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ORIGINAL RESEARCH

Lopsided PMQHSS and double lopsided PMQHSS iteration methods for solving complex symmetric linear equations

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Abstract By applying the lopsided technology to the preconditioned modified quasi-Hermitian and skew-Hermitian splitting (PMQHSS) iteration method, we construct a lopsided PMQHSS (LPMQHSS) iteration method for solving complex symmetric linear equations. We discuss the convergence properties of the LPMQHSS method. Specially, the convergence properties of the LPMQHSS method with $V = T$ are established. In addition, we also give another new iteration method, referred to as double lopsided PMQHSS (DLPMQHSS) iteration method. The convergence conditions of the DLPMQHSS iteration method are analyzed. The proposed LPMQHSS and DLPMQHSS methods have faster convergence rates than the PMQHSS one. Numerical experiments are reported to illustrate the feasibility and effectiveness of the proposed methods.

Keywords LPMQHSS method · DLPMQHSS method · Properties · Positive definite · Eigenvalue

Mathematics Subject Classification 65F10 · 65H10

1 Introduction

In this paper, we consider the system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{n\times n}, \; x, b \in \mathbb{C}^n, \tag{1.1}$$

and $A \in \mathbb{C}^{n\times n}$ is a large sparse complex symmetric matrix satisfying the following form

$$A = W + \mathrm{i}T, \tag{1.2}$$

where $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite (SPSD) matrices with at least one of them being positive definite. Throughout the paper, we assume that T is symmetric positive definite (SPD). Here and in the sequel, we use i to denote the imaginary unit that satisfies $\mathrm{i} = \sqrt{-1}$.

In scientific computing and engineering applications, many problems depend on solving this kind of complex symmetric linear systems (1.1), such as molecular scattering [4], structural dynamics [1], lattice quantum chromodynamics [2], diffuse optical tomography [16], FFT-based solution of certain time-dependent PDEs [5], and quantum mechanics [18]. For more details, please refer to [12].

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The coefficient matrix A naturally has the following Hermitian and skew-Hermitian splitting

$$A = H + S, \tag{3}$$

where $H = \frac{1}{2}(A + A^*) = W$ *and* $S = \frac{1}{2}(A - A^*) = \mathrm{i}T$ with $H = W$, $S = \mathrm{i}T$ and A^* being the Hermitian part, skew-Hermitian part and conjugate transpose of A, respectively. So as to effectively calculate the approximate solution of (1.1), many effective methods for processing complex symmetric linear equations (1.1) have been proposed in recent years. For example, Hermitian and skew-Hermitian splitting (HSS) method [28], quasi-HSS (QHSS) iteration method [24], modified HSS (MHSS) iteration method [29], modified QHSS (MQHSS) iteration method [8], preconditioned MHSS (PMHSS) iteration method [30], preconditioned MQHSS (PMQHSS) iteration method [3], lopsided PMHSS (LPMHSS) iteration method [19], generalized PMHSS (GPMHSS) iteration method [13], accelerated HSS (AHSS) iteration method [26], LHSS iteration method [11], PHSS iteration method [27], combination method of real part and imaginary (CRI) iteration method [17], double-step scale splitting real-valued (DSS) iteration method [10], new double-step scale splitting (NDSS) method [20], efficient block splitting (PBS) iteration method [21], Hermitian normal splitting (HNS) iteration method, simplified HNS (SHNS) method [15], preconditioned SHNS (PSHNS) iteration method [9] and inexact MQHSS (IMQHSS) iteration method [14]. For saddle point problems, Bai et al. [25] constructed the generalized successive overrelaxation (GSOR) iteration method. Bai and Wang [31] subsequent developed the parameterized inexact Uzawa (PIU) iteration method, which contains the GSOR one. By making using of the ideas of [25,31], the generalized SOR (GSOR) iteration method for the complex symmetric linear systems was designed by Salkuyeh et al. in [7], and a series of variants have been proposed. For example, preconditioned generalized SOR (PGSOR) iteration method [6], preconditioned accelerated generalized successive overrelaxation (PAGSOR) iteration method [23], symmetric SOR (SSOR) iteration method [32].

