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ORIGINAL RESEARCH

Biases amongst products of two Beatty primes in arithmetic progressions

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Abstract Motivated by a recently observed bias in products of two prime numbers with congruence conditions by Dummit, Granville and Kisilevsky, we try to observe some bias in the distribution of integers which are product of two distinct primes taken from Beatty sequences with each one is in an arithmetic progression.

Keywords Beatty sequences · Large bias · Prime number

Mathematics Subject Classification 11A63 · 11L03 · 11N05 · 11N69

1 Introduction

Let α be an irrational number and β be an arbitrary real number. We define the Beatty sequence as

$$\mathcal{B}_{\alpha,\beta} = \{n \in \mathbb{N} \,:\, n = \lfloor \alpha m + \beta \rfloor \text{ for some } m \in \mathbb{Z}\},$$

(henceforth, $\lfloor \eta \rfloor$ represents the integer part of real number η). We will require conditions on the type τ of α, which is given by

$$\tau = \tau(\eta) = \sup\left\{t \in \mathbb{R} \;\; \liminf_{n\to\infty} n^t ||\eta n|| = 0\right\}.$$

where $||.||$ denotes the distance to the nearest integer. Beatty sequences appear in a variety of apparently unrelated mathematical settings and because of their versatility, the arithmetic properties of these sequences have been widely studied in the literature; see, for example, [1,9,10].

A study of certain Hamiltonian systems has lead Long [12] to conjecture the existence of infinitely many prime numbers of the form $p = 2\lfloor \alpha n \rfloor + 1$, where $1 < \alpha < 2$ is a fixed irrational number. Motivated by this, Banks and Shparlinski in [2] establish asymptotic formula for the number of primes $p = q\lfloor \alpha n + \beta \rfloor + a$ with $n \leqslant N$, where α, β are real numbers such that α is positive and irrational and a, q are integers with $0 \leqslant a < q \leqslant N^{\kappa}$ and $\gcd(a, q) = 1$, where $\kappa > 0$ depends only on α. They also proved a similar result for primes $p = \lfloor \alpha n + \beta \rfloor$ such that $p \equiv a \pmod{q}$. An explicit version of such result was provided by Banks and Yeager [3] when α is of

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finite type. Indeed, They showed that the set of primes in a Beatty sequence is well distributed over arithmetic progressions.

The study of prime races has attracted the attention of many authors (see [5,7,8,15,17]). Its conception as a subject was originated by the observation of Tchebychev [4] in a letter written to M. Fuss

"There is a notable difference in the splitting of the prime numbers between the two forms 4n + 3, 4n + 1: The first form contains a lot more than the second."

Dummit, Granville and Kisilevsky [5] showed that numbers $n \leqslant x$ with precisely two prime factors p_1, p_2 are far from evenly distributed among the possible residue classes for $p_i \pmod 4$, $1 \leqslant i \leqslant 2$. More precisely, they showed that

$$\frac{\#\{p_1p_2 \leqslant x : p_1 \equiv p_2 \equiv 3 \pmod 4\}}{(1/4)\#\{p_1p_2 \leqslant x\}} = 1 + \frac{c + o(1)}{\log\log x}$$

for some positive constant c. This phenomenon has been extended [13] to general arithmetic progressions $a \pmod q$ with $(a, q) = 1$, and to products of exactly k prime factors, for arbitrary k.

In the present paper, we consider the problem of the distribution of integers which are product of two distinct primes taken from Beatty sequences with each one is in an arithmetic progression. Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Let $\alpha_1, \alpha_2 > 1$ be irrationals of finite type and a_1, a_2, m_1, m_2 be integers such that $(a_1, m_1) = (a_2, m_2) = 1$. If we consider the ratio

$$\mathcal{R}(x) = \mathcal{R}_{\alpha_i,\beta_i,a_i,m_i}(x) = \frac{\#\{p_1p_2 \leqslant x : p_1 \equiv a_1 \pmod{m_1} \in \mathcal{B}_{\alpha_1,\beta_1},\ p_2 \equiv a_2 \pmod{m_2} \in \mathcal{B}_{\alpha_2,\beta_2}\}}{\frac{\alpha_1^{-1}}{\phi(m_1)}\frac{\alpha_2^{-1}}{\phi(m_2)}\#\{p_1p_2 \leqslant x, (p_1, m_1) = (p_2, m_2) = 1\}}$$

