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ORIGINAL RESEARCH

On congruences involving harmonic numbers H_{3n} and H_{3n+r}

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Abstract In this paper, we establish various congruences involving harmonic numbers H_{3n} and H_{3n+r} modulo prime number p, ie., $\sum_{0\leq k\leq[p/3]} H_{3k}^2 \pmod{p}$ and $\sum_{0\leq k\leq[p/3]} \frac{H_{3k+r}}{3k+r} \pmod{p}$. Also, we give the generalization of Meštrović's congruence, ie., for any prime number $p \geq 5$,

$$\sum_{k\equiv r \pmod 3}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k} \pmod{p^2},$$

where $r \in \{1, 2, 3\}$.

Keywords Congruences · Harmonic numbers · Abel sum

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1 Introduction

The harmonic numbers are given by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N}^* = \{1, 2, 3, \cdots\},$$

and for each $m = 2, 3, \cdots$, the harmonic numbers of order m are given by

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}, \quad n \in \mathbb{N}^*.$$

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The Bernoulli polynomial $B_n\left(x\right)$ is defined by

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}, \quad \text{for } n \geq 0,$$

where $B_n := B_n(0)$ is the Bernoulli number.
For a prime p, let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is not divisible by p. For $a \in \mathbb{Z}_p$ and prime number p, $q_p\left(a\right)$ is the Fermat quotient defined by

$$q_p\left(a\right) := \frac{a^{p-1}-1}{p}.$$

Let p be an odd prime and $\gcd\left(a, p\right) = 1$. The Legendre symbol$\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue of } p. \end{cases}$$

If p and q are distinct odd numbers, $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \left(-1\right)^{(p-1)/2}\left(-1\right)^{(q-1)/2}$.
In [17], Wolstenholme discovered that for any prime number $p > 3$,

$$H_{p-1} \equiv 0 \pmod{p^2}. \tag{1}$$

In [11], Sun showed that for odd prime number p and $k \in \{0, 1, ..., p-1\}$,

$$(-1)^k \binom{p-1}{k} \equiv 1 - pH_k + \frac{p^2}{2}\left(H_k^2 - H_{k,2}\right) \pmod{p^3}, \tag{2}$$

$$H_{p-1-k} \equiv H_k \pmod{p}, \tag{3}$$

and

$$\sum_{k=1}^{p-1} H_k^2 \equiv -2 \pmod{p}. \tag{4}$$

In [4], Lehmer gave that for prime number $p > 3$,

$$\sum_{k=1}^{[p/3]} \frac{1}{p-3k} \equiv \frac{1}{2} q_p(3) - \frac{1}{4} p q_p^2(3) \pmod{p^2}. \tag{5}$$

Recently, congruences on Bernoulli numbers and harmonic numbers modulo prime powers have been widely studied [5,8,13,14].
Indeed, in [13], Sun investigated that for odd prime number p and $m \in \{1, 2, 3, \ldots, p-4\}$,

$$H_{p-1,m} \equiv \frac{m\left(m+1\right)}{2} \frac{B_{p-2-m}}{p-2-m} p^2 \pmod{p^3}. \tag{6}$$

In [14], Sun obtained that for any prime number $p > 3$,

$$H_{[p/3],2} \equiv \frac{1}{2}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p}, \tag{7}$$

and in [12], Sun and Zhao established some congruences involving harmonic numbers. For example, for any prime number $p > 3$,

$$\sum_{k=1}^{p-1} \frac{H_{k,2}}{k2^k} \equiv -\frac{3}{8} B_{p-3} \pmod{p} \text{ and } \sum_{k=1}^{p-1} \frac{H_{k,2n}^2}{k^{2n}} \equiv 0 \pmod{p},$$

where n is a positive integer with $p-1 \nmid 6n$.
In [16], Wang and Yang obtained a series of modulo p^2 congruences involving harmonic numbers. For example, for any prime number $p>3$,

$$\sum_{k=1}^{p-1} k^2 H_k^2 \equiv \frac{79}{108}p - \frac{4}{9} \pmod{p^2}, \ \sum_{k=1}^{p-1} k H_k^3 \equiv \frac{27}{8}p - \frac{1}{6}pB_{p-3} - 3 \pmod{p^2}.$$