Next, we give the iteration schemes of the PMHSS, LPMHSS and PMQHSS methods in detail. By preconditioning the linear system (1.1) by choosing a symmetric positive definite matrix V, Bai et al. [30] presented the PMHSS iteration method, and deduced its convergence theory. The PMHSS method can be described as follows:

The PMHSS iteration method [30]: Let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. For $k = 0, 1, 2, \cdots$ until the sequence of iterates $\{x^{(k)}\}$ converges, compute the next iterate $x^{(k+1)}$ according to the following program

$$\begin{cases} (\alpha V + W)x^{(k+\frac{1}{2})} = (\alpha V - \mathrm{i}T)x^{(k)} + b \\ (\alpha V + T)x^{(k+1)} = (\alpha V + \mathrm{i}W)x^{(k+\frac{1}{2})} - \mathrm{i}b \end{cases}, \tag{1.3}$$

where $\alpha > 0$, and V is a SPD matrix. The iteration matrix of the PMHSS method can be expressed as follows

$$L(V, \alpha) = (\alpha V + T)^{-1}(\alpha V + \mathrm{i}W)(\alpha V + W)^{-1}(\alpha V - \mathrm{i}T).$$

When $V = W$, one has

$$L(\alpha) := L(W, \alpha) = \frac{\alpha + \mathrm{i}}{\alpha + 1}(\alpha W + T)^{-1}(\alpha W - \mathrm{i}T).$$

To improve the convergence rate of the PHSS method, Li et al. [19] proposed the lopsided PMHSS (LPMHSS) iteration method, and theoretical analyses show the LPMHSS iteration method converges to the unique solution of the complex symmetric linear system (1.1) under proper conditions. The iteration format of the LPMHSS method is:

The LPMHSS iteration method [19]: Let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. Compute the next iterate $x^{(k+1)}$ ($k = 0, 1, 2, \cdots$) according to the following iteration sequence until $\{x^{(k)}\}$ converges,

$$\begin{cases} Wx^{(k+\frac{1}{2})} = -\mathrm{i}Tx^{(k)} + b \\ (\alpha V + T)x^{(k+1)} = (\alpha V + \mathrm{i}W)x^{(k+\frac{1}{2})} - \mathrm{i}b \end{cases}, \tag{1.4}$$

where V is a SPD matrix and α is a positive constant. The iteration matrix of the LPMHSS method can be expressed as

$$M(V; \alpha) = -\mathrm{i}(\alpha V + T)^{-1}(\alpha V + \mathrm{i}W)W^{-1}T. \tag{1.5}$$

In particular, when $V = W$, the iteration matrix is

$$M(W; \alpha) = (1 - \mathrm{i}\alpha)(\alpha W + T)^{-1}T. \tag{1.6}$$

Very recently, Li et al. [3] introduced the preconditioned MQHSS (PMQHSS) method for solving complex symmetric linear equations (1.1), and theoretically analyzed the properties of the PMQHSS method.

The PMQHSS iteration method [3]: Let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. For $k = 0, 1, 2, \cdots$ until the sequence of iterates $\{x^{(k)}\}$ converges, compute the next iterate $x^{(k+1)}$ according to

$$\begin{cases} (\alpha V + T)x^{(k+\frac{1}{2})} = (\alpha V + \mathrm{i}H(\omega, V))x^{(k)} + \omega T V^{-1} W x^{(k)} - \mathrm{i}(I - \mathrm{i}\omega T V^{-1})b \\ (\alpha V + H(\omega, V))x^{(k+1)} = (\alpha V - \mathrm{i}T)x^{(k+\frac{1}{2})} + \mathrm{i}\omega T V^{-1} W x^{(k)} + (I - \mathrm{i}\omega T V^{-1})b \end{cases}, \tag{1.7}$$

where V is a SPD matrix, α, ω are given positive and nonnegative constants, respectively, I is the identity matrix, and $H(\omega, V) = W + \omega T V^{-1} T$.

