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ORIGINAL RESEARCH

Asymptotic behavior of entire solutions in space and time media

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Abstract In space and time periodic media, the paper mainly focuses on the existence and the asymptotic behavior of entire solutions for reaction-diffusion equation with advection. We prove that there exists an entire solution tending to a time-periodic traveling wave solution of $u_t = u_{xx} + f_1(t, u)$ (as $t \to -\infty$), as t increases to $+\infty$, it converges to a space-periodic traveling wave solution of $u_t = u_{xx} + f_2(x, u)$ locally uniformly. Moreover, the two types of traveling wave solutions may have different speeds, and the time-periodic traveling wave solution is smaller than the space-periodic traveling wave solution (of course, they can be equal at $\pm\infty$). Finally, we give numerical simulations for the main results.

Keywords Reaction-diffusion equation · Periodic traveling wave solution · Asymptotic behavior of solutions · Entire solution

Mathematics Subject Classification 35K57 · 35B08 · 35B40

1 Introduction

In this paper, we consider the entire solutions of the following equation

$$u_t = u_{xx} - \beta u_x + f(t, x, u), \tag{1.1}$$

where $\beta > 0$ is a constant, $f(t, x, u) \in C^{\nu/2, 2+\nu/2, 1+\nu/2}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ (for some $\nu \in (0, 1)$) is asymptotically periodic in t:

$$\lim_{t \to -\infty} f(t, x, u) = f_1(t, u), \quad \lim_{t \to +\infty} f(t, x, u) = f_2(x, u), \tag{1.2}$$

where $f_1(t, u)$ and $f_2(x, u)$ are of Fisher-KPP type. The convergence in (1.2) is uniformly for $t, x \in \mathbb{R}$ and $u \in \mathbb{R}^+$. $f_1(t, \cdot)$ is T-periodic in t, and $f_2(x, \cdot)$ is L-periodic in x, where $T > 0$ and $L > 0$. They satisfy, for $i = 1, 2$,

$$\begin{cases} f_i(\cdot, 0) \equiv 0;\ f_i(\cdot, u) < 0 \text{ for } u > i; \\ f_i(\cdot, u) \in C^{\nu/2, 1+\nu/2} \text{ for some } \nu \in (0, 1); \\ a_i(\cdot) := f_{iu}(\cdot, 0) > 0 \text{ is periodic}; \\ f_i(\cdot, u)/u \text{ is strictly decreasing in } u > 0. \end{cases} \tag{1.3}$$

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This condition implies that $f_i(\cdot, u) > 0$ for small $u > 0$, $f_i(\cdot, u) < 0$ for $u > i$ and $f_i(\cdot, u) \le a_i(\cdot)u$. Hence $f_i(\cdot, u)$ is a Fisher-KPP type of nonlinearity. One example is $f_1(t, u) = u(b_1(t) - b_2(t)u)$ for some positive T-periodic functions $b_1(t)$ and $b_2(t)$. $f_2(x, u) = u(c_1(x) - c_2(x)u)$ for some positive L-periodic functions $c_1(x)$ and $c_2(x)$. To study the asymptotic behavior of the entire solution, we give the following conditions:

$$f_1(t, u) \le f(t, x, u) \le f_2(x, u), \tag{1.4}$$

and, as $t \to -\infty$,

$$-N_1 e^{\sigma_1 t} u \le f_1(t, u) - f(t, x, u) \tag{1.5}$$

for some $N_1 > 0, \sigma_1 > 0$. This condition is weak since $\lim_{t \to -\infty} f(t, x, u) = f_1(t, u)$.

As $t \to +\infty$, we assume

$$f_2(t, u) - f(t, x, u) \le N_2 e^{-\sigma_2 t} u \quad \text{for some } N_2 > 0 \text{ and } \sigma_2 > 0. \tag{1.6}$$

In Remark 1.4, we give an example of $f(t, x, u)$ satisfying (1.5) and (1.6).

In this paper, we aim to construct a different type of entire solution, i.e., the entire solution, as $t \to -\infty$, behaves as a time-periodic traveling wave solution, as t increases to $+\infty$, it eventually changes into a space-periodic traveling wave solution. For this purpose, we need the following solutions.

(1) The unique positive T-periodic solution $P_1(t)$ of the equation $u_t = f_1(t, u)$; The unique L-periodic positive solution $P_2(x)$ of the equation $u'' + f_2(x, u) = 0$.

(2) It is known (cf. [9,12]) that, there exists $c_1^* > 0$ (actually, $c_1^* = \sqrt{\frac{1}{T}\int_0^T a_1(s)ds + \beta}$) such that for each $c \ge c_1^*$, the equation

$$u_t = u_{xx} - \beta u_x + f_1(t, u) \tag{1.7}$$

has a unique periodic traveling wave solution of the form $\phi_1^c(t, x - ct)$, where $\phi_1^c(t, z)$ is the solution of

$$\begin{cases} v_t = v_{zz} + (c - \beta)v_z + f_1(t, v) \text{ and } v_z(t, z) < 0 \text{ for } t \in [0, T], z \in \mathbb{R}, \\ v(t, +\infty) = 0, v(t, -\infty) = P_1(t), \ t \in [0, T], \\ v(0, z) = v(T, z) \text{ for } z \in \mathbb{R}. \end{cases} \tag{1.8}$$

Actually, $\phi_1^c(t, x - ct) = \overline{\phi}(t, x - (c + \beta)t)$, where $\overline{\phi}(t, x - ct)$ is traveling wave solution of $u_t = u_{xx} + f_1(t, u)$.

