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ORIGINAL RESEARCH

Compatibility conditions allowing mono phasic oscillating solutions for the multidimensional incompressible Euler system

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Abstract We are interested in Cauchy's problem formed by a multidimensional incompressible Euler's system and large amplitude oscillating initial data $w(x, \varphi(x)/\varepsilon) \in \mathcal{C}^1(\Omega_r^0, \mathbb{R}^n)$, with $\varepsilon \in]0, 1]$ is a parameter and $\Omega_r^0 \subset \mathbb{R}^n$ the ball of centre zero and radius r. We determine the necessary and sufficient conditions that guarantee a solution on a domain of $\mathbb{R}^+ \times \mathbb{R}^n$ independent of ε for the Cauchy's problem previously mentioned. These conditions are a system of nonlinear partial differential equations uniform in ε involving the couple (φ, w), we show the existence of this couple, and we discuss its propagation over time.

Keywords Burger equations · Euler equations · WKB analysis · Oscillations · Compatibility conditions · Nonlinear geometric optics

Mathematics Subject Classification 35A21 · 35A30 · 53C65 · 35M10 · 35Q35 · 53C12

1 Introduction and main results

We consider the multidimensional incompressible Euler's system with constant pressure defined as

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon = 0, \tag{1}$$

$$div\, u^\varepsilon = 0, \tag{2}$$

such that $\varepsilon \in]0, 1]$ a parameter, $u^\varepsilon = (u_1^\varepsilon, \cdots, u_n^\varepsilon) \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad u^\varepsilon \cdot \nabla = \sum_{i=1}^n u_i^\varepsilon\, \partial_i, \quad div\ u^\varepsilon = \sum_{i=1}^n \frac{\partial u_i^\varepsilon}{\partial x_i}.$$

The study of the oscillating solutions for the quasilinear hyperbolic systems, like (1), consists of involving a WKB (Wentzel-Kramers-Brillouin) expansion, see for instance [11] for the initial data problem and [15] for the boundary value one. In the case of a single phase and an initial data defined on the closed ball $\Omega_r^0 \subset \mathbb{R}^n$ of centre zero and radius r, the WKB method consists to work with $u^\varepsilon(0, x) = u_0^\varepsilon(x, \varphi(x)/\varepsilon)$ such that the phase $\varphi \in \mathcal{C}^1(\Omega_r^0, \mathbb{R})$ satisfies

$$\forall x \in \Omega_r^0 : \quad \nabla\varphi(x) \neq 0, \tag{3}$$

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the function $u_0^\varepsilon(x,\theta) = U_0(\varepsilon, x, \theta)$ with $U_0 \in \mathcal{C}^\infty([0,1], \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}, \mathbb{R}^n))$, periodic in $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and satisfies the following expansion

$$u_0^\varepsilon(x,\theta) = w(x,\theta) + \sum_{l=1}^{k} \varepsilon^l w_l(x,\theta) + o(\varepsilon^{k+1}). \tag{4}$$

The fact that $U_0 \in \mathcal{C}^\infty([0,1], \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}, \mathbb{R}^n))$ ensures that $u^\varepsilon(0,x)$ is bounded on Ω_r^0. Thus for every $\varepsilon \in]0,1]$ fixed, the results that appear in [16,18] ensure that (1)-(4) has a $\mathcal{C}^1-$solution on a domain of $[0, T^\varepsilon] \times \mathbb{R}^n$.

The study of $w(x, \varphi(x)/\varepsilon)$, the first term on the expansions (4), is called the nonlinear geometric optics [11,14]. By one hand, when $\partial_\theta w \equiv 0$, the system (1)-(4) is well described, see for instance [12,13,17]. On the other hand, when $\partial_\theta w \neq 0$ the estimations of T^ε given in [1,7] show that in general $Lim_{\varepsilon \to 0} T^\varepsilon = 0$. Therefore, without more specific conditions on (φ, w), the asymptotic behaviour of (1)-(2) when ε goes to zero is not well-posed.

In the case when $n \leq 3$ and $\partial_\theta w \neq 0$, in [5,10] the authors have found a class of initial data leading to the existence of solutions of (1)-(4) on a domain independent of ε. In the case of dimension $n = 3$ and $\partial_\theta w \neq 0$, see [8], we have a full classification of (φ, w) that ensure the existence of $\mathcal{C}^1-$solutions, on a domain independent of ε, for the problem (1)-(2) completed with

$$\forall (x,\varepsilon) \in \Omega_r^0 \times]0,1] : \quad u^\varepsilon(0,x) = h^\varepsilon(x) = w\Big(x, \frac{\varphi(x)}{\varepsilon}\Big). \tag{5}$$

Without restriction on the space dimension, with the constraint $\partial_\theta w \neq 0$ and without involving the condition (2), a class of initial data h^ε that ensures the existence of some $T_0 > 0$ such that the Cauchy's problem (1)-(5) has a solution on $\Omega_r^{T_0}$ is exposed in [7]. For the interest of studying the multidimensional Euler system in physics, we can refer for instance to [9].

In this paper, we work on $\mathbb{R}^n$ with $n \geq 4$, and we consider the case when

$$\forall (x,\theta) \in \Omega_r^0 \times \mathbb{T} : \quad \partial_\theta w(x,\theta) \neq 0. \tag{6}$$

By using the Theorem 2.6 of [6], we obtain that the solution of (1)-(5) satisfies the compressibilty condition (2) if and only if the Jacobian matrix of $h^\varepsilon(x)$ is nilpotent, precisely

$$\forall (x,\varepsilon) \in \Omega_r^0 \times]0,1] : \quad (D_x h^\varepsilon(x))^n = 0. \tag{7}$$

Thus, the problem (1)-(2)-(5) becomes a Cauchy's one formed by the Burger' system (1) and the initial data (5) that satisfies (7). Our aim is to give the necessary and sufficient conditions to impose on (φ, w) to have (7), we show that these conditions lead to the existence of $T_0 > 0$ such that on $\Omega_r^{T_0}$ the problem (1)-(2)-(5) has a solution. In addition, we discuss the existence, geometrical structure, and propagation of the initial data satisfying (7). In [8], we find a definition of *compatible family* and *compatible couple* corresponding to $n = 3$; in this paper we keep the same definition for $n \geq 4$.

Definition 1 (*compatible family - compatible couple*) The family $\{h^\varepsilon\}_{\varepsilon \in]0,1]}$ is compatible on Ω_r^0 if it is defined as in (5), satisfies (7) and (φ, w) checks (3)-(6). In this case, we say that the couple (φ, w) is a compatible one.

Remark 1 In this paper, we denote by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathbb{N}_p = \{0, 1, 2, \cdots, p\}$, $\mathbb{N}_p^* = \mathbb{N}_p \setminus \{0\}$. For $f(s,\theta) \in \mathcal{C}^1(\mathbb{R}^p \times \mathbb{T}, \mathbb{R}^q)$, such that $s = (s_1, \cdots, s_p)$, the Jacobian matrix of f with respect to s is noted $D_s f$ and

$$\forall k \in \mathbb{N}_p^* : \quad \partial_k f = \frac{\partial f}{\partial s_k}, \quad \partial_\theta f = \frac{\partial f}{\partial \theta}.$$

The sub vector space generated by the set $\mathcal{F} = \{v_1, \cdots, v_p\} \subset \mathbb{R}^n$ is noted $\mathcal{E} = span\{v_1, \cdots, v_p\}$, $dim\, \mathcal{E}$ is the dimension of $\mathcal{E}$.

The transpose of the matrix $\mathcal{A}$ is noted $\mathcal{A}^T$ and for $v = (v_1, \cdots, v_n)^T$, $w = (w_1, \cdots, w_n)^T \in \mathbb{R}^n$ the scalar and the tensor products are defined as

$$v \cdot w = v^T\, w = \sum_{k=1}^{n} v_k w_k, \quad v \otimes w = v\, w^T = \begin{pmatrix} v_1 w_1 & \cdots & v_1 w_n \\ \vdots & \ddots & \vdots \\ v_n w_1 & \cdots & v_n w_n \end{pmatrix},$$

For a given matrix $\mathcal{A}$, we denote by $v \cdot \mathcal{A} w := v \cdot (\mathcal{A} w) = v^T \mathcal{A} w$.

The plan of the paper is as follows:

• Section 2 discusses the structure and the existence of compatible couples. Theorem 1 below exhibits the necessary and sufficient conditions to impose on (φ, w) in order to recover a compatible family $\{h^\varepsilon\}_{\varepsilon\in]0,1]}$, the proof is given at the level of Subsection 2.1.

Theorem 1 *Let (φ, w) checks* (3)-(6) *and h^ε defined as in* (5). *Then, the family $\{h^\varepsilon\}_{\varepsilon\in]0,1]}$ is compatible on Ω_r^0 if and only if over $\Omega_r^0 \times \mathbb{T}$ we have*

$$(D_x w)^n = 0, \tag{8}$$

$$\nabla\varphi \cdot \partial_\theta w = 0, \tag{9}$$

$$\forall k \in \mathbb{N}^*, \quad k \leq \frac{n}{2} : \quad \nabla\varphi \cdot (D_x w)^k \partial_\theta w = 0, \tag{10}$$

$$\sum_{\alpha=0}^{n-1} (D_x w)^\alpha (\partial_\theta w \otimes \nabla\varphi)(D_x w)^{n-1-\alpha} = 0. \tag{11}$$

Examples of initial data with a Jacobian matrix of rank one or two appear in [1,4,6,7]. In [3,20], and without involving the torus $\mathbb{T}$, we find classes of functions defined over $\mathbb{R}^n$ with polynomial components and a nilpotent Jacobian matrix. Subsection 2.2, see Proposition 3 and Example 1, discuss the question of the existence of compatible couples with any rank.

• The propagation problem is exposed by Theorem 2, this one shows that the construction of the solution of (1)-(5) involves a compatible couple. The proof is given at the level of Section 3.

Theorem 2 *Let (φ, w) a compatible couple on $\Omega_r^0 \times \mathbb{T}$, $V = Sup_{(x,\theta)\in\Omega_r^0\times\mathbb{T}} \|w(x,\theta)\|$, $T_0 = r/V$ and*

$$\Omega_r^{T_0} = \left\{ (t, x) \in [0, T_0] \times \mathbb{R}^n; \quad \|x\| + tV \leq r \right\}.$$

Then there exists $\mathbf{W} \in \mathcal{C}^1(\Omega_r^{T_0} \times \mathbb{T}, \mathbb{R}^n)$ and $\Phi \in \mathcal{C}^1(\Omega_r^{T_0}, \mathbb{R})$ solutions on $\Omega_r^{T_0} \times \mathbb{T}$ of

$$\partial_t \mathbf{W} + (\mathbf{W} \cdot \nabla)\mathbf{W} = 0, \qquad \mathbf{W}(0, x, \theta) = w(x, \theta), \tag{12}$$

$$\partial_t \Phi + (\mathbf{W} \cdot \nabla)\Phi = 0, \qquad \Phi(0, x) = \varphi(x). \tag{13}$$

The couple $(\Phi, \mathbf{W})$ is a compatible one on $\Omega_r^t \times \mathbb{T}$ for every $t \in [0, T_0]$, more precisely

$$(D_x \mathbf{W})^n = 0, \tag{14}$$

$$\nabla\Phi \cdot \partial_\theta \mathbf{W} = 0, \tag{15}$$

$$\forall l \in \mathbb{N}^*, \quad l \leq \frac{n}{2} : \quad \nabla\Phi \cdot (D_x \mathbf{W})^l \partial_\theta \mathbf{W} = 0, \tag{16}$$

$$\sum_{\alpha=0}^{n-1} (D_x \mathbf{W})^\alpha (\partial_\theta \mathbf{W} \otimes \nabla\Phi)(D_x \mathbf{W})^{n-1-\alpha} = 0, \tag{17}$$

and the function $u^\varepsilon \in \mathcal{C}^1(\Omega_r^{T_0}, \mathbb{R}^n)$ defined as

$$u^\varepsilon(t, x) = \mathbf{W}\left(t, x, \frac{\Phi(t, x)}{\varepsilon}\right), \tag{18}$$

is a solution on $\Omega_r^{T_0}$ of the Cauchy's problem (1)-(2)-(5) *for every $\varepsilon \in]0, 1]$. Moreover, during the evolution, the rank is preserved along the characteristics, more precisely*

$$rank\ D_y w(y, \theta) = \big(rank\ D_x \mathbf{W}\big)\big(t, y + tw(y, \theta), \theta\big). \tag{19}$$

• Section 4 gives some characteristics (see Lemma 4), the geometrical structure (see Theorem 4) as well as the propagation (see Corollary 1) of the compatible couples satisfying $rank\ D_x w \equiv n - 1$.

In comparison with the case $n = 3$ (see [8]), when $n \geq 4$, the methods used to determine the compatibility conditions (see Theorem 1) and to show that the decomposition of w, when $rank\ D_x w = n - 1$, explicitly involves the phases φ (see Theorem 4), are different from those used when $n = 3$. On the other hand, since the incompressibility condition (2) is equivalent to (7), the analysis used in this paper can not be replicated for the compressible case.

2 The compatibility conditions

In this section, we prove Theorem 1; we start with the following lemma.