Recently, there are some works involving the generalization of Morley's congruence[10]. Some congruences related to the Fermat quotients and the binomial coefficients can be showed in literature[2,6–9].
In [15], Cai and Granville obtained that for any prime number $p \geq 5$,

$$\sum_{k=0}^{p-1} (-1)^{(a-1)k} \binom{p-1}{k}^a \equiv 2^{a(p-1)} \pmod{p^3},$$

and in [1], Pan showed that for any prime number $p \geq 3$,

$$\sum_{k=0}^{p-1} (-1)^{(a-1)k} \binom{p-1}{k}^a \equiv 2^{a(p-1)} + \frac{a(a-1)(3a-4)}{48}p^3 B_{p-3} \pmod{p^4},$$

where a is a positive integer. In [6], Meštrović proved that for any prime number $p \geq 3$,

$$\sum_{k=0}^{p-1} \binom{p-1}{k}^{-1} \equiv 2^{1-p} - \frac{7}{24}p^3 B_{p-3} \pmod{p^4}.$$

In [9], Meštrović proved new congruences involving binomial coefficients

$$\sum_{k \equiv r \pmod 2}^{p-1} \frac{1}{k}\binom{p-1}{k} \pmod{p^2},$$

where $r \in \{0,1\}$. For example, for $r=1$, the author gave the following congruence

$$\sum_{k \equiv 1 \pmod 2}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k} \equiv -q_p(2) \pmod{p^2}.$$

In this paper, motivated by the above works, we establish various congruences involving harmonic numbers modulo prime number p, ie.,

$$\sum_{k=0}^{[p/3]} H_{3k}^2 \pmod{p} \text{ and } \sum_{k=0}^{[p/3]-1} \frac{H_{3k+r}}{3k+r} \pmod{p},$$

where $r \in \{1,2,3\}$. Also, we give the generalization of Meštrović's congruence, ie.,

$$\sum_{k \equiv r \pmod 3}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k} \pmod{p^2} \quad \text{and} \quad \sum_{k \equiv r \pmod 3}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k}^{-1} \pmod{p^2},$$

where $r \in \{1,2,3\}$ and prime number $p \geq 5$.
In [3], Abel's partial summation formula asserts that every pair of families $(a_k)_{k=1}^n$ and $(b_k)_{k=1}^n$ of complex numbers verifies the identity

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1}\left[(a_k - a_{k+1})\left(\sum_{j=1}^{k} b_j\right)\right] + a_n\left(\sum_{j=1}^{n} b_j\right). \tag{8}$$

2 Some Congruences

In this section, we will consider main theorems including congruences with harmonic numbers. Throughout this section, we take $\delta = \frac{1}{2}\left(\left(\frac{p}{3}\right) - 1\right)$. Firstly, we need some lemmas for future use:

Lemma 1 *Let $p > 3$ be prime number. Then*

$$H_{[p/3]} \equiv -\frac{3}{2}q_p(3) + \frac{3}{4}pq_p^2(3) - \frac{p}{6}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2}. \tag{9}$$

Proof Consider that

$$\sum_{k=1}^{[p/3]} \frac{1}{p-3k} = \sum_{k=1}^{[p/3]} \frac{p+3k}{p^2-9k^2} \equiv -\frac{1}{9}\sum_{k=1}^{[p/3]} \frac{p+3k}{k^2} = -\frac{p}{9}\sum_{k=1}^{[p/3]} \frac{1}{k^2} - \frac{1}{3}\sum_{k=1}^{[p/3]} \frac{1}{k} \pmod{p^2}.$$

From congruences (5) and (7), we obtain

$$-\frac{1}{3}H_{[p/3]} \equiv \frac{p}{18}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) + \frac{1}{2}q_p(3) - \frac{p}{4}q_p^2(3) \pmod{p^2},$$

as claimed. □

Lemma 2 *Let p be odd prime number. Then*

$$\sum_{k=0}^{[p/3]-1} \frac{1}{(3k+1)^2} \equiv \delta + \frac{1}{18}(1-\delta)B_{p-2}\left(\frac{1}{3}\right) \pmod{p}, \tag{10}$$

$$\sum_{k=0}^{[p/3]-1} \frac{1}{(3k+2)^2} \equiv -\frac{1}{18}(2+\delta)B_{p-2}\left(\frac{1}{3}\right) \pmod{p}. \tag{11}$$

Proof For $p \equiv 1 \pmod 3$, use the congruence (7) to get

$$\begin{aligned}\sum_{k=0}^{(p-4)/3} \frac{1}{(3k+1)^2} &= \sum_{k=0}^{(p-4)/3} \frac{1}{(p-3-3k)^2}\\ &\equiv \frac{1}{9}\sum_{k=0}^{(p-4)/3} \frac{1}{(k+1)^2} = \frac{1}{9}\sum_{k=1}^{(p-1)/3} \frac{1}{k^2} \equiv \frac{1}{18}B_{p-2}\left(\frac{1}{3}\right) \pmod{p}.\end{aligned}$$