Also from (cf. [9,12]), there exists $c_2^* > 0$ such that for each $c \ge c_2^*$, the equation

$$u_t = u_{xx} - \beta u_x + f_2(x, u) \tag{1.9}$$

has a unique space-periodic traveling wave solution of the form $u(t, x) = \phi_2^c(x, x - ct)$, where $\phi_2^c(x, z)$ satisfies

$$\begin{cases} cv_z - \beta(v_z + v_x) + v_{xx} + v_{zz} + 2v_{xz} + f_2(x, v) = 0 \text{ for } x \in [0, L], z \in \mathbb{R}, \\ v(x, +\infty) = 0, v(x, -\infty) = P_2(x), \ x \in [0, L], \\ v(0, z) = v(L, z) \text{ and } v_z(x, z) < 0 \text{ for } z \in \mathbb{R}, \ x \in [0, L]. \end{cases} \tag{1.10}$$

By the statements on page 358-359 in [9], the formula $u(t, x) = \phi_2^c(x, x - ct)$ can be rewritten as

$$\phi_2^c(x, \xi) := u\left(\frac{x - \xi}{c}, x\right).$$

In the spreading phenomenon, as we know, traveling wave solutions are very important in describing this phenomenon and dynamics of reaction-diffusion equations, but to study the whole dynamics, we need to study a general solution, such as, the entire solutions (i.e, the solutions of (1.1) defined for $t \in \mathbb{R}$). It is the key point to study the dynamics of Cauchy problems. Actually, traveling wave solution belongs to the entire solution. As we know, the ω-limit set of bounded solutions is composed of entire solutions. Moreover, they are mainly characterized by homoclinic orbits or heteroclinic orbits. Many authors studied the entire solutions of reaction-diffusion equation to describe the spreading phenomenon.

In homogeneous media, [7] found some entire solutions for Allen-Cahn equation, one of the entire solutions tends to two traveling wave solutions ($\phi(x - ct - x_1)$ and $\phi(-x - ct + x_2)$ at $x \approx \pm\infty$), and eventually converges to 1 as $t \to +\infty$. [11] obtained an entire solution of KPP equation characterized by two traveling wave solutions coming from both sides of the real axis and then mixing. [3] constructed an entire solution by studying the phenomenon that two traveling wave solutions at $x \approx \pm\infty$ move to each other and then annihilate in a finite time. Besides these, there are many papers studied such like entire solutions (cf. [3,4,7,8,23,30]). Different from above, [19] constructed a new type of entire solution tending to two humps at $t \to -\infty$, but as t increases, the humps move towards the origin, eventually"colliding" and form a one-hump final shape of the solution. There are also entire solutions connecting three or four traveling wave solutions, such as [4]. Recently, [18] constructed a new entire solution to advective Fisher-KPP equation on the half-line with Dirichlet boundary condition, they showed that the entire solution is asymptotically flat as $t \to -\infty$ but $U(t, +\infty) \equiv 1$, where U is the entire solution they obtained.

Since spreading often depends on time and space, there are also entire solutions in heterogeneous media (cf. [2,13,15,16,24,26,29]), where [24] obtained an entire solution connecting two different traveling wave solutions (as $t \to -\infty$) and a time-periodic solution (as $t \to +\infty$). [26] constructed two types of entire solutions, one type of entire solution behaves as two traveling wave solutions with different speeds moving towards to the same direction, as t increases, the slower traveling wave solution will be caught up and disappears. While in space periodic media, [15] proved that there exists an entire solution converging to two different space periodic traveling wave solutions as $t \to -\infty$. For bistable type of equation, [2] got three types of entire solutions connecting different traveling wave solutions.

Some authors also studied the entire solutions of nonlocal/delayed problems (cf. [6,13,14,17,22,28,31] etc.), where [17] constructed the entire solution converging to two traveling wave solutions as $t \to -\infty$. Besides this, they also obtained a new type of entire solution. [14] obtained an entire solution connecting two traveling wave solutions and the solution of ODE. [6,31] proved that there exists an entire solution connecting three traveling wave solutions, while [22] constructed some new types of entire solutions.

However, in the spreading process, there are many external factors affecting the biological populations of spreading behavior, such as spatial distribution of water and food, the species living in water are also affected by the direction of wind, the spreading of bacteria are often affected by the migration of the host, and some species like moving to their favorable habitats which are relevant to climate and stream, and so on. In mathematical model, such biased movements are generally described as the convection term. For example [1,20]. There are also many other examples having convection in spreading phenomenon ([21,25,27]). There are some papers concerned the entire solutions of reaction-advection-diffusion equation (cf. [15,16,18]).

From the spreading phenomena, it is necessary to study the problem in heterogeneous media. In this paper, we will study the entire solutions of Fisher-KPP equation with advection. Here we use a asymptotically periodic nonlinearity term depending on space and time, which is a more general one. We will obtain a different entire solution from before ones (especially, different from ones connecting traveling wave solution and periodic solution), to complete the dynamics of reaction-diffusion equation.

By the property of traveling wave solutions (see Section 2), there are pairs $(c, \hat{c}) \in [c_1^*, +\infty) \times [c_2^*, +\infty)$ such that ϕ_1^c and $\phi_2^{\hat{c}}$ have the same exponential behavior (cf. Section 2) as $\xi_1 := x - ct \to +\infty$ and $\xi_2 := x - \hat{c}t \to +\infty$ respectively. Our main results are as follows.