The iteration matrix of the PMQHSS method is

$$\begin{aligned} L_\omega(V, \alpha) = {} & (\alpha V + H(\omega, V))^{-1}(\alpha V - \mathrm{i}T)(\alpha V + T)^{-1}(\alpha V + \mathrm{i}H(\omega, V)) \\ & + (1 + \mathrm{i})\alpha\omega(\alpha V + H(\omega, V))^{-1} V (\alpha V + T)^{-1} T V^{-1} W. \end{aligned} \tag{1.8}$$

In particular, when $V = T$, one has

$$L_\omega(T, \alpha) = \frac{\alpha - \mathrm{i}}{\alpha + 1}(\alpha T + H_\omega)^{-1}(\alpha T + \mathrm{i}H_\omega) + \frac{1 + \mathrm{i}}{\alpha + 1}\alpha\omega(\alpha T + H_\omega)^{-1} W. \tag{1.9}$$

Theoretical analyses and numerical experiments in [19] show that the LPMHSS method is more effective than the PMHSS one. Therefore, in order to further improve the convergence behavior of the PMQHSS iteration method, and inspired by the ideas in [19], this paper proposes the lopsided PMQHSS (LPMQHSS) method by utilizing the lopsided technology, hoping to get better results. In addition, adopting the lopsided technology in the first step iteration scheme of the PMQHSS method and the second step iteration scheme of the PMHSS method, we establish another new iteration method, called double lopsided PMQHSS (DLPMQHSS) iteration method. The purpose of using the lopsided technology twice is to make that the coefficient matrices of two linear subsystems of the DLPMQHSS method are the same in every iteration step. This makes that the new method needs less computation, which will be verified by numerical experiments in Section 4.

The rest of the paper is organized as follows. In Section 2 we construct the LPMQHSS and DLPMQHSS iteration methods for solving the system (1.1). In Section 3, the convergence conditions of the two new methods are discussed together with the parameters regions in detail. Specially, the properties of the LPMQHSS method with $V = T$ are also established. Section 4 is devoted to numerical experiments to test the feasibility and effectiveness of the LPMQHSS and DLPMQHSS methods for solving complex symmetric linear equations (1.1). Finally, in Section 5, we conclude the article with a short conclusion.

In the rest of this paper, we use $sp(A), \kappa(A), \rho(A), \|A\|_2$ to represent the spectrum, spectral condition number, spectral radius, Euclidean norm of the matrix A, respectively. Furthermore, $\|x\|_2$ represents the Euclidean norm of vector x.

2 The LPMQHSS and DLPMQHSS methods

In this section, inspired by the ideas of the PMQHSS and LPMHSS methods, we establish lopsided PMQHSS (LPMQHSS) and double lopsided PMQHSS (DLPMQHSS) iteration methods by utilizing the lopsided technology.

2.1 The LPMQHSS iteration method

It follows from iteration format (1.7) that the first step iteration of the PMQHSS method is

$$(\alpha V + T)x^{(k+\frac{1}{2})} = (\alpha V + \mathrm{i}H(\omega, V) + \omega T V^{-1} W)x^{(k)} - \mathrm{i}(I - \mathrm{i}\omega T V^{-1})b. \tag{2.1}$$

Adopting the lopsided technology applied in [11,19] to (2.1), we can get

$$T x^{(k+\frac{1}{2})} = (\mathrm{i}H(\omega, V) + \omega T V^{-1} W)x^{(k)} - \mathrm{i}(I - \mathrm{i}\omega T V^{-1})b. \tag{2.2}$$

The LPMQHSS iteration method can be obtained by combining Equation (2.2) and the second step iteration of iteration scheme (1.7).

The LPMQHSS iteration method: Let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. For $k = 0, 1, 2, \cdots$ until the sequence of iterates $\{x^{(k)}\}$ converges, compute the next iterate $x^{(k+1)}$ according to the following program

$$\begin{cases} Tx^{(k+\frac{1}{2})} = (\mathrm{i}H(\omega, V) + \omega TV^{-1}W)x^{(k)} - \mathrm{i}(I - \mathrm{i}\omega TV^{-1})b \\ (\alpha V + H(\omega, V))x^{(k+1)} = (\alpha V - \mathrm{i}T)x^{(k+\frac{1}{2})} + \mathrm{i}\omega TV^{-1}Wx^{(k)} + (I - \mathrm{i}\omega TV^{-1})b \end{cases}, \tag{2.3}$$

where V is a SPD matrix, α, ω are given positive and nonnegative constants, respectively. I is the identity matrix, and $H(\omega, V) = W + \omega TV^{-1}T$.