in the particular case where $(\alpha_1, \beta_1) = (\alpha_2, \beta_2) = (\sqrt{2}, 0)$, $(a_1, m_1) = (1, 3)$, $(a_2, m_2) = (2, 5)$, we found that:

$$\mathcal{R}(10^3) \approx 1.57, \quad \mathcal{R}(10^4) \approx 1.40, \quad \mathcal{R}(3 * 10^4) \approx 1.37, \quad \mathcal{R}(10^5) \approx 1.35, \quad \mathcal{R}(10^6) \approx 1.32.$$

This reveals a bias that seems to be converging to 1 surprisingly slowly and indicated the existence of a large secondary term, as indeed can be proved:

Theorem 1.1 *Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Let $\alpha_1, \alpha_2 > 1$ be irrationals of finite type. Then, for any coprime integers (a_i, m_i), $(1 \leqslant i \leqslant 2)$, we have the following*

$$\frac{\#\{p_1p_2 \leqslant x : p_1 \equiv a_1 \pmod{m_1} \in \mathcal{B}_{\alpha_1,\beta_1},\ p_2 \equiv a_2 \pmod{m_2} \in \mathcal{B}_{\alpha_2,\beta_2}\}}{\frac{\alpha_1^{-1}}{\phi(m_1)}\frac{\alpha_2^{-1}}{\phi(m_2)}\#\{p_1p_2 \leqslant x, (p_1, m_1) = (p_2, m_2) = 1\}} = 1 + \frac{C + o(1)}{\log\log x},$$

where the constant C depends on $\alpha_1, \alpha_2, a_1, m_1, m_2$.

To this end, our paper is organized as follows: Section 2 is devoted to some preliminary results that will be needed in the sequel. Then, Section 3 concerns the distribution of primes in beatty sequences in which we prove a Mertnes like result . Finally, in Section 4 we give the proof of Theorem 1.1.

2 Some tools

2.1 Notation

We denote by $\lfloor x \rfloor$ and $\{x\}$ the integral part and the fractional part of x, respectively. Considering a real number x, we set $e(x) = \exp(2\pi i x)$ and $\{x\} = x - \lfloor x \rfloor$. We make considerable use the sawtooth function defined by $\psi(x) = x - \lfloor x \rfloor - \frac{1}{2} = \{x\} - \frac{1}{2}$. The letters p_1, p_2 will denote prime numbers. If $\mathcal{A}$ is a finite set, we denote by $\#\mathcal{A}$ the number of its elements. Given two integers a and b, we denote by (a, b) their greatest common divisor.we use $\Lambda(.)$ and $\phi(.)$ to denote the von Mangoldt and Euler functions, respectively. $\pi(x)$ is the function counting the number of prime numbers up to some real number x and let $\pi(x; q, a)$ the number of prime up to x congruent to $a \pmod q$.

Given two complex valued functions f and g, if $f = O(g)$, then we write $f \ll g$ where the implied constants in the symbols "O", "$\ll$" are absolute. If the implied constants depend on certain parameters $\alpha, \beta, \ldots$ (but on no other parameters), then we write $f = O_{\alpha,\beta,\ldots}(g)$ and $f \ll_{\alpha,\beta,\ldots} g$. We recall that for functions f and g the notations $f \ll g$ and $f = O(g)$ are equivalent to the statement that the inequality $|f| \leqslant C|g|$ holds for some constant $C > 0$.

2.2 Numbers in Beatty sequences

Let α, β be in $\mathbb{R}$ with $\alpha > 1$. An integer m has the form $\lfloor \alpha n + \beta \rfloor$ for some integer n if and only if $\alpha^{-1}(m-\beta) \leqslant n < \alpha^{-1}(m+1-\beta)$. Thus, we give the following result [2, Lemma 3.2] which is standard in characterizing the elements of the Beatty sequence $\mathcal{B}_{\alpha,\beta}$.

