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ORIGINAL RESEARCH

Further results on alternating two-stage iterative method

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Abstract Matrix splitting is an efficient and readily used technique for study of solution of linear systems, iteratively. Migallón *et al.* [Adv. Eng. Softw. 41:13-21, 2010] proposed alternating two-stage methods in which the inner iterations are accomplished by an alternating method. However, the convergence theory of an alternating two-stage iteration scheme for various class of matrix splittings is a literature gap. In this article, we establish convergence theory of alternating two-stage iterative methods for nonsingular, consistent singular and inconsistent rectangular (or singular) linear systems for different class of matrix splittings. Finally, numerical computations are performed which illustrate that this method has some advantages over simple two-stage iterative method.

Keywords Linear system · Alternating two-stage iteration · Convergence · Nonnegativity · Matrix splittings

1 Introduction

Let us consider the problem of finding a numerical solution of linear systems of the form

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Such systems occur on many occasions including discretization of Fredholm integral equations of the first kind. Similarly, the case when $m = n$ and A is nonsingular appears while discretizing an elliptic partial differential equation (see [24,26]). Likewise, singular linear systems of equations (i.e., where A is singular) arise in many problems that include finite difference representation of Neumann problems, analysis of finite element electromagnetic using edge elements and computing stationary probability vectors of stochastic matrices in the analysis of Markov chains, to name a few. Matrix splitting is a classical and popular method (due to its simple implementation) for solving such problems, iteratively. One of the most used matrix splitting method is the two-stage iterative method. The advantage of such a procedure is that they can easily be extended for nonlinear equations (see Douglas [9]) and hence useful for a large class of engineering problems. Particularly, Wachspress [26] studied the practical applications of iterative methods to the numerical solution of neutron group diffusion equations of reactor physics. Two-stage method is also applicable in PageRank problems (see [16]). One is referred to the recent article [13] for numerical investigation of treated brain glioma model using two-stage method.

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When $m = n$ and A is nonsingular, a matrix splitting $A = B - C$ (B is nonsingular) leads to the iterative method of the form

$$x_{k+1} = B^{-1}Cx_k + B^{-1}b, \quad k = 0, 1, 2, \ldots. \tag{1.2}$$

This method (1.2) converges for any initial guess x_o if $\rho(B^{-1}C) < 1$ [25]. Here, ρ stands for the spectral radius (maximum of moduli of eigenvalues of a matrix). For nonsingular matrix $A \in \mathbb{R}^{n\times n}$, a splitting $A = B - C$ is called *a regular splitting* [25] if B^{-1} exists, $B^{-1} \geq 0$ and $C \geq 0$ (entrywise comparison). It is called *a weak regular splitting of type I (or type II)* [7] if B^{-1} exists, $B^{-1} \geq 0$ and $B^{-1}C \geq 0$ (or $CB^{-1} \geq 0$). Convergence of (1.2) for these classes of matrix splittings are well established (see [3,25]). Nichols [20] proposed the notion of a two-stage iterative method. Later, Frommer and Szyld [10] considered the two-stage method that comprises approximating (1.2) (called outer iteration) by another iterative method (called the inner iteration) using another matrix splitting $B = X - Y$ (called the inner splitting), and the method performs $s(k)$ inner iterations at each k^{th} outer step that leads to iterative method of the form

$$x_{k+1} = (X^{-1}Y)^{s(k)}x_k + \sum_{j=0}^{s(k)-1} (X^{-1}Y)^j X^{-1}(Cx_k + b), \quad k = 0, 1, 2, \ldots. \tag{1.3}$$

In 2010, Migallón *et al.* [15] proposed an alternating two-stage method in which at each k^{th} outer iteration, the linear system (1.2) is approximated by using an alternating iterative method proposed by Benzi and Szyld [1]. In particular, if $B = U - V = X - Y$ are two inner splittings and $A = B - C$ is an outer splitting, then the alternating two-stage method is given as

$$x_{k+1} = (X^{-1}YU^{-1}V)^{s(k)}x_k + \sum_{j=0}^{s(k)-1} (X^{-1}YU^{-1}V)^j X^{-1}(YU^{-1} + I)(Cx_k + b), k = 0, 1, 2, \ldots. \tag{1.4}$$

The authors of [15] established the convergence of (1.4) by considering $A = B - C$ as a convergent regular splitting and $B = U - V = X - Y$ as a convergent weak regular splitting of type I. In this article, we establish the convergence of the alternating two-stage iterative method (1.4) when $A = B - C$ is a regular splitting and $B = U - V = X - Y$ is weak regular splittings of type II, thus expanding the convergence theory of alternating two-stage iterative method.

Now, when the system (1.1) is inconsistent, we want to get the least-squares solution of the minimum norm, i.e., $A^\dagger b$, where $A^\dagger$ denotes the Moore-Penrose inverse of the matrix A, using a matrix splitting. For this purpose, Berman and Plemmons [2] proposed proper splitting. A splitting $A = B - C$ of $A \in \mathbb{R}^{m\times n}$ is called *a proper splitting* [2] if $R(B) = R(A)$ and $N(B) = N(A)$, where $R(E)$ and $N(E)$ denote the range space and the null space of a matrix E, respectively. Using a proper splitting, the authors introduced the following iterative method

$$x_{k+1} = B^\dagger Cx_k + B^\dagger b, \quad k = 0, 1, 2, \ldots, \tag{1.5}$$

and showed that the iterative method (1.5) converges to $A^\dagger b$ for every initial vector x_o if and only if $\rho(B^\dagger C) < 1$ (Corollary 1, [2]). Later, several authors introduced new sub-classes of proper splittings and established their convergence (see [8,18]). Climent and Perea [6] first generalized the notion of the two-stage iterative technique in rectangular matrix setting. The author considered the following iterative method

$$x_{k+1} = (X^\dagger Y)^{s(k)}x_k + \sum_{j=0}^{s(k)-1} (X^\dagger Y)^j X^\dagger(Cx_k + b), \quad k = 0, 1, 2, \ldots \tag{1.6}$$

and established that it converges to $A^\dagger b$ for any initial vector x_o, where $A = B - C$ is a proper regular splitting and $B = X - Y$ is a proper weak regular splitting of type I. A proper splitting $A = B - C$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper regular splitting* if $B^\dagger \geq 0$ and $C \geq 0$, it is called *proper weak regular splitting of type I (or type II)* if $B^\dagger \geq 0$ and $B^\dagger C \geq 0$ (or $CB^\dagger \geq 0$) [6,8,14]. Motivated by the previous works on two-stage iterative method, we present alternating two-stage iterative method in rectangular matrix setting obtained using the inner proper splittings $B = U - V = X - Y$ which takes the form

$$x_{k+1} = (X^\dagger YU^\dagger V)^{s(k)}x_k + \sum_{j=0}^{s(k)-1} (X^\dagger YU^\dagger V)^j X^\dagger(YU^\dagger + I)(Cx_k + b), \quad k = 0, 1, 2, \ldots. \tag{1.7}$$

We obtain convergence criteria of (1.7) when $A = B - C$ is proper regular and $B = U - V = X - Y$ are proper weak regular splittings of type I or II.

Next, let us consider the case when $m = n$ and A is singular and the system (1.1) is consistent. If A is singular, then clearly $\rho(B^{-1}C) \nless 1$ in (1.2), as in this case, 1 is an eigenvalue of $B^{-1}C$ and so convergence of (1.2) fails, i.e., $B^{-1}C$ is not zero-convergent, and we look for some conditions so that (1.2) converges to a solution of (1.1). In this case, we say that the method (1.2) is semiconvergent. Semiconvergence of (1.2) has been studied for different class of matrix splittings in [19]. However, semiconvergence of (1.4) when the inner iteration splittings $B = U - V = X - Y$ are weak regular of type II has not been considered yet. In Section 4, we will deal with this problem.

