\mathfrak{C}_n\,(\mathfrak{z}(t), \upsilon_n, \varkappa) - \mathfrak{z}(\varkappa)|$ with $\upsilon_n = 2,\ \forall\, n$, and different values of n.

$\varkappa$	$n = 10$	$n = 50$	$n = 100$
0	1.71E−05	3.70E−08	2.40E−09
0.1	9.45E−04	9.23E−05	4.05E−05
0.2	0.0036	5.31E−04	2.53E−04
0.3	0.0078	0.0014	6.87E−04
0.4	0.0129	0.0026	0.0013
0.5	0.0171	0.0036	0.0018
0.6	0.0179	0.004	0.002
0.7	0.0126	0.0029	0.0015
0.8	0.0021	5.11E−04	2.62E−04
0.9	0.0287	0.0068	0.0035
1	0.0686	0.0161	0.0082

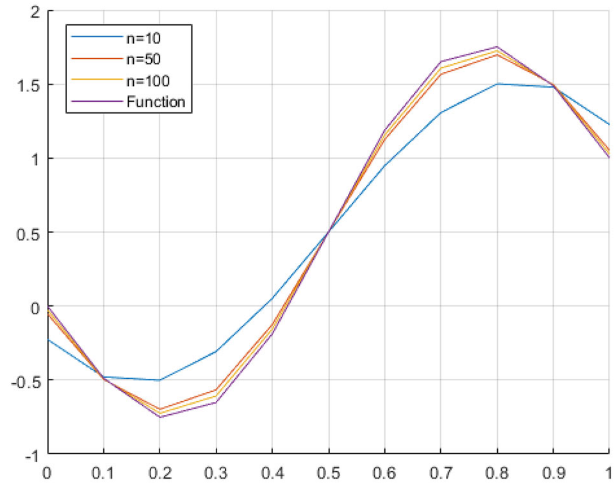

Fig. 3 Approximation Process by $\mathfrak{C}_n\,(\mathfrak{z}(t), \upsilon_n, \varkappa)$ for $n = 10, 50, 100$ and $\upsilon_n = 2,\ \forall\, n$

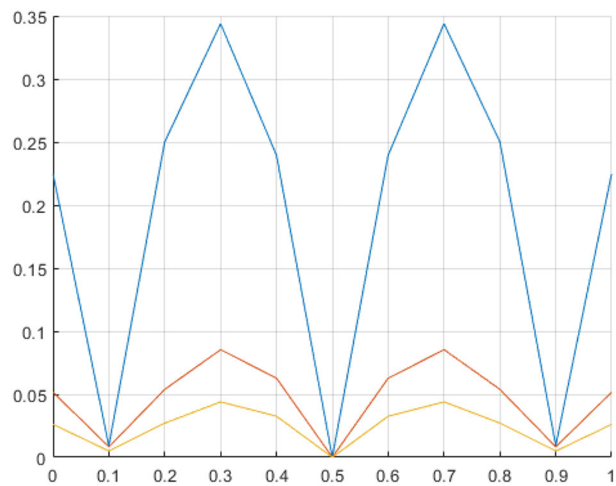

Fig. 4 Error of Approximation by $\mathfrak{C}_n\,(\mathfrak{z}(t), \upsilon_n, \varkappa)$ for $n = 10, 50, 100$ and $\upsilon_n = 2,\ \forall\, n$

Table 2 Maximum absolute errors $|\mathfrak{C}_n\,(\mathfrak{z}(t), \upsilon_n, \varkappa) - \mathfrak{z}(\varkappa)|$ with $\upsilon_n = 2,\ \forall\, n$, and different values of n

$\varkappa$	$n = 10$	$n = 50$	$n = 100$
0	0.2249	0.0516	0.0261
0.1	0.0092	0.0083	0.005
0.2	0.2501	0.0539	0.0271
0.3	0.3438	0.0853	0.044
0.4	0.2395	0.0627	0.0326
0.5	1.11E−16	5.55E−17	2.22E−17
0.6	0.2395	0.0627	0.0326
0.7	0.3438	0.0853	0.044
0.8	0.2501	0.0539	0.0271
0.9	0.0092	0.0083	0.005
1	0.2249	0.0516	0.0261

Example 2 To assess the operators (1.2)'s adaptability to different functions, we applied them to $\mathfrak{z}(\varkappa) = \varkappa - \sin(2\pi\varkappa)$. Using $n = 10, 50, 100$ and the sequence $\upsilon_n = 2,\ \forall\, n \in \mathbb{N}$, we conducted numerical experiments. The resulting graph, presented in Fig. 3, visually demonstrates the operators (1.2)'s convergence to the function $\mathfrak{z}(\varkappa) = \varkappa - \sin(2\pi\varkappa)$.

Table 2 and Fig. 4 present the maximum absolute errors of approximations by the operators (1.2) for $\upsilon_n = 2,\ \forall\, n$ and varying n. The data clearly demonstrates that increasing the value of n consistently improves approximation accuracy, aligning with the general trend observed in the previous analysis. In conclusion, the data supports the convergence of the operators (1.2) to the function $\mathfrak{z}(\varkappa) = \varkappa - \sin(2\pi\varkappa)$. The rate of convergence is influenced by the value of $\varkappa$, with faster convergence for smaller values.

8 Conclusion

We investigated a generalized Bernstein operator introduced by Cao [5] and established its approximation properties. Our analysis included moduli of continuity, Peetre's K-functional, and Voronovskaya theorems. Numerical experiments validated the operator's convergence and efficiency. The results demonstrate the operator's effectiveness in approximating functions, with faster convergence for smaller input values.

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Data availibility Not applicable.

Declarations

Competing interest The authors have no competing interests to declare. All co-authors have seen and agree with the contents of the manuscript and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

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