Lemma 1 *Let M a square matrix of size n and $V, W \in \mathbb{R}^n$. Then*

$$\forall m \in \mathbb{N}, \quad m \geq 3: \quad \Big(M + \frac{1}{\varepsilon} V \otimes W\Big)^m = \sum_{k=0}^{m} \frac{1}{\varepsilon^k} A_{m,k} \tag{20}$$

such that

$$A_{m,0} = M^m \tag{21}$$

$$A_{m,1} = \sum_{\alpha=0}^{m-1} M^{\alpha} (V \otimes W) M^{m-1-\alpha} \tag{22}$$

$$A_{m,m} = (V \otimes W)^m = (W \cdot V)^{m-1}\, V \otimes W \tag{23}$$

and for all $k \in \mathbb{N}^*_{m-1} \setminus \{1\}$:

$$\left.\begin{aligned} A_{m,k} &= \sum_{(\alpha_0,\cdots,\alpha_k)\in\mathcal{S}_{m,k}} C_{W,M,V}^{\alpha_1,\cdots,\alpha_{k-1}}\, M^{\alpha_0}(V \otimes W) M^{\alpha_k} \\ \mathcal{S}_{m,k} &= \Big\{(\alpha_0, \cdots, \alpha_k) \in \mathbb{N}^{k+1}: \quad \sum_{i=0}^{k} \alpha_i = m - k\Big\} \\ C_{W,M,V}^{\alpha_1,\cdots,\alpha_{k-1}} &= \prod_{i=1}^{k-1} W \cdot M^{\alpha_i} V \end{aligned}\right\} \tag{24}$$

Proof By induction on the integer m. For $m = 3$, we have

$$\Big(M + \frac{1}{\varepsilon} V \otimes W\Big)^3 = \sum_{k=0}^{3} \frac{1}{\varepsilon^k} F_k, \tag{25}$$

such that

$$F_0 = M^3, \tag{26}$$

$$F_1 = M^2(V \otimes W) + M(V \otimes W)M + (V \otimes W)M^2, \tag{27}$$

$$F_2 = M(V \otimes W)^2 + (V \otimes W)M(V \otimes W) + (V \otimes W)^2 M, \tag{28}$$

$$F_3 = (V \otimes W)^3. \tag{29}$$

The conditions (21)-(22)-(23)-(24) gives $F_0 = A_{3,0}$, $F_1 = A_{3,1}$, $F_2 = A_{3,2}$ and $F_3 = A_{3,3}$. This with (25) ensures (20) for $m = 3$.

We suppose that (20)- $\cdots$ -(24) s$m \geq 3$, we use (20) we get

$$\begin{aligned} (M + \varepsilon^{-1} V \otimes W)^{m+1} =& M A_{m,0} + \sum_{k=2}^{m} \frac{1}{\varepsilon^k}\Big[M A_{m,k} + (V \otimes W) A_{m,k-1}\Big] + \\ & \frac{1}{\varepsilon^{m+1}} (V \otimes W) A_{m,m}. \end{aligned} \tag{30}$$

We use the induction hypothesis, precisely (21)-(22)-(23), we obtain

$$\left.\begin{aligned} M A_{m,0} &= A_{m+1,0} \\ (V \otimes W) A_{m,m} &= A_{m+1,m+1} \\ M A_{m,1} + (V \otimes W) A_{m,0} &= A_{m+1,1} \end{aligned}\right\} \tag{31}$$

Let k be an integer so that $k \geq 2$, $\mathcal{S}^*_{m+1,k}$ and $\mathcal{S}^\circ_{m+1,k}$ be the sets defined as

$$\mathcal{S}^*_{m+1,k} = \Big\{(s_0, \cdots, s_k) \in \mathbb{N}^{k+1} : \quad s_0 \neq 0, \quad \sum_{i=0}^{k} s_i = m+1-k\Big\}, \tag{32}$$

$$\mathcal{S}^\circ_{m+1,k} = \Big\{(s_0, \cdots, s_k) \in \mathbb{N}^{k+1} : \quad s_0 = 0, \quad \sum_{i=0}^{k} s_i = m+1-k\Big\}, \tag{33}$$

the conditions (32)-(33) ensure that $\mathcal{S}^*_{m+1,k}$ and $\mathcal{S}^\circ_{m+1,k}$ satisfy

$$\mathcal{S}_{m+1,k} = \mathcal{S}^*_{m+1,k} \cup \mathcal{S}^\circ_{m+1,k}\,, \quad \mathcal{S}^*_{m+1,k} \cap \mathcal{S}^\circ_{m+1,k} = \emptyset. \tag{34}$$

• *Calculus of* $MA_{m,k}$ *for* $k \geq 2$: The constraint (24) yields

$$MA_{m,k} = \sum_{(\alpha_0,\cdots,\alpha_k)\in\mathcal{S}_{m,k}} C^{\alpha_1,\cdots,\alpha_{k-1}}_{W,M,V}\, M^{\alpha_0+1}(V \otimes W)M^{\alpha_k}. \tag{35}$$

Let $(\gamma_0, \gamma_1, \cdots, \gamma_k) = (\alpha_0 + 1, \alpha_1, \cdots, \alpha_k)$, then $(\alpha_0, \cdots, \alpha_k) \in \mathcal{S}_{m,k}$ if and only if $(\gamma_0, \cdots, \gamma_k) \in \mathcal{S}^*_{m+1,k}$, this transforms (35) to

$$MA_{m,k} = \sum_{(\gamma_0,\cdots,\gamma_k)\in\mathcal{S}^*_{m+1,k}} C^{\gamma_1,\cdots,\gamma_{k-1}}_{W,M,V}\, M^{\gamma_0}(V \otimes W)M^{\gamma_k}. \tag{36}$$

• *Calculus of* $(V\otimes W)A_{m,k-1}$ *for* $k \geq 2$: For $k = 2$, the condition (22), the change $(\alpha_0, \alpha_1, \alpha_2) = (0, \alpha, m-1-\alpha)$, the information (32) and the expression of $C^{\alpha_1,\cdots,\alpha_{k-1}}_{W,M,V}$ that appear at the level of (24) give

$$(V \otimes W)A_{m,1} = \sum_{(\alpha_0,\alpha_1,\alpha_2)\in\mathcal{S}^\circ_{m+1,2}} C^{\alpha_2}_{W,M,V}\, M^{\alpha_0}(V \otimes W)M^{\alpha_2}. \tag{37}$$

In the case when $k \geq 3$, we use (24) to obtain

$$\begin{aligned}&(V \otimes W)A_{m,k-1}\\ &= \sum_{(\alpha_0,\cdots,\alpha_{k-1})\in\mathcal{S}_{m,k-1}} C^{\alpha_1,\cdots,\alpha_{k-2}}_{W,M,V}\, [W \cdot M^{\alpha_0} V](V \otimes W)M^{\alpha_{k-1}}.\end{aligned} \tag{38}$$

Let $(\beta_0, \beta_1, \cdots, \beta_k) = (0, \alpha_0, \cdots, \alpha_{k-1})$, this yields "$(\alpha_0, \cdots, \alpha_{k-1}) \in \mathcal{S}_{m,k-1}$ *if and only if* $(\beta_0, \cdots, \beta_k) \in \mathcal{S}^\circ_{m+1,k}$" as well as $C^{\alpha_1,\cdots,\alpha_{k-2}}_{W,M,V}\, [W \cdot M^{\alpha_0} V] = C^{\beta_1,\cdots,\beta_{k-1}}_{W,M,V}$. Therefore, (38) becomes as

$$\begin{aligned}&(V \otimes W)A_{m,k-1}\\ &= \sum_{(\beta_0,\cdots,\beta_k)\in\mathcal{S}^\circ_{m+1,k}} C^{\beta_1,\cdots,\beta_{k-1}}_{W,M,V}\, M^{\beta_0}(V \otimes W)M^{\beta_k}.\end{aligned} \tag{39}$$

See that (37) is (39) for $k = 2$, so (39) holds for $k \geq 2$. The information (34)-(36)-(39) lead to $MA_{m,k} + (V \otimes W)A_{m,k-1} = A_{m+1,k}$ for $k \geq 2$, this with (30)-(31) gives

$$\Big(M + \frac{1}{\varepsilon} V \otimes W\Big)^{m+1} = \sum_{k=0}^{m+1} \frac{1}{\varepsilon^k} A_{m+1,k}.$$

In conclusion: we have (20)- $\cdots$ -(24) for every integer $m \geq 3$. □

We recall here a proof by induction typically used for demonstrating assertions defined over a finite subset of $\mathbb{N}$.

Lemma 2 *Let* $j_0, j, m \in \mathbb{N}$ *with* $j_0 \leq j \leq m$, *and* $\mathcal{P}(j)$ *a logic expression such as*

a) $\mathcal{P}(j_0)$ *is true.*

b) *For every* $j \in \mathbb{N}$ *such that* $j_0 + 1 \le j \le m$*: if* $\mathcal{P}(l)$ *is true for every* $l \in \mathbb{N}$ *such as* $j_0 \le l \le j-1$*, then* $\mathcal{P}(j)$ *is true.*

Then $\mathcal{P}(j)$ *holds for every* $j \in \mathbb{N}$ *such that* $j_0 \le j \le m$.

Proof We suppose that

$$G = \{s \in \mathbb{N} : \ j_0 \le s \le m, \ P(s) \ false\} \neq \emptyset,$$

then, $j_1 = min\, G$ exists and $j_1 - 1 \notin G$. The condition $a)$ ensures that $j_1 \neq j_0$, so $j_0 + 1 \le j_1 \le m$, hence $j_0 \le j_1 - 1 \le m-1$. We use the fact that $j_1 = min\, G$ to obtain that for every integer l such as $j_0 \le l \le j_1 - 1$ we have $l \notin G$. Therefore, $P(l)$ is true for $j_0 \le l \le j_1 - 1$. We use $b)$, with $j = j_1$, we obtain $P(j_1)$ true. This is a contradiction because $j_1 \in G$, so $G = \emptyset$. In conclusion $P(j)$ holds for every integer j satisfies $j_0 \le j \le m$. □

2.1 Proof of Theorem 1

Let (φ, w) satisfies (3)-(6) and h^ε as in (5). We use Lemma 1 with the choice

$$m = n, \quad M = D_x w, \quad V = \partial_\theta w, \quad W = \nabla\varphi,$$

to obtain

$$\forall \varepsilon \in]0,1] : \quad (D_x h^\varepsilon)^n = \left(D_x w + \frac{1}{\varepsilon}\partial_\theta w \otimes \nabla\varphi\right)^n = \sum_{k=0}^{n} \frac{1}{\varepsilon^k} A_{n,k}, \tag{40}$$

such that w, $\partial_\theta w$, $A_{n,k}$ are evaluated at $(x, \varphi(x)/\varepsilon)$ and

$$A_{n,0} = (D_x w)^n \tag{41}$$

$$A_{n,1} = \sum_{\alpha=0}^{n-1} (D_x w)^\alpha (\partial_\theta w \otimes \nabla\varphi)(D_x w)^{n-1-\alpha} \tag{42}$$

$$A_{n,n} = (\partial_\theta w \otimes \nabla\varphi)^n = (\nabla\varphi \cdot \partial_\theta w)^{n-1} (\partial_\theta w \otimes \nabla\varphi) \tag{43}$$

and for $k \in \mathbb{N}^*_{n-1} \setminus \{1\}$:

$$\left.\begin{aligned}
A_{n,k} &= \sum_{(\alpha_0,\cdots,\alpha_k)\in\mathcal{S}_{n,k}} C^{\alpha_1,\cdots,\alpha_{k-1}}_{\nabla\varphi, D_x w, \partial_\theta w} \, (D_x w)^{\alpha_0} \, (\partial_\theta w \otimes \nabla\varphi) \, (D_x w)^{\alpha_k} \\
\mathcal{S}_{n,k} &= \left\{(\alpha_0, \cdots, \alpha_k) \in \mathbb{N}^{k+1} : \ \sum_{i=0}^{k} \alpha_i = n-k\right\}, \\
C^{\alpha_1,\cdots,\alpha_{k-1}}_{\nabla\varphi, D_x w, \partial_\theta w} &= \prod_{i=1}^{k-1} \nabla\varphi \cdot (D_x w)^{\alpha_i} \partial_\theta w.
\end{aligned}\right\} \tag{44}$$

2.1.1 First sense of the proof (the if part).

The condition (3) yields $\varphi(x) \neq 0$ almost everywhere on Ω_r^0. Let $x_0 \in \Omega_r^0$ such that $\varphi(x_0) \neq 0$, $\varepsilon \in]0,1]$, $s = +1$ if $\varphi(x_0) > 0$ and $s = -1$ if $\varphi(x_0) < 0$. We set $I = [\varphi(x_0), +\infty[$ if $\varphi(x_0) > 0$ and $I =]-\infty, \varphi(x_0)]$ if $\varphi(x_0) < 0$. Now, consider

$$\forall (j, v, X) \in \mathbb{N} \times I \times \mathbb{R} : \quad \varepsilon_j = \frac{\varphi(x_0)}{v + sj}, \quad P(X) = \sum_{k=0}^{n} A_{n,k}(x_0, v) X^k.$$