To achieve our aim, the article is organized as follows. In Section 2, we introduce some basic results which are helpful in proving the main results. Section 3 establishes convergence result for (1.7) in a rectangular setting when the inner iteration is accomplished by proper weak regular splittings of type I (or type II). Convergence and comparison results for alternating two-stage iterative method (1.4) where the inner splittings are weak regular splittings of type II are obtained as corollaries. The comparison tables demonstrating the efficiency of alternating two-stage method over two-stage method are also shown in this section. Finally, we establish semiconvergence of (1.4) for a singular linear system in Section 4.

2 Preliminaries

Let $\mathbb{R}^n$ denotes an n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ denotes the set of all real matrices of order $m \times n$. A^T denotes the transpose of the matrix A. For $A \in \mathbb{R}^{m\times n}$, the unique matrix $X \in \mathbb{R}^{n\times m}$ is called the *Moore-Penrose inverse* of A if it satisfies the following four equations:

$$AXA = A, \quad XAX = X, \quad (AX)^T = AX \text{ and } (XA)^T = XA,$$

and is denoted by $A^\dagger$. The Moore-Penrose inverse exists for every $A \in \mathbb{R}^{m\times n}$. When A is nonsingular, then the Moore-Penrose inverse of A coincides with the usual matrix inverse, i.e., $A^\dagger = A^{-1}$. It is well-known that $A \in \mathbb{R}^{n\times n}$ is *monotone* if and only if A^{-1} exists and $A^{-1} \geq 0$ (see [4]). A matrix $A \in \mathbb{R}^{m\times n}$ is called *semimonotone* if $A^\dagger \geq 0$ (see [3]). Let $L \oplus M = \mathbb{R}^n$, then $P_{L,M}$ is the projection onto L along M. So, $P_{L,M}A = A$ if and only if $R(A) \subseteq L$, and $AP_{L,M} = A$ if and only if $N(A) \supseteq M$. If $L \perp M$, then $P_{L,M}$ is simply denoted by P_L. A few properties of $A^\dagger$ which will be frequently used in this article are: $R(A^\dagger) = R(A^T)$; $N(A^\dagger) = N(A^T)$; $AA^\dagger = P_{R(A)}$ and $A^\dagger A = P_{R(A^T)}$. The unique matrix $X \in \mathbb{R}^{n\times n}$ satisfying the matrix equations $AXA = A$, $XAX = X$ and $AX = XA$ is called the *group inverse* of a matrix $A \in \mathbb{R}^{n\times n}$, and is denoted by $A^\#$. It exists if and only if A has index 1, i.e., $rank(A^2) = rank(A)$. Some useful properties of the group inverse are: $R(A) = R(A^\#)$; $N(A) = N(A^\#)$; $AA^\# = P_{R(A),N(A)} = A^\#A$. In particular, if $x \in R(A)$, then $x = A^\#Ax$.

A matrix $A \in \mathbb{R}^{n\times n}$ is called *convergent* (or *zero-convergent*) if $\lim\limits_{k\to\infty} A^k = O$ and *semiconvergent* if $\lim\limits_{k\to\infty} A^k$ exists, although it may not be the zero matrix. A is called an M-matrix if $A = sI - B$ such that $s > 0$ and $s \geq \rho(B)$. It is a nonsingular M-matrix if $s > \rho(B)$ and singular M-matrix if $s = \rho(B)$. An M-matrix A is said to have *property c* if $s^{-1}B$ is semiconvergent for some s. The next two results recalled below deal with a nonnegative matrix and its spectral radius.

Lemma 2.1 ([3, Theorem 2.1.11]) *Let $A \geq 0$. Then, $Ax \leq \beta x$, $x > 0$, implies $\rho(A) \leq \beta$.*

Theorem 2.2 ([25, Theorem 3.15]) *Let $A \in \mathbb{R}^{n\times n}$ and $A \geq 0$. Then $\rho(A) < 1$ if and only if $(I - A)^{-1}$ exists and $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \geq 0$.*

The following is an important result obtained by Climent and Perea [5]. The result allows us to prove some main results of this article.

Theorem 2.3 ([5, Theorem 3]) *Let $B = U - V = X - Y$ be two weak regular splittings of type II of a monotone matrix B, then $\rho(X^{-1}YU^{-1}V) < 1$. Assume that $H = X^{-1}YU^{-1}V$ and $S = YX^{-1}VU^{-1}$, then H and S induce same unique splitting $B = F - G$ which is also a weak regular splitting of type II.*

3 Convergence Results for Nonsingular and Rectangular Linear Systems

This section is divided into two subsections. In the first subsection, we discuss convergence of two-stage iterative method and properties of the splittings induced by its iteration matrix in case of rectangular systems. The results proved here will be further utilized in the next subsection to establish convergence of alternating two-stage iterative method and to show that alternating two-stage iteration converges faster than two-stage iteration. The nonsingular version of the results are provided in Subsection 3.2 as corollaries.

3.1 Two-stage method

Rewriting (1.6) in the standard form gives

$$x_{k+1} = (T_{TS})_{s(k)} x_k + \mathbb{P}^{\dagger}_{s(k)} b, \quad k = 0, 1, 2, \ldots, \tag{3.1}$$

where

$$(T_{TS})_{s(k)} = (X^{\dagger}Y)^{s(k)} + \sum_{j=0}^{s(k)-1} (X^{\dagger}Y)^j X^{\dagger} C \tag{3.2}$$

and

$$\mathbb{P}^{\dagger}_{s(k)} = \sum_{j=0}^{s(k)-1} (X^{\dagger}Y)^j X^{\dagger} = (I - (X^{\dagger}Y)^{s(k)})(I - X^{\dagger}Y)^{-1} X^{\dagger} = (I - (X^{\dagger}Y)^{s(k)}) B^{\dagger}. \tag{3.3}$$

Note that for type II proper weak regular splitting, the matrix given as

$$\mathbb{T}_{s(k)} = (YX^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1} (XY^{\dagger})^j C X^{\dagger} \tag{3.4}$$

is nonnegative.

The following lemma guarantees that $\rho(\mathbb{T}_{s(k)}) = \rho((T_{TS})_{s(k)})$. So, $\rho((T_{TS})_{s(k)}) < 1$ whenever $\rho(\mathbb{T}_{s(k)}) < 1$.

Lemma 3.1 *Let $\mathbb{T}_{s(k)}$ be defined as (3.4) and $T_{s(k)}$ be defined as (3.2) for $s(k) = 1, 2, \ldots$. If $CF^{\dagger}G = GF^{\dagger}C$, then $A(T_{TS})_{s(k)}A^{\dagger} = \mathbb{T}_{s(k)}$. Further, $\rho(\mathbb{T}_{s(k)}) = \rho((T_{TS})_{s(k)})$.*