After straightforward derivations we can reformulate the LPMQHSS iteration format as

$$x^{(k+1)} = S_\omega(\alpha, V)x^{(k)} + K_\omega(\alpha, V)b. \tag{2.4}$$

Here,

$$S_\omega(\alpha, V) = \mathrm{i}(\alpha V + H(\omega, V))^{-1}(\alpha V - \mathrm{i}T)T^{-1}H(\omega, V) + \alpha\omega(\alpha V + H(\omega, V))^{-1}W \tag{2.5}$$

and

$$K_\omega(\alpha, V) = -\mathrm{i}\alpha(\alpha V + H(\omega, V))^{-1}VT^{-1}(I - \mathrm{i}\omega TV^{-1}).$$

Note that the matrix $S_\omega(\alpha, V)$ is the iteration matrix of the LPMQHSS method. Actually, the LPMQHSS iteration scheme (2.4) can also be derived from the splitting

$$A = B_\omega(\alpha, V) - C_\omega(\alpha, V) \tag{2.6}$$

with

$$B_\omega(\alpha, V) = \frac{\mathrm{i}}{\alpha}(I - \mathrm{i}\omega TV^{-1})^{-1}TV^{-1}(\alpha V + H(\omega, V))$$

and

$$C_\omega(\alpha, V) = \frac{\mathrm{i}}{\alpha}(I - \mathrm{i}\omega TV^{-1})^{-1}[\mathrm{i}(\alpha V - \mathrm{i}T)V^{-1}H(\omega, V) + \alpha\omega TV^{-1}W].$$

Note that $S_\omega(\alpha, V) = B_\omega(\alpha, V)^{-1}C_\omega(\alpha, V)$. The matrix $B_\omega(\alpha, V)$ can be viewed as a preconditioner for the coefficient matrix A. We call it as the LPMQHSS preconditioner.

In particular, when $V = T$, one has

$$S_\omega(\alpha, T) = (1 + \mathrm{i}\alpha)(\alpha T + H_\omega)^{-1}H_\omega + \alpha\omega(\alpha T + H_\omega)^{-1}W \tag{2.7}$$

and

$$K_\omega(\alpha, T) = -\mathrm{i}\alpha(1 - \mathrm{i}\omega)(\alpha T + H_\omega)^{-1},$$

with $H_\omega = W + \omega T$. The LPMQHSS iteration method with $V = T$ can also be derived from the splitting

$$A = B_\omega(\alpha, T) - C_\omega(\alpha, T) \tag{2.8}$$

with

$$B_\omega(\alpha, T) = \frac{\mathrm{i}(\alpha T + H_\omega)}{\alpha(1 - \mathrm{i}\omega)}$$

and

$$C_\omega(\alpha, T) = \frac{\mathrm{i}}{\alpha(1 - \mathrm{i}\omega)}[(1 + \mathrm{i}\alpha)H_\omega + \alpha\omega W].$$

It follows from iteration schemes (1.7) and (2.3) that the coefficient matrix of the first sublinear system of the LPMQHSS iteration method is T, yet the coefficient matrix of the first sublinear system of the PMQHSS iteration method is $\alpha V + T$. As we all known, it is undoubtedly difficult to select the positive definite matrix V and the positive parameter α. And the first iteration scheme of the LPMQHSS method avoids to select them. In addition, it follows from [19] that the LPMHSS method proposed by the lopsided technology has advantages over the PMHSS one under proper conditions. Thus the LPMQHSS method established by applying the lopsided technology to the PMQHSS one is expected to get better results than the PMQHSS one, and this expectation will be verified by numerical experiments in Section 4.

