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ORIGINAL RESEARCH

On comparing the coefficients of general product L-functions

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Abstract Let f and g be two distinct primitive holomorphic cusp forms of even integral weights k_1 and k_2 for the full modular group $\Gamma = SL(2,\mathbb{Z})$, respectively. Denote by $\lambda_{f\otimes f\otimes\cdots\otimes_l f}(n)$ and $\lambda_{g\otimes g\otimes\cdots\otimes_l g}(n)$ the nth normalized coefficients of the l-fold product product L-functions attached to f and g, respectively. In this paper, we establish a lower bound for the analytic density of the set

$$\left\{p \;:\; \lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) < \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p)\right\},$$

where $l \geqslant 4$ is any fixed integer. By analogy, we also establish some similar density results of the above supported on certain binary quadratic form.

Keywords Hecke eigenforms · Fourier coefficients · Automorphic L-functions · Binary quadratic form · Langlands program

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1 Introduction

The Fourier coefficients of modular forms are important and interesting objects in modern analytic number theory, which have been extensively investigated by lots of scholars (see e.g. [8, 11, 12, 19, 28]). In this paper, we consider the holomorphic cusp form for the full modular group $\Gamma = SL(2,\mathbb{Z})$, which are eigenfunction of all the Hecke operators T_n. And let H_k^* denote the set of normalized primitive holomorphic Hecke cusp forms of even integral weight k for $\Gamma = SL(2,\mathbb{Z})$. The Fourier expansion of $f \in H_k^*$ at the cusp ∞ takes the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \qquad \Im(z) > 0,$$

where $e(z) = e^{2\pi i z}$ and $\lambda_f(n)$ denotes the nth normalized Fourier coefficient (Hecke eigenvalue) of f. It is well-known from the theory of Hecke operator that $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d\mid(m,n)} \lambda_f\left(\frac{mn}{d^2}\right), \tag{1}$$

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where $m, n \geqslant 1$ are integers. In 1974, P. Deligne [8] proved the celebrated Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leqslant d(n). \tag{2}$$

By (2), Deligne's result is equivalent to the fact that there exist $\alpha_f(p), \beta_f(p) \in \mathbb{C}$ satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{3}$$

More generally, for all positive integers $l \geqslant 1$ one has

$$\lambda_f(p^l) = \alpha_f(p)^l + \alpha_f(p)^{l-1}\beta_f(p) + \cdots + \alpha_f(p)\beta_f(p)^{l-1} + \beta_f(p)^l.$$

Before proceeding further, we introduce the definition of analytic density (or Dirichlet density). A set of primes $\mathscr{A}$ has analytic density $\delta > 0$ if it satisfies

$$\sum_{p\in\mathscr{A}} \frac{1}{p^\sigma} \sim (1+o(1))\delta \sum_p \frac{1}{p^\sigma} = -(1+o(1))\delta \log(\sigma - 1), \quad \text{as } \sigma \to 1^+.$$

Density results involving the Hecke eigenvalues of cusp forms are important in the theory of autoorphic forms. In the work, Kowalski et al. [18] considered the estimate of the analytic density of primes at which two Hecke eigenforms have the same signs. As a corollary, they have given a criterion for the signs of $\lambda_f(p)$ for which determines f uniquely. In [25], Meher et al. investigated the distribution of signs of $\{\lambda_f(p^m)\}_p$ for each fixed $m \in \mathbb{N}$ and the natural density results. Furthermore, they have obtained the natural density of the sets $\{p : \lambda_f(p^m) > 0\}$ and $\{p : \lambda_f(p^m) < 0\}$ explicitly, using the celebrated Sato-Tate conjecture.

Let $\lambda_f(n)$ and $\lambda_g(n)$ be the nth Hecke eigenvalues attached to two distinct Hecke eigenforms f and g, respectively. On the other hand, Chiriac [5] shown that the set

$$\{p : \lambda_f(p) < \lambda_g(p)\}$$

has analytic density at least $\frac{1}{16}$, which indicates that the eigenvalues $\lambda_f(p)$ are not less than or equal to $\lambda_g(p)$ for almost all primes p never occurs. Assuming that f and g have not complex multiplication, and that neither is a quadratic twist of each other, in the same paper, Chiriac also established that the set

$$\left\{p : \lambda_f^2(p) < \lambda_g^2(p)\right\}$$

has a lower bound for analytic density of $\frac{1}{16}$. Using the joint Sato-Tate distribution associated to two distinct cusp forms, Chiriac [6] in another paper shown that the set $\{p : \lambda_f(p) < \lambda_g(p)\}$ has Dirichlet density precisely $\frac{1}{2}$. By the using the similar approach as in Chiriac [5], along with the automorphy of certain symmetric power L-functions and the Rankin-Selberg L-functions, on assuming some natural conditions, Lao [21] extended the results of Chiriac by showing the sets