Lemma 2.1 *A natural number m has the form $\lfloor \alpha n + \beta \rfloor$ for some integer n if and only if $\mathbb{1}_{\alpha,\beta} = 1$, where $\mathbb{1}_{\alpha,\beta} = \lfloor -\alpha^{-1}(m-\beta) \rfloor - \lfloor -\alpha^{-1}(m+1-\beta) \rfloor$.*

For an irrational number θ, we define its type τ by the relation

$$\tau = \tau(\theta) = \sup \left\{ t \in \mathbb{R} \ \liminf_{n\to\infty} n^t ||\theta n|| = 0 \right\}.$$

Dirichlet's approximation theorem implies that $\tau \geqslant 1$ for every irrational number θ. Moreover, if θ has finite type τ, then θ^{-1} also has type τ. Actually, It suffices to show that if for some η we have $\liminf_{n\to\infty} n^\eta ||\theta n|| = 0$ then the same is true when θ is replaced with θ^{-1}. By assumption, for any $\epsilon > 0$, there are positive integers m, n for which $|\theta n - m| < \epsilon n^{-\eta}$. Thus,

$$m^\eta ||\theta^{-1} m|| = m^\eta ||\theta^{-1} m - n|| < m^\eta \theta^{-1} (\epsilon n^{-\eta}).$$

As $m^\eta n^{-\eta}$ converges to 0 as ϵ becomes small enough and n becomes large enough, we get $\liminf_{n\to\infty} n^\eta ||\theta^{-1} n|| = 0$.

2.3 Technical lemmas

We need the well-known approximation of Vaaler [16]

Lemma 2.2 *For any $H \geqslant 1$, there exist numbers a_h, b_h such that*

$$\left| \psi(t) - \sum_{0<|h|\leqslant H} a_h e(th) \right| \leqslant \sum_{|h|\leqslant H} b_h e(th), \quad a_h \ll \frac{1}{|h|}, \ \ b_h \ll \frac{1}{H}.$$

Also, the following of [6, page 48]

Lemma 2.3 *For a bounded arithmetic function f and $N < N' \leqslant 2N$, we have*

$$\sum_{N<p<N'} f(p) \ll \frac{1}{\log N} \max_{N<N_1<2N} \left| \sum_{N<p<N_1} \Lambda(n) f(n) \right| + N^{1/2}.$$

3 Primes in Beatty sequences

We start this paragraph by citing a result of Banks and Yeager [3] about the distribution of primes in a Beatty sequence over arithmetic progressions.

Theorem 3.1 *Let $\alpha > 1$ be a fixed irrational number of finite type τ and β be a fixed real. Then there is a constant $\varepsilon > 0$ such that for all integers c, d with $\gcd(c,d) = 1$, we have*

$$\sum_{\substack{n\leqslant x, n\in\mathcal{B}_{\alpha,\beta} \\ n\equiv c \pmod d}} \Lambda(n) = \frac{1}{\alpha} \sum_{\substack{n\leqslant x \\ n\equiv c \pmod d}} \Lambda(n) + O(x^{1-1/(4\tau+2)+\varepsilon}),$$

where the implied constant depends only on the parameters α, β and ε.

In order to establish an analogue to a theorem of Mertnes for primes in a Beatty sequence over arithmetic progressions, we use the lemma bellow.

Lemma 3.2 *Let θ be an irrational number of finite type τ and fix $A \in]0,1[$ and $\varepsilon > 0$. For any coprime integers c and d with $0 \leqslant c < d$ and any non-zero integer k such that $|k| \leqslant x^A$, we have the upper bound*

$$\sum_{\substack{n\leqslant x\\ n\equiv\ c\ (\mathrm{mod}\, d)}} \Lambda(n)e(k\theta n) \ll x^{\frac{A+2+1/\tau}{2+2/\tau}+\varepsilon} + x^{4/5}(\log x)^3,$$

where the implied constant depends only on the parameters θ, A and ε.