2.2 The DLPMQHSS iteration method

In order to make that two linear subsystems of two step iteration schemes are the same, we adopt the lopsided technology in both the first iteration scheme of the PMQHSS method and the second iteration scheme of the PMHSS method to establish double lopsided PMQHSS iteration method, called the DLPMQHSS iteration method for short. Firstly, adopting the lopsided technology to the first iteration scheme in (1.7), one has

$$Tx^{(k+\frac{1}{2})} = (\mathrm{i}H(\omega, V) + \omega TV^{-1}W)x^{(k)} - \mathrm{i}(I - \mathrm{i}\omega TV^{-1})b.$$

Let $V = T$, then the first half-step iteration format can be expressed as

$$Tx^{(k+\frac{1}{2})} = \mathrm{i}H_\omega x^{(k)} + \omega Wx^{(k)} - \mathrm{i}(1 - \mathrm{i}\omega)b, \tag{2.9}$$

where $H(\omega, V) = W + \omega TV^{-1}T$, $H_\omega = W + \omega T$.

Then it follows from (1.3) that the iteration format of the second step of the PMHSS method is

$$(\alpha V + T)x^{(k+1)} = (\alpha V + \mathrm{i}W)x^{(k+\frac{1}{2})} - \mathrm{i}b.$$

The lopsided technique used in [11, 19] is applied to the above equation, we construct a second half-step iteration format

$$Tx^{(k+1)} = \mathrm{i}Wx^{(k+\frac{1}{2})} - \mathrm{i}b. \tag{2.10}$$

The double Lopsided PMQHSS (DLPMQHSS) iteration method is proposed by alternating between above iteration schemes (2.9) and (2.10) for solving the complex symmetric linear equations (1.1), which is described as follows:

The DLPMQHSS iteration method: Let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. For $k = 0, 1, 2, \cdots$ until the sequence of iterates $\{x^{(k)}\}$ converges, compute the next iterate $x^{(k+1)}$ according to

$$\begin{cases} Tx^{(k+\frac{1}{2})} = \mathrm{i}H_\omega x^{(k)} + \omega Wx^{(k)} - \mathrm{i}(1 - \mathrm{i}\omega)b \\ Tx^{(k+1)} = \mathrm{i}Wx^{(k+\frac{1}{2})} - \mathrm{i}b \end{cases}, \tag{2.11}$$

where ω is a nonnegative constant and $H_\omega = W + \omega T$.

By eliminating the vector $x^{(k+\frac{1}{2})}$ in the DLPMQHSS iteration method above, it has

$$x^{(k+1)} = L(\omega)x^{(k)} + G(\omega)b. \tag{2.12}$$

Here, $L(\omega)$ is the iteration matrix of the DLPMQHSS iteration method. The matrices $L(\omega)$ and $G(\omega)$ have forms

$$L(\omega) = \mathrm{i}T^{-1}WT^{-1}(\omega W + \mathrm{i}H_\omega) \tag{2.13}$$

and

$$G(\omega) = T^{-1}(W - \mathrm{i}\omega W - \mathrm{i}T)T^{-1}.$$

We give the following two matrices

$$M(\omega) = T(W - \mathrm{i}\omega W - \mathrm{i}T)^{-1}T,$$

and

$$N(\omega) = T(T + \omega W + \mathrm{i}W)^{-1}WT^{-1}(\omega W + \mathrm{i}H_\omega),$$

then it holds that $A = M(\omega) - N(\omega)$ and $L(\omega) = M(\omega)^{-1}N(\omega)$. Therefore, the DLPMQHSS iteration method is induced by the matrix splitting $A = M(\omega) - N(\omega)$.

Compared with the PMQHSS and PMHSS ones, the DLPMQHSS iteration method only needs to solve two sublinear systems with coefficient matrix T in each iteration. And from the iteration schemes (1.7) and (2.3), the PMQHSS and LPMQHSS iteration methods need to determine two parameters α and ω, while the DLPMQHSS iteration method only involves a nonnegative parameter ω, which needs undoubtedly less computation. Furthermore, in the numerical experiments of many references the matrix T usually is chosen as kI with I being identity matrix. Therefore, applied to solve (1.1), the DLPMQHSS method only requires the computing of the matrix-vector multiplications and scalar-vector multiplications. There is no doubt that the calculation amount of the DLPMQHSS method is very small, and this fact will be verified by numerical experiments in Section 4.