$$\left\{p : \lambda_f(p^j) < \lambda_g(p^j)\right\}, \quad 1 \leqslant j \leqslant 8,$$

and

$$\left\{p : \lambda_f^2(p^j) < \lambda_g^2(p^j)\right\}, \quad 1 \leqslant j \leqslant 4$$

has analytic density at least $\frac{1}{16[\frac{j+1}{2}]^2}$ and $\frac{1}{4j(j+1)^2}$, respectively. Very recently, the density result of Lao has been extended by Vaishya [31] accordingly for any $j \geqslant 1$, by using the recent breakthrough of Newton and Thorne [26,27] that the automorphy of $\mathrm{sym}^j f$ for all $j \in \mathbb{N}$. In [33], Zou et al. generalized the results in [21] in another direction by considering the analytic density for the set of primes where the product $\lambda_f(p^i)\lambda_f(p^j)$ is strictly less than $\lambda_g(p^i)\lambda_g(p^j)$ for all $j \geqslant 1$, $0 \leqslant i \leqslant j$.

For $\Re(s) > 1$, the Hecke L-function $L(f, s)$ attached to $f \in H_k^*$ is defined as

$$L(f,s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}.$$

Let $f_1, f_2, \cdots, f_l \in H_k^*$ be Hecke eigenforms, following Garret and Harris [10], for $\Re(s) > 1$, we can define the L-function $L(f_1 \otimes f_2 \otimes \cdots \otimes f_l, s)$ in the following

$$\begin{aligned} L(f_1 \otimes f_2 \otimes \cdots \otimes f_l, s) &= \sum_{n=1}^{\infty} \frac{\lambda_{f_1 \otimes f_2 \otimes \cdots \otimes f_l}(n)}{n^s} \\ &= \prod_p \prod_\eta \left(1 - \alpha_{f_1}^{\eta(1)}(p)\alpha_{f_2}^{\eta(2)}(p)\cdots\alpha_{f_l}^{\eta(l)}(p)p^{-s}\right)^{-1}, \end{aligned}$$

where η ranges over the set of maps $\{1, 2, \cdots, l\} \to \{1, 2\}$ with

$$\alpha_{f_i}^{\eta(i)}(p) = \begin{cases} \alpha_{f_i}(p), \text{ if } \eta(i) = 1, \\ \beta_{f_i}(p), \text{ if } \eta(i) = 2, \end{cases}$$

here $\alpha_{f_i}(p), \beta_{f_i}(p)$ satisfy the relation (3) corresponding to f_i. In particular, for $f_1 = f_2 = \cdots = f_l$, one has the following L-function

$$L(f \otimes f \otimes \cdots \otimes_l f, s) = \sum_{n=1}^{\infty} \lambda_{f \otimes f \otimes \cdots \otimes_l f}(n) n^{-s}. \tag{4}$$

Clearly, the L-function $L(f \otimes f \otimes \cdots \otimes_l f, s)$ is of degree 2^l. For $4 \leqslant l \leqslant 8$, Venkatasubbareddy and Sankaranarayanan [32] investigated the error term $E(x)$ in the asymptotic formula

$$\sum_{n \leqslant x} \lambda_{f \otimes f \otimes \cdots \otimes_l f}(n) = \underbrace{M_l(x)}_{\text{main term}} + \underbrace{E_l(x)}_{\text{error term}}. \tag{5}$$

More precisely, they derived the upper bounds

$$\begin{aligned} &E_4(x) \ll x^{\frac{79}{91}+\varepsilon}, \qquad E_5(x) \ll x^{\frac{40}{43}+\varepsilon}, \qquad E_6(x) \ll x^{\frac{88}{91}+\varepsilon}, \\ &E_7(x) \ll x^{\frac{184}{187}+\varepsilon}, \qquad E_8(x) \ll x^{\frac{374}{377}+\varepsilon}. \end{aligned}$$

Inspired by the above results, the first purpose in this paper is to establish the following result.

Theorem 1.1 *Let $l \geqslant 4$ be any fixed integers. And let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms such that $sym^m \pi_f \not\cong sym^m \pi_g$ for $m \geqslant 1$. Then, the set*

$$D_l := \left\{p \ : \ \lambda_{f \otimes f \otimes \cdots \otimes_l f}(p) < \lambda_{g \otimes g \otimes \cdots \otimes_l g}(p)\right\}$$

has analytic density at least e_l, where

$$e_l = \begin{cases} \dfrac{A_l - A_j^2}{16\left(\sum_{1 \leqslant r \leqslant j-1} C_j(r) r + j\right)^2}, & \text{if } l = 2j \text{ is even,} \\ \dfrac{A_l}{16\left(B_j + \sum_{1 \leqslant r \leqslant j-1} D_j(r)(r+1) + (j+1)\right)^2}, & \text{if } l = 2j+1 \text{ is odd.} \end{cases} \tag{6}$$

Here A_j, B_j and $D_j(r)$ are constants defined as in (20).