Theorem 1.1 *Assume* (1.2)-(1.6). *For each pair* $(c, \hat{c})$ *above, any* $\theta \in \mathbb{R}$*, the problem* (1.1) *has an entire solution* $\mathcal{U}^c(t, x)$ *satisfying*

$$\lim_{t \to -\infty} |\mathcal{U}^c(t, x) - \phi_1^c(t, x - ct - \theta)| = 0 \quad for \quad x \in \mathbb{R}. \tag{1.11}$$

For each $m \in (0, \max_{x \in [0, L]} P_2(x))$*, let* $t_m(n)$ *be the* m*-level set of* $\mathcal{U}^c(\cdot, nL)$*, then* $t_m(n) \to +\infty$ *as* $n \to +\infty$*, and*

$$\mathcal{U}^c(t + t_m(n), x + nL) \to \phi_2^{\hat{c}}(x, x - \hat{c}t - \theta_m) \text{ as } n \to \infty \text{ in } C_{loc}^{1,2}(\mathbb{R} \times \mathbb{R}), \tag{1.12}$$

where $\phi_2^{\hat{c}}(x, x - \hat{c}t - \theta_m)$ *is the traveling wave solution of* (1.9) *with* $\phi_2^{\hat{c}}(0, -\theta_m) = m$.

From this theorem, we see that when $t \approx -\infty$, the entire solution has a shape similar to the time periodic traveling wave solution $\phi_1^c(t, x - ct)$. As t goes to $+\infty$, the shape of the entire solution is turning into the space periodic traveling wave solution $\phi_2^{\hat{c}}(x, x - \hat{c}t)$ (cf. Fig. 1).

Remark 1.2 Especially, when $f_{1u}(t, 0) = f_{2u}(x, 0)$ (i.e, $a_1(t) = a_2(x) = constant$), then the property of traveling wave solutions implies that $c = \hat{c}$. For example, $f_1(t, u) = u(1 - a(t)u)$, $f_2(x, u) = u(1 - b(x)u)$. Then for each $c \geq c_1^*$, there exists an entire solution $\mathcal{U}^c(t, x)$ satisfies (1.11) and

$$\mathcal{U}^c(t + t_m(n), x + nL) \to \phi_2^c(x, x - ct - \theta_m) \text{ as } n \to \infty \text{ in } C_{loc}^{1,2}(\mathbb{R} \times \mathbb{R}),$$

with $\phi_2^c(0, -\theta_m) = m$.

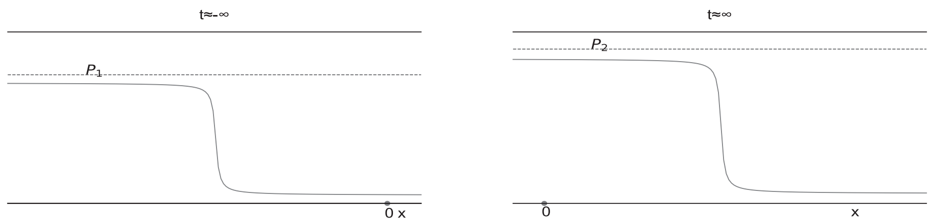

Fig. 1 The graph of the entire solution $\mathcal{U}^c$ for $t \approx -\infty$, $t \approx +\infty$ (left to right)

Remark 1.3 Assume that $f_1(t, u)$ satisfies (1.3) and $f_2(x, u)$ is the bistable function, we can obtian (1.11) for some $c := \widetilde{c} \geq c_1^*$, and (1.12) holds with $\phi_2^{\hat{c}}$ replaced by $\phi_B(x, x - c^*t + \theta_B)$ for some $\theta_B \in \mathbb{R}$ (with $\phi_B(0, \theta_B) = m$), where $\phi_B(x, x - c^*t + \theta_B)$ is the unique traveling wave solution of (1.9) with bistable nonlinearity f_2. $\widetilde{c}$ is the speed of the traveling wave $\phi_1^{\widetilde{c}}(t, x - \widetilde{c}t)$ which has the same exponential behavior as $\phi_B(x, x - c^*t + \theta_B)$.

Remark 1.4 Now, we give an example of $f(t, x, u)$ as follows. Define

$$\nu(t) := \frac{e^t}{e^t + e^{-t}}.$$

Then $\lim_{t\to+\infty} \nu(t) \to 1$ and $\lim_{t\to-\infty} \nu(t) \to 0$. Set

$$f(t, x, u) := (1 - \nu(t)) f_1(t, u) + \nu(t) f_2(x, u).$$

By calculation, f satisfies (1.4)-(1.6).

The paper is organized as follows. In section 2, we give the exponential asymptotically behavior of traveling wave solutions. Then in section 3, we prove the existence of entire solutions and their properties. In section 4, we give numerical simulations for the main results.

2 Preliminaries

To prove the existence of entire solutions, lower and upper solutions will be constructed by traveling wave solutions. So, we give the following asymptotic behavior of traveling wave solutions. For simplicity, we only give the properties of ϕ_1^c, since $\phi_2^{\hat{c}}$ is similar. From [9, Sec. 1.3], for each $\lambda \in \mathbb{R}$, call $k(\lambda)$ the principal eigenvalue of the operator,

$$\psi \to \psi_t - (\lambda^2 + a_1(t))\psi, \ t > 0.$$

Denote by ψ_λ the unique positive principal eigenfunction satisfying $||\psi_\lambda||_{L^\infty(\mathbb{R})} = 1$. For each $c \in (c_1^*, \infty)$, there is a constant $A_c > 0$ such that

$$0 < \phi_1^c(t, z) \leq A_c e^{-\lambda_c z} \psi_{\lambda_c}(t), \ 0 \leq z < \infty,$$

where $\lambda_c := \min\{\lambda > 0 | k(\lambda) + \lambda c = 0\}$.

From [9], we obtain the following exponential behavior of the traveling wave solution $\phi_1^c(t, x - ct)$.

Lemma 2.1 *Assume* (1.3). *Then there is a constant* $A_1 > 0$ *such that*

$$\phi_1^c(t, z) \sim A_1 e^{-\lambda_c z} \psi_{\lambda_c}(t) \text{ as } z \to +\infty \tag{2.1}$$

for $c > c_1^*$*; and*

$$\phi_1^c(t, z) \sim A_0 |s|^{2m+1} e^{-\lambda_{c_0} z} \psi_{\lambda_{c_0}}(t) \text{ as } z \to +\infty$$

for $c = c_0$. λ_{c_0} *is the unique root of the equation* $k(\lambda) + c_0\lambda = 0$, *and* $2m + 1$ ($m \in \mathbb{N}$) *is the multiplicity of the solution* λ_{c_0}.