Proposition 3.3 *Let $\alpha > 1$ be a fixed irrational number of finite type τ and β be a fixed real. Then there is a constant $\varepsilon > 0$ such that for all integer c, d with $\gcd(c,d) = 1$, we have*

$$\sum_{\substack{p\leqslant x,\ p\in\mathcal{B}_{\alpha,\beta}\\ p\equiv c\ (\mathrm{mod}\ d)}} \frac{1}{p} = \frac{1}{\alpha} \sum_{\substack{p\leqslant x\\ p\equiv c\ (\mathrm{mod}\ d)}} \frac{1}{p} + O\big(x^{-\frac{1}{(4\tau+2)}+\varepsilon}\log x.\big),$$

where the implied constant depends only on the parameters α and β.

Proof Define

$$C_{\alpha,\beta,c,d}(x) = \sum_{\substack{p\leqslant x,\ p\in\mathcal{B}_{\alpha,\beta}\\ p\equiv c\ (\mathrm{mod}\ d)}} \frac{1}{p} = \sum_{\substack{p\leqslant x\\ p\equiv c\ (\mathrm{mod}\ d)}} \frac{\mathbb{1}_{\alpha,\beta}(p)}{p}. \tag{3.1}$$

It is easy to see from Lemma 2.1

$$\mathbb{1}_{\alpha,\beta}(p) = \alpha^{-1} + \psi\Big(-\alpha^{-1}(p+1-\beta)\Big) - \psi\Big(-\alpha^{-1}(p-\beta)\Big). \tag{3.2}$$

If we putt (3.2) in (3.1), we obtain

$$C_{\alpha,\beta,c,d}(x) = \frac{1}{\alpha} \sum_{\substack{p\leqslant x\\ p\equiv c\ (\mathrm{mod}\ d)}} \frac{1}{p} + \sum_{\substack{p\leqslant x\\ p\equiv c\ (\mathrm{mod}\ d)}} \frac{\Big(\psi\big(-\alpha^{-1}(p+1-\beta)\big) - \psi\big(-\alpha^{-1}(p-\beta)\big)\Big)}{p}.$$

In order to get the desired result, it is sufficient to show that

$$S(x) = \sum_{\substack{p\leqslant x\\ p\equiv c\ (\mathrm{mod}\ d)}} p^{-1}\Big(\psi\big(-\alpha^{-1}(p+1-\beta)\big) - \psi\big(-\alpha^{-1}(p-\beta)\big)\Big) \ll x^{-\frac{1}{(4\tau+2)}+\varepsilon}\log x. \tag{3.3}$$

By using the Vaaler's approximation, i.e. Lemma 2.2, and taking $H = x^A$, we write

$$\begin{aligned}\psi\big(-\alpha^{-1}(p+1-\beta)\big) - \psi\big(-\alpha^{-1}(p-\beta)\big) = &\sum_{0<|h|\leqslant H} a(h)\left(e\Big(\alpha^{-1}h(p+1-\beta)\Big) - e\Big(\alpha^{-1}h(p-\beta)\Big)\right)\\ &+ O\left(\sum_{|h|\leqslant H} b(h)e\Big(\alpha^{-1}h(p+1-\beta)\Big) - e\Big(\alpha^{-1}h(p-\beta)\Big)\right).\end{aligned}$$

with $a(h) \ll |h|^{-1}$ and $b(h) \ll H^{-1}$. If we let $\phi(h) = e(\alpha^{-1}h) - 1$, it follows that

$$\begin{aligned}\psi\big(-\alpha^{-1}(p+1-\beta)\big) - \psi\big(-\alpha^{-1}(p-\beta)\big) = &\sum_{0<|h|\leqslant H} a(h)\phi(h)e\big(\alpha^{-1}h(p-\beta)\big)\\ &+ O\Big(\sum_{|h|\leqslant H} b(h)\phi(h)e\big(\alpha^{-1}h(p-\beta)\big)\Big).\end{aligned}$$

The last equality together with (3.3), gives $S(x) = S_1(x) + O\big(S_2(x)\big)$, where

$$S_1(x) = \sum_{\substack{p\leqslant x\\ p\equiv c\ (\mathrm{mod}\ d)}} p^{-1} \sum_{0<|h|\leqslant H} a(h)\phi(h)e\big(\alpha^{-1}h(p-\beta)\big),$$

and

$$S_2(x) = \sum_{\substack{p\leqslant x\\ p\equiv c \pmod d}} p^{-1} \sum_{|h|\leqslant H} b(h)\phi(h)e\big(\alpha^{-1}h(p-\beta)\big).$$