Also, from the above congruence, we can write

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{1}{k^2} &= \sum_{k=0}^{(p-4)/3} \frac{1}{(3k+1)^2} + \sum_{k=0}^{(p-4)/3} \frac{1}{(3k+2)^2} + \sum_{k=0}^{(p-4)/3} \frac{1}{(3k+3)^2}\\ &\equiv \frac{1}{18}B_{p-2}\left(\frac{1}{3}\right) + \sum_{k=0}^{(p-4)/3} \frac{1}{(3k+2)^2} + \frac{1}{18}B_{p-2}\left(\frac{1}{3}\right) \pmod{p}.\end{aligned}$$

Thus, by congruence (6), the desired result is given. Similarly, for $p \equiv 2 \pmod 3$, the other congruences are obtained. Thus, the proof is complete. □

Lemma 3 *Let p be an odd prime number. Then*

$$\sum_{k=0}^{[p/3]-1} \frac{1}{3k+1} \equiv (1+\delta)\left(\frac{1}{2}q_p(3) - \frac{p}{4}q_p^2(3)\right) - \delta\left(1 + p - \frac{p}{18}B_{p-2}\left(\frac{1}{3}\right)\right) \pmod{p^2}, \tag{12}$$

and

$$\sum_{k=0}^{[p/3]-1} \frac{1}{3k+2} \equiv (1+\delta)\frac{p}{18}B_{p-2}\left(\frac{1}{3}\right) - \delta\left(\frac{1}{2}q_p(3) - \frac{p}{4}q_p^2(3)\right) \pmod{p^2}. \tag{13}$$

Proof We will give the proof of (12). Firstly, for $p \equiv 1 \pmod 3$, we have

$$\begin{aligned}\sum_{k=0}^{(p-4)/3} \frac{1}{3k+1} &= \sum_{k=0}^{(p-4)/3} \frac{1}{p-3k-3} = \sum_{k=0}^{(p-4)/3} \frac{p+3k+3}{p^2-(3k+3)^2} \\ &\equiv -p \sum_{k=0}^{(p-4)/3} \frac{1}{(3k+3)^2} - \sum_{k=0}^{(p-4)/3} \frac{1}{3k+3} \pmod{p^2}.\end{aligned}$$

With the help of the congruences (7) and (9), we get

$$\begin{aligned}\sum_{k=0}^{(p-4)/3} \frac{1}{3k+1} &\equiv -p \sum_{k=0}^{(p-4)/3} \frac{1}{(3k+3)^2} - \sum_{k=0}^{(p-4)/3} \frac{1}{3k+3} \\ &= -\frac{p}{9} \sum_{k=1}^{(p-1)/3} \frac{1}{k^2} - \frac{1}{3} \sum_{k=1}^{(p-1)/3} \frac{1}{k} \\ &\equiv -\frac{p}{18} B_{p-2}\left(\frac{1}{3}\right) - \frac{1}{3}\left(-\frac{p}{6} B_{p-2}\left(\frac{1}{3}\right) - \frac{3}{2} q_p(3) + \frac{3}{4} p q_p^2(3)\right) \\ &= \frac{1}{2} q_p(3) - \frac{1}{4} p q_p^2(3) \pmod{p^2}.\end{aligned}$$

Also, for $p \equiv 2 \pmod 3$, we write

$$\begin{aligned}\sum_{k=0}^{(p-5)/3} \frac{1}{3k+1} &= \sum_{k=1}^{(p-2)/3} \frac{1}{p-3k-1} = \sum_{k=1}^{(p-2)/3} \frac{p+3k+1}{p^2-(3k+1)^2} \\ &\equiv -p \sum_{k=1}^{(p-2)/3} \frac{1}{(3k+1)^2} - \sum_{k=1}^{(p-2)/3} \frac{1}{3k+1} \\ &= -p\left(\sum_{k=0}^{(p-5)/3} \frac{1}{(3k+1)^2} - 1 + \frac{1}{(p-1)^2}\right) - \left(\sum_{k=0}^{(p-5)/3} \frac{1}{3k+1} - 1 + \frac{1}{p-1}\right) \\ &= -p \sum_{k=0}^{(p-5)/3} \frac{1}{(3k+1)^2} + p - \frac{p}{(p-1)^2} - \sum_{k=0}^{(p-5)/3} \frac{1}{3k+1} + 1 - \frac{1}{p-1} \\ &\equiv p + 2 - p \sum_{k=0}^{(p-5)/3} \frac{1}{(3k+1)^2} - \sum_{k=0}^{(p-5)/3} \frac{1}{3k+1} \pmod{p^2},\end{aligned}$$

and with some necessary arrangements, we use the congruence (10) to obtain

$$2 \sum_{k=0}^{(p-5)/3} \frac{1}{3k+1} \equiv p + 2 - p \sum_{k=0}^{(p-5)/3} \frac{1}{(3k+1)^2} \equiv 2 + 2p - \frac{p}{9} B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2}.$$