Moreover, there are constants $K_i > 0$ *such that*

$$K_1 e^{-\lambda_c z} \leq \phi_1^c(t, z) \leq K_2 e^{-\lambda_c z}, \quad \forall z \geq 0, \ t \in \mathbb{R}.$$

By the above lemma, we have the following conclusions which will be used in constructing lower and upper solutions.

Lemma 2.2 *Under all the assumptions in Lemma 2.1, for any $c \geq c_i^*$ $(i = 1, 2)$, there are some positive constants η_i^c such that*

$$-\phi_{1z}^c(t, z) \geq \eta_1^c \phi_1^c(t, z), \quad \forall z \geq 0,\ t \in \mathbb{R}$$

and

$$-\phi_{2z}^c(x, z) \geq \eta_2^c \phi_2^c(x, z), \quad \forall z \geq 0,\ x \in \mathbb{R}, \tag{2.2}$$

where (2.2) *is proved in [9, Proposition 2.2].*

3 Existence of the entire solution

We mainly use upper-lower solution method to prove the existence of the entire solution. To construct the upper solution, for any given constants $c > 0$, $M > 0$ and $\gamma > 0$, we first consider the following problem

$$r'(t) = c + Me^{\gamma r(t)}, \tag{3.1}$$

whose solution is

$$r(t) = r(0) + ct - \vartheta(t),$$

where $\vartheta(t) = \ln\left[c + M(e^{\gamma r(0)} - e^{\gamma(r(0)+ct)})\right]/\gamma - \ln c/\gamma$. For simplicity, we write $\widetilde{r}$ as

$$\widetilde{r} := r(0) - \ln\left(c + Me^{\gamma r(0)}\right)/\gamma - \ln c/\gamma.$$

Then, for some constant N_0, we have

$$0 \leq r(t) - ct - \widetilde{r} \leq N_0 e^{c\gamma t} \quad \text{for all } \ t \leq 0. \tag{3.2}$$

Obviously, $r(t) \to -\infty$ as $t \to -\infty$.

3.1 Existence of entire solution

In this section, we prove Theorem 1.1. We first show the existence of the entire solution. By the property $r(t) \to -\infty$ as $t \to -\infty$, for large $m > 0$, there holds $r(-m) < 0$. Let $n > m$, we consider the following Cauchy problem to obtain the entire solution,

$$\begin{cases} (u_n)_t + \beta(u_n)_x - (u_n)_{xx} - f(t, x, u_n) = 0,\ t > -n,\ x \in \mathbb{R}, \\ u_{n,0}(x) = u_n(-n, x) := \phi_1^c(-n, x + cn - \widetilde{r}),\ x \in \mathbb{R}. \end{cases} \tag{3.3}$$

Now we prove that $\overline{u}(t, x) := \phi_1^c(t, x - r(t))$ is an upper solution of (3.3). Firstly, for $x - r(t) \geq 0$,

$$\begin{aligned} &\overline{u}_t + \beta\overline{u}_x - \overline{u}_{xx} - f(t, x, \overline{u}) \\ =\ &\phi_{1t}^c - r'(t)\phi_{1z}^c + \beta\phi_{1z}^c - \phi_{1zz}^c - f(t, x, \phi_1^c) \\ =\ &-Me^{\gamma r(t)}\phi_{1z}^c + f_1(t, \phi_1^c) - f(t, x, \phi_1^c) \\ \geq\ &-Me^{\gamma r(t)}\phi_{1z}^c - N_1 e^{\sigma_1 t}\phi_1^c \\ =\ &\phi_1^c\left(\frac{-\phi_{1z}^c}{\phi_1^c} Me^{\gamma r(t)} - N_1 e^{\sigma_1 t}\right) \\ \geq\ &\phi_1^c\left(\eta_1^c Me^{\gamma r(t)} - N_1 e^{\sigma_1 t}\right) \\ \geq\ &\phi_1^c\left(\eta_1^c Me^{\gamma(ct+\widetilde{r})} - N_1 e^{\sigma_1 t}\right) \\ \geq\ &0 \end{aligned} \tag{3.4}$$

if we choose $\gamma := \sigma_1/c$, and $M > 0$ large satisfying $M\eta_1^c e^{\frac{\sigma_1 \widetilde{r}}{c}} > N_1$.

Moreover, from the property $\phi_{1z}^c < 0$ and $r(-n) + cn - \widetilde{r} > 0$ we have

$$\overline{u}(-n, x) = \phi_1^c(-n, x - r(-n)) > \phi_1^c(-n, x + cn - \widetilde{r}) = u_{n,0}(x) \text{ for } x \in \mathbb{R}.$$

It follows from the maximum principle that $u_n(t, x) \leq \overline{u}(t, x)$ for $t \in [-n, -m)$ and $x \in [r(-m), +\infty)$.