3 Properties of the LPMQHSS and DLPMQHSS methods

In this section, we give the upper bounds of the spectral radiuses of the iteration matrices of the LPMQHSS and DLPMQHSS methods. Then based on the upper bounds, the convergence properties of the two methods are presented.

3.1 Properties of the LPMQHSS method

Theorem 3.1 *Let $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$ be a SPSD matrix, $T \in \mathbb{R}^{n\times n}$ be a SPD matrix. Let $\mu_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalues of the matrices $P = V^{-\frac{1}{2}}TV^{-\frac{1}{2}}$ and $L = V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}} = V^{-\frac{1}{2}}WV^{-\frac{1}{2}} + \omega P^2$, respectively. And ω and α be nonnegative and positive constants, respectively. Then, the spectral radius $\rho(S_\omega(\alpha, V))$ of the iteration matrix of the LPMQHSS method satisfies $\rho(S_\omega(\alpha, V)) \le \sigma(\alpha, \omega)$, where*

$$\sigma(\alpha,\omega) = \frac{\lambda_{\max}}{\alpha+\lambda_{\max}}\left(\frac{\sqrt{\alpha^2+\mu_{\min}^2}}{\mu_{\min}} + \alpha\omega\eta\right), \tag{3.1}$$

with V being a SPD matrix, $\eta = \kappa(V^{-\frac{1}{2}})\left\|WH(\omega, V)^{-1}\right\|_2$ and $H(\omega, V) = W + \omega TV^{-1}T$. If $\sigma(\alpha,\omega) < 1$, then the LPMQHSS iteration method is convergent.

Proof From (2.5), we can obtain the iteration matrix of the LPMQHSS iteration method of the form

$$S_\omega(\alpha, V) = \mathrm{i}(\alpha V + H(\omega, V))^{-1}(\alpha V - \mathrm{i}T)T^{-1}H(\omega, V) + \alpha\omega(\alpha V + H(\omega, V))^{-1}W,$$

then

$$\begin{aligned}
\rho(S_\omega(\alpha, V)) &= \rho(\mathrm{i}(\alpha V + H(\omega, V))^{-1}(\alpha V - \mathrm{i}T)T^{-1}H(\omega, V) + \alpha\omega(\alpha V + H(\omega, V))^{-1}W)\\
&= \rho(V^{-\frac{1}{2}}(\alpha I + V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}})^{-1}(\alpha I - \mathrm{i}V^{-\frac{1}{2}}TV^{-\frac{1}{2}})V^{\frac{1}{2}}T^{-1}V^{\frac{1}{2}}V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}}V^{\frac{1}{2}}\\
&\quad + \alpha\omega V^{-\frac{1}{2}}(\alpha I + V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}})^{-1}V^{-\frac{1}{2}}WV^{-\frac{1}{2}}V^{\frac{1}{2}})\\
&= \rho((\alpha I - \mathrm{i}V^{-\frac{1}{2}}TV^{-\frac{1}{2}})V^{\frac{1}{2}}T^{-1}V^{\frac{1}{2}}V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}}(\alpha I + V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}})^{-1}\\
&\quad + \alpha\omega V^{-\frac{1}{2}}WV^{-\frac{1}{2}}(V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}})^{-1}V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}}(\alpha I + V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}})^{-1}\\
&\le \left\|(\alpha I - \mathrm{i}V^{-\frac{1}{2}}TV^{-\frac{1}{2}})V^{\frac{1}{2}}T^{-1}V^{\frac{1}{2}}\right\|_2\left\|V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}}(\alpha I + V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}})^{-1}\right\|_2\\
&\quad + \alpha\omega\left\|V^{-\frac{1}{2}}WV^{-\frac{1}{2}}(V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}})^{-1}\right\|_2\left\|V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}}(\alpha I + V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}})^{-1}\right\|_2\\
&= \max_{\mu_j\in sp(P)}\frac{\sqrt{\alpha^2+\mu_j^2}}{\mu_j}\max_{\lambda_j\in sp(L)}\frac{\lambda_j}{\alpha+\lambda_j} + \alpha\omega\left\|V^{-\frac{1}{2}}WV^{-\frac{1}{2}}(V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}})^{-1}\right\|_2\max_{\lambda_j\in sp(L)}\frac{\lambda_j}{\alpha+\lambda_j}.
\end{aligned}$$