For S_1 by Lemma (2.3) and a splitting argument, it suffices to provide an upper bound for

$$S_1^*(x) = \sum_{\substack{x/2<n\leqslant x\\ n\equiv c \pmod d}} \sum_{0<|h|\leqslant H} n^{-1}a(h)\phi(h)\Lambda(n)e\big(\alpha^{-1}h(n-\beta)\big).$$

It follows by partial summation and the trivial estimate $\phi(h) \ll 1$, that

$$S_1^*(x) \ll x^{-1} \max_{x/2<u\leqslant x} \sum_{0<|h|\leqslant H} h^{-1}\Big| \sum_{\substack{x/2<n\leqslant x\\ n\equiv c \pmod d}} \Lambda(n)e\big(\alpha^{-1}hn\big)\Big|.$$

Using Lemma 3.2, we write

$$\begin{aligned} S_1^*(x) &\ll x^{-1} \sum_{0<|h|\leqslant H} h^{-1}\Big(x^{\frac{A+2+1/\tau}{2+2/\tau}+\varepsilon} + x^{4/5}(\log x)^3\Big)\\ &\ll x^{\frac{A-1/\tau}{2+2/\tau}+\varepsilon}\log x + x^{-1/5}(\log x)^4. \end{aligned}$$

and the implied constant depends only on τ.

The contribution to S_2 from $h = 0$ is

$$\ll \sum_{p\leqslant x} p^{-1}H^{-1} \ll x^{-A}\log\log x.$$

Moreover, for $h \neq 0$ the same arguments as in S_1 gives the result.

Finally, taking $A = \frac{1}{2\tau+1}$ we obtain the desired bound.

4 Proof of Theorem 1.1

Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Let $\alpha_1, \alpha_2 > 1$ be irrational of finite type and a_1, a_2, m_1, m_2 be integers such that $(a_1, m_1) = (a_2, m_2) = 1$. One can write any such integer $p_1p_2 \leqslant x$ with $p_1 \leqslant p_2 \leqslant \frac{x}{p_1}$, so that $p_1 \leqslant \sqrt{x}$. Therefore,

$$\begin{aligned} \mathcal{T}_{\alpha_i,\beta_i,a_i,m_i}(x) &= \#\{p_1p_2 \leqslant x : p_1 \equiv a_1(\mathrm{mod}\, m_1) \in \mathcal{B}_{\alpha_1,\beta_1},\ p_2 \equiv a_2(\mathrm{mod}\, m_2) \in \mathcal{B}_{\alpha_2,\beta_2}\}\\ &= \sum_{\substack{p_1\leqslant\sqrt{x},\ p_1\in\mathcal{B}_{\alpha_1,\beta_1}\\ p_1\equiv a_1(\mathrm{mod}\, m_1)}} \sum_{\substack{p_1\leqslant p_2\leqslant\frac{x}{p_1},\ p_2\in\mathcal{B}_{\alpha_2,\beta_2}\\ p_2\equiv a_2(\mathrm{mod}\, m_2)}} 1\\ &= \sum_{\substack{p_1\leqslant\sqrt{x},\ p_1\in\mathcal{B}_{\alpha_1,\beta_1}\\ p_1\equiv a_1(\mathrm{mod}\, m_1)}} \Big(\sum_{\substack{p_2\leqslant\frac{x}{p_1},\ p_2\in\mathcal{B}_{\alpha_2,\beta_2}\\ p_2\equiv a_2(\mathrm{mod}\, m_2)}} 1 + O\Big(\frac{p_1}{\log p_1}\Big)\Big), \end{aligned} \tag{4.1}$$

where the big O term is derived from the prime number theorem. Thus, by partial summation we write

$$\sum_{\substack{p_2\leqslant\frac{x}{p_1},\ p_2\in\mathcal{B}_{\alpha_2,\beta_2}\\ p_2\equiv a_2(\mathrm{mod}\, m_2)}} 1 = \frac{1}{\log x} \sum_{\substack{p_2\leqslant\frac{x}{p_1},\ p_2\in\mathcal{B}_{\alpha_2,\beta_2}\\ p_2\equiv a_2(\mathrm{mod}\, m_2)}} \log p_2 + \int_2^{\frac{x}{p_1}} \Big(\sum_{\substack{p_2\leqslant t,\ p_2\in\mathcal{B}_{\alpha_2,\beta_2}\\ p_2\equiv a_2(\mathrm{mod}\, m_2)}} \log p_2\Big)\frac{dt}{t\log^2 t}. \tag{4.2}$$