Proof Since $CF^{\dagger}G = GF^{\dagger}C$, we have $C(F^{\dagger}G)^{s(k)} = (GF^{\dagger})^{s(k)}C$ for any nonnegative integer $s(k)$. Also, we observe that $B^{\dagger}GF^{\dagger} = F^{\dagger}GB^{\dagger}$. Therefore, $B^{\dagger}(GF^{\dagger})^j = (F^{\dagger}G)^j B^{\dagger}$ for any nonnegative integer j. Similarly, it is easy to see that $BB^{\dagger}(GF^{\dagger})^{s(k)} = (GF^{\dagger})^{s(k)}$. Now,

$$\begin{aligned}
A(T_{TS})_{s(k)}A^{\dagger} &= A[I - (I - (F^{\dagger}G)^{s(k)})(I - B^{\dagger}C)]A^{\dagger} \\
&= AA^{\dagger} - A(I - (F^{\dagger}G)^{s(k)})B^{\dagger} \\
&= AA^{\dagger} - A(B^{\dagger} - (F^{\dagger}G)^{s(k)}B^{\dagger}) \\
&= AA^{\dagger} - AB^{\dagger} + AB^{\dagger}(GF^{\dagger})^{s(k)} \\
&= CB^{\dagger} + AB^{\dagger}(GF^{\dagger})^{s(k)} \\
&= CB^{\dagger} + (B - C)B^{\dagger}(GF^{\dagger})^{s(k)} \\
&= CB^{\dagger} + BB^{\dagger}(GF^{\dagger})^{s(k)} - CB^{\dagger}(GF^{\dagger})^{s(k)} \\
&= (GF^{\dagger})^{s(k)} + CB^{\dagger}(I - (GF^{\dagger})^{s(k)}) \\
&= (GF^{\dagger})^{s(k)} + CB^{\dagger} \sum_{j=0}^{s(k)-1} (GF^{\dagger})^j (I - GF^{\dagger})
\end{aligned}$$

$$= (GF^{\dagger})^{s(k)} + C\sum_{j=0}^{s(k)-1}(F^{\dagger}G)^{j}B^{\dagger}(I-GF^{\dagger})$$
$$= (GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1}(GF^{\dagger})^{j}CB^{\dagger}(I-GF^{\dagger})$$
$$= (GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1}(GF^{\dagger})^{j}CF^{\dagger} = \mathbb{T}_{s(k)}.$$

Thus, $A(T_{TS})_{s(k)}A^{\dagger} = \mathbb{T}_{s(k)}$.

Now, let $\lambda(\neq 0)$ be an eigenvalue of $\mathbb{T}_{s(k)}$. Then, there exists an eigenvector corresponding to λ, x such that $\mathbb{T}_{s(k)}x = \lambda x$, i.e.,

$$[(GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1}(GF^{\dagger})^{j}CF^{\dagger}]x = \lambda x.$$

Pre-multiplying by $F^{\dagger}$ in the above equation, we get

$$F^{\dagger}[(GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1}(GF^{\dagger})^{j}CF^{\dagger}]x = \lambda F^{\dagger}x. \tag{3.5}$$

Therefore,

$$[(F^{\dagger}G)^{s(k)} + \sum_{j=0}^{s(k)-1}(F^{\dagger}G)^{j}F^{\dagger}C]y = \lambda y.$$

We claim that $y = F^{\dagger}x \neq 0$. Suppose that $y = F^{\dagger}x = 0$, then $x \in N(F^T) = N(A^T)$. Since $x \in R(\mathbb{T}_{s(k)}) \subseteq R(A)$, we have $x \in R(A) \cap N(A^T)$. Thus, $x = 0$, a contradiction. Hence, $\sigma(\mathbb{T}_{s(k)})\backslash\{0\} \subseteq \sigma((T_{TS})_{s(k)})\backslash\{0\}$.

Conversely, let $\mu(\neq 0)$ be an eigenvalue of $T_{s(k)}$. Then, there exists an eigenvector corresponding to μ, x^T such that $x^T(T_{TS})_{s(k)} = \mu x^T$, i.e.,

$$x^T[(F^{\dagger}G)^{s(k)} + \sum_{j=0}^{s(k)-1}(F^{\dagger}G)^{j}F^{\dagger}C] = \mu x^T.$$

Post-multiplying by $F^{\dagger}$ in the above equation, we get

$$x^T[(F^{\dagger}G)^{s(k)} + \sum_{j=0}^{s(k)-1}(F^{\dagger}G)^{j}F^{\dagger}C]F^{\dagger} = \mu x^T F^{\dagger},$$

i.e.,

$$z^T[(GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1}(GF^{\dagger})^{j}CF^{\dagger}] = \mu z^T,$$

where $z^T = x^T F^{\dagger} \neq 0$ by similar argument as before. Therefore, $\sigma((T_{TS})_{s(k)})\backslash\{0\} \subseteq \sigma(\mathbb{T}_{s(k)})\backslash\{0\}$. Thus, $\sigma(T_{s(k)})\backslash\{0\} = \sigma(\mathbb{T}_{s(k)})\backslash\{0\}$. Hence, $\rho((T_{TS})_{s(k)}) = \rho(\mathbb{T}_{s(k)})$. □

Theorem 3.2 *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix A and $B = U - V = X - Y$ be two proper weak regular splittings of type II such that $CU^{\dagger}V = VU^{\dagger}C$ and $CX^{\dagger}Y = YX^{\dagger}C$. Then, the two-stage method (3.1) converges, for any initial vector x_o and for any sequence of inner iterations.*

Proof We have $\mathbb{T}_{s(k)} \geq 0$ such that

$$\begin{aligned}
\mathbb{T}_{s(k)} &= (GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{\dagger})^{j} CF^{\dagger} \\
&= (GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{\dagger})^{j} CB^{\dagger}(I - GF^{\dagger}) \\
&= (GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{\dagger})^{j} (CB^{\dagger} - CB^{\dagger}GF^{\dagger}) \\
&= (GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{\dagger})^{j} (CB^{\dagger} - CF^{\dagger}GF^{\dagger}) \\
&= (GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{\dagger})^{j} (CB^{\dagger} - GF^{\dagger}VB^{\dagger}) \\
&= (GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{\dagger})^{j} (I - GF^{\dagger})CB^{\dagger} \\
&= I - (I - (GF^{\dagger})^{s(k)}) + (I - (GF^{\dagger})^{s(k)})CB^{\dagger} \\
&= I - (I - (GF^{\dagger})^{s(k)})(I - CB^{\dagger}) \\
&= I - \sum_{j=0}^{s(k)-1} (GF^{\dagger})^{j} (I - GF^{\dagger})(I - CB^{\dagger}).
\end{aligned}$$

Let $y > 0$. Then, $x = (I - CB^{\dagger})^{-1}(I - GF^{\dagger})^{-1} y > 0$. Now, post-multiplying $\mathbb{T}_{s(k)} = I - \sum_{j=0}^{s(k)-1} (GF^{\dagger})^{j} (I - GF^{\dagger})(I - CB^{\dagger})$ by x, we get $\mathbb{T}_{s(k)} x \geq 0$ such that $x > x - \sum_{j=0}^{s(k)-1} (GF^{\dagger})^{j} y = \mathbb{T}_{s(k)} x$. By Lemma 2.1, we have $\rho(\mathbb{T}_{s(k)}) < 1$. Hence $\rho((T_{TS})_{s(k)}) < 1$ by Lemma 3.1. □

Now, we provide some results which guarantee that the unique splitting induced by the iteration matrix $(T_{TS})_{s(k)}$ of the two-stage iterative scheme (3.2) is proper weak regular splitting of type II and regular splitting, respectively. The first one generalizes [23, Theorem 3.6], while the second one is motivated by [22, Theorem 3.6].