We now prove that $\underline{u}(t, x) := \phi_1^c(t, x - ct - \widetilde{r})$ is a lower solution, by (1.4) and the fact that ϕ_1^c is the traveling wave solution of (1.7), we deduce that

$$\underline{u}_t - \underline{u}_{xx} + \beta \underline{u}_x - f(t, x, \underline{u}) \leq \underline{u}_t - \underline{u}_{xx} + \beta \underline{u}_x - f_1(t, \underline{u}) = 0.$$

This and $\underline{u}(-n, x) = u_{n,0}(x)$ imply that $\underline{u}(t, x)$ is a lower solution for $t \in [-n, +\infty)$ and $x \in \mathbb{R}$. Since there is a lower solution and an upper solution for $u_n(t, x)$, by the standard parabolic estimates and passing $n \to +\infty$, we obtain an solution $\mathcal{U}_m^c(t, x)$ defined for $t \in (-\infty, -m]$ and $x > r(-m)$, and it can be extended to the whole time by the theorem of Cauchy problem (which is similar with Cauchy Lipschitz Theorem, cf. [32, Theorem 8.3.6, 8.7.4]), that is, $\mathcal{U}_m^c$ is extended to $t \in \mathbb{R}$. Then by parabolic estimates and letting $m \to +\infty$ we can obtain an entire solution $\mathcal{U}^c(t, x)$ for $x \in \mathbb{R}$. Moreover, $\mathcal{U}^c(t, x)$ satisfies $\mathcal{U}^c(t, x) \leq \overline{u}(t, x)$ for $t \in \mathbb{R}$ and $x \geq r(t)$, and

$$\underline{u}(t, x) \leq \mathcal{U}^c(t, x), \quad t \in \mathbb{R},\ x \in \mathbb{R}. \tag{3.5}$$

3.2 The asymptotic properties of entire solution

We first show the asymptotic behavior of entire solution $\mathcal{U}^c(t, x)$ as $t \to -\infty$. Follows from (3.2) and $r(t) \to -\infty$ (as $t \to -\infty$), we have

$$\begin{aligned}
0 &< \mathcal{U}^c(t, x) - \phi_1^c(t, x - ct - \widetilde{r}) \\
&\leq \phi_1^c(t, x - r(t)) - \phi_1^c(t, x - ct - \widetilde{r}) \\
&\leq \sup_{z \in \mathbb{R}} |\phi_{1z}^c(t, z)| \big(r(t) - ct - \widetilde{r}\big) \\
&\leq \sup_{z \in \mathbb{R}} |\phi_{1z}^c(t, z)| N_0 e^{c\gamma t} \\
&\to 0 \quad \text{as } t \to -\infty.
\end{aligned}$$

Combining this and the fact that $r(0)$ can be chosen arbitrarily (so is $\widetilde{r}$), we can prove (1.11).

Next, we focus on the asymptotic behavior of the entire solution as $t \to +\infty$.

Lemma 3.1 *Assume* (1.4). *Let $\mathcal{U}^c(t, x)$ be the entire solution as above, then*

$$\lim_{t \to +\infty} \lim_{s \to +\infty} \mathcal{U}^c(t + s, x) = P_2(x) \quad \textit{locally uniformly in } \mathbb{R}. \tag{3.6}$$

Proof For any $X > 0$, $\widehat{T} > 0$, $\alpha \in (0, 1)$, by the parabolic estimates, we have

$$||\mathcal{U}^c(t + nT, x)||_{C^{1+\alpha/2, 2+\alpha}([-\widehat{T}, \widehat{T}] \times [-X, X])} \leq C$$

for some constant $C > 0$ depending on X and $\widehat{T}$ but not on n. Therefore, there is a sequence $\{n_i\}$ and a function $\eta(t, x)$ such that

$$\mathcal{U}^c(t + n_i T, x) \to \eta(t, x) \quad \text{as } n_i \to \infty \quad in\ C_{loc}^{1,2}(\mathbb{R} \times \mathbb{R}),$$

where η is a nonnegative solution of

$$\eta_t + \beta \eta_x - \eta_{xx} - f_2(x, \eta) = 0. \tag{3.7}$$

Actually, we next prove that $\eta(t, x)$ is a positive solution. Consider the problem

$$\begin{cases} w_t + \beta w_x - w_{xx} - f_1(t, w) = 0,\ t > 0,\ x \in \mathbb{R}, \\ w(0, x) = \mathcal{U}^c(0, x) > 0,\ x \in \mathbb{R}. \end{cases}$$

By the comparison principle we have

$$\mathcal{U}^c(t,x) \geq w(t,x) \to P_1(t) \text{ as } t \to +\infty \text{ locally uniformly in } \mathbb{R}.$$

This means that $\eta(t,x) \geq P_1(t)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}$.

Consider the Cauchy problem of (3.7) with positive initial data $\eta(0,x) > 0$. By the convergence of the solution, we have $\eta(t,x) \to P_2(x)$ as $t \to \infty$ locally uniformly in $\mathbb{R}$. More precisely, by the parabolic estimates, $\eta(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R})$ is uniformly bounded. Therefore, there is a subsequence $t_n \to +\infty$ $(n \to \infty)$ such that $\eta(t_n,x) \to P_2(x)$ as $n \to +\infty$. By the uniqueness of $P_2(x)$ (since $f_2(x,u)$ is of Fisher-KPP type), we have $\eta(t,x) \to P_2(x)$ as $t \to +\infty$. This proves (3.6).

Theorem 3.2 *Assume* (1.4)-(1.6). *Let* $\mathcal{U}^c(t,x)$ *be the entire solution obtained above, then there exist constants* ε, $\xi_1 \in \mathbb{R}$, *for large* $t > 0$,

$$(1 - \varepsilon e^{-\sigma_2 t})\phi_2^{\hat{c}}(\cdot, x - \hat{c}t - \delta e^{-\sigma_2 t} + \xi_1) \leq \mathcal{U}^c(\cdot, x), \; x \geq \hat{c}t \tag{3.8}$$

when $\delta > 0$ *is large. And, for* $t \in \mathbb{R}$, *some* $\theta^* \in \mathbb{R}$,

$$\mathcal{U}^c(t,x) \leq \phi_2^{\hat{c}}(x, x - \hat{c}t - \theta^*) \text{ for } t \in \mathbb{R}, \; x \in \mathbb{R}, \tag{3.9}$$

where $\hat{c}$ *is defined in Theorem 1.1.*

Proof Since ϕ_1^c and $\phi_2^{\hat{c}}$ have the same property, especially the decay rate, combine this, (1.4) and the choice of the initial data in (3.3), there exists a constant $\theta^* \in \mathbb{R}$ such that

$$\mathcal{U}^c(t,x) \leq \phi_2^{\hat{c}}(x, x - \hat{c}t - \theta^*) \text{ for } t \in \mathbb{R}. \tag{3.10}$$