In this paper, we always take $Q(\mathbf{x})$, $(\mathbf{x} \in \mathbb{Z}^2)$ be a primitive integral positive-definite binary quadratic form of fixed discriminant $D < 0$ with the class number $h(D) = 1$. In [31], Vaishya established the lower bounds for

$$A_j := \left\{p \ : \ p = Q(\mathbf{x}),\ \lambda_f(p^j) < \lambda_g(p^j)\right\}$$

and

$$B_j := \left\{p \ : \ p = Q(\mathbf{x}),\ \lambda_f^2(p^j) < \lambda_g^2(p^j)\right\},$$

respectively. More accurately, Vaishya [31] proved that the analytic density for A_j and B_j are at least $\frac{1}{32[\frac{j+1}{2}]^2}$ and $\frac{1}{8j(j+1)^2}$, respectively.

Following the similar methods as in Vaishya [31] and Zou et al. [33], in combination with nice properties of the associated automorphic L-functions, we also establish the following result.

Theorem 1.2 *Let $l \geqslant 4$ be any fixed integers. And let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms such that $\mathrm{sym}^m\pi_f \not\cong \mathrm{sym}^m\pi_g$ for $m \geqslant 1$. Then, the set*

$$\widetilde{D}_l := \left\{p \;:\; p = Q(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{Z}^2,\ \lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) < \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p)\right\}$$

has analytic density at least $\tilde{e}_l = \frac{1}{2}e_l$, where e_l is defined as in (6).

Throughout the paper, we always assume that $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms, respectively. Denote by $\varepsilon > 0$ an arbitrarily small positive constant that may vary in different occurrences. The symbol p always denotes a prime number.

2 Preliminaries

In this section, we collect some facts and lemmas which would be useful in the proof of the main results in this paper.

Let $f \in H_{k_1}^*$ be a Hecke eigenform. The jth symmetric power L-function associated to f is defined as

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} \left(1 - \alpha_f(p)^{j-m}\beta_f(p)^m p^{-s}\right)^{-1}, \quad \Re(s) > 1.$$

We may expand it into a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \\ &= \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^m)}{p^{ms}} + \cdots\right), \quad \Re(s) > 1. \end{aligned}$$

Obviously, $\lambda_{\mathrm{sym}^j f}(n)$ is real multiplicative function. For $j = 1$, we have $L(\mathrm{sym}^1 f, s) = L(f, s)$. It is standard to find that

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m, \tag{7}$$

which can also be written as

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = U_j(\lambda_f(p)/2),$$

where $U_j(x)$ is the jth Chebyshev polynomial of the second kind.

Let χ be a Dirichlet character modulo q. Then we can define the twisted jth symmetric power L-function by the Euler product with degree $j + 1$

$$\begin{aligned} L(\mathrm{sym}^j f \otimes \chi, s) &= \prod_p \prod_{m=0}^{j} \left(1 - \alpha_f(p)^{j-m}\beta_f(p)^m \chi(p) p^{-s}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)\chi(n)}{n^s}, \quad \Re(s) > 1. \end{aligned}$$

Let $g \in H_{k_2}^*$ be another Hecke eigenform. The Rankin-Selberg L-function attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined as

$$\begin{aligned} &L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) \\ &= \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} \left(1 - \alpha_f(p)^{i-m}\beta_f(p)^m \alpha_g(p)^{j-m'}\beta_g(p)^{m'} p^{-s}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)}{n^s}, \quad \Re(s) > 1. \end{aligned}$$

The twisted Rankin-Selberg L-function $L(\mathrm{sym}^i f \times \mathrm{sym}^j g \otimes \chi, s)$ can be defined in a similar manner. For any prime number p, we also have

$$\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p) = \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j g}(p) = \lambda_f(p^i)\lambda_g(p^j). \tag{8}$$

From (3), it is not hard to find that

$$|\lambda_{\mathrm{sym}^j f}(n)| \leqslant d_{j+1}(n), \quad |\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p)| \leqslant d_{(i+1)(j+1)}(n),$$

for any $j \geqslant 1$, where $d_\nu(n)$ denotes the ν-dimensional divisor function, which is defined as the number of ordered representations $n = n_1 \cdots n_\nu$ with integers $n_1, \cdots, n_\nu \geqslant 1$.