Thus, the proof is complete. Thus, the proof of (13) is similar to the proof of (12). □

Proposition 1 *For positive integers n and m, we have*

$$\sum_{k=1}^{n} H_{3k,m} = \frac{1}{3} \sum_{k=0}^{n-1} \frac{1}{(3k+2)^m} - \frac{1}{3} H_{3n,m-1} + \frac{2}{3^{m+1}} H_{n,m} + \left(n + \frac{1}{3}\right) H_{3n,m}. \tag{14}$$

Proof By Abel's sum, by taking $a_k = H_{3k,m}$ and $b_k = 1$ in (8), we can write

$$\begin{aligned}
\sum_{k=1}^{n} H_{3k,m} &= -\sum_{k=1}^{n-1}\left(\frac{k}{(3k+3)^m}+\frac{k}{(3k+2)^m}+\frac{k}{(3k+1)^m}\right)+nH_{3n,m}\\
&= -\frac{1}{3}\sum_{k=1}^{n-1}\left(\frac{3k}{(3k+3)^m}+\frac{3k}{(3k+2)^m}+\frac{3k}{(3k+1)^m}\right)+nH_{3n,m}\\
&= -\frac{1}{3}\sum_{k=1}^{n-1}\left(\frac{1}{(3k+3)^{m-1}}+\frac{1}{(3k+2)^{m-1}}+\frac{1}{(3k+1)^{m-1}}\right)\\
&\quad +\frac{1}{3}\sum_{k=1}^{n-1}\left(\frac{3}{(3k+3)^m}+\frac{2}{(3k+2)^m}+\frac{1}{(3k+1)^m}\right)+nH_{3n,m}\\
&= -\frac{1}{3}\sum_{k=4}^{3n}\frac{1}{k^{m-1}}+\frac{1}{3}\sum_{k=1}^{n-1}\frac{2}{(3k+3)^m}+\frac{1}{3}\sum_{k=1}^{n-1}\frac{1}{(3k+2)^m}\\
&\quad +\frac{1}{3}\sum_{k=1}^{n-1}\left(\frac{1}{(3k+3)^m}+\frac{1}{(3k+2)^m}+\frac{1}{(3k+1)^m}\right)+nH_{3n,m}\\
&= -\frac{1}{3}\sum_{k=4}^{3n}\frac{1}{k^{m-1}}+\frac{2}{3^{m+1}}\sum_{k=1}^{n-1}\frac{1}{(k+1)^m}+\frac{1}{3}\sum_{k=1}^{n-1}\frac{1}{(3k+2)^m}\\
&\quad +\frac{1}{3}\sum_{k=4}^{3n}\frac{1}{k^m}+nH_{3n,m},
\end{aligned}$$

and after necessary arrangements, equals

$$\begin{aligned}
&-\frac{1}{3}\sum_{k=1}^{3n}\frac{1}{k^{m-1}}+\frac{1}{3}\left(1+\frac{1}{2^{m-1}}+\frac{1}{3^{m-1}}\right)+\frac{2}{3^{m+1}}\sum_{k=1}^{n}\frac{1}{k^m}-\frac{2}{3^{m+1}}\\
&\quad +\frac{1}{3}\sum_{k=1}^{n-1}\frac{1}{(3k+2)^m}+\frac{1}{3}\sum_{k=1}^{3n}\frac{1}{k^m}-\frac{1}{3}\left(1+\frac{1}{2^m}+\frac{1}{3^m}\right)+nH_{3n,m}\\
&= \frac{1}{3\times 2^m}-\frac{1}{3}H_{3n,m-1}+\frac{2}{3^{m+1}}H_{n,m}+\frac{1}{3}H_{3n,m}+\frac{1}{3}\sum_{k=1}^{n-1}\frac{1}{(3k+2)^m}+nH_{3n,m}\\
&= \frac{1}{3\times 2^n}-\frac{1}{3}H_{3n,m-1}+\frac{2}{3^{m+1}}H_{n,m}+\left(n+\frac{1}{3}\right)H_{3n,m}\\
&\quad +\frac{1}{3}\sum_{k=0}^{n}\frac{1}{(3k+2)^m}-\frac{1}{3\times 2^m},
\end{aligned}$$