Note that there exists $T_0 > T^*$, such that (1.6) holds for $t \geq T_0$. Choose $\varepsilon > 0$ and $\hat{\xi} > 0$ such that

$$(1 - \varepsilon e^{-\sigma_2 T_0})\phi_2^{\hat{c}}(x, x - \hat{c}T_0 + \hat{\xi}) < \phi_1^c(T_0, x - cT_0 - \widetilde{r}). \tag{3.11}$$

This is possible since ϕ_1^c and $\phi_2^{\hat{c}}$ have the same exponential behavior. Let $\delta > 0$ be large such that

$$\frac{\varepsilon}{\eta_2^{\hat{c}}} + \frac{N_2 + \varepsilon \max\limits_{u \in [0, P_2(x)], x \in [0,L]} |f_{2u}(x,u)|}{\sigma_2 \eta_2^{\hat{c}}} - \delta(1 - \varepsilon e^{-\sigma_2 T_0}) < 0. \tag{3.12}$$

Then choose ξ_1 large such that

$$-\delta e^{-\sigma_2 T_0} + \xi_1 > \hat{\xi}.$$

This condition and the property $\phi_{2z}^{\hat{c}} < 0$ imply that

$$\phi_2^{\hat{c}}(x, x - \hat{c}t - \delta e^{-\sigma_2 T_0} + \xi_1) < \phi_2^{\hat{c}}(x, x - \hat{c}t + \hat{\xi}). \tag{3.13}$$

Construct a lower solution of (1.1) as follows

$$\underline{V}(t,x) := (1 - \varepsilon e^{-\sigma_2 t})\phi_2^{\hat{c}}(x, x - \hat{c}t - \delta e^{-\sigma_2 t} + \xi_1) \quad \text{for } t \geq T_0, \; x \geq \hat{c}t.$$

We next show that $\underline{V}(t,x)$ is a lower solution of (1.1) when $x \geq \hat{c}t$ and $t \geq T_0$. Firstly, by (3.5), (3.11) and (3.13),

$$\underline{V}(T_0,x) = (1 - \varepsilon e^{-\sigma_2 T_0})\phi_2^{\hat{c}}(x, x - \hat{c}T_0 - \delta e^{-\sigma_2 T_0} + \xi_1) \leq \mathcal{U}^c(T_0,x), \; x \in \mathbb{R}.$$

Moreover, combine the definition of $\phi_2^{\hat{c}}$ and (1.6), we prove that the following inequality holds when $t \geq T_0$ and $x \geq \hat{c}t$,

$$
\begin{aligned}
&\underline{V}_t + \beta \underline{V}_x - \underline{V}_{xx} - f(t, x, \underline{V}) \\
&= \varepsilon\sigma_2 e^{-\sigma_2 t}\phi_2^{\hat{c}} + \sigma_2\delta e^{-\sigma_2 t}(1 - \varepsilon e^{-\sigma_2 t})\phi_{2z}^{\hat{c}} - \hat{c}(1 - \varepsilon e^{-\sigma_2 t})\phi_{2z}^{\hat{c}} + \beta(1 - \varepsilon e^{-\sigma_2 t})(\phi_{2x}^{\hat{c}} + \phi_{2z}^{\hat{c}}) \\
&\quad - (1 - \varepsilon e^{-\sigma_2 t})(\phi_{2xx}^{\hat{c}} + \phi_{2zz}^{\hat{c}} + 2\phi_{2xz}^{\hat{c}}) - f(t, \xi + ct, \underline{V}) \\
&= \varepsilon\sigma_2 e^{-\sigma_2 t}\phi_2^{\hat{c}} + \sigma_2\delta e^{-\sigma_2 t}(1 - \varepsilon e^{-\sigma_2 t})\phi_{2z}^{\hat{c}} + (1 - \varepsilon e^{-\sigma_2 t}) f_2(x, \phi_2^{\hat{c}}) - f(t, \xi + ct, \underline{V}) \\
&\leq \varepsilon\sigma_2 e^{-\sigma_2 t}\phi_2^{\hat{c}} + \sigma_2\delta e^{-\sigma_2 t}(1 - \varepsilon e^{-\sigma_2 t})\phi_{2z}^{\hat{c}} + e^{-\sigma_2 t}\left(N_2 + \varepsilon \max_{u\in[0,P_2(x)],x\in[0,L]} |f_{2u}(x, u)|\right)\phi_2^{\hat{c}} \\
&= -\phi_{2z}^{\hat{c}} e^{-\sigma_2 t}\left[\varepsilon\sigma_2 \frac{\phi_2^{\hat{c}}}{-\phi_{2z}^{\hat{c}}} - \sigma_2\delta(1 - \varepsilon e^{-\sigma_2 t}) + \left(N_2 + \varepsilon \max_{u\in[0,P_2(x)],x\in[0,L]} |f_{2u}(x, u)|\right)\frac{\phi_2^{\hat{c}}}{-\phi_{2z}^{\hat{c}}}\right] \\
&\leq -\phi_{2z}^{\hat{c}} e^{-\sigma_2 t}\left[\varepsilon\sigma_2 \frac{1}{\eta_2^{\hat{c}}} - \sigma_2\delta(1 - \varepsilon e^{-\sigma_2 t}) + \left(N_2 + \varepsilon \max_{u\in[0,P_2(x)],x\in[0,L]} |f_{2u}(x, u)|\right)\frac{1}{\eta_2^{\hat{c}}}\right] \\
&\leq 0.
\end{aligned}
$$

The penultimate inequality is proved by Lemma 2.2, and the last inequality can be proved by (3.12). Hence $\underline{V}$ is a lower solution of (1.1) for $t \geq T_0$ and $x \geq \hat{c}t$.