Using the standard estimate

$$\sum_{n\leqslant x}\Lambda(n) = \sum_{p\leqslant x}\log p + O(\sqrt{x}),$$

we deduce from Theorem 3.1, for sufficiently large x the following

$$\sum_{\substack{p_2\leqslant \frac{x}{p_1},\ p_2\in\mathcal{B}_{\alpha_2,\beta_2}\\ p_2\equiv a_2 \,(\mathrm{mod}\, m_2)}} \log p_2 = \frac{1}{\alpha_2} \sum_{\substack{p_2\leqslant \frac{x}{p_1},\\ p_2\equiv a_2 \,(\mathrm{mod}\, m_2)}} \log p_2 + O\Big((x/p_1)^{1-1/(4\tau+2)+\varepsilon}\Big).$$

Since,

$$\sum_{\substack{p_2\leqslant t,\ p_2\in\mathcal{B}_{\alpha_2,\beta_2}\\ p_2\equiv a_2 \,(\mathrm{mod}\, m_2)}} \log p_2 \leqslant \sum_{p_2\leqslant t} \log p_2 = O(t),$$

the integral in (4.2) is $O\Big(\int_2^{\frac{x}{p_1}} (\log t)^{-2} dt\Big)$. We consider $\big(2 \leqslant t \leqslant \sqrt{x/p_1}\big)$ and $\big(\sqrt{x/p_1} \leqslant t \leqslant (x/p_1)\big)$ separately. In the first range the integrand is uniformly bounded, thus the first portion contributes an amount that is $O(\sqrt{x/p_1})$ and in the integral over the second range is $O\Big(\frac{x}{p_1 \log^2(x/p_1)}\Big)$. Now, if we put what we have proved in (4.2), we write

$$\begin{aligned}\sum_{\substack{p_2\leqslant \frac{x}{p_1},\ p_2\in\mathcal{B}_{\alpha_2,\beta_2}\\ p_2\equiv a_2 \,(\mathrm{mod}\, m_2)}} \log p_2 &= \frac{1}{\alpha_2} \sum_{\substack{p_2\leqslant \frac{x}{p_1},\\ p_2\equiv a_2 \,(\mathrm{mod}\, m_2)}} \log p_2 + O\Big(\frac{x}{p_1 \log^2(x/p_1)}\Big)\\ &= \frac{1}{\alpha_2} \log x \ \ \pi\big(\frac{x}{p_1}; m_2, a_2\big) + O\Big(\frac{x}{p_1 \log^2(x/p_1)}\Big).\end{aligned}$$

The last equality is derived again by partial summation. Furthermore, (4.1) becomes

$$\begin{aligned}\mathcal{T}_{\alpha_i,\beta_i,a_i,m_i}(x) &= \sum_{\substack{p_1\leqslant \sqrt{x},\ p_1\in\mathcal{B}_{\alpha_1,\beta_1}\\ p_1\equiv a_1 \,(\mathrm{mod}\, m_1)}} \Big(\frac{1}{\alpha_2}\pi\big(\frac{x}{p_1}; m_2, a_2\big) + O\Big(\frac{x}{p_1 \log^2(x/p_1)}\Big) + O\Big(\frac{p_1}{\log p_1}\Big)\Big)\\ &= \frac{1}{\alpha_2} \sum_{\substack{p_1\leqslant \sqrt{x},\ p\in\mathcal{B}_{\alpha_1,\beta_1}\\ p_1\equiv a_1 \,(\mathrm{mod}\, m_1)}} \pi\big(\frac{x}{p_1}; m_2, a_2\big) + O\Big(\frac{x}{\log^2 x} \sum_{\substack{p_1\leqslant \sqrt{x},\ p_1\in\mathcal{B}_{\alpha_1,\beta_1}\\ p_1\equiv a_1 \,(\mathrm{mod}\, m_1)}} \frac{1}{p_1}\Big)\\ &\quad + O\Big(\sum_{\substack{p_1\leqslant \sqrt{x},\ p_1\in\mathcal{B}_{\alpha_1,\beta_1}\\ p_1\equiv a_1 \,(\mathrm{mod}\, m_1)}} \frac{p_1}{\log p_1}\Big). \qquad (4.3)\end{aligned}$$