as claimed. □

Corollary 1 *Let p be an odd prime number. Then*

$$\begin{aligned}
\sum_{k=1}^{[p/3]} H_{3k} \equiv \frac{1}{3}-\frac{p}{3}+\frac{1}{12}(2+\delta)\left(pq_p^2(3)-2q_p(3)\right)&\\
-\frac{p}{54}(1+3\delta)B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2},& \qquad (15)
\end{aligned}$$

and

$$\sum_{k=1}^{[p/3]} H_{3k,2} \equiv \frac{1}{18}\delta B_{p-2}\left(\frac{1}{3}\right) \pmod{p}. \qquad (16)$$

Proof By taking $n=[p/3]$ and $m=1$ in (14), we can write

$$\sum_{k=1}^{[p/3]} H_{3k} = -[p/3] + \frac{2}{9} H_{[p/3]} + \left([p/3] + \frac{1}{3}\right) H_{3[p/3]} + \frac{1}{3} \sum_{k=0}^{[p/3]-1} \frac{1}{3k+2}.$$

For $p \equiv 1 \pmod 3$, upon using the congruences (1), (9) and (13), we obtain

$$\begin{aligned}\sum_{k=1}^{(p-1)/3} H_{3k} &\equiv -\frac{p-1}{3} - \frac{p}{27} B_{p-2}\left(\frac{1}{3}\right) - \frac{1}{3} q_p(3) + \frac{p}{6} q_p^2(3) + \frac{p}{54} B_{p-2}\left(\frac{1}{3}\right) \\ &= -\frac{p-1}{3} - \frac{1}{3} q_p(3) + \frac{p}{6} q_p^2(3) - \frac{p}{54} B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2}.\end{aligned}$$

Also, for $p \equiv 2 \pmod 3$, upon using the congruences (1), (9) and (13), we obtain

$$\sum_{k=1}^{(p-2)/3} H_{3k} \equiv -\frac{p-1}{3} - \frac{1}{6} q_p(3) + \frac{p}{12} q_p^2(3) + \frac{p}{27} B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2}.$$

Thus, the proof of (15) is complete. Similarly, the proof of the other congruence is obtained. □

Lemma 4 *For all integer number n, we have*

$$\sum_{k=0}^{[n/3]} (-1)^k \binom{n}{3k} = 2 \times 3^{(n-2)/2} \cos\left(\frac{n\pi}{6}\right). \tag{17}$$

Proof From Binomial theorem, it is known that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(1 + j^k + j^{2k}\right) = (1-j)^n + \left(1-j^2\right)^n, \quad j = e^{2\pi i/3}, \tag{18}$$

and since

$$1 + j^k + j^{2k} = \begin{cases} 3, & \text{if } k = 3m \\ 0, & \text{if } k = 3m+1 \\ 0, & \text{if } k = 3m+2 \end{cases},$$

it follows

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(1 + j^k + j^{2k}\right) = 3 \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{3k}.$$

Also, we have

$$\begin{aligned}&(1-j)^n + \left(1-j^2\right)^n \\ &= \left(1 - e^{2\pi i/3}\right)^n + \left(1 - e^{4\pi i/3}\right)^n = 3^{n/2}\left(e^{-n\pi i/6} + e^{n\pi i/6}\right) = 2 \times 3^{n/2} \cos\left(\frac{n\pi}{6}\right).\end{aligned}$$

From equality of the two sides of sum in (18), the result is given. □

Theorem 2 *For prime number $p \geq 5$, we have*

$$\begin{aligned}\sum_{k=0}^{[p/3]} H_{3k}^2 &\equiv -\frac{2}{3} + \frac{4 \times 3^{(p-3)/2}}{p^2} \cos\frac{(p-1)\pi}{6} \\ &\quad + (2+\delta)\left(-\frac{1}{3p} q_p(3) - \frac{2}{3p^2} + \frac{1}{6} q_p^2(3) - \frac{1}{54}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right)\right) \pmod p.\end{aligned} \tag{19}$$