See that $\varepsilon_j \in]0,1]$, $v = (\varphi(x_0)/\varepsilon_j) - sj$, and $A_{n,k}(x,\theta)$ is periodic with respect to $\theta \in \mathbb{T}$ with 1 as period. Combining this with (7)-(40) we get $P\left(1/\varepsilon_j\right) = 0$. Thus, $P(X)$ is a polynomial with an infinite number of roots; hence, $A_{n,k}(x_0, v) = 0$ for every $(k, v) \in \mathbb{N}_n \times I$. Using the periodicity of $A_{n,k}(x,\theta)$ with respect to θ and the expression of I, we get $A_{n,k}(x_0,\theta) = 0$ for every $(k,\theta) \in \mathbb{N}_n \times \mathbb{T}$. Recall that this result holds for every $x_0 \in \Omega_r^0$

such that $\varphi(x_0) \neq 0$, or $\varphi(x) \neq 0$ almost everywhere on Ω_r^0, thus $A_{n,k}(x,\theta) = 0$ almost everywhere on Ω_r^0 for every $(k,\theta) \in \mathbb{N}_n \times \mathbb{T}$. Since $A_{n,k}(x,\theta)$ is continuous with respect to x over Ω_r^0, then

$$\forall (x,\theta) \in \Omega_r^0 \times \mathbb{T}, \quad \forall k \in \mathbb{N}_n : \quad A_{n,k}(x,\theta) = 0. \tag{45}$$

The condition (45) for $k = 0$ with (41) gives (8), (45) for $k = 1$ with (42) gives (11). We use the information that $v \otimes w \neq 0$ if and only if $v \neq 0$ and $w \neq 0$, the condition (45) for $k = n$ with (3)-(6)-(43) gives (9). Now we prove (10): For $j \geq 1$ the integer $n \geq 4$ can be decomposed as

$$n = q(j+1) + \delta_0, \quad q \in \mathbb{N}, \quad \delta_0 \in \mathbb{N}_j. \tag{46}$$

Let $\mathbf{e} = [n/2]$, the integer part of $n/2$, so

$$n = 2\mathbf{e} + \delta, \quad \delta \in \{0,1\}, \quad \mathbf{e} \in \mathbb{N}, \quad \mathbf{e} \geq 2, \tag{47}$$

$$\mathbf{e} \leq \frac{n}{2} < \mathbf{e} + 1. \tag{48}$$

For $1 \leq j \leq \mathbf{e} - 1$ the condition (46) gives $0 \leq \delta_0 \leq \mathbf{e} - 1$, this with (47) leads to

$$q = \frac{n - \delta_0}{j+1} \geq \frac{(2\mathbf{e} + \delta) - (\mathbf{e} - 1)}{\mathbf{e}} > 1, \quad q = \frac{n - \delta_0}{j+1} \leq \frac{n}{2} \leq n - 1,$$

then $2 \leq q \leq n-1$, so we can work with $A_{n,q}$ defined as in (44). Let $(\gamma_1, \gamma_2) \in \mathbb{N}^2$ and $B_q = \nabla\varphi^T (D_x w)^{\gamma_1} A_{n,q} (D_x w)^{\gamma_2} \partial_\theta w$, we use (44)-(45)-(46) we obtain

$$\left.\begin{aligned}
&B_q = \sum_{(\alpha_0,\cdots,\alpha_q) \in \mathcal{S}_{n,q}} C^{\beta_0,\cdots,\beta_q}_{\nabla\varphi, D_x w, \partial_\theta w} = 0,\\
&\beta_0 = \alpha_0 + \gamma_1, \quad \beta_q = \alpha_q + \gamma_2, \quad \forall i \in \mathbb{N}^*_{q-1} : \quad \beta_i = \alpha_i,\\
&\mathcal{S}_{n,q} = \left\{(\alpha_0, \cdots, \alpha_q) \in \mathbb{N}^{q+1} : \quad \sum_{i=0}^{q} \alpha_i = qj + \delta_0\right\},\\
&C^{\beta_0,\cdots,\beta_q}_{\nabla\varphi, D_x w, \partial_\theta w} = \prod_{i=0}^{q} \nabla\varphi \cdot (D_x w)^{\beta_i} \partial_\theta w.
\end{aligned}\right\} \tag{49}$$

• *First step.* We prove by induction (see Lemma 2) that

$$\forall j \in \mathbb{N}^*_{\mathbf{e}-1} : \quad \nabla\varphi \cdot (D_x w)^j \partial_\theta w = 0. \tag{50}$$

Let $j = 1$, the condition (46) gives $n = 2q + \delta_0$ with $\delta_0 \in \{0,1\}$. We select (γ_1, γ_2) such as $\gamma_1 + \gamma_2 + \delta_0 = 1$, we combine this with the expression of $\mathcal{S}_{n,q}$ and β_i (that appears at the level of (49)) we obtain

$$\sum_{i=0}^{q} \beta_i = q + 1. \tag{51}$$

If there exists $l \in \mathbb{N}_q$ such that $\beta_l > 1$, the condition (51) ensures the existence of $k \in \mathbb{N}_q$ such that $\beta_k = 0$. This with (9) gives $C^{\beta_0,\cdots,\beta_q}_{\nabla\varphi, D_x w, \partial_\theta w} = 0$. There remains one case corresponding to $\beta_i = 1$ for every $i \in \mathbb{N}_q$ (*see Remark* 2 *part 1. for the explication of the existence of such case*). Thus, the condition (49) becomes as

$$0 = B_q = C^{1,\cdots,1}_{\nabla\varphi, D_x w, \partial_\theta w} = [\nabla\varphi \cdot (D_x w) \partial_\theta w]^{q+1},$$

hence (50) for $j = 1$.

Let $j \in \mathbb{N}$ satisfying $2 \leq j \leq \mathbf{e} - 1$, we suppose that

$$\forall l \in \mathbb{N}^*_{j-1} : \quad \nabla\varphi \cdot (D_x w)^l \partial_\theta w = 0. \tag{52}$$

We select (γ_1, γ_2) such that $\gamma_1 + \gamma_2 + \delta_0 = j$ and we use the constraint (49), precisely the expression of $\mathcal{S}_{n,q}$ and β_i, to obtain

$$\sum_{i=0}^{q} \beta_i = (q+1)j. \tag{53}$$

If there exists $l \in \mathbb{N}_q$ such that $\beta_l > j$, the condition (53) ensures the existence of $k \in \mathbb{N}_q$ such that $\beta_k < j$. This with (9)-(52) gives $C^{\beta_0,\cdots,\beta_q}_{\nabla\varphi,D_x w,\partial_\theta w} = 0$. There remains one case corresponding to $\beta_i = j$ for every $j \in \mathbb{N}_q$ (*see Remark* 2 *part 1. for the explication of the existence of such case*), so the condition (49) becomes as

$$0 = B_q = C^{j,\cdots,j}_{\nabla\varphi,D_x w,\partial_\theta w} = \Big[\nabla\varphi \cdot (D_x w)^j \partial_\theta w\Big]^{q+1}.$$

This leads to $\nabla\varphi \cdot (D_x w)^j \partial_\theta w = 0$. In conclusion, we proved that for $j = 1$ we have $\nabla\varphi \cdot (D_x w)^j \partial_\theta w = 0$, for $2 \le j \le \mathbf{e} - 1$ we supposed (52) and we obtained $\nabla\varphi \cdot (D_x w)^j \partial_\theta w = 0$. Lemma 2 ensures that (50) holds.
• *Second step.* Let $(\gamma_1, \gamma_2) \in \mathbb{N}^2$ and $B_2 = \nabla\varphi^T (D_x w)^{\gamma_1} A_{n,2} (D_x w)^{\gamma_2} \partial_\theta w$. We use the conditions (44)-(45)-(47) to obtain

$$\left.\begin{aligned}
&B_2 = \sum_{(\alpha_0,\alpha_1,\alpha_2)\in\mathcal{S}_{n,2}} C^{\beta_0,\beta_1,\beta_2}_{\nabla\varphi,D_x w,\partial_\theta w} = 0,\\
&\beta_0 = \alpha_0 + \gamma_1, \quad \beta_1 = \alpha_1, \quad \beta_2 = \alpha_2 + \gamma_2,\\
&\mathcal{S}_{n,2} = \Big\{(\alpha_0,\alpha_1,\alpha_2) \in \mathbb{N}^3 : \sum_{i=0}^{2} \alpha_i = 2\mathbf{e} + \delta - 2\Big\},\\
&C^{\beta_0,\beta_1,\beta_2}_{\nabla\varphi,D_x w,\partial_\theta w} = \prod_{i=0}^{2} \nabla\varphi \cdot (D_x w)^{\beta_i} \partial_\theta w.
\end{aligned}\right\} \tag{54}$$

We select (γ_1, γ_2) such that $\gamma_1 + \gamma_2 + \delta = \mathbf{e} + 2$, we use (54), precisely the expressions of $\mathcal{S}_{n,2}$ and β_i and we obtain

$$\sum_{i=0}^{2} \beta_i = 3\mathbf{e}. \tag{55}$$

If for some $l \in \{0, 1, 2\}$ we have $\beta_l > \mathbf{e}$, the condition (55) ensures the existence of $k \in \{0, 1, 2\}$ such that $\beta_k < \mathbf{e}$. This with (9)-(50) gives $C^{\beta_0,\beta_1,\beta_2}_{\nabla\varphi,D_x w,\partial_\theta w} = 0$. There remains one case corresponding to β_0=β_1=β_2=$\mathbf{e}$ (*see Remark* 2 *part 2. for the explication of the existence of such case*). Then (54) becomes $0 = B_2 = C^{\mathbf{e},\mathbf{e},\mathbf{e}}_{\nabla\varphi,D_x w,\partial_\theta w} = \big[\nabla\varphi \cdot (D_x w)^{\mathbf{e}} \partial_\theta w\big]^3$, this with (48)-(50) yields (10).

Remark 2 1. For $j \in \mathbb{N}^*_{\mathbf{e}-1}$, the existence of the case $\beta_i = j$ for $i \in \mathbb{N}_q$ is possible by the following construction: Once (j, q, δ_0) fixed at the level of the decomposition (46), we select α_0 such that $\delta_0 \le \alpha_0 \le j$. Let $\gamma_1 = j - \alpha_0$, $\gamma_2 = \alpha_0 - \delta_0$ and $\alpha_q = \delta_0 + j - \alpha_0$. With this construction we have $\gamma_1 \ge 0$, $\gamma_2 \ge 0$, $\alpha_q \ge 0$ and $\gamma_1 + \gamma_2 + \delta_0 = j$. Then there exist $\gamma_1, \gamma_2, \alpha_1, \cdots, \alpha_{q-1} \in \mathbb{N}$ such that

$$\beta_0 = \alpha_0 + \gamma_1 = j, \quad \beta_q = \alpha_q + \gamma_2 = j, \quad \forall i \in \mathbb{N}^*_{q-1} : \quad \beta_i = \alpha_i = j.$$

2. For $j = \mathbf{e}$, the existence of the case $\beta_1 = \beta_2 = \beta_3 = \mathbf{e}$ is possible by the following construction: See (47), let $0 \le \alpha_0 \le \mathbf{e}$, $\alpha_1 = \mathbf{e}$, $\gamma_1 = \mathbf{e} - \alpha_0$, $\gamma_2 = 2 - \delta + \alpha_0$ and $\alpha_2 = \mathbf{e} - \alpha_0 + \delta - 2$. Then, $\gamma_1 \ge 0$, $\gamma_2 \ge 0$, $\alpha_2 \ge 0$ and

$$\beta_0 = \alpha_0 + \gamma_1 = \mathbf{e}, \quad \beta_2 = \alpha_2 + \gamma_2 = \mathbf{e}, \quad \beta_1 = \alpha_1 = \mathbf{e}, \quad \sum_{i=0}^{2} \beta_i = 3\mathbf{e}.$$

2.1.2 Second sense of the proof (the only if part).

We suppose that on $\Omega^0_r \times \mathbb{T}$ the couple (φ, w) satisfies the conditions (8)-$\cdots$-(11). The aim is to prove (7); we start with the following lemma.

Lemma 3 *Let* (φ, w) *satisfies* (8)-$\cdots$-(11) *on* $\Omega^0_r \times \mathbb{T}$. *Then on* $\Omega^0_r \times \mathbb{T}$ *we have*

$$\forall k \in \mathbb{N}^*_{n-1} : \quad \nabla\varphi \cdot (D_x w)^k \partial_\theta w = 0. \tag{56}$$

Proof Let $i \in \mathbb{N}$ such that $i \leq (n-1)/2$, so $n-1-2i \geq 0$, we multiply the equation (11) at the left by $\nabla\varphi^T (D_x w)^{n-1-2i}$ and at the right by $\partial_\theta w$ we get

$$\forall i \in \mathbb{N}, \quad i \leq (n-1)/2 :$$
$$\sum_{\alpha=0}^{n-1} \Big[\nabla\varphi \cdot (D_x w)^{n-1-(2i-\alpha)} \partial_\theta w\Big]\Big[\nabla\varphi \cdot (D_x w)^{n-1-\alpha} \partial_\theta w\Big] = 0. \tag{57}$$

For every integer $\alpha \geq 2i+1$ we have $n-1-(2i-\alpha) \geq n$, we use (8) we obtain $(D_x w)^{n-1-(2i-\alpha)} = 0$. This transforms (57) to

$$\forall i \in \mathbb{N}, \quad i \leq (n-1)/2 :$$
$$\sum_{\alpha=0}^{2i} \Big[\nabla\varphi \cdot (D_x w)^{n-1-(2i-\alpha)} \partial_\theta w\Big]\Big[\nabla\varphi \cdot (D_x w)^{n-1-\alpha} \partial_\theta w\Big] = 0. \tag{58}$$

We prove by induction (see Lemma 2) that

$$\forall j \in \mathbb{N}, \quad 0 \leq j \leq n/2 - 1 : \quad \nabla\varphi \cdot (D_x w)^{n-1-j} \partial_\theta w = 0. \tag{59}$$

The equation (58) with the choice $i = 0$ gives $\nabla\varphi \cdot (D_x w)^{n-1} \partial_\theta w = 0$. This is (59) for $j = 0$. We suppose now that for some integer $i \leq (n/2) - 1$, we have

$$\forall j \in \mathbb{N}_{i-1} : \quad \nabla\varphi \cdot (D_x w)^{n-1-j} \partial_\theta w = 0. \tag{60}$$

The condition (60), with the change $j = \alpha$, transforms the equation (58) into

$$\sum_{\alpha=i}^{2i} \Big[\nabla\varphi \cdot (D_x w)^{n-1-(2i-\alpha)} \partial_\theta w\Big]\Big[\nabla\varphi \cdot (D_x w)^{n-1-\alpha} \partial_\theta w\Big] = 0. \tag{61}$$

For $i+1 \leq \alpha \leq 2i$ we have $0 \leq 2i-\alpha \leq i-1$, this with (60) (for $j = 2i-\alpha$) gives

$$\forall \alpha \in \mathbb{N}, \quad i+1 \leq \alpha \leq 2i : \quad \nabla\varphi \cdot (D_x w)^{n-1-(2i-\alpha)} \partial_\theta w = 0. \tag{62}$$

The conditions (62)-(61) yield $[\nabla\varphi \cdot (D_x w)^{n-1-i} \partial_\theta w]^2 = 0$, hence (59) for $j = i$.