3.3 Completion of proof of theorem 1.1.

It remains to prove (1.12) in Theorem 1.1. For convenience, set $\xi := x - \hat{c}t$, $\mathcal{V}^c(x, \xi) := \mathcal{U}^c(\frac{x-\xi}{\hat{c}}, x)$. Then $\mathcal{V}^c(x, \xi)$ satisfies

$$
v_{\xi\xi} + v_{xx} - \beta(v_x + v_\xi) + \hat{c}v_\xi + 2v_{x\xi} - f((x - \xi)/\hat{c}, x, v) = 0.
$$

The above inequality (3.10) implies that

$$
\mathcal{V}^c(x, \xi) \leq \phi_2^{\hat{c}}(x, \xi - \theta^*) \tag{3.14}
$$

For any $m \in \left(0, \max_{x\in[0,L]} P_2(x)\right)$, $\phi_2^{\hat{c}}(x, \xi - \theta^*)$ may have m-level set. For each $x > 0$, there is $\widetilde{\xi}_m(x)$ such that $\phi_2^{\hat{c}}(x, \widetilde{\xi}_m(x) - \theta^*) = m$. Since $\phi_2^{\hat{c}}(x, \cdot)$ is periodic in x, so is $\widetilde{\xi}_m(x)$, this implies that it is bounded. Denote $\widetilde{\xi}_0 := \max_{x\in[0,L]} \widetilde{\xi}_m(x)$.

By Lemma 3.1, for any $\varepsilon_0 > 0$ and any fixed $X > 0$, there is $T^* > 0$ such that, when $t > T^*$,

$$
\mathcal{U}^c(t, x) \geq P_2(x) - \varepsilon_0,\ x \in [-X, X]. \tag{3.15}
$$

From this inequality, the definition of $\mathcal{V}^c(x, \xi)$ and $\xi := x - \hat{c}t \to -\infty$ (as $t \to +\infty$) deduce that

$$
\mathcal{V}^c(x, \xi) \geq P_2(x) - \varepsilon_0, \quad \mathcal{V}^c(x, -\infty) \geq P_2(x) - \varepsilon_0,\ x \in [-X, X],\ \xi < x - \hat{c}T^*. \tag{3.16}
$$

It follows from (3.14) and (3.16) that, for any $m \in (0, \max_{x\in[0,L]} P_2(x))$, $\mathcal{V}^c$ may have m-level set. For each $s \in \mathbb{R}$, there is $\xi_m(s)$ such that $\mathcal{V}^c(s, \xi_m(s)) = m$.

We show that $\{\xi_m(s)\}$ is bounded above. Actually, by (3.14),

$$
\begin{aligned}
0 &= \mathcal{V}^c(s, \xi_m(s)) - \phi_2^{\hat{c}}(s, \widetilde{\xi}_m(s) - \theta^*) \\
&\leq \phi_2^{\hat{c}}(s, \xi_m(s) - \theta^*) - \phi_2^{\hat{c}}(s, \widetilde{\xi}_m(s) - \theta^*) \\
&= \phi_{2z}^{\hat{c}}(s, \varrho(s))[\xi_m(s) - \widetilde{\xi}_m(s)],
\end{aligned}
$$

where $\varrho(s)$ is a function between $\xi_m(s)$ and $\widetilde{\xi}_m(s)$. The property $\phi_{2z}^{\hat{c}} < 0$ implies that $\xi_m(s) \leq \widetilde{\xi}_m(s)$. Therefore, the boundedness of $\widetilde{\xi}_m(s)$ implies that $\xi_m(s)$ has an upper bound.

For any $\overline{L} > 0$ and $\overline{K} > 0$, the parabolic estimates imply that

$$||\mathcal{V}^c(x + nL, \xi + \xi_m(nL))||_{C^{1+\alpha/2,2+\alpha}([-\overline{L},\overline{L}]\times[-\overline{K},\overline{K}])} \le C$$

for some constant $C > 0$ depending on $\overline{L}$ and $\overline{K}$ but not on n. Therefore, there is a sequence $\{n_i\} \subset \mathbb{R}$ ($n_i \to +\infty$ as $i \to \infty$) and a function $W_m(x, \xi) \in C^{1+\alpha/2,2+\alpha}_{loc}(\mathbb{R} \times \mathbb{R})$ such that

$$\mathcal{V}^c(x + n_i L, \xi + \xi_m(n_i L)) \to W_m(x, \xi) \quad \text{as } n_i \to \infty \quad \text{in } C^{1+\alpha/2,2+\alpha}_{loc}(\mathbb{R} \times \mathbb{R}). \tag{3.17}$$

From this and (3.16), $W_m(x, -\infty) \ge P_2(x) - \varepsilon_0/2$. Moreover, we can derive from (3.8), (3.14) that $W_m(x, +\infty) = 0$. The definition of $\xi_m(s)$ implies that $W_m(0, 0) = m$.