As is well-known, to a primitive form f is associated an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_\mathbb{Q})$ and hence an automorphic L-function $L(\pi_f, s)$ which coincides with $L(f, s)$. For $1 \leqslant j \leqslant 8$, the special Langlands functoriality conjecture which states that $\mathrm{sym}^j f$ is automorphic cuspidal has been established in a series of important works of Gelbart and Jacquet [9], Kim [15], Kim and Shahidi [16,17], Shahidi [30], Clozel and Thorne [2–4]. Very recently, Newton and Thorne [26,27] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_\mathbb{Q})$ for all $j \geqslant 1$ (with f being a holomorphic cusp form). Furthermore, by combining the works of Newton-Thorne and Cogdell-Michel [1], we know that the L-function $L(\mathrm{sym}^j f, s)$ has an analytic continuation as an entire function in the whole complex plane and satisfies a certain functional equation of Riemann zeta-type. From the Rankin-Selberg theory associated to two automorphic representations, we also know that the Rankin-Selberg L-function $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ can be extend to the whole complex plane as an entire function (except for $\mathrm{sym}^j \pi_f \cong \mathrm{sym}^j \pi_g$ where it has simples poles at $s = 0, 1$) and satisfies nice functional equation of Riemann zeta-type. For a more comprehensive investigation, the interested readers can refer to [14, Chap.5].

Let $Q(\mathbf{x})$ be a primitive integral positive-definite binary quadratic form of fixed discriminant $D < 0$ with the class number $h(D) = 1$. And let r_Q be defined as

$$r_Q(n) := \{\mathbf{x} \in \mathbb{Z}^2 \ : \ n = Q(\mathbf{x})\}.$$

The generating function $\theta_Q(\tau)$ associated to $Q(\mathbf{x})$ is given by

$$\theta_Q(\tau) = \sum_{\mathbf{x} \in \mathbb{Z}^2} q^{Q(\mathbf{x})} = \sum_{n=0}^{\infty} r_Q(n) q^n, \quad q = e(\tau), \ \Im(\tau) > 0.$$

The generating function $\theta_Q(\tau) \in M_1(\Gamma_0(|D|), \chi_D)$ (see e.g. [13, Theorme 10.9]), where χ_D is the Dirichlet character modulo $|D|$ which is given by Jacobi symbol $\chi_D(d) = (\frac{D}{d})$. From Weil's bound, we have $r_Q(n) \ll n^\varepsilon$.

The character sum $r(n; D)$ is given explicitly in terms of Jacobi symbol χ_D (see [13, (11.9),(11.10)]), i.e.,

$$r(n; D) = w_D \sum_{d|n} \chi_D(d), \quad \text{where} \ \ w_D = \begin{cases} 6, \text{ if } D = -3, \\ 4, \text{ if } D = -4, \\ 2, \text{ if } D < -4. \end{cases}$$

It is known that $r(n; D)$ counts the number of representations of a positive integer n by all the reduced forms of fixed discriminant D. Since we are considering the quadratic form $Q(\mathbf{x})$ of discriminant $D < 0$ with the class number $h(D) = 1$, we have

$$r_Q(n) = r(n; D) = w_D \sum_{d|n} \chi_D(d).$$

In particular, $r_Q(p) = w_D(1 + \chi_D(p))$. Hence, the prime p is represented by $Q(\mathbf{x})$ if and only if $\chi_D(p) = 1$. Motivated by this fact, we define the characteristic function

$$\mathbf{1}_Q(p) = \begin{cases} \frac{r_Q(p)}{2w_D}, & \text{if } p = Q(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{Z}^2, \\ 0, & \text{otherwise.} \end{cases}$$

This indicate that $\frac{r_Q(p)}{w_D}$ is 2 or 0 according to p is represented by $Q(\mathbf{x})$ or not.

Lemma 2.1 [31, Lemma 3.1] *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then, for any fixed $i, j \geqslant 1$, we have*

$$\sum_p \frac{\lambda_f^2(p^j)}{p^\sigma} = \sum_p \frac{1}{p^\sigma} + O(1), \quad \sigma \to 1^+ \tag{9}$$

and

$$\sum_p \frac{\lambda_f(p^i)\lambda_g(p^j)}{p^\sigma} = O(1), \quad \sigma \to 1^+. \tag{10}$$

Furthermore,

$$\sum_p \frac{\lambda_f^2(p^j)\chi_D(p)}{p^\sigma} = O(1), \quad \sum_p \frac{\lambda_f(p^i)\lambda_g(p^j)\chi_D(p)}{p^\sigma} = O(1), \tag{11}$$

as $\sigma \to 1^+$.