In conclusion: we proved that $\nabla\varphi \cdot (D_x w)^{n-1-j} \partial_\theta w = 0$ for $j = 0$, we supposed that $\nabla\varphi \cdot (D_x w)^{n-1-j} \partial_\theta w = 0$ holds for $j \in \mathbb{N}^*_{i-1}$ such that $i \leq (n/2) - 1$ and we proved that $\nabla\varphi \cdot (D_x w)^{n-1-j} \partial_\theta w = 0$ for $j = i$. Lemma 2 ensures that (59) holds. Now we do the change $k = n-1-j$, the condition (59) becomes as

$$\forall k \in \mathbb{N}, \quad \frac{n}{2} \leq k \leq n-1 : \quad \nabla\varphi \cdot (D_x w)^k \partial_\theta w = 0,$$

this with the condition (10) gives (56). □

Now we give the proof of the second sense (the only if part) of Theorem 1. We use Lemma 3, the condition (9) and the expression of $A_{n,k}$ that appears at the level of the condition (44) we obtain

$$\forall k \in \mathbb{N}^*_{n-1} \setminus \{1\} : \quad A_{n,k} = 0. \tag{63}$$

The conditions (8)-(41) give $A_{n,0} = 0$, (11)-(42) give $A_{n,1} = 0$ and (9)-(43) give $A_{n,n} = 0$. This with (63) transform (40) to (7).

2.2 Existence of compatible couples

Let $\Phi_1, \cdots, \Phi_{n-3} \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{T}, \mathbb{R})$, we define the matrix Φ^* as

$$\Phi^* = \big(a_{ij}\big)_{1\le i,j\le n-3}, \quad a_{ij} = \partial_j \Phi_i. \tag{64}$$

Proposition 3 *Let $h_0, h_1, M_0 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and $\mathcal{X}(s,t) \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ such that*

$$\forall r \in \mathbb{R}: \quad M_0'(r) \neq 0. \tag{65}$$

Then there exists $r_0 > 0$ and $\widetilde{\varphi}(u,v,w) \in \mathcal{C}^1(B(r_0), \mathbb{R})$, with $B(r_0) \subset \mathbb{R}^3$ the ball of centre zero and radius r_0, such that on $B(r_0)$ we have

$$\partial_v \widetilde{\varphi} + h_1(M_0(\widetilde{\varphi}))\partial_u \widetilde{\varphi} = 0, \tag{66}$$

$$\partial_w \widetilde{\varphi} + [h_0'(M_0(\widetilde{\varphi})) + h_1'(M_0(\widetilde{\varphi}))\mathcal{X}(w, \widetilde{\varphi})]\partial_u \widetilde{\varphi} = 0, \tag{67}$$

$$\partial_u \widetilde{\varphi}(u,v,w) \neq 0. \tag{68}$$

Let $\Phi_1, \cdots, \Phi_{n-3} \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{T}, \mathbb{R})$ and Φ^* as in (64) such that

$$\forall (\tau, \theta) \in \mathbb{R}^n \times \mathbb{T}: \quad [\Phi^*(\tau,\theta)]^{n-3} = 0, \tag{69}$$

$$\exists i \in \mathbb{N}^*_{n-3}: \quad \partial_\theta \Phi_i(\tau, \theta) \neq 0. \tag{70}$$

Let $a_1, \cdots, a_{n-3} \in \mathbb{R}$, $\varphi \in \mathcal{C}^1(\Omega_r^0, \mathbb{R})$ and $\psi_1, \cdots, \psi_{n-2} \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}, \mathbb{R})$ such as

$$r = r_0(1 + a_1^2 + \cdots + a_{n-3}^2)^{-1/2}, \tag{71}$$

$$\varphi(x) = \widetilde{\varphi}\Big(x_n - \sum_{j=1}^{n-3} a_j x_j, x_{n-2}, x_{n-1}\Big), \tag{72}$$

$$\psi_{n-2}(x,\theta) = \mathcal{X}(x_{n-1}, \varphi(x)), \tag{73}$$

$$\forall i \in \mathbb{N}^*_{n-3}: \quad \psi_i(x,\theta) = \Phi_i(x_1, \cdots, x_{n-2}, \psi_{n-2}(x,\theta), \varphi(x), \theta). \tag{74}$$

Let the vectorial function $w \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}, \mathbb{R}^n)$ be defined as

$$w = \mathcal{W}(\psi), \quad \psi = (\psi_1, \cdots, \psi_{n-2}, \psi_{n-1}), \quad \psi_{n-1} = M_0(\varphi), \tag{75}$$

such that

$$\mathcal{W}(y_1, \cdots, y_{n-1}) = (y_1, \cdots, y_{n-2}, y_{n-1}, L(y_1, \cdots, y_{n-2}, y_{n-1}))^T, \tag{76}$$

$$L(y_1, \cdots, y_{n-1}) = h_0(y_{n-1}) + \Big(\sum_{k=1}^{n-3} a_k y_k\Big) + h_1(y_{n-1})y_{n-2}. \tag{77}$$

Then the couple (φ, w) is a compatible one on $\Omega_r^0 \times \mathbb{T}$.

Proof Let $z \in \mathbb{R}$ a parameter, X_z and Y_z the vector fields defined over $\mathbb{R}^3$ as

$$X_z = \partial_v + h_1(M_0(z))\partial_u, \quad Y_z = \partial_w + [h_0'(M_0(z)) + h_1'(M_0(z))\mathcal{X}(w,z)]\partial_u. \tag{78}$$

The vector fields X_z, Y_z are linearly independent in each point of $\mathbb{R}^3$. Let the Lie bracket $[X_z, Y_z] = X_z Y_z - Y_z X_z$. The calculus gives $[X_z, Y_z] = 0$ so $X_z, Y_z \in Span\{X_z, Y_z\}$. Frobenius Theorem [1] ensures the existence of $\mathcal{C}^1$ no constant function $F_z(u,v,w)$ defined on $B(r_1) \subset \mathbb{R}^3$, for some $r_1 > 0$, satisfying

$$X_z F_z = 0, \quad Y_z F_z = 0. \tag{79}$$

The function $F_z(u,v,w) + f(z)$ satisfies (79) for any function f. Thus, we can consider the case where $G(u,v,w,z) = F_z(u,v,w) + f(z)$ and $\partial_z G(u,v,w,z) \neq 0$. On the other hand, the information that F_z is not constant combined with (78)-(79) ensures that $\partial_u F_z(u, v, w) \neq 0$, thus the function $G \in \mathcal{C}^1(B(r_1) \times \mathbb{R}, \mathbb{R})$ satisfying

$$X_z G(u, v, w, z) = 0, \quad Y_z G(u, v, w, z) = 0, \tag{80}$$

$$\partial_u G(u, v, w, z) \neq 0, \quad \partial_z G(u, v, w, z) \neq 0. \tag{81}$$

The second part of (81) and the implicit function theorem leads to the existence of $r_0 \in \mathbb{R}$, with $0 < r_0 < r_1$, and $\widetilde{\varphi} \in \mathcal{C}^1(B(r_0), \mathbb{R})$ satisfies

$$\forall (u, v, w) \in B(r_0) : \quad G(u, v, w, \widetilde{\varphi}(u, v, w)) = 0. \tag{82}$$

We apply X_z (respectively Y_z) defined as in (78) to the equation (82), we replace z by $\widetilde{\varphi}(u, v, w)$ and we use the first part (respectively the second one) of (80) to recover (66) (respectively (67)). Applying ∂_u to (82) we obtain $\partial_u G + \partial_z G \partial_u \widetilde{\varphi} = 0$. This with (81) gives (68). Let $u_1 = (1, -a_1, \cdots, -a_{n-3})^T$ and $u_2 = (x_n, x_1, \cdots, x_{n-3})^T$. The Cauchy-Schwartz inequality gives $|u_1 \cdot u_2|^2 \leq \|u_1\|^2 \|u_2\|^2$, hence

$$\Big| x_n - \sum_{k=1}^{n-3} a_j x_j \Big|^2 \leq \Big(1 + \sum_{k=1}^{n-3} a_k^2\Big)\Big(x_n^2 + \sum_{k=1}^{n-3} x_k^2\Big), \tag{83}$$

thus, for $x \in \Omega_r^0$ the conditions (71)-(83) give $(x_n - \sum_{j=1}^{n-3} a_j x_j, x_{n-2}, x_{n-1}) \in B(r_0)$. This guarantee that (72) is well defined on Ω_r^0.

In the following: $\partial_k \Phi_i$ computed at $(x_1, \cdots, x_{n-2}, \psi_{n-2}(x, \theta), \varphi(x), \theta)$, $\partial_k \mathcal{W}$ computed at $\psi(x, \theta)$ such that ψ as in (75), $\partial_t \mathcal{X}$ and $\partial_s \mathcal{X}$ computed at $(x_{n-1}, \varphi(x))$. Let $V_1, \cdots, V_{n-2} \in \mathbb{R}^n$ be defined as

$$\forall i \in \mathbb{N}_{n-3}^* : \quad V_i = (\partial_1 \Phi_i, \cdots, \partial_{n-2} \Phi_i, 0, 0)^T, \tag{84}$$

$$V_{n-2} = (0, \cdots, 0, \partial_s \mathcal{X}, 0)^T. \tag{85}$$

The constraints (73)-(74) give

$$\forall i \in \mathbb{N}_{n-3}^* : \quad \nabla \psi_i = V_i + \partial_{n-1} \Phi_i \nabla \psi_{n-2} + \partial_n \Phi_i \nabla \varphi, \tag{86}$$

$$\nabla \psi_{n-2} = V_{n-2} + \partial_t \mathcal{X} \nabla \varphi. \tag{87}$$

The conditions (66)-(67)-(72)-(73)-(76)-(77) yield

$$\forall k \in \mathbb{N}_{n-1}^* : \quad \nabla \varphi \cdot \partial_k \mathcal{W} = 0. \tag{88}$$

The conditions (76)-(85)-(86)-(88) give

$$\forall j \in \mathbb{N}_{n-2}^* : \quad \nabla \psi_{n-2} \cdot \partial_j \mathcal{W} = 0, \quad \nabla \psi_{n-2} \cdot \partial_{n-1} \mathcal{W} = \partial_s \mathcal{X}. \tag{89}$$

The conditions (76)-(84)-(87)-(88)-(89) lead to

$$\forall i \in \mathbb{N}_{n-3}^*, \quad \forall j \in \mathbb{N}_{n-2}^* : \quad \nabla \psi_i \cdot \partial_j \mathcal{W} = \partial_j \Phi_i, \tag{90}$$

$$\forall i \in \mathbb{N}_{n-3}^* : \quad \nabla \psi_i \cdot \partial_{n-1} \mathcal{W} = \partial_s \mathcal{X} \partial_{n-1} \Phi_i. \tag{91}$$

Let ψ as in (75) and $\mathcal{B} = D_x \psi(x, \theta) D_y \mathcal{W}(\psi(x, \theta))$. We use (88)- $\cdots$ -(91) to obtain

$$\mathcal{B} = \begin{pmatrix} \partial_1 \Phi_1 & \cdots & \partial_{n-3} \Phi_1 & \partial_{n-2} \Phi_1 & \partial_s \mathcal{X} \partial_{n-1} \Phi_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \partial_1 \Phi_{n-3} & \cdots & \partial_{n-3} \Phi_{n-3} & \partial_{n-2} \Phi_{n-3} & \partial_s \mathcal{X} \partial_{n-1} \Phi_{n-3} \\ 0 & \cdots & 0 & 0 & \partial_s \mathcal{X} \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

The conditions (64)-(69) affirm that Φ^* is of size $n-3$ and nilpotent, then its characteristic polynomial is $(-\lambda)^{n-3}$. Thus, the characteristic polynomial of $\mathcal{B}$ is $P(\lambda) = (-\lambda)^{n-1}$. We use the Cayley-Hamilton theorem [19] we get $P(\mathcal{B})$=0, therefore $[D_x \psi D_y \mathcal{W}(\psi)]^{n-1} = 0$. On the other hand, the first part of (75) gives $(D_x w)^n = D_y \mathcal{W}(\psi)[D_x \psi D_y \mathcal{W}(\psi)]^{n-1} D_x \psi$, thus

$$(D_x w)^n = 0. \tag{92}$$

We use the expression (75) to obtain

$$D_x w = \sum_{k=1}^{n-1} \partial_k \mathcal{W}(\psi) \otimes \nabla \psi_k, \quad \partial_\theta w = \sum_{k=1}^{n-2} \partial_k \mathcal{W}(\psi) \partial_\theta \psi_k. \tag{93}$$

The first part of (93) with (88) yields

$$\nabla \varphi^T D_x w = 0. \tag{94}$$

The information that $\partial_\theta w \otimes \nabla \varphi = \partial_\theta w \, \nabla \varphi^T$ and the condition (94) give

$$\sum_{\alpha=0}^{n-1} (D_x w)^\alpha (\partial_\theta w \otimes \nabla \varphi)(D_x w)^{n-1-\alpha} = [(D_x w)^{n-1} \partial_\theta w] \, \nabla \varphi^T. \tag{95}$$

Let $(x_0, \theta_0) \in \Omega_r^0 \times \mathbb{T}$. The condition (92) gives $rank\ D_x w(x_0, \theta_0) \leq n-1$.