We have,

$$\begin{cases} \displaystyle\sum_{p_1\leqslant x} \frac{1}{p_1} = \log\log x + O(1),\\ \displaystyle\sum_{p_1\leqslant x} \frac{1}{\log p_1} = \frac{x}{\log^2 x} + O\Big(\frac{x\log\log x}{\log^3 x}\Big).\end{cases}$$

We obtain,

$$\sum_{\substack{p_1\leqslant \sqrt{x},\ p_1\in\mathcal{B}_{\alpha_1,\beta_1}\\ p_1\equiv a_1 \,(\mathrm{mod}\, m_1)}} \frac{1}{p_1} \ll \sum_{p_1\leqslant\sqrt{x}} \frac{1}{p_1} \ll \log\log x. \qquad (4.4)$$

$$\sum_{\substack{p_1\leqslant \sqrt{x},\ p_1\in\mathcal{B}_{\alpha_1,\beta_1}\\ p_1\equiv a_1 \,(\mathrm{mod}\, m_1)}} \frac{p_1}{\log p_1} \ll \sum_{p_1\leqslant\sqrt{x}} \frac{p_1}{\log p_1} \ll \frac{x}{\log^2 x}. \qquad (4.5)$$

Assembling (4.3), (4.4) and (4.5) yields

$$\mathcal{T}_{\alpha_i,\beta_i,a_i,m_i}(x) = \frac{1}{\alpha_2} \sum_{\substack{p_1\leqslant \sqrt{x},\ p_1\in\mathcal{B}_{\alpha_1,\beta_1}\\ p_1\equiv a_1 \,(\mathrm{mod}\, m_1)}} \pi\big(\frac{x}{p_1}; m_2, a_2\big) + O\Big(\frac{x\log\log x}{\log^2 x}\Big). \qquad (4.6)$$

Since $(a_2, m_2) = 1$, then we have asymptotically

$$\pi\big(\frac{x}{p_1}; m_2, a_2\big) = \frac{\pi(x/p_1)}{\phi(m_2)}\Big(1 + O\big(\log \frac{x}{p_1}\big)^{-1}\Big) = \frac{1}{\phi(m_2)}\frac{x}{p_1 \log\big(\frac{x}{p_1}\big)}\Big(1 + O\big(\log \frac{x}{p_1}\big)^{-1}\Big).$$

Thus,

$$\mathcal{T}_{\alpha_i,\beta_i,a_i,m_i}(x) = \frac{1}{\alpha_2\phi(m_2)} \sum_{\substack{p_1 \leqslant \sqrt{x},\ p_1 \in \mathcal{B}_{\alpha_1,\beta_1} \\ p_1 \equiv a_1 \,(\mathrm{mod}\, m_1)}} \frac{x}{p_1 \log\big(\frac{x}{p_1}\big)} + O\Big(\frac{x \log\log x}{\log^2 x}\Big).$$

The difference between the sum in the last equality and the same sum with $\log(x/p_1)$ is replaced by $\log x$, is

$$\frac{x}{\log x} \sum_{\substack{p_1 \leqslant \sqrt{x},\ p_1 \in \mathcal{B}_{\alpha_1,\beta_1} \\ p_1 \equiv a_1 \,(\mathrm{mod}\, m_1)}} \frac{\log p_1}{p_1 \log(x/p_1)} \ll \frac{x \log\log x}{(\log x)^2}.$$

Then, we write

$$\mathcal{T}_{\alpha_i,\beta_i,a_i,m_i}(x) = \frac{1}{\alpha_2\phi(m_2)}\frac{x}{\log x} \sum_{\substack{p_1 \leqslant \sqrt{x},\ p_1 \in \mathcal{B}_{\alpha_1,\beta_1} \\ p_1 \equiv a_1 \,(\mathrm{mod}\, m_1)}} \frac{1}{p_1} + O\Big(\frac{x \log\log x}{\log^2 x}\Big).$$