Lemma 2.2 [33, Lemma 4] *Let $\{h(p)\}$ be a set of real numbers, and $|h(p)| \leqslant B$ with a absolute bound B. Suppose that there exists two absolute constants a, A such that*

$$\sum_p \frac{h(p)^2}{p^\sigma} = a \sum_p \frac{1}{p^\sigma} + O(1) \tag{12}$$

and

$$\sum_p \frac{h(p)}{p^\sigma} = A \sum_p \frac{1}{p^\sigma} + O(1), \tag{13}$$

as $\sigma \to 1^+$. Then the set $\mathscr{B} := \{p \ : \ h(p) < 0\}$ has analytic density at least $\frac{(a-AB)}{2B^2}$.

Lemma 2.3 *Let $\{k(p)\}$ be a set of real numbers, and $|k(p)| \leqslant B$ with a absolute bound B. Suppose that there exists two absolute constants c, C such that*

$$\sum_p \frac{k(p)^2 \mathbf{1}_Q(p)}{p^\sigma} = c \sum_p \frac{1}{p^\sigma} + O(1) \tag{14}$$

and

$$\sum_p \frac{k(p)\mathbf{1}_Q(p)}{p^\sigma} = C \sum_p \frac{1}{p^\sigma} + O(1), \tag{15}$$

as $\sigma \to 1^+$. Then the set $\widetilde{\mathscr{B}} := \{p \ : \ p = Q(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{Z}^2, \ k(p) < 0\}$ has analytic density at least $\frac{c-BC}{2B^2}$.

Proof We prove this result by following the similar argument in [33, Lemma 4] with some modifications.

If $p \notin \widetilde{\mathscr{B}}$, then $k(p) \geqslant 0$ or p cannot be represented by $Q(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{Z}^2$. In the second case, $r_Q(p) = 0$. This leads to the following estimate

$$\begin{aligned}
\sum_{p \notin \widetilde{\mathscr{B}}} \frac{k(p)^2}{p^\sigma} &= \sum_{\substack{p \notin \widetilde{\mathscr{B}}, \ p = Q(\mathbf{x}) \\ \text{for some } \mathbf{x} \in \mathbb{Z}^2}} \frac{k(p)^2}{p^\sigma} = \frac{1}{2w_D} \sum_{p \notin \widetilde{\mathscr{B}}} \frac{k(p)^2 r_Q(p)}{p^\sigma} \\
&= B \frac{1}{2w_D} \sum_{p \notin \widetilde{\mathscr{B}}} \frac{k(p) r_Q(p)}{p^\sigma} \\
&\leqslant B \frac{1}{2w_D} \left(\sum_p \frac{k(p) r_Q(p)}{p^\sigma} - \sum_{p \in \widetilde{\mathscr{B}}} \frac{k(p) r_Q(p)}{p^\sigma} \right) \\
&\leqslant BC \sum_p \frac{1}{p^\sigma} + B^2 \sum_{p \in \widetilde{\mathscr{B}}} \frac{1}{p^\sigma} + O(1), \quad \text{as } \sigma \to 1^+.
\end{aligned} \tag{16}$$

From (16) and the bound $|k(p)| \leqslant B$, we get

$$\begin{aligned}\sum_{\substack{p=Q(\mathbf{x})\\ \text{for some } \mathbf{x}\in\mathbb{Z}^2}} \frac{k(p)^2}{p^\sigma} &= \sum_{p\in\widetilde{\mathscr{B}}} \frac{k(p)^2}{p^\sigma} + \sum_{p\notin\widetilde{\mathscr{B}}} \frac{k(p)^2}{p^\sigma}\\ &\leqslant B^2 \sum_{p\in\widetilde{\mathscr{B}}} \frac{1}{p^\sigma} + BC\sum_p \frac{1}{p^\sigma} + B^2 \sum_{p\in\widetilde{\mathscr{B}}} \frac{1}{p^\sigma} + O(1)\\ &= 2B^2 \sum_{p\in\widetilde{\mathscr{B}}} \frac{1}{p^\sigma} + BC\sum_p \frac{1}{p^\sigma} + O(1). \end{aligned} \tag{17}$$

Combining (14) and (17), we obtain

$$\begin{aligned} c\sum_p \frac{1}{p^\sigma} &\leqslant \sum_{\substack{p=Q(\mathbf{x})\\ \text{for some } \mathbf{x}\in\mathbb{Z}^2}} \frac{k(p)^2}{p^\sigma}\\ &\leqslant 2B^2 \sum_{p\in\widetilde{\mathscr{B}}} \frac{1}{p^\sigma} + BC\sum_p \frac{1}{p^\sigma} + O(1), \quad \text{as } \sigma \to 1^+. \end{aligned}$$

This shows that the analytic density of the set $\widetilde{\mathscr{B}}$ is at least $\frac{c-BC}{2B^2}$. □

3 Proofs of Theorems 1.1 and 1.2

In this section, we give the proof of Theorems 1.1 and 1.2, by utilizing the results formulated in Section 2. In order to establish Theorems 1.1 and 1.2, we need to prove the following two lemmas.