Theorem 3.3 *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix $A \in \mathbb{R}^{m \times n}$. Let $B = X - Y$ be a proper weak regular splitting of type II such that $CX^{\dagger}Y = YX^{\dagger}C$. Then, the matrices $(T_{TS})_{s(k)}$ and $\mathbb{T}_{s(k)}$ induce the same splitting $A = U - V$, where $U = A(I - (T_{TS})_{s(k)})^{-1}$. Further, the unique splitting $A = F - G$ induced by the matrix $\mathbb{T}_{s(k)}$ is also a proper weak regular splitting of type II.*

Proof We are given that $U = A(I - (T_{TS})_{s(k)})^{-1}$ and $V = U - A$. Let $F = (I - \mathbb{T}_{s(k)})^{-1} A$ and $G = F - A$. We will show that $T_{s(k)}$ and $\mathbb{T}_{s(k)}$ induce the same splitting $A = U - V$. Since $\mathbb{T}_{s(k)} = A(T_{TS})_{s(k)} A^{\dagger}$, $YA^{\dagger}A = Y$ and $CA^{\dagger}A = C$, it is easy to see that $(T_{TS})_{s(k)} A^{\dagger}A = (T_{TS})_{s(k)}$, and therefore, $\mathbb{T}^{i}_{s(k)} = A(T_{TS})^{i}_{s(k)} A^{\dagger}$ for each nonnegative integer i. Similarly, we can check that $A^{\dagger}A(T_{TS})_{s(k)} = (T_{TS})_{s(k)}$ and so, $F = (I - \mathbb{T}_{s(k)})^{-1} A = \sum_{j=0}^{\infty} \mathbb{T}^{j}_{s(k)} A = \sum_{j=0}^{\infty} A(T_{TS})^{j}_{s(k)} A^{\dagger} A = A \sum_{j=0}^{\infty} (T_{TS})^{j}_{s(k)} = A(I - (T_{TS})_{s(k)})^{-1} = U$. We further have

$$
\begin{aligned}
F^{\dagger} &= A^{\dagger}(I - \mathbb{T}_{s(k)}) \\
&= A^{\dagger}(I - A(T_{TS})_{s(k)}A^{\dagger}) \\
&= (I - (T_{TS})_{s(k)})A^{\dagger} \\
&= (I - (X^{\dagger}Y)^{s(k)})(I - B^{\dagger}C)A^{\dagger} \\
&= (I - (X^{\dagger}Y)^{s(k)})B^{\dagger} \\
&= B^{\dagger}(I - (YX^{\dagger})^{s(k)}) \\
&= X^{\dagger}(I - YX^{\dagger})^{-1}(I - (YX^{\dagger})^{s(k)}) \\
&= X^{\dagger}\sum_{j=0}^{s(k)-1}(YX^{\dagger})^{j} \geq 0.
\end{aligned}
$$

Clearly, $N(U) = N(F) = N((I - \mathbb{T}_{s(k)})^{-1}A) = N(A)$ and the relation $U = A(I - T_{s(k)})^{-1}$ yields $R(A) \subset R(U)$ and $r(A) = r(U)$ implying $R(A) = R(U) = R(F)$. Thus, $A = U - V$ and $A = F - G$ are same proper splittings. Additionally, $AA^{\dagger} = FF^{\dagger}$ implies $FF^{\dagger}\mathbb{T}_{s(k)} = \mathbb{T}_{s(k)}$ which further gives $GF^{\dagger} = (F - A)F^{\dagger} = I - AA^{\dagger}(I - \mathbb{T}_{s(k)}) = \mathbb{T}_{s(k)} \geq_K 0$. Thus, $A = F - G$ is a proper weak regular splitting of type II. Finally, let $A = F_1 - G_1$ be another splitting induced by $\mathbb{T}_{s(k)}$ such that $\mathbb{T}_{s(k)} = G_1 F_1^{\dagger}$. Then, $A = F_1 - \mathbb{T}_{s(k)}F_1 = (I - \mathbb{T}_{s(k)})F_1$ which implies $F_1 = (I - \mathbb{T}_{s(k)})^{-1}A = F$. Hence, $A = F - G$ is a unique weak regular splitting of type II induced by $\mathbb{T}_{s(k)}$. Additionally, one can observe that the induced splitting takes the form $A = \mathbb{P}_{s(k)} - \mathbb{T}_{s(k)}\mathbb{P}_{s(k)}$. □

Theorem 3.4 *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. Let $B = X - Y$ be a proper weak regular splitting of type II such that $CX^{\dagger}Y = YX^{\dagger}C$. If $Y \geq YX^{\dagger}Y$, then the induced splitting $A = \mathbb{P}_{s(k)} - \mathbb{T}_{s(k)}\mathbb{P}_{s(k)}$ is a proper regular splitting.*

Proof By Theorem 3.3, the induced splitting $A = \mathbb{P}_{s(k)} - \mathbb{T}_{s(k)}\mathbb{P}_{s(k)}$ is a proper weak regular splitting of type II. To show that this is a proper regular splitting, we only need to prove that $\mathbb{T}_{s(k)}\mathbb{P}_{s(k)} \geq 0$. We observe that

$$
\begin{aligned}
\mathbb{T}_{s(k)}\mathbb{P}_{s(k)} &= \mathbb{P}_{s(k)} - A \\
&= (I - (YX^{\dagger})^{s(k)})^{-1}B - B + C \\
&= [(I - (YX^{\dagger})^{s(k)})^{-1} - I]B + C \\
&= (I - (YX^{\dagger})^{s(k)})^{-1}(YX^{\dagger})^{s(k)}B + C.
\end{aligned}
$$

Now, to show $(YX^{\dagger})^{s(k)}B \geq 0$, we prove $(YX^{\dagger})^{2}B \geq 0$. Clearly,

$$
\begin{aligned}
(YX^{\dagger})^{2}B &= (YX^{\dagger})(YX^{\dagger})B \\
&= YX^{\dagger}YX^{\dagger}(F - G) \\
&= YX^{\dagger}(Y - YX^{\dagger}G) \geq 0.
\end{aligned}
$$

As $YX^{\dagger} \geq 0$ and $(YX^{\dagger})^{2}B \geq 0$, we have $(YX^{\dagger})^{s(k)}B \geq 0$. Hence, $\mathbb{T}_{s(k)}\mathbb{P}_{s(k)} \geq 0$. □

Before we conclude this section, let us recall two results proved in [22]. These results will be crucial for proving some results in the next section.

Theorem 3.5 (Theorem 3.4, [22]) *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. Let $B = X - Y$ be a proper regular splitting. If $(X^{\dagger}Y)^{2} \leq X^{\dagger}Y$, then the two-stage splitting $A = \mathbb{P}_{s(k)} - \mathbb{P}_{s(k)}(T_{TS})_{s(k)}$ is a proper regular splitting.*

Comparison results help us to choose one splitting over others. Let $(T_{TS})_{s(k)}$ be defined as in (3.2) and $(\overline{T}_{TS})_{s(k)} = (\overline{X}^{\dagger}\overline{Y})^{s(k)} + \sum_{j=0}^{s(k)-1}(\overline{X}^{\dagger}\overline{Y})^{j}\overline{X}^{\dagger}C$. Next result compares the rate of convergence of two-stage iterative methods arising from two different inner splittings.

Theorem 3.6 ([22, Theorem 3.17]) *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. Let $B = X - Y = \overline{X} - \overline{Y}$ be proper weak regular splittings of type I. Consider the iterative scheme defined by (3.1) with corresponding iteration matrices $(T_{TS})_{s(k)}$ and $(\overline{T}_{TS})_{q(k)}$. If the following conditions hold:*

(i) $\overline{\mathbb{P}}_{q(k)}(\overline{T}_{TS})_{q(k)} \geq 0$, $k = 0, 1, 2, \ldots$,

(ii) $\overline{X}^{\dagger} \leq X^{\dagger}$ *and* $q(k) \geq s(k)$, $k = 0, 1, 2, \ldots$,

(iii) $\overline{Y}\,\overline{X}^{\dagger} \geq 0$,
then $\rho((T_{TS})_{s(k)}) \leq \rho((\overline{T}_{TS})_{q(k)}) < 1$.

3.2 Alternating two-stage method

In this subsection we consider a linear system of the form (1.1). As a special case, we obtain analogous results for nonsingular linear systems as corollaries.

We have a proper splitting $A = B - C$, so that we get

$$Bx_{k+1} = Cx_k + b, \quad k = 0, 1, 2, \ldots.$$

For solving the above equation, let $B = U - V = X - Y$ be two proper weak regular splitting of type I (or type II), then alternating iterations give rise to

$$\begin{aligned} y_{l+1/2} &= U^{\dagger}Vy_l + U^{\dagger}\tilde{b}, \\ y_{l+1} &= X^{\dagger}Yy_{l+1/2} + X^{\dagger}\tilde{b}, \quad l = 0, 1, \ldots, s(k) - 1 \end{aligned}$$

where $\tilde{b} = Cx_k + b$. Eliminating $y_{l+1/2}$, we can equivalently write the iterations in the form of a single iterative method given as

$$y_{l+1} = X^{\dagger}YU^{\dagger}Vy_l + X^{\dagger}(YU^{\dagger} + I)\tilde{b}, \quad l = 0, 1, \ldots, s(k) - 1. \tag{3.6}$$

Now, we recall a result which guarantees that the iteration matrix in (3.6) induces splitting of same type as in $B = U - V = X - Y$.