By Proposition 3.3, we have

$$\sum_{\substack{p_1 \leqslant \sqrt{x},\ p_1 \in \mathcal{B}_{\alpha_1,\beta_1} \\ p_1 \equiv a_1 \,(\mathrm{mod}\, m_1)}} \frac{1}{p_1} = \sum_{\substack{p_1 \leqslant \sqrt{x} \\ p_1 \equiv a_1 \ (\mathrm{mod}\ m_1)}} \frac{1}{p} + O\Big(x^{-1/(4\tau+2)+\varepsilon} \log x\Big),$$

The Mertens theorem [14] for arithmetic progressions, assert that

$$\sum_{\substack{p_1 \leqslant \sqrt{x} \\ p_1 \equiv a_1 \ (\mathrm{mod}\ m_1)}} \frac{1}{p_1} = \frac{\log\log\sqrt{x}}{\phi(m_1)} + M(m_1; a_1) + O_q\big(\frac{1}{\log x}\big).$$

Languasco and Zaccagnini [11] investigated the value of $M(m_1; a_1)$ and other related constants. Letting $x \longrightarrow \infty$, we have

$$\sum_{\substack{a_1 \ (\mathrm{mod}\ m_1) \\ (a_1, m_1) = 1}} M(m_1; a_1) = \gamma + B - \sum_{p \mid m_1} \frac{1}{p}.$$

Where γ is the Euler function and $B = \sum_p \Big(\log\big(1 - \frac{1}{p}\big) + \frac{1}{p}\Big)$.

$$\begin{aligned}
\mathcal{T}_{\alpha_i,\beta_i,a_i,m_i}(x) &= \frac{1}{\alpha_2\phi(m_2)}\frac{x}{\log x}\Big(\frac{\log\log\sqrt{x}}{\alpha_1\phi(m_1)} + \frac{M(m_1; a_1)}{\alpha} + O_q\big(\frac{1}{\log x}\big)\Big) + O\Big(\frac{x\log\log x}{\log^2 x}\Big). \\
&= \frac{1}{\alpha_2\phi(m_2)}\frac{x}{\log x}\Big(\frac{\log(1/2)}{\alpha_1\phi(m_1)} + \frac{\log\log x}{\alpha_1\phi(m_1)} + \frac{M(m_1; a_1)}{\alpha_1}\Big) + O\Big(\frac{x\log\log x}{\log^2 x}\Big). \\
&= \frac{1}{\alpha_1\alpha_2\phi(m_1)\phi(m_2)}\frac{x\log\log x}{\log x} + \frac{x}{\log x}\Big(\frac{\log(1/2)}{\alpha_1\alpha_2\phi(m_1)\phi(m_2)} + \frac{M(m_1; a_1)}{\alpha_1\alpha_2\phi(m_2)}\Big) \\
&\quad + O\Big(\frac{x\log\log x}{\log^2 x}\Big).
\end{aligned}$$

It is well known that

$$\#\{p_1p_2 \leqslant x, (p_1, m_1) = (p_2, m_2) = 1\} = \frac{x}{\log x}\big(\log\log x + O(1)\big).$$

So, we deduce that

$$\frac{\#\{p_1p_2 \leqslant x : p_1 \equiv a_1 (\mathrm{mod}\, m_1) \in \mathcal{B}_{\alpha_1,\beta_1},\, p_2 \equiv a_2 (\mathrm{mod}\, m_2) \in \mathcal{B}_{\alpha_2,\beta_2}\}}{\frac{\#\{p_1p_2\leqslant x,(p_1,m_1)=(p_2,m_2)=1\}}{\alpha_1\alpha_2\phi(m_1)\phi(m_2)}} = 1 + \frac{C + o(1)}{\log\log x},$$

with $C = \Big(\dfrac{\log(1/2)}{\alpha_1\alpha_2\phi(m_1)\phi(m_2)} + \dfrac{M(m_1; a_1)}{\alpha_1\alpha_2\phi(m_2)}\Big)$ as claimed. □

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