Lemma 3.1 *Let $l \geqslant 4$ be any fixed integer, and let $f \in H^*_{k_1}$ and $g \in H^*_{k_2}$ be two distinct Hecke eigenforms. Then*

(i) For $l = 2j$ even integer, we have

$$\begin{aligned}&\sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))^2 p^{-\sigma}\\ &= \sum_p \frac{\sigma_l}{p^\sigma} + O(1), \quad as \quad \sigma \to 1^+, \end{aligned} \tag{18}$$

where $\sigma_l = 2(A_l - A_j^2)$.

(ii) For $l = 2j+1$ odd integer, we have

$$\begin{aligned}&\sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))^2 p^{-\sigma}\\ &= \sum_p \frac{\tilde{\sigma}_l}{p^\sigma} + O(1), \quad as \quad \sigma \to 1^+, \end{aligned} \tag{19}$$

where $\tilde{\sigma}_l = 2A_l$. Here the constants $A_j, B_j, C_j(r), D_j(r)$ $(1 \leqslant r \leqslant j-1)$ are given by

$$\begin{aligned} A_j &= \frac{(2j)!}{j!(j+1)!}, \qquad B_j = 2\frac{(2j+1)!}{j!(j+2)!},\\ C_j(r) &= \frac{(2j)!(2r+1)}{(j-r)!(j+r+1)!}, \qquad D_j(r) = \frac{(2j+1)!(2r+2)}{(j-r)!(j+r+2)!}. \end{aligned} \tag{20}$$

Proof We observe that

$$\lambda_{f\otimes f}(p)=\lambda_f^2(p).$$

By iteration, we get

$$\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p)=\lambda_f^l(p).$$

We first consider the case $l=2j$ even. By comparing the pth coefficients in [19, Lemma 7.1], for $l=2j$,

$$\lambda_f^l(p)=A_j+\sum_{1\leqslant r\leqslant j-1}C_j(r)\lambda_{\mathrm{sym}^{2r}f}(p)+\lambda_{\mathrm{sym}^{2j}f}(p), \tag{21}$$

where A_j, $C_j(r)$ are defined as in (20). Hence, by (21), we have

$$\begin{aligned}
&\sum_p(\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p)-\lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))^2p^{-\sigma}\\
&=\sum_p\frac{(\lambda_f^l(p)-\lambda_g^l(p))^2}{p^\sigma}\\
&=\sum_p\frac{\lambda_f^{2l}(p)+\lambda_g^{2l}(p)-2\lambda_f^l(p)\lambda_g^l(p)}{p^\sigma}\\
&=2\sum_p\frac{A_l}{p^\sigma}-2\sum_p\frac{A_j^2}{p^\sigma}+\sum_l\sum_p\frac{\lambda_{\mathrm{sym}^l f}(p)}{p^\sigma}+\sum_{l'}\sum_p\frac{\lambda_{\mathrm{sym}^{l'}g}(p)}{p^\sigma}\\
&\quad+\sum_{l,l'}\sum_p\frac{\lambda_{\mathrm{sym}^l f\times\mathrm{sym}^{l'}g}(p)}{p^\sigma}\\
&=\sum_p\frac{2(A_l-A_j^2)}{p^\sigma}+O(1),\quad \text{as}\quad \sigma\to 1^+,
\end{aligned}$$

where l, l' ranges over certain positive integers.

Now we turn to the case $l=2j+1$. By comparing the pth coefficients in [19, Lemma 7.1], for $l=2j+1$,

$$\lambda_f^l(p)=B_j\lambda_f(p)+\sum_{1\leqslant r\leqslant j-1}D_j(r)\lambda_{\mathrm{sym}^{2r+1}f}(p)+\lambda_{\mathrm{sym}^{2j+1}f}(p), \tag{22}$$

where B_j, $D_j(r)$ are defined as in (20). Therefore, using (21) and (22), we obtain