Since $X > 0$ can be arbitrary large, and ε_0 can be arbitrary small, we see that $W_m(x, \xi)$ satisfies, for $x \in \mathbb{R}$,

$$\begin{cases} (W_m)_{\xi\xi} + (W_m)_{xx} - \beta[(W_m)_x + (W_m)_\xi] + \hat{c}(W_m)_\xi + 2(W_m)_{x\xi} + f_2(x, W_m) = 0, \\ W_m(0, 0) = m, \\ W_m(x, -\infty) = P_2(x), \quad W_m(x, +\infty) = 0. \end{cases} \tag{3.18}$$

Let $v^*(t, x) := W_m(x, \xi)$ with $\xi = x - \hat{c}t$. Then $v^*(t, x)$ satisfies (1.9) with $c := \hat{c}$. By the definition of space-periodic traveling wave solution (cf. [9, page 358-359]) and its uniqueness, we can derive that $W_m(x, \xi)$ is nothing but $\phi_2^{\hat{c}}(x, \xi - \theta_m)$ for some $\theta_m \in \mathbb{R}$ with $\phi_2^{\hat{c}}(0, -\theta_m) = m$. Note that $\xi = x - \hat{c}t$ and $\mathcal{V}^c(x, \xi) := \mathcal{U}^c(\frac{x-\xi}{\hat{c}}, x)$. Denoted by $t_m(n_i) := \frac{n_i L - \xi_m(n_i L)}{\hat{c}}$, then $t_m(n_i) \to +\infty$ as $n_i \to +\infty$ (note that $\xi_m(n_i L)$ has an upper bound). Combining these, the definition of $\mathcal{V}^c(x, \xi)$ and (3.17) we have

$$\mathcal{U}^c(t + t_m(n_i), x + n_i L) \to \phi_2^{\hat{c}}(x, x - \hat{c}t - \theta_m) \text{ as } n_i \to \infty \text{ in } C^{1,2}_{loc}(\mathbb{R} \times \mathbb{R}).$$

It follows from the uniqueness of the solution of (3.18) and the traveling wave solution, we prove (1.12).

3.4 Numerical simulations

In this section, we give the numerical simulations for the results in Theorem 1.1.

Set

$$f_1(t, u) = u(1 - u), \quad f_2(x, u) = u\,(1 - 0.5u)\,, \quad \nu(t) := \frac{e^t}{e^t + e^{-t}}.$$

and

$$f(t, x, u) := (1 - \nu(t)) f_1(t, u) + \nu(t) f_2(x, u).$$

In this special case, for each $c \ge 2$, the traveling waves of (1.7) and (1.9) have the form $\phi(x - ct)$. More precisely, the traveling wave $\phi_1^c(x - ct)$ of (1.7) satisfies

$$\phi_1^c(+\infty) = 0, \quad \phi_1^c(-\infty) = 1, \quad \phi_{1z}^c < 0.$$

And, the traveling wave $\phi_2^{\hat{c}}(x - ct)$ of (1.9) satisfies

$$\phi_2^{\hat{c}}(+\infty) = 0, \quad \phi_2^{\hat{c}}(-\infty) = 2, \quad \phi_{2z}^{\hat{c}} < 0.$$

Case 1. The case $t < 0$. From Table 1, we see that when $t < 0$ and $|t|$ large, the entire solution tends to 0 as x increases, at the same time Table 2 says that the value tends to 1 as $x < 0$ decreases. Moreover, for any fixed x (such as $x = 1000$ in Table 1), the value of the entire solution is decreasing in $t < 0$. These are consistent with (1.11) in Theorem 1.1 and the property $\phi_{1z}^c < 0$.

Case 2. The case $t > 0$. For any large $t > 0$, Table 3 shows that the value of the entire solution increases to 2 as $x < 0$ decreases, while Table 4 says that the value decreases to 0 as $x > 0$ increases; Moreover, for any fixed x (such as $x = 1000$ in Table 4), the value is increasing in $t > 0$. These are consistent with (1.12) in Theorem 1.1 and the property $\phi_{1z}^{\hat{c}} < 0$.

Table 1 Asymptotic behavior of entire solution for large $x > 0$ and large $|t|(t < 0)$

$t \setminus x$	1000	1300	1500	1700	2000	2200
-100	0.0003151	0.0002424	0.0002100	0.00018537	0.0001575	0.0001432
-500	0.0003028	0.0002329	0.0002019	0.0001781	0.0001514	0.0001376
-700	0.0002875	0.0002231	0.0001971	0.0001719	0.0001444	0.0001358

Table 2 Asymptotic behavior of entire solution for large $|x|(x < 0)$ and large $|t|(t < 0)$

$t \setminus x$	-1000	-1300	-1500	-1700	-2000	-2200
-100	0.9994909	0.9995651	0.9995980	0.9996232	0.9996516	0.9996662
-500	0.9994701	0.9995473	0.9995816	0.9996079	0.9996374	0.9996527
-600	0.9994622	0.9995435	0.9995779	0.9996031	0.9996354	0.9996527

Table 3 Asymptotic behavior of entire solution for large $|x|(x < 0)$ and large $t > 0$

$t \setminus x$	-500	-600	-1000	-1500	-2000	-2500
5900	1.9945211	1.9945222	1.9945246	1.9945257	1.9945263	1.9945266
6000	1.9950414	1.9950425	1.9950446	1.9950456	1.9950461	1.9950465
6300	1.9963248	1.9963256	1.9963271	1.9963279	1.9963283	1.9963285
6900	1.9979819	1.9979824	1.9979832	1.9979836	1.9979839	1.9979840

Table 4 Asymptotic behavior of entire solution for large $x > 0$ and large $t > 0$

$t \setminus x$	1000	2300	2400	2800	3000	3900
1500	0.0003698	0.0001608	0.0001541	0.0001321	0.0001233	0.0000948
2300	0.0004006	0.0001742	0.0001669	0.0001431	0.0001335	0.0001027
2600	0.0004128	0.0001795	0.0001720	0.0001474	0.0001376	0.0001059
2900	0.0004254	0.0001850	0.0001772	0.0001519	0.0001418	0.0001091
3000	0.0004297	0.0001868	0.0001790	0.0001535	0.0001432	0.0001102

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