Theorem 3.7 ([17, Theorem 4.4] & [11, Theorem 4.9]) *Let $B = U - V = X - Y$ be two proper weak regular splittings of type I (or type II) of a semimonotone matrix $B \in \mathbb{R}^{m\times n}$. Then the unique splitting $B = F - G$ induced by $H = X^{\dagger}YU^{\dagger}V$ with $F = U(U + X - A)^{\dagger}X$ is a proper weak regular splitting of the same type if $R(U + X - B) = R(B)$ and $N(U + X - B) = N(B)$.*

Remark 3.1 From the above theorem, (3.6) can be re-written as

$$y_{l+1} = F^{\dagger}Gy_l + F^{\dagger}\tilde{b}, \quad l = 0, 1, 2, \ldots, \tag{3.7}$$

where $\tilde{b} = Cx_k + b$ and $B = F - G$ has same type of splitting as that of $B = U - V = X - Y$.

Next result provides convergence criteria for two-stage iterative method of the form (3.1).

Theorem 3.8 ([22, Theorem 2]) *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. Let $B = F - G$ be a proper weak regular splitting of type I. Then, the two-stage iterative method (3.1) is convergent for any sequence $s(k) \geq 1$, $k = 0, 1, 2, \ldots$.*

Remark 3.2 Using iteration (3.7) as inner iteration in two-stage iterative method, the alternating two-stage iterative method (1.7) equivalently becomes

$$x_{k+1} = (T_{ALT-TS})_{s(k)}x_k + \sum_{j=0}^{s(k)-1}(F^{\dagger}G)^jF^{\dagger}\tilde{b}, \tag{3.8}$$

where

$$(T_{ALT-TS})_{s(k)} = (F^{\dagger}G)^{s(k)} + \sum_{j=0}^{s(k)-1}(F^{\dagger}G)^jF^{\dagger}C. \tag{3.9}$$

So, convergence of (1.7) is equivalent to convergence of (3.8).

We are now ready to present a convergence result when the splittings for inner iterations are proper weak regular splittings of type I.

Theorem 3.9 *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix A and $B = U - V = X - Y$ be two proper weak regular splittings of type I. Suppose that $R(U + X - B) = R(B)$ and $N(U + X - B) = N(B)$. Then, the alternating two-stage method (3.8) converges, for any initial vector x_o and for any sequence of inner iterations.*

Proof By Theorem 3.7, the unique splitting induced by the iteration matrix of (3.6) is given by $B = F - G$ which is also a proper weak regular splitting matrix of type I. Now, applying Theorem 3.8 on proper regular splitting $A = B - C$ and proper weak regular splitting $B = F - G$, we get $\rho((T_{ALT-TS})_{s(k)}) < 1$. Hence, (3.8) converges. □

As in the previous subsection, for proper weak regular splittings of type II, the nonnegativity of iteration matrix of (3.8), i.e., nonnegativity of (3.9) is not guaranteed, so we seek help of the nonnegative matrix $\mathcal{T}_{s(k)} = (GF^{\dagger})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{\dagger})^j CF^{\dagger}$.

Theorem 3.10 *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix A and $B = U - V = X - Y$ be two proper weak regular splittings of type II such that $CU^{\dagger}V = VU^{\dagger}C$ and $CX^{\dagger}Y = YX^{\dagger}C$. Suppose that $R(U + X - B) = R(B)$ and $N(U + X - B) = N(B)$. Then, the alternating two-stage method (3.8) converges, for any initial vector x_o and for any sequence of inner iterations.*

Proof By Theorem 3.7, the unique splitting induced by the iteration matrix of (3.6) is given by $B = F - G$ which is also a proper weak regular splitting matrix of type II. Now, applying Theorem 3.2 on proper regular splitting $A = B - C$ and proper weak regular splitting of type II $B = F - G$, we get $\rho((T_{ALT-TS})_{s(k)}) < 1$. Hence, (3.8) converges. □

Re-writing (1.4), we have

$$x_{k+1} = (T_{alt-ts})_{s(k)} x_k + \sum_{j=0}^{s(k)-1} (X^{-1}YU^{-1}V)^j X^{-1}(YU^{-1} + I)b, \quad k = 0, 1, 2, \ldots, \tag{3.10}$$

where

$$(T_{alt-ts})_{s(k)} = (X^{-1}YU^{-1}V)^{s(k)} + \sum_{j=0}^{s(k)-1} (X^{-1}YU^{-1}V)^j X^{-1}(YU^{-1} + I)C \tag{3.11}$$

is the iteration matrix of (3.10). The following result establishes convergence of alternating two-stage iterative method (3.10) in case of a nonsingular linear system which is a special case of Theorem 3.10.

Corollary 3.11 *Let $A = B - C$ be a regular splitting of a monotone matrix A and $B = U - V = X - Y$ be two weak regular splittings of type II with $CU^{-1}V = VU^{-1}C$ and $CX^{-1}Y = YX^{-1}C$. Then, the alternating two-stage method (3.10) converges, for any initial vector x_o and for any sequence of inner iterations.*

Criteria for faster convergence of alternating two-stage over two-stage is obtained next.

Theorem 3.12 *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix $A \in \mathbb{R}^{n\times n}$. Let $B = U - V = X - Y$ be proper weak regular splittings of type II such that $CU^{\dagger}V = VU^{\dagger}C$ and $CX^{\dagger}Y = YX^{\dagger}C$. Suppose that $R(U+X-B) = R(B)$ and $N(U+X-B) = N(B)$. Then, $\rho((T_{ALT-TS})_{s(k)}) \leq \rho((T_{TS})_{s(k)}) < 1$, provided the following conditions hold:*

(i) $V \geq VU^{\dagger}V$,
(ii) $U^{\dagger}V \geq 0$.

Proof Clearly, by Theorem 3.10, we have $\rho((T_{ALT-TS})_{s(k)}) < 1$. By Remark 3.2, the iteration matrix of alternating two-stage iterative method can be given as

$$(T_{ALT-TS})_{s(k)} = (F^{\dagger}G)^{s(k)} + \sum_{j=0}^{s(k)-1} (F^{\dagger}G)^j F^{\dagger}C$$

such that $F^{\dagger}G = X^{\dagger}YU^{\dagger}V$ and $F^{\dagger} = X^{\dagger}(YU^{\dagger} + I)$. The induced splitting $B = F - G$ is also a weak regular splitting of type II. Now, consider the two-stage method corresponding to the matrix splittings $A = B - C$ and $B = X - Y$, then its iteration matrix is given by

$$(T_{TS})_{s(k)} = (U^{\dagger}V)^{s(k)} + \sum_{j=0}^{s(k)-1} (U^{\dagger}V)^{j}U^{\dagger}C.$$

Observe that $F^{\dagger} = X^{\dagger}YU^{\dagger} + X^{\dagger} = X^{\dagger}(X - U + V)U^{\dagger} + X^{\dagger} = U^{\dagger} + X^{\dagger}VU^{\dagger} \geq U^{\dagger}$, since $X^{\dagger}VU^{\dagger} \geq 0$. As $V \geq VU^{\dagger}V$, by Theorem 3.4, the splitting induced by $(T_{TS})_{s(k)}$, $A = \mathbb{P}_{s(k)} - \mathcal{T}_{s(k)}\mathbb{P}_{s(k)}$ is regular and thus $\mathcal{T}_{s(k)}\mathbb{P}_{s(k)} \geq 0$. Hence, the claim directly follows from [23, Theorem 3.11]. □

The above result reduces to the following corollary in case of nonsingular linear system.