$$\begin{aligned}
&\sum_p\left(\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p)-\lambda_{g\otimes g\otimes\cdots\otimes_l g}(p)\right)^2p^{-\sigma}\\
&=\sum_p\frac{(\lambda_f^l(p)-\lambda_g^l(p))^2}{p^\sigma}\\
&=\sum_p\frac{\lambda_f^{2l}(p)+\lambda_g^{2l}(p)-2\lambda_f^l(p)\lambda_g^l(p)}{p^\sigma}\\
&=2\sum_p\frac{A_l}{p^\sigma}+\sum_l\sum_p\frac{\lambda_{\mathrm{sym}^l f}(p)}{p^\sigma}+\sum_{l'}\sum_p\frac{\lambda_{\mathrm{sym}^{l'}g}(p)}{p^\sigma}\\
&\quad+\sum_{l,l'}\sum_p\frac{\lambda_{\mathrm{sym}^l f\times\mathrm{sym}^{l'}g}(p)}{p^\sigma}\\
&=\sum_p\frac{2A_l}{p^\sigma}+O(1),\quad \text{as}\quad \sigma\to 1^+,
\end{aligned}$$

where l, l' runs over suitable positive integers. □

Lemma 3.2 *Let $l \geqslant 4$ be any fixed integer, and let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Then*

(i) For $l = 2j$ even integer, we have

$$\begin{aligned}&\sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))^2 \mathbf{1}_Q(p) p^{-\sigma}\\ &\quad = \sum_p \frac{\kappa_l}{p^\sigma} + O(1), \quad \textit{as} \quad \sigma \to 1^+,\end{aligned} \tag{23}$$

where $\kappa_l = A_l - A_j^2$.

(ii) For $l = 2j + 1$ odd integer, we have

$$\begin{aligned}&\sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))^2 \mathbf{1}_Q(p) p^{-\sigma}\\ &\quad = \sum_p \frac{\tilde{\kappa}_l}{p^\sigma} + O(1), \quad \textit{as} \quad \sigma \to 1^+,\end{aligned} \tag{24}$$

where $\tilde{\kappa}_l = A_l$. Here the constant A_l ($l \geqslant 2$) is given by (20).

Proof We only consider the case $l = 2j$, since the second case follows the similar argument. By (11) and Lemma 3.1, one has

$$\begin{aligned}&\sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))^2 \mathbf{1}_Q(p) p^{-\sigma}\\ &= \frac{1}{2w_D} \sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))^2 r_Q(p) p^{-\sigma}\\ &= \frac{1}{2} \sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))^2 (1 + \chi_D(p)) p^{-\sigma}\\ &= \frac{1}{2} \sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))^2 p^{-\sigma} + O(1)\\ &= \frac{1}{2} \sum_p \frac{\sigma_l}{p^\sigma} + O(1), \quad \text{as} \quad \sigma \to 1^+.\end{aligned}$$

where $\sigma_l = 2(A_l - A_j^2)$ is defined as in Lemma 3.1. □

Now we are in the stage to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1 Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Let $l \geqslant 4$ be any fixed integer, and set

$$h_l(p) := \lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p). \tag{25}$$

We first consider the case $l = 2j$ even. From [21, Lemma 3.1], for $j' = 2l'$ ($l \in \mathbb{N}$), we know that

$$\left|\lambda_f(p^{2l'}) - \lambda_g(p^{2l'})\right| \leqslant 4l'.$$

Hence,

$$\begin{aligned}\left|h_l(p)\right| &= \left|\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p)\right| = \left|\lambda_f^l(p) - \lambda_g^l(p)\right|\\ &\leqslant \sum_{1\leqslant r\leqslant j-1} C_j(r)\left|\lambda_{\mathrm{sym}^{2r} f}(p) - \lambda_{\mathrm{sym}^{2r} g}(p)\right| + \left|\lambda_{\mathrm{sym}^{2j} f}(p) - \lambda_{\mathrm{sym}^{2j} g}(p)\right|\\ &\leqslant 4 \sum_{1\leqslant r\leqslant j-1} C_j(r) r + 4j.\end{aligned} \tag{26}$$

We infer from Lemma 2.1 and (21) that

$$\begin{aligned}
&\sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))p^{-\sigma}\\
&= \sum_p \frac{\lambda_f^l(p) - \lambda_g^l(p)}{p^\sigma}\\
&= \sum_{1\leqslant r\leqslant j-1}\left(C_j(r)\sum_p \frac{\lambda_{\mathrm{sym}^{2r}f}(p) - \lambda_{\mathrm{sym}^{2r}g}(p)}{p^\sigma}\right) + \sum_p \frac{\lambda_{\mathrm{sym}^{2j}f}(p) - \lambda_{\mathrm{sym}^{2j}g}(p)}{p^\sigma}\\
&= O(1), \quad \text{as } \sigma \to 1^+. \qquad (27)
\end{aligned}$$

Applying Lemma 2.2 with $|h_l(p)| \leqslant 4\sum_{1\leqslant r\leqslant j-1} C_j(r)r + 4j$ in (26) in place of $h(p)$, and $a = \sigma_l = 2(A_l - A_j^2)$ by (18) and $A = 0$, we obtain

$$\frac{2(A_l - A_j^2)}{2\cdot\left(4\sum_{1\leqslant r\leqslant j-1} C_j(r)r + 4j\right)^2}\sum_p \frac{1}{p^\sigma} + O(1) \leqslant \sum_{p\in D_l}\frac{1}{p^\sigma} + O(1), \quad \text{as } \sigma \to 1^+.$$

i.e., the set D_l has analytic density at least e_l, where e_l is defined as in (6).