Proof Using the congruence (2) to get

$$\sum_{k=0}^{[p/3]}(-1)^k\binom{p-1}{3k}\equiv\sum_{k=0}^{[p/3]}\left(1-pH_{3k}+\frac{p^2}{2}\left(H_{3k}^2-H_{3k,2}\right)\right)$$
$$=\sum_{k=0}^{[p/3]}1-p\sum_{k=0}^{[p/3]}H_{3k}+\frac{p^2}{2}\sum_{k=0}^{[p/3]}H_{3k}^2-\frac{p^2}{2}\sum_{k=0}^{[p/3]}H_{3k,2}\pmod{p^3},$$

and hence

$$\sum_{k=0}^{[p/3]}H_{3k}^2\equiv\frac{2}{p^2}\sum_{k=0}^{[p/3]}(-1)^k\binom{p-1}{3k}-\frac{2}{p^2}\sum_{k=0}^{[p/3]}1+\frac{2}{p}\sum_{k=0}^{[p/3]}H_{3k}+\sum_{k=0}^{[p/3]}H_{3k,2}\pmod{p^3}.$$

For $p\equiv 1\pmod 3$, with the help of the relation (17), the congruences (15) and (16), it follows

$$\sum_{k=0}^{(p-1)/3}H_{3k}^2\equiv\frac{1}{p^2}\left(4\times 3^{(p-3)/2}\cos\frac{(p-1)\pi}{6}-\frac{4}{3}-\frac{2p}{3}q_p(3)\right)$$
$$-\frac{2}{3}+\frac{1}{3}q_p^2(3)-\frac{1}{27}B_{p-2}\left(\frac{1}{3}\right)\pmod p.$$

Similarly, for $p\equiv 2\pmod 3$, the result is obtained. Hence the proof is complete. □

Theorem 3 *For prime number $p\geq 7$, we have*

$$\sum_{k=0}^{[p/3]-1}\frac{H_{3k+1}}{3k+1}\equiv\frac{(1+\delta)}{p^2}\left(1-3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}\right)$$
$$+\frac{1}{18}B_{p-2}\left(\frac{1}{3}\right)+(1+\delta)\left(\frac{q_p(3)}{2p}-\frac{1}{4}q_p^2(3)\right)\pmod p, \tag{20}$$

$$\sum_{k=0}^{[p/3]-1}\frac{H_{3k+2}}{3k+2}\equiv-\frac{\delta}{p^2}\left(1-2\times 3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}\right)$$
$$-\frac{1}{18}B_{p-2}\left(\frac{1}{3}\right)-\delta\left(\frac{q_p(3)}{2p}-\frac{1}{4}q_p^2(3)\right)\pmod p, \tag{21}$$

and

$$\sum_{k=0}^{[p/3]-1}\frac{H_{3k+3}}{3k+3}$$
$$\equiv-\frac{1}{p^2}-\frac{1}{2p}q_p(3)+\frac{1}{4}q_p^2(3)+(1-\delta)\frac{3^{(p-1)/2}}{p^2}\cos\frac{(p-1)\pi}{6}\pmod p. \tag{22}$$

Proof For $p\equiv 1\pmod 3$, exchanging sums and using the congruences (1) and (3), we have

$$\sum_{k=0}^{(p-5)/3}\frac{H_{3k+1}}{3k+1}$$
$$=\sum_{k=0}^{(p-5)/3}\frac{H_{p-4-3k}}{p-4-3k}=\sum_{k=0}^{(p-5)/3}\frac{H_{p-1-3(k+1)}}{p-1-3(k+1)}$$
$$\equiv-\sum_{k=0}^{(p-5)/3}\frac{H_{3k+3}}{1+3(k+1)}=-\sum_{k=1}^{(p-2)/3}\frac{H_{3k}}{3k+1}=-\sum_{k=0}^{(p-5)/3}\frac{H_{3k}}{3k+1}-\frac{H_{p-2}}{p-1}$$
$$\equiv-\sum_{k=0}^{(p-5)/3}\frac{H_{3k+1}}{3k+1}+\sum_{k=0}^{(p-5)/3}\frac{1}{(3k+1)^2}+1\pmod p,$$

which gives

$$2\sum_{k=0}^{(p-5)/3}\frac{H_{3k+1}}{3k+1}\equiv 1+\sum_{k=0}^{(p-5)/3}\frac{1}{(1+3k)^2}\pmod{p},$$

and by the congruence (10), this last congruence becomes

$$\sum_{k=0}^{(p-5)/3}\frac{H_{3k+1}}{3k+1}\equiv\frac{1}{18}B_{p-2}\left(\frac{1}{3}\right)\pmod{p}.$$

For $p\equiv 2\pmod 3$, we have

$$\sum_{k=0}^{(p-5)/3}H_{3k}^2+\sum_{k=0}^{(p-5)/3}H_{3k+1}^2+\sum_{k=0}^{(p-5)/3}H_{3k+2}^2=\sum_{k=0}^{p-3}H_k^2.$$