Corollary 3.13 *Let $A = B - C$ be a regular splitting of a monotone matrix $A \in \mathbb{R}^{n\times n}$. Let $B = U - V = X - Y$ be weak regular splittings of type II such that $CU^{-1}V = VU^{-1}C$ and $CX^{-1}Y = YX^{-1}C$. Then, $\rho((T_{alt-ts})_{s(k)}) \leq \rho((T_{ts})_{s(k)}) < 1$, provided the following conditions hold:*

(i) $V \geq VU^{-1}V$,

(ii) $U^{-1}V \geq 0$.

Comparison of convergence rate of alternating two-stage iterative method and two-stage iterative method when both the splittings are regular is provided next.

Theorem 3.14 *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. Let $B = U - V = X - Y$ be proper regular splittings. Suppose that $R(U + X - B) = R(B)$ and $N(U + X - B) = N(B)$. If $(X^{\dagger}Y)^2 \leq X^{\dagger}Y$, then the alternating two-stage iterative method converges faster than the two-stage iterative method.*

Proof By Remark 3.2, the iteration matrix of alternating two-stage iterative method is given as

$$(T_{ALT-TS})_{s(k)} = (F^{\dagger}G)^{s(k)} + \sum_{j=0}^{s(k)-1} (F^{\dagger}G)^{j}F^{\dagger}C$$

such that $F^{\dagger}G = X^{\dagger}YU^{\dagger}V$ and $F^{\dagger} = X^{\dagger}(YU^{\dagger} + I)$. The splitting $B = F - G$ is also a regular splitting. Let the iteration matrix given the two-stage splitting corresponding to the splittings $A = B - C$ and $B = X - Y$ be given as

$$(T_{TS})_{s(k)} = (X^{\dagger}Y)^{s(k)} + \sum_{j=0}^{s(k)-1} (X^{\dagger}Y)^{j}X^{\dagger}C.$$

By the hypothesis, it is clear that $\rho((T_{ALT-TS})_{s(k)}) < 1$ and $\rho((T_{TS})_{s(k)}) < 1$. Observe that $F^{\dagger} = X^{\dagger}YU^{\dagger} + X^{\dagger} \geq X^{\dagger}$, since $X^{\dagger}YU^{\dagger} \geq 0$. As $B = X - Y$ is a regular splitting, we have $YX^{\dagger} \geq 0$. Since $(X^{\dagger}Y)^2 \leq X^{\dagger}Y$, the splitting induced by $(T_{TS})_{s(k)}$, $A = \mathbb{P}_{s(k)} - \mathbb{P}_{s(k)}(T_{TS})_{s(k)}$ is a regular splitting by [22, Theorem 3.4]. Therefore, $\mathbb{P}_{s(k)}(T_{TS})_{s(k)} \geq 0$. Finally, by Theorem 3.6, we conclude that $\rho((T_{ALT-TS})_{s(k)}) \leq \rho((T_{TS})_{s(k)}) < 1$. □

Theorem 3.14 reduces to the following corollary when we consider iterative scheme (3.10).

Corollary 3.15 *Let $A = B - C$ be a regular splitting of a monotone matrix $A \in \mathbb{R}^{n\times n}$. Let $B = U - V = X - Y$ be regular splittings. If $(X^{-1}Y)^2 \leq X^{-1}Y$, then the alternating two-stage iterative method converges faster than the two-stage iterative method.*

Theorem 3.16 *Let $A = B - C$ be a proper regular splitting of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. Further, assume that $B = U - V = X - Y$ be proper weak regular of type I and $s(k) \geq 1$, $k = 0, 1, 2\ldots$ be the inner iteration sequence. If x_o and y_o are initial values that hold*

$$x_1 \geq x_o,\ \ y_o \geq y_1 \text{ and } y_o \geq A^{\dagger}b \geq x_o.$$

Then, the sequences $\{x_k\}$ *and* $\{y_k\}$ *generated by*

$$x_{k+1} = (T_{ALT-TS})_{s(k)} x_k + \sum_{j=0}^{s(k)-1} (X^\dagger Y U^\dagger V)^j X^\dagger (Y U^\dagger + I) b,$$

$$y_{k+1} = (T_{ALT-TS})_{s(k)} y_k + \sum_{j=0}^{s(k)-1} (X^\dagger Y U^\dagger V)^j X^\dagger (Y U^\dagger + I) b,$$

$k = 0, 1, 2, \ldots$ *satisfy*

(i) $y_k \geq y_{k+1} \geq x_{k+1} \geq x_k,\ k = 0, 1, 2, \ldots,$
(ii) $\lim_{k\to\infty} x_k = A^\dagger b = \lim_{k\to\infty} y_k$ *and* $y_k \geq y_{k+1} \geq A^\dagger b \geq x_{k+1} \geq x_k,\ k = 0, 1, 2, \ldots.$

Proof By the Remark 3.1, $B = F - G$ is also proper weak regular splitting of type I and convergence of (3.6) is equivalent to convergence of (3.7). Thus, employing $B = F - G$ as inner splitting and $A = B - C$ as outer iteration, we get a two-stage iterative scheme of the form (3.1). Hence, by the Monotone Convergence Theorem [22, Theorem 4.1], we get the desired result. □

The corollary produced below is a special case of Theorem 3.16.

Corollary 3.17 *Let* $A = B - C$ *be a regular splitting of a monotone matrix* $A \in \mathbb{R}^{n\times n}$*. Further, assume that* $B = U - V = X - Y$ *be weak regular of type I and* $s(k) \geq 1$*,* $k = 0, 1, 2\ldots$ *be the inner iteration sequence. If* x_o *and* y_o *are initial values that hold*

$$x_1 \geq x_o,\ \ y_o \geq y_1 \text{ and } y_o \geq A^{-1} b \geq x_o.$$

Then, the sequences $\{x_k\}$ *and* $\{y_k\}$ *generated by*

$$x_{k+1} = (T_{alt-ts})_{s(k)} x_k + \sum_{j=0}^{s(k)-1} (X^{-1} Y U^{-1} V)^j X^{-1} (Y U^{-1} + I) b,$$

$$y_{k+1} = (T_{alt-ts})_{s(k)} y_k + \sum_{j=0}^{s(k)-1} (X^{-1} Y U^{-1} V)^j X^{-1} (Y U^{-1} + I) b,$$

$k = 0, 1, 2, \ldots$ *satisfy*

(i) $y_k \geq y_{k+1} \geq x_{k+1} \geq x_k,\ k = 0, 1, 2, \ldots,$
(ii) $\lim_{k\to\infty} x_k = A^{-1} b = \lim_{k\to\infty} y_k$ *and* $y_k \geq y_{k+1} \geq A^{-1} b \geq x_{k+1} \geq x_k,\ k = 0, 1, 2, \ldots.$

For type II proper weak regular splitting, we need additional conditions $CU^\dagger V = VU^\dagger C$, $CX^\dagger Y = YX^\dagger C$ to guarantee convergence. This is shown next.