• If $rank\ D_x w(x_0, \theta_0) = n-1$: The condition (92) and the first part of (93) leads to

$$\forall v \in \mathbb{R}^n: \quad \sum_{k=1}^{n-1} [v \cdot (D_x w)^{n-1} \partial_k \mathcal{W}(\psi)] \, \nabla \psi_k^T = 0. \tag{96}$$

The conditions (75)-(76) and the fact that $rank\ D_x w(x_0, \theta_0) = n-1$ lead to $\nabla \psi_1(x_0, \theta_0), \cdots, \nabla \psi_{n-2}(x_0, \theta_0), \nabla \psi_{n-1}(x_0)$ linearly independent. Thus (96) gives $(D_x w(x_0, \theta_0))^{n-1} \partial_k \mathcal{W}(\psi(x_0, \theta_0)) = 0$ for $k \in \mathbb{N}_{n-1}^*$. This with the second part of (93) yields $(D_x w(x_0, \theta_0))^{n-1} \partial_\theta w(x_0, \theta_0) = 0$.

• If $rank\ D_x w(x_0, \theta_0) \leq n-2$: The condition (92) gives $D_x w(x_0, \theta_0)$ nilpotent, then $D_x w(x_0, \theta_0)$ has only one eigenvalue $\lambda = 0$ and $E_0 = Ker\ D_x w(x_0, \theta_0)$ is the corresponding eigenspace. We use the rank theorem and the fact that $rank\ D_x w(x_0, \theta_0) \leq n-2$ to get $dim\ E_0 \geq 2$. The integer $dim\ E_0$ indicates the number of the Jordan blocks in the Jordan canonical form of $D_x w(x_0, \theta_0)$, see for instance [19], so there are at least two blocks. Therefore, the largest block is of size $\gamma \leq n-1$. The integer γ, in the Jordan canonical form, represents the degree of the minimal polynomial of $D_x w(x_0, \theta_0)$, hence $(D_x w(x_0, \theta_0))^\gamma = 0$. Or $\gamma \leq n-1$, then $(D_x w(x_0, \theta_0))^{n-1} \partial_\theta w(x_0, \theta_0) = 0$.

The previous discussion leads to $(D_x w)^{n-1} \partial_\theta w \equiv 0$ on $\Omega_r^0 \times \mathbb{T}$. This transforms (95) to (11), (68)-(72) gives (3), (70)-(74)-(75) give (6), (92) is same as (8), (75)-(88) ensure (9), (94) ensures (10). Thus (φ, w) is compatible on $\Omega_r^0 \times \mathbb{T}$. □

Example 1 Let $\Phi_1, \cdots, \Phi_{n-3} \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{T}, \mathbb{R})$ satisfy on $\mathbb{R}^n \times \mathbb{T}$ the condition

$$\forall i, j \in \mathbb{N}_{n-3}^*, \quad 1 \leq j \leq i: \quad \partial_j \Phi_i(\tau, \theta) = 0, \quad \partial_\theta \Phi_1(\tau, \theta) \neq 0. \tag{97}$$

We use the first part of (97) and the notation (64) to obtain that the matrix Φ^* is a triangular one with zero diagonal, so it is nilpotent. This gives (69). We select any function $h_0, h_1, M_0 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and $\mathcal{X} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ such that M_0 satisfy (65), we select a solution $\widetilde{\varphi}$ of (66)-(67)-(68), we select $a_1, \cdots, a_{n-3} \in \mathbb{R}$ and we build $\varphi, \psi_1, \cdots, \psi_{n-2}$ as in (72)-(73)-(74). Then (φ, w), such that w as in (75)-(76)-(77), is a compatible couple. In addition, the conditions (65)-(75)-(76) give $rank\ D_x w = dim\ Span\ \{\nabla \psi_1, \cdots, \nabla \psi_{n-2}, \nabla \varphi\}$. This with (86)-(87) leads to

$$rank\ D_x w = dim\ Span\ \{V_1, \cdots, V_{n-2}, \nabla \varphi\}, \tag{98}$$

such that $V_1, \cdots, V_{n-2}$ as in (84)-(85). Thus, $rank\ D_x w$ depends on the choice of $\Phi_1, \cdots, \Phi_{n-3}, \mathcal{X}$. For instance, let $\mathcal{X}$ satisfies $\partial_s \mathcal{X}(s, t) \neq 0$, $m \in \mathbb{N}_{n-3}^* \setminus \{1\}$ and

$$\forall (\tau, \theta) \in \mathbb{R}^n \times \mathbb{T}, \quad \forall i \in \mathbb{N}_{m-1}^*: \quad \Phi_i(\tau, \theta) = 0, \tag{99}$$

$$\forall (\tau, \theta) \in \mathbb{R}^n \times \mathbb{T}, \quad \forall i \in \mathbb{N}, \quad m \leq i \leq n-3: \quad \partial_{i+1} \Phi_i(\tau, \theta) \neq 0. \tag{100}$$

We get $rank\ D_x w = n - m$. Without involving the condition (99), we consider the condition (100) with $m = 1$ we get $rank\ D_x w = n - 1$.

3 Propagation problem, proof of Theorem 2

Let $\theta \in \mathbb{T}$, the characteristics $X_\theta(t, y)$ of the Cauchy's problem (12) are given by the expression $X_\theta(t, y) = y + tw(y, \theta)$. Using (8), we deduce that $D_y X_\theta(t, y)$ is invertible for $y \in \Omega_r^0$ with $(\theta, t) \in \mathbb{T} \times \mathbb{R}_+$. Thus, the $X_{(\theta,t)}$ defined as

$$\begin{aligned} X_{(\theta,t)} \;:\; \Omega_r^0 &\longrightarrow X_\theta(t, \Omega_r^0) \\ y &\longmapsto X_\theta(t, y), \end{aligned}$$

is a $\mathcal{C}^1-$diffeomorphism. Let $V = Sup_{(y,\theta)\in\Omega_r^0\times\mathbb{T}}\|w(y, \theta)\|$, $T_0 = r/V$ and for $(t, x, \theta) \in \Omega_r^{T_0} \times \mathbb{T}$ fixed we define $K_{(t,x,\theta)}(y) = x - tw(y, \theta)$ with $y \in \Omega_r^0$. See that $\|K_{(t,x)}(y)\| \leq \|x\| + tV \leq r$, thus $K_{(t,x)}(y) \in \Omega_r^0$, Brouwer fixed point theorem leads to the existence of $y \in \Omega_r^0$ such that $y = x - tw(y, \theta)$ and then $x = y + tw(y, \theta) = X_{(\theta,t)}(y)$. We combine this with the fact that $X_{(\theta,t)}$ is a $\mathcal{C}^1-$diffeomorphism, we deduce that for every $(t, x, \theta) \in \Omega_r^{T_0} \times \mathbb{T}$ there exists a unique $(t, y, \theta) \in [0, T_0] \times \Omega_r^0 \times \mathbb{T}$ satisfies $x = X_{(\theta,t)}(y)$. Hence, the solution of (12) is defined as $\mathbf{W}(t, x, \theta) = w\left((X_{(\theta,t)})^{-1}(x), \theta\right)$ for $(t, x, \theta) \in \Omega_r^{T_0} \times \mathbb{T}$. On the other hand, (13) involves a linear equation and an initial data $\varphi \in \mathcal{C}^1(\Omega_r^0; \mathbb{R})$, by integrating (12)-(13) along the associated characteristics we obtain, on $\Omega_r^{T_0} \times \mathbb{T}$, the expressions

$$\mathbf{W}(t, x, \theta) = w(x - t\mathbf{W}(t, x, \theta), \theta), \quad \Phi(t, x) = \varphi(x - t\mathbf{W}(t, x, \theta)). \tag{101}$$

The condition (8) ensures that the matrix $I + tD_x w$ is invertible for $(t, x, \theta) \in \Omega_r^{T_0} \times \mathbb{T}$, then on $\Omega_r^{T_0} \times \mathbb{T}$ we have

$$[I + tD_x w]^{-1} = \sum_{k=0}^{n-1}(-t)^k(D_x w)^k. \tag{102}$$

On $\Omega_r^{T_0} \times \mathbb{T}$, the expressions (101)-(102) lead to the following expressions

$$D_x\mathbf{W} = [I + tD_x w]^{-1}\, D_x w = \sum_{k=0}^{n-1}(-t)^k(D_x w)^{k+1}, \tag{103}$$

$$\partial_\theta\mathbf{W} = [I + tD_x w]^{-1}\, \partial_\theta w = \sum_{k=0}^{n-1}(-t)^k(D_x w)^k\partial_\theta w, \tag{104}$$

$$\nabla\Phi^T = \nabla\varphi^T\; [I - t\, D_x\mathbf{W}]\,. \tag{105}$$

The the second part of (101) gives $\partial_\theta\Phi = -t\,\nabla\varphi \cdot \partial_\theta\mathbf{W}$. We use (104) we get

$$\partial_\theta\Phi = \sum_{k=0}^{n-1}(-t)^{k+1}\,\nabla\varphi \cdot (D_x w)^k\partial_\theta w. \tag{106}$$

Such that: At the level of (103)- $\cdots$ -(106) the functions $\partial_\theta\Phi$, $\partial_\theta\mathbf{W}$, $D_x\mathbf{W}$ are evaluated at (t, x, θ); $D_x w$, $\partial_\theta w$ are evaluated at $(x - t\mathbf{W}(t, x, \theta), \theta)$; and $\nabla\varphi$ evaluated at $(x - t\mathbf{W}(t, x, \theta))$.

- *We prove that* Φ *is independent of* θ. The information (106) combined with (9) and the Lemma 3 (condition (56)) leads to $\partial_\theta\Phi \equiv 0$.
- *We prove that* u^ε *defined as in* (18) *is solution of* (1)-(2)-(5). The couple (φ, w) is a compatible one, thus h^ε defined as in (5) satisfies (7) (see Definition 1). On the other hand, the systems (12)-(13) ensure that u^ε defined as in (18) is a solution of (1)-(5) on $\Omega_r^{T_0}$. We combine this with (7) and Theorem 2.6 of [6] to obtain that $u^\varepsilon(t, x)$ satisfies (2).
- *We prove that we have* (14)- $\cdots$ -(17). The expression (102) ensures that the matrices $[I + tD_x w]^{-1}$ and $D_x w$ commute, thus the condition (103) gives $(D_x\mathbf{W})^n = ([I + tD_x w]^{-1})^n(D_x w)^n$. This with (8) leads to

$$\forall(t, x, \theta) \in \Omega_r^{T_0} \times \mathbb{T} : \quad (D_x\mathbf{W})^n(t, x, \theta) = 0. \tag{107}$$

  The condition (107) implied that on $\Omega_r^{T_0} \times \mathbb{T}$ all the eigenvalues of $I + t\, D_x\mathbf{W}$ are equal to one, thus $I + t\, D_x\mathbf{W}$ is invertible on $\Omega_r^{T_0} \times \mathbb{T}$. This with (3)-(105) leads to $\nabla\Phi(t, x) \neq 0$.

The information that $[I + tD_x w]^{-1}$ is invertible combined with (6) and (104) leads to $\partial_\theta \mathbf{W}(t, x, \theta) \neq 0$. Theorem 2.6 of [6] ensures that the Jacobian matrix of u^ε, the solution of (1)-(2)-(5) defined as in (18), is nilpotent.
In conclusion: for every $t \in [0, T_0]$ and on the domain $\Omega_r^t \times \mathbb{T}$ we have

$$\nabla\Phi(t, x) \neq 0, \quad \partial_\theta \mathbf{W}(t, x, \theta) \neq 0, \quad \big(D_x u^\varepsilon(t, x)\big)^n = 0.$$

Thus $\{u^\varepsilon(t, \cdot)\}_{\varepsilon\in]0,1]}$ is a compatible family on $\Omega_r^t \times \mathbb{T}$ (see the Definition 1). Consequently, the couple $(\Phi(t, \cdot), \mathbf{W}(t, \cdot, \cdot))$ is a compatible one. Using Theorem 1, we get that on $\Omega_r^t \times \mathbb{T}$ (with $t \in [0, T_0]$) we have (14)- $\cdots$ -(17). On the other hand, the solution of (12) satisfies $w(y, \theta) = \mathbf{W}(t, z, \theta)$ with $z = y + tw(y, \theta)$; thus, $[I - tD_x\mathbf{W}(t, z, \theta)]\, D_y w(y, \theta) = D_x \mathbf{W}(t, z, \theta)$, by using (107) we get (19).