Then,

$$\begin{aligned}\sum_{k=0}^{p-3}H_k^2&=\sum_{k=0}^{(p-5)/3}H_{3k}^2+\sum_{k=0}^{(p-5)/3}\left(H_{3k}+\frac{1}{3k+1}\right)^2+\sum_{k=0}^{(p-5)/3}\left(H_{3k}+\frac{1}{3k+1}+\frac{1}{3k+2}\right)^2\\&=3\sum_{k=0}^{(p-5)/3}H_{3k}^2+2\sum_{k=0}^{(p-5)/3}\frac{H_{3k+2}}{3k+2}+\sum_{k=0}^{(p-5)/3}\left(\frac{4H_{3k+1}}{3k+1}-\frac{2}{(3k+1)^2}-\frac{1}{(3k+2)^2}\right),\end{aligned}$$

and from here,

$$\begin{aligned}\sum_{k=0}^{(p-5)/3}\frac{H_{3k+2}}{3k+2}&=\frac{1}{2}\sum_{k=0}^{p-3}H_k^2-\frac{3}{2}\sum_{k=0}^{(p-5)/3}H_{3k}^2-2\sum_{k=0}^{(p-5)/3}\frac{H_{3k+1}}{3k+1}+\sum_{k=0}^{(p-5)/3}\frac{1}{(3k+1)^2}\\&\quad+\frac{1}{2}\sum_{k=0}^{(p-5)/3}\frac{1}{(3k+2)^2}.\end{aligned}$$

Using the congruences (4), (10) and (11), we have

$$\sum_{k=0}^{(p-5)/3}\frac{H_{3k+2}}{3k+2}\equiv-\frac{3}{2}\sum_{k=0}^{(p-5)/3}H_{3k}^2-2\sum_{k=0}^{(p-5)/3}\frac{H_{3k+1}}{3k+1}-\frac{5}{2}+\frac{1}{12}B_{p-2}\left(\frac{1}{3}\right)\pmod{p}.$$

From the congruences (19) and (20), we prove the result.

For $p\equiv 2\pmod 3$, with the help of congruence $\sum_{k=1}^{p-2}\frac{H_k}{k}\equiv 0\pmod{p}$, we get

$$\begin{aligned}\sum_{k=0}^{(p-5)/3}\frac{H_{3k+3}}{3k+3}&=\sum_{k=1}^{p-2}\frac{H_k}{k}-\sum_{k=0}^{(p-5)/3}\frac{H_{3k+1}}{3k+1}-\sum_{k=0}^{(p-5)/3}\frac{H_{3k+2}}{3k+2}\\&\equiv-\left(\sum_{k=0}^{(p-5)/3}\frac{H_{3k+1}}{3k+1}+\sum_{k=0}^{(p-5)/3}\frac{H_{3k+2}}{3k+2}\right)\pmod{p}.\end{aligned}$$

By the congruences (20) and (21), the result is given. Similarly, for $p\equiv 1\pmod 3$, the proofs of the other congruences are clearly obtained. □

Theorem 4 *For prime number $p \geq 5$, we have*

$$\sum_{k\equiv r \pmod 3}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k} \equiv \begin{cases} A(p) & \pmod{p^2}, \quad \text{if } r=1 \\ B(p) & \pmod{p^2}, \quad \text{if } r=2 \\ C(p) & \pmod{p^2}, \quad \text{if } r=3 \end{cases},$$

where

$$A(p) = (\delta-1)\frac{p}{18}B_{p-2}\left(\frac{1}{3}\right) - \frac{1+\delta}{p}\left(1-3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}\right),$$

$$B(p) = (\delta+2)\frac{p}{18}B_{p-2}\left(\frac{1}{3}\right) + \frac{\delta}{p}\left(1-2\times 3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}\right),$$

and

$$C(p) = -(2\delta+1)\frac{p}{18}B_{p-2}\left(\frac{1}{3}\right) + \frac{1}{p} + (\delta-1)\frac{3^{(p-1)/2}}{p}\cos\frac{(p-1)\pi}{6}.$$

Proof For $r=1$, from congruences (1) and (2), we have

$$\sum_{k\equiv 1 \pmod 3}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k} \equiv \sum_{k=0}^{[p/3]-1}\frac{1}{3k+1} - p\sum_{k=0}^{[p/3]-1}\frac{H_{3k+1}}{3k+1} - \frac{\delta}{p-1}.$$