Theorem 3.18 *Let* $A = B - C$ *be a regular splitting of a nonnegative and monotone matrix* $A \in \mathbb{R}^{n\times n}$*. Further, assume that* $B = U - V = X - Y$ *be weak regular splittings of type II such that* $CU^\dagger V = VU^\dagger C$*,* $CX^\dagger Y = YX^\dagger C$ *and* $s(k) \geq 1$*,* $k = 0, 1, 2\ldots$ *be the inner iteration sequence. If* x_o *and* y_o *are initial values that hold*

$$x_1 \geq x_o,\ \ y_o \geq y_1 \text{ and } y_o \geq A^\dagger b \geq x_o.$$

Then, the sequences $\{x_k\}$ *and* $\{y_k\}$ *generated by*

$$x_{k+1} = (T_{ALT-TS})_{s(k)} x_k + \sum_{j=0}^{s(k)-1} (X^\dagger Y U^\dagger V)^j X^\dagger (Y U^\dagger + I) b,$$

$$y_{k+1} = (T_{ALT-TS})_{s(k)} y_k + \sum_{j=0}^{s(k)-1} (X^\dagger Y U^\dagger V)^j X^\dagger (Y U^\dagger + I) b,$$

$k = 0, 1, 2, \ldots$ *satisfy*

(i) $y_k \geq y_{k+1} \geq x_{k+1} \geq x_k,\ k = 0, 1, 2, \ldots,$
(ii) $\lim_{k\to\infty} x_k = A^\dagger b = \lim_{k\to\infty} y_k$ *and* $y_k \geq y_{k+1} \geq A^\dagger b \geq x_{k+1} \geq x_k,\ k = 0, 1, 2, \ldots.$

Proof Since the splitting $B = F - G$ induced by the iteration matrix of method (3.7) is also a proper weak regular type II, we only need to show that $CF^\dagger G = GF^\dagger C$. The conditions $CU^\dagger V = VU^\dagger C$ and $CX^\dagger Y = YX^\dagger C$ ensure that $CF^\dagger G = GF^\dagger C$. Then, the proof follows using similar steps as in Theorem 3.16. □

The following result is a nonsingular version of Theorem 3.18.

Corollary 3.19 *Let $A = B - C$ be a regular splitting of a nonnegative and monotone matrix $A \in \mathbb{R}^{n\times n}$. Further, assume that $B = U - V = X - Y$ be weak regular splittings of type II such that $CU^{-1}V = VU^{-1}C$, $CX^{-1}Y = YX^{-1}C$ and $s(k) \geq 1$, $k = 0, 1, 2\ldots$ be the inner iteration sequence. If x_o and y_o are initial values that hold*

$$x_1 \geq x_o,\ y_0 \geq y_1 \text{ and } y_0 \geq A^{-1}b \geq x_o.$$

Then, the sequences $\{x_k\}$ and $\{y_k\}$ generated by

$$x_{k+1} = (T_{alt-ts})_{s(k)}x_k + \sum_{j=0}^{s(k)-1}(X^{-1}YU^{-1}V)^j X^{-1}(YU^{-1} + I)b,$$
$$y_{k+1} = (T_{alt-ts})_{s(k)}y_k + \sum_{j=0}^{s(k)-1}(X^{-1}YU^{-1}V)^j X^{-1}(YU^{-1} + I)b,$$

$k = 0, 1, 2, \ldots$ *satisfy*

(i) $y_k \geq y_{k+1} \geq x_{k+1} \geq x_k,\ k = 0, 1, 2, \ldots,$
(ii) $\lim_{k\to\infty} x_k = A^{-1}b = \lim_{k\to\infty} y_k$ *and* $y_k \geq y_{k+1} \geq A^{-1}b \geq x_{k+1} \geq x_k,\ k = 0, 1, 2, \ldots.$

We conclude this section with four tables to show the effectiveness of alternating two-stage iterative method over simple two-stage method. For this we first consider the two-dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \leq x \leq 1,\ 0 \leq y \leq 1 \tag{3.12}$$

whose boundary conditions is given by

$$u(x, y)|_{\partial R} = x + y + xy.$$

Let the square region $R = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ be covered by a grid with sides parallel to the coordinate axis and with an equal grid spacing $h = \Delta x = \Delta y$. If $Nh = 1$, then the number of internal grid points is $(N-1)^2$. The finite difference method using the $\mathcal{O}(h^2)$ central difference discretization on uniform grids generates the linear system $Ax = b$, where A is of order $(N-1)^2 \times (N-1)^2$ and b is the right-hand side vector derived from the Dirichlet boundary conditions. The coefficient matrix A is of the form $A = I \otimes J + J \otimes I$. Here, $\otimes$ is the Kronecker product, I is an identity matrix of order $(N-1)\times(N-1)$ and $J = tridiagonal\ (-1, 4, -1)$ is of order $(N-1)\times(N-1)$. For computations, MATLAB R2023a has been used. All the computations are run on the laptop with configuration: 12th Generation Intel(R) Core(TM) i5, 16 GB RAM. IT and IT Diff denote the 'Iteration' and 'Iteration Difference', respectively. Similarly, 'Time Diff' is used to calculate Time Differences (in Seconds) of the iterations and 'ρ' stands for the spectral radius of the iteration matrix of the corresponding iterative method.

Let $A = B - C$ and $B = U - V = X - Y$ be the splittings used to formulate the alternating two-stage iterative method, while $A = B - C$ and $B = X - Y$ are used to formulate the two-stage iterative method.

For Table 1, we take $B = 1.1diag(A) + tril(A, -1)$ (lower triangular part of A with diagonals multiplied by 1.1), $U = 2diag(B)$ (a diagonal matrix with diagonal entries twice as that of diagonal entries of B) and $X = diag(B)$ (a diagonal matrix whose diagonal entries are same as that of B), so that $A = B - C$ and $B = U - V = X - Y$ are regular splittings. In Table 1, it is clear that the iterations and amount of time consumed by alternate two-stage method is lower than the two-stage method. The difference in amount of time and iterations significantly increase as we increase the size of the coefficient matrix.

Table 1 Comparison between alternating two-stage and two-stage iteration method

Size	Splitting	IT	$IT\,Diff$	$Time(s)$	Time Diff	ρ
400×400	Alt Two-stage	939	530	0.0282	0.0354	0.9821
	Two-stage	1469		0.0636		0.9886
1156×1156	Alt Two-stage	2689	1519	1.1814	1.1401	0.9935
	Two-stage	4208		2.3215		0.9959
1936×1936	Alt Two-stage	4508	2549	7.0036	4.8085	0.9961
	Two-stage	7057		11.8121		0.9975
4096×4096	Alt Two-stage	9601	5431	112.8019	60.8316	0.9981
	Two-stage	15032		173.6335		0.9988

Table 2 Convergence alternating two-stage when two-stage iteration method fails

Size	Splitting	IT	$Time(s)$	ρ
100×100	Alt Two-stage	207	0.0020	0.9423
	Two-stage	-	1.0102	1.0049
225×225	Alt Two-stage	465	0.0076	0.9761
	Two-stage	-	1.9531	1.0369
400×400	Alt Two-stage	988	0.0361	0.9895
	Two-stage	-	3.8829	1.0492
900×900	Alt Two-stage	62655	30.5494	0.9990
	Two-stage	-	48.1192	1.0584

For Table 2, we take $B = 1.1diag(A) + 2tril(A, -1)$, $U = diag(B)$ and $X = 2diag(B)$ so that B is not a regular or weak regular splitting of type I or II. So, convergence of both the methods are not guaranteed. On computation, it shows that the two-stage method fails. However, if we choose a suitable splitting $B = X - Y$ then the alternating two-stage method works even when the two-stage method fails. That is, applying alternate splittings method in two-stage gives an iteration matrix whose spectral radius is significantly less than 1 even when the spectral radius corresponding to two-stage method is greater than 1. This indicates the importance of second splitting $B = X - Y$ and the alternating two-stage method. The tolerance is fixed at 10^{-6} for all the computations done in this article.