4 The case of $n - 1$ rank

The Theorem 1, precisely the condition (8), affirms that on $\Omega_r^0 \times \mathbb{T}$ the matrix $D_x w$ is a nilpotent one, thus $rank\ D_x w = R(x, \theta) \leq n - 1$. In the case when $R(x, \theta) \equiv R \in \mathbb{N}^*_{n-1}$ on $\Omega_r^0 \times \mathbb{T}$, the constant rank theorem [2] and the compactness of the torus $\mathbb{T}$ ensures the existence of $f_1, \cdots, f_R \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}, \mathbb{R})$ and $W \in \mathcal{C}^1(\mathbb{R}^R \times \mathbb{T}, \mathbb{R}^n)$ such that w can be factorized as $w = W(f_1, \cdots, f_R, \theta)$. The Theorem 4 shows that when $R(x, \theta) = n - 1$ on $\Omega_r^0 \times \mathbb{T}$, the factorisation of the compatible couples explicitly involves the function φ.

Lemma 4 *Let (φ, w) a compatible couple on $\Omega_r^0 \times \mathbb{T}$ satisfies*

$$\forall (x, \theta) \in \Omega_r^0 \times \mathbb{T}: \quad rank\ D_x w(x, \theta) = n - 1. \tag{108}$$

Then on $\Omega_r^0 \times \mathbb{T}$ we have

$$(D_x w)^{n-1}(x, \theta) \neq 0, \quad (D_x w)^{n-1}\partial_\theta w \equiv 0, \quad \nabla\varphi^T (D_x w)^{n-1} \equiv 0. \tag{109}$$

Proof The matrix $D_x w$ is a nilpotent one, see (8), then all its eigenvalues are zero. Thus, the minimal polynomial of $D_x w$ has the expression $m(X) = X^\alpha$. Let $E_0 = Ker\ D_x w$ the eigenspace corresponding to the eigenvalue $\lambda = 0$. The condition (108) and the rank theorem lead to $dim\ E_0 = 1$. In the Jordan canonical form [19] of the matrix $D_x w$, the integer $dim\ E_0$ indicates the number of the Jordan blocks, and α (the degree of the minimal polynomial) indicates the size of the largest block. Thus, the Jordan canonical form of $D_x w$ is formed by one block, or $D_x w$ is of n size, hence $\alpha = n$. Then, the minimal polynomial of $D_x w$ is X^n, so X^{n-1} cannot annihilate $D_x w$. Hence the first part of (109). We use now the condition (9) and the Lemma 3 to we obtain that on $\Omega_r^0 \times \mathbb{T}$ we have

$$\nabla\varphi \in (span\ \mathcal{F})^\perp, \quad \mathcal{F} = \{\partial_\theta w, D_x w \partial_\theta w, \cdots, (D_x w)^{n-1}\partial_\theta w\}. \tag{110}$$

The condition (3) and the first part of (110) give $dim\ (span\ \mathcal{F})^\perp \geq 1$, or $span\ \mathcal{F} \subset \mathbb{R}^n$, then $dim\ span\ \mathcal{F} \leq n-1$. This and the fact that $\mathcal{F}$ is formed by n vectors lead to conclude that the family $\mathcal{F}$ is linearly dependent on $\Omega_r^0 \times \mathbb{T}$. Thus, for $(x_0, \theta_0) \in \Omega_r^0 \times \mathbb{T}$ fixed there exists $\alpha_0, \cdots, \alpha_{n-1} \in \mathbb{R}$, not all zeros, such that

$$\sum_{k=0}^{n-1} \alpha_k\, (D_x w(x_0, \theta_0))^k\, \partial_\theta w(x_0, \theta_0) = 0. \tag{111}$$

Let α_i the first nonzero element in $(\alpha_k)_{k=0,\cdots,n-1}$. The condition (111) becomes as

$$\sum_{k=i}^{n-1} \alpha_k\, (D_x w(x_0, \theta_0))^k \partial_\theta w(x_0, \theta_0) = 0, \quad \alpha_i \neq 0. \tag{112}$$

The information that $i \in \mathbb{N}^*_{n-1}$ gives the possibility to multiply the first part of the constraint (112) at the left by $(D_x w(x_0, \theta_0))^{n-i-1}$, this leads to

$$\sum_{k=i}^{n-1} \alpha_k\, (D_x w(x_0, \theta_0))^{n+k-i-1}\, \partial_\theta w(x_0, \theta_0) = 0, \quad \alpha_i \neq 0. \tag{113}$$

The condition (8) and the fact that $n+k-i-1 \geq n$ for $k \geq i+1$, transform (113) to $(D_x w(x_0, \theta_0))^{n-1}\, \partial_\theta w(x_0, \theta_0) = 0$. This with the information that (x_0, θ_0) is chosen arbitrarily in $\Omega_r^0 \times \mathbb{T}$ ensure the second part of (109). Now, let $\mathcal{I} \subset \mathbb{N}$ be defined as

$$\mathcal{I} = \{k \in \mathbb{N} : \quad \forall (x, \theta) \in \Omega_r^0 \times \mathbb{T}, \quad (D_x w(x, \theta))^k \partial_\theta w(x, \theta) \neq 0\}.$$

The condition (6) confirms that $0 \in \mathcal{I}$, then $\mathcal{I} \neq \emptyset$, and the second part of (109) ensures that $\mathcal{I}$ is bounded by $n-1$. Thus $\beta = max(\mathcal{I})$ exists, so on $\Omega_r^0 \times \mathbb{T}$ we have

$$(D_x w(x, \theta))^\beta \partial_\theta w(x, \theta) \neq 0, \tag{114}$$

$$\forall k \in \mathbb{N}, \quad k \geq \beta + 1 : \quad (D_x w(x, \theta))^k \partial_\theta w(x, \theta) = 0. \tag{115}$$

Recall that $\partial_\theta w \otimes \nabla\varphi = \partial_\theta w\, \nabla\varphi^T$, we multiply (11) at the left by $(D_x w)^\beta$ to obtain

$$\sum_{\alpha=0}^{n-1} [(D_x w)^{\beta+\alpha} \partial_\theta w][\, \nabla\varphi^T (D_x w)^{n-1-\alpha}] = 0. \tag{116}$$

For $\alpha \geq 1$ we have $k = \beta + \alpha \geq \beta + 1$, this with (115) transforms (116) to

$$[(D_x w)^\beta \partial_\theta w]\, [\, \nabla\varphi^T (D_x w)^{n-1}] = 0. \tag{117}$$

We multiply the condition (117) at the left by $[(D_x w)^\beta \partial_\theta w]^T$ we get

$$\|(D_x w)^\beta \partial_\theta w\|^2 [\nabla\varphi^T (D_x w)^{n-1}] = 0,$$

this with (114) leads to the third part of (109). □

Theorem 4 *Let* (φ, w) *be a compatible couple on* $\Omega_r^0 \times \mathbb{T}$ *satisfying* (108). *Then, by reducing* r*, there exists* $\mathcal{W} \in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{T}, \mathbb{R}^n)$ *satisfies*

$$dim\, span\, \{\partial_1 \mathcal{W}, \cdots, \partial_{n-1} \mathcal{W}\} = n - 1, \tag{118}$$

there exists $M \in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{T}, \mathbb{R})$ *such that*

$$\partial_{n-1} M(y_1, \cdots, y_{n-1}, \theta) \neq 0, \tag{119}$$

there exists $\psi_1, \cdots, \psi_{n-2} \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}, \mathbb{R})$ *satisfying*

$$dim\ span\ \{\nabla\psi_1, \cdots, \nabla\psi_{n-2}, \nabla\varphi\} = n - 1, \tag{120}$$

and on $\Omega_r^0 \times \mathbb{T}$ *we have*

$$w = \mathcal{W}\big(\psi_1, \cdots, \psi_{n-2}, M(\psi_1, \cdots, \psi_{n-2}, \varphi, \theta), \theta\big). \tag{121}$$

Proof Let $w = (w_1, \cdots, w_n)^T = W(w_1, \cdots, w_n)$, the condition (108) leads to

$$dim\ span\ \{\nabla w_1, \cdots, \nabla w_n\} = n - 1. \tag{122}$$

On the other hand, the fact that $D_x w = \sum_{k=1}^n \partial_k W \otimes \nabla w_k$ combined with the condition (8) gives

$$\forall v \in \mathbb{R}^n : \quad (D_x w)((D_x w)^{n-1} v) = \sum_{k=1}^{n} [\nabla w_k \cdot (D_x w)^{n-1} v]\, \partial_k W = 0. \tag{123}$$

The set $\{\partial_1 W, \cdots, \partial_n W\}$ is a basis of $\mathbb{R}^n$, so (123) gives $\nabla w_k^T (D_x w)^{n-1} \equiv 0$ for all $k \in \mathbb{N}_n^*$. We combine this with the third part of (109) we obtain

$$\{\nabla\varphi, \nabla w_1, \cdots, \nabla w_n\} \subset (Im\ (D_x w)^{n-1})^\perp. \tag{124}$$

The first part of (109) gives $dim\, Im\, (D_x w)^{n-1} = rank\, (D_x w)^{n-1} \geq 1$. We use (124) we obtain $dim\ span\{\nabla\varphi, \nabla w_1, \cdots, \nabla w_n\} \leq n - 1$. This with (122) leads to

$$\forall (x, \theta) \in \Omega_r^0 \times \mathbb{T} : \quad \nabla\varphi(x) \in span\{\nabla w_1(x, \theta), \cdots, \nabla w_n(x, \theta)\}. \tag{125}$$

Let $(0, \widetilde{\theta}) \in \Omega_r^0 \times \mathbb{T}$. The condition (122) gives the existence of a permutation $\mathfrak{P}(\widetilde{\theta}, \cdot)$ of $\mathbb{N}_n^*$ (that depend of the choice of $\widetilde{\theta}$) such that

$$w_{\mathfrak{P}(\widetilde{\theta},n)}(0, \widetilde{\theta}) \in span\,\{\nabla w_{\mathfrak{P}(\widetilde{\theta},1)}(0, \widetilde{\theta}), \cdots, \nabla w_{\mathfrak{P}(\widetilde{\theta},n-1)}(0, \widetilde{\theta})\}, \tag{126}$$

and

$$dim\ span\,\{\nabla w_{\mathfrak{P}(\widetilde{\theta},1)}(0, \widetilde{\theta}), \cdots, \nabla w_{\mathfrak{P}(\widetilde{\theta},n-1)}(0, \widetilde{\theta})\} = n - 1. \tag{127}$$

The conditions (125)-(126) give the existence of $\alpha_{\mathfrak{P}(\widetilde{\theta},1)}, \cdots, \alpha_{\mathfrak{P}(\widetilde{\theta},n-1)} \in \mathbb{R}$ such as

$$\nabla\varphi(0) = \sum_{k=1}^{n-1} \alpha_{\mathfrak{P}(\widetilde{\theta},k)} \nabla w_{\mathfrak{P}(\widetilde{\theta},k)}(0, \widetilde{\theta}). \tag{128}$$

If all $\alpha_{\mathfrak{P}(\widetilde{\theta},k)}$ are zeros the condition (128) gives $\nabla\varphi(0) = 0$ and this is a contradiction with (3). Thus, there exists $k \in \mathbb{N}_{n-1}^*$ such that $\alpha_{\mathfrak{P}(\widetilde{\theta},k)} \neq 0$. The permutation $\mathfrak{P}(\widetilde{\theta}, \cdot)$ can always be adjusted to satisfies $\alpha_{\mathfrak{P}(\widetilde{\theta},n-1)} \neq 0$. This transforms (128) to

$$\nabla w_{\mathfrak{P}(\widetilde{\theta},n-1)}(0, \widetilde{\theta}) = \frac{1}{\alpha_{\mathfrak{P}(\widetilde{\theta},n-1)}} \nabla\varphi(0) - \sum_{k=1}^{n-2} \frac{\alpha_{\mathfrak{P}(\widetilde{\theta},k)}}{\alpha_{\mathfrak{P}(\widetilde{\theta},n-1)}} \nabla w_{\mathfrak{P}(\widetilde{\theta},k)}(0, \widetilde{\theta}). \tag{129}$$

If $\nabla\varphi(0) \in span\{\nabla w_{\mathfrak{P}(\widetilde{\theta},1)}(0, \widetilde{\theta}), \cdots, \nabla w_{\mathfrak{P}(\widetilde{\theta},n-2)}(0, \widetilde{\theta})\}$ the condition (129) leads to $\nabla w_{\mathfrak{P}(\widetilde{\theta},n-1)}(0, \widetilde{\theta}) \in span\{\nabla w_{\mathfrak{P}(\widetilde{\theta},1)}(0, \widetilde{\theta}), \cdots, \nabla w_{\mathfrak{P}(\widetilde{\theta},n-2)}(0, \widetilde{\theta})\}$ and this is a contradiction with (127), hence

$$\nabla\varphi(0) \notin span\{\nabla w_{\mathfrak{P}(\widetilde{\theta},1)}(0, \widetilde{\theta}), \cdots, \nabla w_{\mathfrak{P}(\widetilde{\theta},n-2)}(0, \widetilde{\theta})\}. \tag{130}$$