Now we consider the case $l = 2j+1$. Let $h_l(p)$ be defined as in (25). From [21, Lemma 3.1], for $j'' = 2l''+1$ ($l'' \in \mathbb{N}\cup\{0\}$), we know that

$$\left|\lambda_f(p^{2l''+1}) - \lambda_g(p^{2l''+1})\right| \leqslant 4l'' + 4. \qquad (28)$$

By using (22) and (28), we obtain

$$\begin{aligned}
\left|h_l(p)\right| &= \left|\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p)\right| = \left|\lambda_f^l(p) - \lambda_g^l(p)\right|\\
&\leqslant B_j\left|\lambda_f(p) - \lambda_g(p)\right| + \sum_{1\leqslant r\leqslant j-1} D_j(r)\left|\lambda_{\mathrm{sym}^{2r+1}f}(p) - \lambda_{\mathrm{sym}^{2r+1}g}(p)\right|\\
&\quad + \left|\lambda_{\mathrm{sym}^{2j+1}f}(p) - \lambda_{\mathrm{sym}^{2j+1}g}(p)\right|\\
&\leqslant 4B_j + 4\sum_{1\leqslant r\leqslant j-1} D_j(r)(r+1) + 4(j+1). \qquad (29)
\end{aligned}$$

Invoking Lemma 2.1 and using (22), we get

$$\begin{aligned}
&\sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))p^{-\sigma}\\
&= \sum_p \frac{\lambda_f^l(p) - \lambda_g^l(p)}{p^\sigma}\\
&= B_j\sum_p \frac{\lambda_f(p) - \lambda_g(p)}{p^\sigma} + \sum_{1\leqslant r\leqslant j-1}\left(D_j(r)\sum_p \frac{\lambda_{\mathrm{sym}^{2r+1}f}(p) - \lambda_{\mathrm{sym}^{2r+1}g}(p)}{p^\sigma}\right)\\
&\quad + \sum_p \frac{\lambda_{\mathrm{sym}^{2j+1}f}(p) - \lambda_{\mathrm{sym}^{2j+1}g}(p)}{p^\sigma}\\
&= O(1), \quad \text{as } \sigma \to 1^+. \qquad (30)
\end{aligned}$$

Now we apply Lemma 2.2 with $|h_l(p)| \leqslant 4B_j + 4\sum_{1\leqslant r\leqslant j-1} D_j(r)(r+1) + 4(j+1)$ in (29) in place of $h(p)$, and $a = \sigma_l = 2A_l$ by (19) and $A = 0$, we obtain

$$\begin{aligned}
&\frac{2A_l}{2\cdot\left(4B_j + 4\sum_{1\leqslant r\leqslant j-1} D_j(r)(r+1) + 4(j+1)\right)^2}\sum_p \frac{1}{p^\sigma} + O(1)\\
&\leqslant \sum_{p\in D_l}\frac{1}{p^\sigma} + O(1), \quad \text{as } \sigma \to 1^+. \qquad (31)
\end{aligned}$$

i.e., the set D_l has analytic density at least e_l, where e_l is defined as in (6). □

Proof of Theorem 1.2 Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Set $h_l(p)$ as defined in (25). We only consider the case $l = 2j$ even integer, since the other case can be treated in the similar manner.

We learn from Lemma 2.1 and (27) that

$$\begin{aligned}
&\sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))\mathbf{1}_Q(p)p^{-\sigma} \\
&= \frac{1}{2w_D}\sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))r_Q(p)p^{-\sigma} \\
&= \frac{1}{2}\sum_p (\lambda_{f\otimes f\otimes\cdots\otimes_l f}(p) - \lambda_{g\otimes g\otimes\cdots\otimes_l g}(p))(1+\chi_D(p))p^{-\sigma} \\
&= O(1), \quad \text{as } \sigma \to 1^+.
\end{aligned}$$

Applying Lemma 2.3 with $h_l(p)$ in place of $k(p)$, and $c = \kappa_l = A_l - A_j^2$ in Lemma 3.2 and $C = 0$, we obtain that the set $\tilde{D}_l$ has analytic density at least $\tilde{e}_l = \frac{1}{2}e_l$, where e_l is defined as in (6). □

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Declarations

Conflict of Interest The author declares that there is no conflict of interests regarding the publication of this paper.

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