Using the congruences (12) and (20), we get

$$\begin{aligned}
&\sum_{k\equiv 1 \pmod 3}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k} \\
&\equiv (1+\delta)\left(\frac{1}{2}q_p(3) - \frac{p}{4}q_p^2(3)\right) - \delta\left(1+p-\frac{p}{18}B_{p-2}\left(\frac{1}{3}\right)\right) + \delta(p+1) \\
&-p\left((1+\delta)\left(\frac{1-3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}}{p^2} + \frac{q_p(3)}{2p} - \frac{1}{4}q_p^2(3)\right) + \frac{1}{18}B_{p-2}\left(\frac{1}{3}\right)\right) \\
&= (\delta-1)\frac{p}{18}B_{p-2}\left(\frac{1}{3}\right) - \frac{1+\delta}{p}\left(1-3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}\right) \pmod{p^2}.
\end{aligned}$$

For $r=2$, from congruences (2), (13) and (21), we have

$$\begin{aligned}
&\sum_{k\equiv 2 \pmod 3}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k} \\
&\equiv \sum_{k=0}^{[p/3]-1}\frac{1}{3k+2} - p\sum_{k=0}^{[p/3]-1}\frac{H_{3k+2}}{3k+2} \\
&\equiv (1+\delta)\frac{p}{18}B_{p-2}\left(\frac{1}{3}\right) - \delta\left(\frac{1}{2}q_p(3) - \frac{p}{4}q_p^2(3)\right) \\
&-p\left(-\delta\left(\frac{1-2\times 3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}}{p^2} + \frac{1}{2p}q_p(3) - \frac{1}{4}q_p^2(3)\right) - \frac{1}{18}B_{p-2}\left(\frac{1}{3}\right)\right) \\
&= (\delta+2)\frac{p}{18}B_{p-2}\left(\frac{1}{3}\right) + \frac{\delta}{p}\left(1-2\times 3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}\right) \pmod{p^2}.
\end{aligned}$$

For $r=3$, from congruences (2), (9) and (22), we obtain

$$\begin{aligned}
&\sum_{k\equiv 3 \pmod 3}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k}\\
&\equiv \sum_{k=0}^{[p/3]-1} \frac{1}{3k+3} - p\sum_{k=0}^{[p/3]-1} \frac{H_{3k+3}}{3k+3} = \frac{1}{3}\sum_{k=1}^{[p/3]} \frac{1}{k} - p\sum_{k=0}^{[p/3]-1} \frac{H_{3k+3}}{3k+3}\\
&\equiv \frac{1}{3}\left(-\frac{3}{2}q_p(3) + \frac{3}{4}pq_p^2(3) - \frac{p}{6}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)\right)\\
&-p\left(-\frac{1}{p^2} - \frac{1}{2p}q_p(3) + \frac{1}{4}q_p^2(3) + (1-\delta)\frac{3^{(p-1)/2}}{p^2}\cos\frac{(p-1)\pi}{6}\right)\\
&= -(2\delta+1)\frac{p}{18}B_{p-2}\left(\frac{1}{3}\right) + \frac{1}{p} + (\delta-1)\frac{3^{(p-1)/2}}{p}\cos\frac{(p-1)\pi}{6} \pmod{p^2}.
\end{aligned}$$

Thus, the desired result is obtained. □

Theorem 5 *For prime number $p\geq 5$, we have*

$$\sum_{k\equiv r \pmod 3}^{p-1} \frac{(-1)^k}{k}\binom{p-1}{k}^{-1} \equiv \begin{cases} D(p) & \pmod{p^2}, \quad \text{if } r=1 \\ E(p) & \pmod{p^2}, \quad \text{if } r=2 \\ F(p) & \pmod{p^2}, \quad \text{if } r=3 \end{cases},$$

where

$$\begin{aligned}
D(p) &= (\delta+1)\left(q_p(3) - \frac{p}{2}q_p^2(3) + \frac{p}{18}B_{p-2}\left(\frac{1}{3}\right)\right) + \frac{1}{p}\left(1 - 3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}\right),\\
E(p) &= \delta\left(-q_p(3) + \frac{p}{2}q_p^2(3) + \frac{p}{18}B_{p-2}\left(\frac{1}{3}\right)\right) - \frac{1}{p}\left(1 - 2\times 3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}\right),
\end{aligned}$$

and

$$F(p) = -q_p(3) + \frac{p}{2}q_p^2(3) - \frac{p}{18}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) - \frac{1}{p}\left(1 - (1-\delta)3^{(p-1)/2}\cos\frac{(p-1)\pi}{6}\right).$$

Proof With the help of the congruence $(-1)^k\binom{p-1}{k}^{-1} \equiv 1 + pH_k \pmod{p^2}$, the proof of Theorem 5 is obtained similar to the proof of Theorem 4. □

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