Let $\mathcal{A}(x, \theta)$, $\mathcal{B}(x, \theta)$ two matrices defined as

$$\mathcal{A}(x, \theta) = (\nabla\varphi(x), \nabla w_{\mathfrak{P}(\widetilde{\theta},1)}(x, \theta), \cdots, \nabla w_{\mathfrak{P}(\widetilde{\theta},n-2)}(x, \theta)), \tag{131}$$
$$\mathcal{B}(x, \theta) = (\nabla w_{\mathfrak{P}(\widetilde{\theta},1)}(x, \theta), \cdots, \nabla w_{\mathfrak{P}(\widetilde{\theta},n-1)}(x, \theta)). \tag{132}$$

The conditions (127)-(130) ensure that $rank\,\mathcal{A}(0, \widetilde{\theta}) = rank\,\mathcal{B}(0, \widetilde{\theta}) = n - 1$. We use the information that the application that associates to a matrix its rank is lower-semi continuous to obtain the existence of $r(\widetilde{\theta}) \in \mathbb{R}_+^*$ and $s(\widetilde{\theta}) \in]0, 1[$ such that

$$\forall (x, \theta) \in \Omega^0_{r(\widetilde{\theta})} \times T^{\widetilde{\theta}}_{s(\widetilde{\theta})} : \quad rank\,\mathcal{A}(x, \theta) \geq n - 1, \ \ rank\,\mathcal{B}(x, \theta) \geq n - 1, \tag{133}$$

and $T^{\widetilde{\theta}}_{s(\widetilde{\theta})} =]\widetilde{\theta} - s(\widetilde{\theta}), \widetilde{\theta} + s(\widetilde{\theta})[$. The torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is compact and $\{T^{\widetilde{\theta}}_{s(\widetilde{\theta})} : \ \widetilde{\theta} \in \mathbb{T}\}$ is an open cover of $\mathbb{T}$, so we can extract a finite open cover of the form

$$\mathbb{T} \subset \bigcup_{i=1}^{m} T^{\theta(i)}_{s(\theta(i))}.$$

We use (131)-(132) to get $rank\,\mathcal{A}(x, \theta) \leq n - 1$ and $rank\,\mathcal{B}(x, \theta) \leq n - 1$. This with (133) for $\widetilde{\theta} \in \{\theta(1), \cdots, \theta(m)\}$ and $r = min\{r(\theta(1)), \cdots, r(\theta(m))\} > 0$ gives

$$\forall i \in \mathbb{N}_m^*, \quad \forall (x, \theta) \in \Omega_r^0 \times T^{\theta(i)}_{s(\theta(i))} : \quad rank\,\mathcal{A}(x, \theta) = rank\,\mathcal{B}(x, \theta) = n - 1. \tag{134}$$

For $i \in \mathbb{N}_m^*$, on $\Omega_r^0 \times T^{\theta(i)}_{s(\theta(i))}$ the conditions (131)-(132)-(134) ensure that

$$dim\,span\,\{\nabla w_{\mathfrak{P}(\theta(i),1)}, \cdots, \nabla w_{\mathfrak{P}(\theta(i),n-2)}, \nabla\varphi\} = n - 1, \tag{135}$$
$$dim\,span\,\{\nabla w_{\mathfrak{P}(\theta(i),1)}, \cdots, \nabla w_{\mathfrak{P}(\theta(i),n-1)}\} = n - 1. \tag{136}$$

We consider $\{\mathcal{X}_{\theta(i)}\}_{i=1,\cdots,m}$ a partition of the unity of the torus $\mathbb{T}$ satisfying

$$\mathcal{X}_{\theta(i)} \in \mathcal{C}^\infty(\mathbb{T}, \mathbb{R}^+), \quad supp\,\mathcal{X}_{\theta(i)} \subset T^{\theta(i)}_{s(\theta(i))}, \quad \sum_{i=1}^{m} \mathcal{X}_{\theta(i)} \equiv 1.$$

We define the functions $\psi_1, \cdots, \psi_{n-1} \in \mathcal{C}^1(\Omega_r^0 \times \mathbb{T}, \mathbb{R}^n)$ as

$$\forall k \in \mathbb{N}_{n-1}^*, \quad \forall (x,\theta) \in \Omega_r^0 \times \mathbb{T}: \quad \psi_k(x,\theta) = \sum_{i=1}^{m} \mathcal{X}_{\theta(i)}(\theta) w_{\mathfrak{P}(\theta(i),k)}(x,\theta). \tag{137}$$

The fact that $\mathfrak{P}(\theta(i),k) \in \mathbb{N}_n^*$ and the conditions (135)-(136)-(137) yield (120) and

$$dim\, span\{\nabla\psi_1, \cdots, \nabla\psi_{n-1}\} = n-1, \tag{138}$$

as well as $span\{\nabla\psi_k\}_{k=1,\cdots,n-1} \subset span\{\nabla w_k\}_{k=1,\cdots,n}$. This with (122)-(138) gives

$$span\{\nabla w_1, \cdots, \nabla w_n\} = span\{\nabla\psi_1, \cdots, \nabla\psi_{n-1}\}. \tag{139}$$

The condition (139) ensures the existence of $\mathcal{W} \in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{T}, \mathbb{R}^n)$ such that

$$w = \mathcal{W}(\psi_1, \cdots, \psi_{n-1}, \theta), \tag{140}$$

and this with (108)-(138) gives (118). On the other hand, the conditions (125)-(139) affirm that $\nabla\varphi \in span\{\nabla\psi_1, \cdots, \nabla\psi_{n-1}\}$. This with (120) (previously recovered) leads to $\nabla\psi_{n-1} \in span\{\nabla\psi_1, \cdots, \nabla\psi_{n-2}, \nabla\varphi\}$. Hence the existence of $M \in \mathcal{C}^1(\mathbb{R}^{n-1} \times \mathbb{T}, \mathbb{R})$ such that

$$\psi_{n-1} = M(\psi_1, \cdots, \psi_{n-2}, \varphi, \theta). \tag{141}$$

The conditions (140)-(141) give (121). If $\partial_{n-1} M = 0$, the condition (141) leads to $\nabla\psi_{n-1} \in span\{\nabla\psi_1, \cdots, \nabla\psi_{n-2}\}$ which contradicts (138), hence (119). □

For the existence of compatible couples satisfying (108) we can refer to the Proposition 3 and the example 1. The corollary below explains how the compatible couple of $n-1$ rank propagate via the system (1)-(2). During this propagation, the function $\mathcal{W}$ stays unchanged, and the rank is preserved.

Corollary 1 *Let (φ, w) be a compatible couple on $\Omega_r^0 \times \mathbb{T}$ satisfying (118)-$\cdots$-(121). Then there exists $T_0 > 0$, $\widetilde{\Psi}_1, \cdots, \widetilde{\Psi}_{n-2} \in \mathcal{C}^1(\Omega_r^{T_0} \times \mathbb{T}, \mathbb{R}^n)$ and $\Phi \in \mathcal{C}^1(\Omega_r^{T_0}, \mathbb{R})$ independent of θ solutions on $\Omega_r^{T_0} \times \mathbb{T}$ of the system*

$$\forall k \in \mathbb{N}_{n-2}^*: \quad \partial_t \widetilde{\Psi}_k + (\mathcal{W} \cdot \nabla)\widetilde{\Psi}_k = 0, \quad \widetilde{\Psi}_k(0,x,\theta) = \psi_k(x,\theta), \tag{142}$$

$$\partial_t \Phi + (\mathcal{W} \cdot \nabla)\Phi = 0, \quad \Phi(0,x) = \varphi(x), \tag{143}$$

such as $\mathcal{W}$ is evaluated at $(\widetilde{\Psi}_1, \cdots, \widetilde{\Psi}_{n-2}, \widetilde{\Psi}_{n-1}, \theta)$ and

$$\widetilde{\Psi}_{n-1}(t,x,\theta) = M\big(\widetilde{\Psi}_1(t,x,\theta), \cdots, \widetilde{\Psi}_{n-2}(t,x,\theta), \Phi(t,x), \theta\big). \tag{144}$$

Moreover, on $\Omega_r^{T_0} \times \mathbb{T}$ we have

$$dim\, Span\{\nabla\widetilde{\Psi}_1, \cdots, \nabla\widetilde{\Psi}_{n-2}, \nabla\Phi\} = n-1, \tag{145}$$

and the function $u^\varepsilon(t,x) = \mathbf{W}\big(t,x,\Phi(t,x)/\varepsilon\big)$ such that

$$\mathbf{W}(t,x,\theta) = \mathcal{W}\big(\widetilde{\Psi}_1(t,x,\theta), \cdots, \widetilde{\Psi}_{n-2}(t,x,\theta), \widetilde{\Psi}_{n-1}(t,x,\theta), \theta\big), \tag{146}$$

is a solution of the Cauchy's problem (1)-(2)-(5) on $\Omega_r^{T_0} \times \mathbb{T}$ for every $\varepsilon \in]0,1]$.

Proof Let $\widetilde{\Psi}$ and ψ defined as

$$\widetilde{\Psi} = (\widetilde{\Psi}_1, \cdots, \widetilde{\Psi}_{n-2}, \Phi)^T, \quad \psi = (\psi_1, \cdots, \psi_{n-2}, \varphi)^T. \tag{147}$$

We use (144)-(147), we can denote

$$S(\widetilde{\Psi}(t,x,\theta),\theta) = \mathcal{W}\big(\widetilde{\Psi}_1(t,x,\theta), \cdots, \widetilde{\Psi}_{n-2}(t,x,\theta), \widetilde{\Psi}_{n-1}(t,x,\theta), \theta\big), \tag{148}$$

this with (147) transforme the system (142)-(143) to

$$\partial_t \widetilde{\Psi} + (S(\widetilde{\Psi},\theta) \cdot \nabla)\widetilde{\Psi} = 0, \quad \widetilde{\Psi}(0,x,\theta) = \psi(x,\theta). \tag{149}$$

The condition (121) with (148) for $t = 0$ give $S(\psi(y,\theta),\theta) = w(y,\theta)$, thus for $\theta \in \mathbb{T}$ the characteristics of the Cauchy's problem (149) are given by the expression $X_\theta(t,y) = y + tw(x,\theta)$. We use the same analysis as in the section 3, we conclude that (149) (and thus (142)-(143)) has a $\mathcal{C}^1-$solution on $\Omega_r^{T_0} \times \mathbb{T}$. On the other hand, the couple (φ, w) is compatible on $\Omega_r^0 \times \mathbb{T}$ and the conditions (142)-(143)-(146) ensure that $(\Phi, \mathbf{W})$ is a solutions of (12)-(13) on $\Omega_r^{T_0} \times \mathbb{T}$. We use the Theorem 2 to obtain that Φ is independent of θ and $u^\varepsilon(t,x) = \mathbf{W}\big(t, x, \Phi(t,x)/\varepsilon\big)$ is a solution of (1)-(2)-(5). The integration of (149) along the characteristics (such that $\mathbf{W}$ satisfying (146)) yields $\widetilde{\Psi}(t,x,\theta) = \psi\big(x - t\mathbf{W}(t,x,\theta),\theta\big)$ on $\Omega_r^{T_0} \times \mathbb{T}$, hence

$$D_x\widetilde{\Psi} = D_x\psi\,[I - t\,D_x\mathbf{W}]\,. \tag{150}$$

The couple (φ, w) is a compatible one and $(\Phi, \mathbf{W})$ is solution of (12)-(13), this with the Theorem 2 ensures that $D_x\mathbf{W}$ satisfies (14), hence $I - t\,D_x\mathbf{W}$ is invertible. Thus (150) gives $rank\ D_x\widetilde{\Psi} = rank\ D_x\psi$. This with (120)-(147) ensures (145). □

Example 2 Let (φ, w) be the compatible couple constructed at Example 1. We set

$$K_1 = \underset{B(r_0)}{Sup}|M_0'(\widetilde{\varphi})||\partial_w\widetilde{\varphi} + h_0'(M_0(\widetilde{\varphi}))\partial_u\widetilde{\varphi}|, \tag{151}$$

$$K_2 = \underset{B(r_0)}{Sup}\,\|(h_0(M_0(\widetilde{\varphi})), 0, M_0(\widetilde{\varphi}))\|, \tag{152}$$

$$s_1 = \underset{B(r_0)}{Sup}|\widetilde{\varphi}(u,v,w)|, \quad I_1 = [-s_1, s_1], \tag{153}$$

$$s_2 = \underset{[-r_0,r_0]\times I_1}{Sup}|\mathcal{X}(\lambda,\mu)|, \quad I_2 = [-s_2, s_2] \tag{154}$$

Let $B' \subset \mathbb{R}^{n-2}$ be the ball of center zero and radius r such that r as in (71), and

$$K_3 = \underset{(\tau,\theta)\in B'\times I_2\times I_1\times\mathbb{T}}{Sup}\|\big(\Phi_1(\tau,\theta), \cdots, \Phi_{n-3}(\tau,\theta), \tau_{n-1}\big)\|, \tag{155}$$

$$K_4 = Sup\,\{\,K_2, K_3\,\}. \tag{156}$$

We select $T_1 \in]0, 1/K_1[$ and we set $T_2 = Inf\{T_1, r_0/K_2\}$, this provides

$$T_2 \le T_1 < 1/K_1, \quad T_2K_2 \le r_0, \tag{157}$$

thus, for every $(u,v,w) \in B(r_0)$ and $t \in [0, T_2]$ we have

$$|1 + tM_0'(\widetilde{\varphi})(\partial_w\widetilde{\varphi} + h_0'(M_0(\widetilde{\varphi}))\partial_u\widetilde{\varphi})| \ge 1 - T_2K_1 > 0. \tag{158}$$