Next, we present the Google's search engine problem and demonstrate a motivation for the given framework. In the PageRank algorithm [12], we find the positive eigenvector corresponding to the simple eigenvalue 1 of the Google PageRank matrix defined by

$$A = \alpha P + (1-\alpha)ve^T, \tag{3.13}$$

where P is a column stochastic matrix, e is the column vector of all ones, v is a nonnegative column vector such that $e^T v = 1$ and $\alpha \in (0, 1)$ is the damping factor which determines the weight given to the web link graph in the model. The matrix A is irreducible, here. Assuming $e^T x = 1$, the above problem (3.13) takes the following form

$$(I - \alpha P)x = (1-\alpha)v. \tag{3.14}$$

When the α is close to one, one prefers splitting method to solve (3.14). For this consider splittings $I - \alpha P = B - C$ and $B = U - V = X - Y$, where $B = I - \beta P$, $U = I - \beta_1 P$ and $X = I - \beta_2 P$ such that $0 < \beta_1 \le \beta_2 \le \beta < \alpha < 1$. Since $I - \alpha P$ is nearly a singular matrix, one may choose suitable values of β_1, β_2, β that makes the computation easier. In particular, when $\beta_1 = 0.1$, $\beta_2 = 0.2$ and $\beta = 0.5$, we have the following comparison tables for different values of α.

4 Semiconvergence of Consistent Singular Systems

This section is devoted to consistent singular linear systems, i.e., the coefficient matrix A is singular and the system has infinite solutions. Here, we discuss the semiconvergence of alternating two-stage iterative method (3.10) when the inner splitting is regular and outer splittings are weak regular of type II. But, first we recall some established results from the literature. The first result connects an M-matrix with property c with nonnegativity of the group inverse of the matrix.

Table 3 Comparison of the alternating two-stage and two-stage method ($\alpha = 0.9$)

Matrix	Alternating Two-Stage		Two-Stage Method	
	IT	Time(s)	IT	Time(s)
SNAP/p2p-Gnutella05–8846	29	103	36	132
SNAP/wiki-Vote–8297	08	22	09	26
Gleich/wb-cs-stanford–9914	33	163	40	200
California–9664	43	391	53	506

Table 4 Comparison of the alternating two-stage and two-stage method ($\alpha = 0.99$)

Matrix	Alternating Two-Stage		Two-Stage Method	
	IT	Time(s)	IT	Time(s)
SNAP/p2p-Gnutella05–8846	261	983	316	1182
SNAP/wiki-Vote–8297	11	35	12	73
Gleich/wb-cs-stanford–9914	354	3311	429	3624
California–9664	345	3094	417	3964

Theorem 4.1 ([21, Theorem 2]) *Let $A \in \mathbb{R}^{n\times n}$. Then, A is an M-matrix with property c if and only if $A^{\#}$ exists and nonnegative on $R(A)$.*

The next two results give sufficient conditions for semiconvergence of (1.2).

Theorem 4.2 ([3]) *Let $A = B - C$ be a regular splitting. Then, A is an M-matrix with property c if and only if $\rho(H) \le 1$ and $N(I - H) \oplus R(I - H) = \mathbb{R}^n$, where $H = CB^{-1}$.*

Theorem 4.3 ([3]) *Let $H \in R^{n\times n}$, with $H \ge 0$, and let M be a $\{1, 2\}$-inverse of $I - H$ with $R(M)$ complementary to $N(I - H)$, such that M is nonnegative on $R(I - H)$. Then, $\rho(H) \le 1$ and $ind(I - H) \le 1$.*

Let $\mathfrak{T}_{s(k)} = (GF^{-1})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{-1})^j CF^{-1}$. Then observe that $\mathfrak{T}_{s(k)} \ge 0$ whenever $A = B - C$ is regular and $B = F - G$ is weak regular of type II. Now, we are ready to provide sufficient conditions for the semiconvergence of alternating two-stage iterative method.

Theorem 4.4 *Let A be an M-matrix with property c. Let the splitting $A = B - C$ be regular, and the splittings $B = U - V = X - Y$ be weak regular splittings of type II with $CU^{-1}V = VU^{-1}C$ and $CX^{-1}Y = YX^{-1}C$. Then, $\rho((T_{alt-ts})_{s(k)}) \le 1$ and $ind(I - (T_{alt-ts})_{s(k)}) \le 1$.*

Proof We have

$$\begin{aligned}
\mathfrak{T}_{s(k)} &= (GF^{-1})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{-1})^j CF^{-1} \\
&= (GF^{-1})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{-1})^j CB^{-1}(I - GF^{-1}) \\
&= (GF^{-1})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{-1})^j (CB^{-1} - CB^{-1}GF^{-1}) \\
&= (GF^{-1})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{-1})^j (CB^{-1} - CF^{-1}GU^{-1})
\end{aligned}$$

$$
\begin{aligned}
&= (GF^{-1})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{-1})^j (CB^{-1} - GF^{-1}CB^{-1}) \\
&= (GF^{-1})^{s(k)} + \sum_{j=0}^{s(k)-1} (GF^{-1})^j (I - GF^{-1})CB^{-1} \\
&= I - (I - (GF^{-1})^{s(k)}) + (I - (GF^{-1})^{s(k)})CB^{-1} \\
&= I - (I - (GF^{-1})^{s(k)})(I - CB^{-1}).
\end{aligned}
$$

So, $I - \mathfrak{T}_{s(k)} = (I - (GF^{-1})^{s(k)})(I - CB^{-1})$. As $B = U - V = X - Y$ are weak regular splittings of type II, thus the induced splitting $B = F - G$ is also a weak regular splittings of type II by Theorem 2.3. Since, $B^{-1} \geq 0$, we have $\rho(F^{-1}G) < 1$ and therefore, $(I - (GF^{-1})^{s(k)})^{-1} \geq 0$ by Theorem 2.2. Also, $(I - CB^{-1})^{\#}$ exists by Theorem 4.2. Clearly, $M = (I - CB^{-1})^{\#}(I - (GF^{-1})^{s(k)})^{-1}$ exists. The matrix M is an {1,2}-inverse of $I - \mathfrak{T}_{s(k)}$. Since $I - CB^{-1}$ has index 1, we have $N(I - CB^{-1}) \oplus R(I - CB^{-1}) = \mathbb{R}^n$. Further, $R(I - CB^{-1}) = R((I - CB^{-1})^{\#}) = R(M)$ and $N(I - CB^{-1}) = N(I - \mathfrak{T}_{s(k)})$, therefore, $N(I - \mathfrak{T}_{s(k)}) \oplus R(M) = \mathbb{R}^n$. So, $N(I - \mathfrak{T}_{s(k)})$ and $R(M)$ are complementary. Now, suppose that $x \in R(I - \mathfrak{T}_{s(k)})$ and $x \geq 0$, then $x = (I - \mathfrak{T}_{s(k)})y$ for some y. Pre-multiplying by $(I - (GF^{-1})^{s(k)})^{-1}$, we get $(I - CB^{-1})y = (I - (GF^{-1})^{s(k)})^{-1}x$ and thus $0 \leq (I - (GF^{-1})^{s(k)})^{-1}x \in R(I - CB^{-1})$. As $(I - CB^{-1})^{\#} \geq 0$ over $R(I - CB^{-1})$ by Theorem 4.1. We finally obtain $Mx \geq 0$. Thus, $M \geq 0$ in $R(I - \mathfrak{T}_{s(k)})$. Hence, by Theorem 4.3, $\rho((T_{alt-ts})_{s(k)}) = \rho(\mathfrak{T}_{s(k)}) \leq 1$ and $ind(I - (T_{alt-ts})_{s(k)}) \leq 1$. □

5 Conclusion

Main findings of this article are summarized below:

1. Convergence result for alternating two-stage method has been established for nonsingular linear system where the inner splittings are governed by type II weak regular splittings. Computational advantage of alternating two-stage method over two-stage method has been shown.
2. For rectangular linear systems, first we have established convergence criteria for two-stage iterative method (3.1) for type II proper weak regular splitting, and then the results are utilized to establish the convergence of alternating two-stage iterative method (3.8).
3. We have analyzed and obtained sufficient conditions for semiconvergence of the alternating two-stage iterative method (3.10) when the coefficient matrix is singular, and the inner splittings $B = U - V = X - Y$ are type II weak regular splittings.

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