We set

$$\Omega_r^{T_2} = \Big\{\,(t,x) \in [0,T_2] \times \mathbb{R}^n : \quad \|x\|^2 + tK_4 \le r\Big\}.$$

• We consider the following Cauchy problem

$$\left.\begin{array}{r}\partial_t\widetilde{\Phi} + h_0(M_0(\widetilde{\Phi}))\partial_u\widetilde{\Phi} + M_0(\widetilde{\Phi})\partial_w\widetilde{\Phi} = 0\\ \forall(u,v,w) \in B(r_0): \quad \widetilde{\Phi}(0,u,v,w) = \widetilde{\varphi}(u,v,w)\end{array}\right\} \tag{159}$$

The characteristics of (159) are given by

$$C(t,z) = z + t\big(h_0(M_0(\widetilde{\varphi})), 0, M_0(\widetilde{\varphi})\big), \quad z = (u,v,w) \in B(r_0),$$

such that $\widetilde{\varphi}$ evaluated at z. For any $t \in [0, T_2]$ we define

$$\begin{aligned} X_t\ :\ B(r_0) &\longrightarrow C(t, B(r_0))\\ z &\longmapsto C(t,z). \end{aligned}$$

The condition (158) yields $|\det D_zX_t(z)| > 0$ for any $(t,z) \in [0,T_2] \times B(r_0)$; thus, X_t is $\mathcal{C}^1-$diffeomorphism. Now we set

$$I = \Big\{\,(t,z) \in [0,T_2] \times \mathbb{R}^3 : \quad \|z\| + tK_2 \le r_0\Big\},$$

and for every $(t,z)\in I$, we define $H_{(t,z)}$ by

$$\forall Z\in B(r_0):\quad H_{(t,z)}(Z)=z-t\big(h_0(M_0(\widetilde{\varphi}(Z))),0,M_0(\widetilde{\varphi}(Z))\big).$$

On the one hand, by using (152) we get $\|H_{(t,z)}(Z)\|\leq\|z\|+tK_2\leq r_0$; so, $H_{(t,z)}(Z)\in B(r_0)$. By using the Brouwer fixed point theorem, we get the existence of $Z\in B(r_0)$ so that $Z=H_{(t,z)}(Z)$, and this provides $z=X_t(Z)=C(t,Z)$. Combining this with the fact that X_t is a $\mathcal{C}^1-$diffeomorphism, we conclude that (159) has a solution defined over I by $\widetilde{\Phi}(t,z)=\tilde{\varphi}(X_t^{-1}(z))$ and

$$\forall(t,z)\in I:\quad|\widetilde{\Phi}(t,z)|\leq s_1,\tag{160}$$

such that s_1 as in (153). For $(t,z)\in I$, with $z=(u,v,w)$, we have $|w-M_0(\widetilde{\Phi}(t,z))|\leq|z|+tK_2\leq r_0$, combining this with (160) we get

$$\forall(t,z)\in I:\quad|\mathcal{X}(w-tM_0(\widetilde{\Phi}(t,z)),\widetilde{\Phi}(t,z))|\leq s_2,\tag{161}$$

such that s_2 as in (154). On the other hand, for any $(t,x)\in\Omega_r^{T_2}$ the conditions (83)-(156) provide

$$\begin{aligned}\Big\|\Big(x_n-\sum_{j=1}^{n-3}a_jx_j,x_{n-2},x_{n-1}\Big)\Big\|^2+tK_2&\leq\Big(1+\sum_{k=1}^{n-3}a_k^2\Big)^{1/2}\Big[\|x\|+tK_4\Big]\\&\leq\Big(1+\sum_{k=1}^{n-3}a_k^2\Big)^{1/2}r,\end{aligned}$$

this with the condition (71) yields $(t,x_n-\sum_{j=1}^{n-3}a_jx_j,x_{n-2},x_{n-1})\in I$; so, we can define $\Phi(t,x)$,$\widetilde{\Psi}_{n-1}(t,x,\theta)$ and $\widetilde{\Psi}_{n-2}(t,x,\theta)$ by

$$\forall(t,x)\in\Omega_r^{T_2}:\quad\Phi(t,x)=\widetilde{\Phi}\Big(t,x_n-\sum_{j=1}^{n-3}a_jx_j,x_{n-2},x_{n-1}\Big),\tag{162}$$

$$\forall(t,x)\in\Omega_r^{T_2}\times\mathbb{T}:\quad\widetilde{\Psi}_{n-2}(t,x,\theta)=\mathcal{X}\Big(x_{n-1}-tM_0(\Phi(t,x)),\Phi(t,x)\Big),\tag{163}$$

$$\forall(t,x)\in\Omega_r^{T_2}\times\mathbb{T}:\quad\widetilde{\Psi}_{n-1}(t,x,\theta)=M_0(\Phi(t,x)).\tag{164}$$

Since for $(t,x)\in\Omega_r^{T_2}$ we have $(t,x_n-\sum_{j=1}^{n-3}a_jx_j,x_{n-2},x_{n-1})\in I$, the conditions (160)-(161)-(162)-(163) provide

$$\forall(t,x)\in\Omega_r^{T_2}\times\mathbb{T}:\quad|\Phi(t,x)|\leq s_1,\quad|\Psi_{n-2}(t,x,\theta)|\leq s_2.\tag{165}$$

• Let $\eta=(\tau_{n-1},\tau_n,\theta)\in I_2\times I_1\times\mathbb{T}$ be fixed airbitrarly and

$$\forall k\in\mathbb{N}_{n-3}^*:\quad f_k^\eta(h)=\Phi_k(\tau,\theta),\quad h=(\tau_1,\cdots,\tau_{n-2}).\tag{166}$$

We set $F^\eta(t,h)=(F_1^\eta(t,h),\cdots,F_{n-3}^\eta(t,h))$, $f^\eta(h)=(f_1^\eta(h),\cdots,f_{n-3}^\eta(h))$, and we consider the following Cauchy problem

$$\partial_tF^\eta+\Big(\sum_{k=1}^{n-3}F_k^\eta\partial_kF^\eta\Big)+\tau_{n-1}\partial_{n-2}F^\eta=0,\quad F^\eta(0,h)=f^\eta(h).\tag{167}$$

The characteristics of (167) are given by

$$K^\eta(t,h)=h+t\big(f_1^\eta(h),\cdots,f_{n-3}^\eta(h),\tau_{n-1}\big).$$

We set

$$J=\Big\{(t,h)\in\mathbb{R}_+\times\mathbb{R}^{n-2}:\quad\|h\|+tK_3\leq r\Big\},$$

and we define Y_t^η by

$$\begin{aligned}Y_t^\eta\;:\;B'&\longrightarrow K^\eta(t,B')\\h&\longmapsto K^\eta(t,h).\end{aligned}$$

By using (97) we get $det\, D_h Y_t^{\eta}(h) = 1$; thus, Y_t^{η} is $\mathcal{C}^1-$diffeomorphism. Now, for every $(t, h) \in J$ we define $Q^{\eta}_{t,h}$ by

$$\forall z \in B' : \quad Q^{\eta}_{t,h}(z) = h - t\big(f_1^{\eta}(z), \cdots, f_{n-3}^{\eta}(z), \tau_{n-1}\big).$$

On the one hand, the conditions (155)-(156)-(166) give $\|Q^{\eta}_{t,h}(z)\| \leq \|h\| + tK_3 \leq r$; then, $Q^{\eta}_{t,h}(z) \in B'$. By using Brouwer fixed point theorem, we get the existence of $z \in B'$ so that $z = Q^{\eta}_{t,h}(z)$, and this provides $h = Y_t^{\eta}(z) = K^{\eta}(t, z)$. By using the information Y_t^{η} is a $\mathcal{C}^1-$diffeomorphism, we conclude that (167) has a solution defined over J by $F^{\eta}(t, h) = h^{\eta}((Y_t^{\eta})^{-1}(h))$. Let h as in (166) and $\widetilde{\Phi}_k(t, \tau, \theta) = F^{\eta}(t, h)$, by using (167) we conclude that the system

$$\forall j \in \mathbb{N}^*_{n-3} : \quad \begin{cases} \partial_t \widetilde{\Phi}_j + \Big(\displaystyle\sum_{k=1}^{n-3} \widetilde{\Phi}_k \partial_k \widetilde{\Phi}_j\Big) + \tau_{n-1}\partial_{n-2}\widetilde{\Phi}_j = 0, \\ \qquad\qquad\qquad\qquad \widetilde{\Phi}_j(0, \tau, \theta) = \Phi_j(\tau, \theta), \end{cases} \tag{168}$$

has a solution over $J \times I_2 \times I_1 \times \mathbb{T}$. On the other hand, for $(t, x) \in \Omega_r^{T_2}$ and by using (156) we get

$$\|(x_1, \cdots, x_{n-2})\| + tK_3 \leq \|x\| + tK_4 \leq r,$$

thus, $(t, x_1, \cdots, x_{n-2}) \in J$; combining this with (165) we get

$$\forall (t, x) \in \Omega_r^{T_2} : \quad (t, x_1, \cdots, x_{n-2}, \widetilde{\Psi}_{n-2}(t, x), \widetilde{\Phi}(t, x), \theta) \in J \times I_2 \times I_1 \times \mathbb{T}.$$

Hence, over $\Omega_r^{T_2} \times \mathbb{T}$ we can definie $\widetilde{\Psi}_k$ by

$$\forall k \in \mathbb{N}^*_{n-3} : \quad \widetilde{\Psi}_k(t, x, \theta) = \widetilde{\Phi}_i\big(t, x_1, \cdots, x_{n-2}, \widetilde{\Psi}_{n-2}(t, x), \Phi(t, x), \theta\big). \tag{169}$$

• The solution of (159) satisfies

$$\widetilde{\Phi} = \widetilde{\varphi}\Big(u - th_0(M_0(\widetilde{\Phi})), v, w - tM_0(\widetilde{\Phi})\Big), \tag{170}$$

the calculus of $\partial_v \widetilde{\Phi} + h_1(M_0(\widetilde{\Phi}))\partial_u \widetilde{\Phi}$ such that $\widetilde{\Phi}$ as in (170) provides

$$\begin{aligned} &[\, 1 + tM_0'(\widetilde{\varphi})(\partial_w \widetilde{\varphi} + h_0'(M_0(\widetilde{\varphi}))\partial_u \widetilde{\varphi})\,]\,[\,\partial_v \widetilde{\Phi} + h_1(M_0(\widetilde{\Phi}))\partial_u \widetilde{\Phi}\,] \\ &= \partial_v \widetilde{\varphi} + h_1(M_0(\widetilde{\varphi}))\partial_u \widetilde{\varphi}, \end{aligned}$$

combining this with (66)-(158) yields

$$\partial_v \widetilde{\Phi} + h_1(M_0(\widetilde{\Phi}))\partial_u \widetilde{\Phi} = 0. \tag{171}$$

Let $\mathcal{W}$ as in (76) and $\widetilde{\Psi} = (\widetilde{\Psi}_1, \cdots, \widetilde{\Psi}_{n-1})$, by using (72)-(159)-(162)-(171) we get

$$\partial_t \Phi + (\mathcal{W}(\widetilde{\Psi}) \cdot \nabla)\Phi = 0, \quad \Phi(0, x) = \varphi(x). \tag{172}$$

By using (73)-(75)-(163)-(164)-(172) we get

$$\partial_t \widetilde{\Psi}_{n-1} + (\mathcal{W}(\widetilde{\Psi}) \cdot \nabla)\widetilde{\Psi}_{n-1} = 0, \quad \widetilde{\Psi}_{n-1}(0, x, \theta) = \psi_{n-1}(x, \theta), \tag{173}$$

$$\partial_t \widetilde{\Psi}_{n-2} + (\mathcal{W}(\widetilde{\Psi}) \cdot \nabla)\widetilde{\Psi}_{n-2} = 0, \quad \widetilde{\Psi}_{n-2}(0, x, \theta) = \psi_{n-2}(x, \theta). \tag{174}$$

By using (74)–(168)-(169)-(172)-(174), we get

$$\forall k \in \mathbb{N}^*_3 : \quad \partial_t \widetilde{\Psi}_k + (\mathcal{W}(\widetilde{\Psi}) \cdot \nabla)\widetilde{\Psi}_k = 0, \quad \widetilde{\Psi}_k(0, x, \theta) = \psi_k(x, \theta). \tag{175}$$

For any $\varepsilon \in]0, 1]$, we define u^{ε} by

$$\forall (t, x) \in \Omega_r^{T_2} : \quad u^{\varepsilon}(t, x) = \mathcal{W}\Big(\widetilde{\Psi}\Big(t, x, \frac{\Phi(t, x)}{\varepsilon}\Big)\Big).$$

By using (172)-(173)-(174)-(175) we get

$$\partial_t u^{\varepsilon} + (u^{\varepsilon} \cdot \nabla)u^{\varepsilon} = 0, \quad u^{\varepsilon}(0, x) = w\Big(x, \frac{\varphi(x)}{\varepsilon}\Big),$$

Since (φ, w) is compatible couple over $\Omega_r^0 \times \mathbb{T}$; then, $div\, u^{\varepsilon} = 0$.

Declarations

Conflict of interest The author declares that there are no conflicts of interest (financial or non-financial) related directly or indirectly to this work.

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