Let $f(x) = \frac{\sqrt{\alpha^2+x^2}}{x}(x > 0)$ and $g(x) = \frac{x}{\alpha+x}(x \ge 0)$. Owing to that the facts that $\frac{df(x)}{dx} = \frac{-\alpha^2}{x^2\sqrt{\alpha^2+x^2}} < 0$ and $\frac{dg(x)}{dx} = \frac{\alpha}{(\alpha+x)^2} > 0$, then the two functions $f(x)$ and $g(x)$ are monotonically decreasing and increasing with respect to x, respectively. Thus, we can get

$$\max_{\mu_j\in sp(P)}\frac{\sqrt{\alpha^2+\mu_j^2}}{\mu_j} = \frac{\sqrt{\alpha^2+\mu_{\min}^2}}{\mu_{\min}} \quad\text{and}\quad \max_{\lambda_j\in sp(L)}\frac{\lambda_j}{\alpha+\lambda_j} = \frac{\lambda_{\max}}{\alpha+\lambda_{\max}}. \tag{3.2}$$

It follows from the property of matrix norm that

$$\left\|V^{-\frac{1}{2}}WV^{-\frac{1}{2}}(V^{-\frac{1}{2}}H(\omega, V)V^{-\frac{1}{2}})^{-1}\right\|_2 = \left\|V^{-\frac{1}{2}}WH(\omega, V)^{-1}V^{\frac{1}{2}}\right\|_2 \le \kappa(V^{-\frac{1}{2}})\left\|WH(\omega, V)^{-1}\right\|_2. \tag{3.3}$$

According to (3.2) and (3.3), one has

$$\rho(S_\omega(\alpha, V)) \le \frac{\lambda_{\max}}{\alpha + \lambda_{\max}} \left(\frac{\sqrt{\alpha^2 + \mu_{\min}^2}}{\mu_{\min}} + \alpha\omega\eta \right) \triangleq \sigma(\alpha, \omega),$$

where

$$\eta = \kappa(V^{-\frac{1}{2}}) \left\| W H(\omega, V)^{-1} \right\|_2.$$

If the upper bound $\sigma(\alpha, \omega) < 1$, then $\rho(S_\omega(\alpha, V)) < 1$ holds, and the LPMQHSS method is convergent. □

Theorem 3.2 *Let the conditions of Theorem 3.1 be satisfied. We denote by*

$$\begin{aligned}
\alpha_1 &= \frac{2\mu_{\min}^2 \lambda_{\max}(1 - \omega\eta\lambda_{\max})}{\lambda_{\max}^2 - \mu_{\min}^2 + \omega\eta\mu_{\min}^2\lambda_{\max}(2 - \omega\eta\lambda_{\max})},\\
\omega_1 &= \frac{2}{\eta\lambda_{\max}}, \omega_3 = \frac{\mu_{\min} - \lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}},\\
\omega_2 &= \frac{1}{\eta\lambda_{\max}}, \omega_4 = \frac{\mu_{\min} + \lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}.
\end{aligned}$$

Then

(1) when $\lambda_{\max} \le \mu_{\min}$, the LPMQHSS method is convergent if parameters α and ω satisfy one of the following conditions:
- *(a) $\omega > \omega_1$, and $\alpha > \alpha_1$;*
- *(b) $\omega_3 \le \omega < \omega_2$, and $\alpha < \alpha_1$,*
- *(c) $0 \le \omega < \omega_3$, and any $\alpha > 0$;*
- *(d) $\omega_4 < \omega \le \omega_1$, and $\alpha > \alpha_1$.*

(2) when $\lambda_{\max} > \mu_{\min}$, the LPMQHSS method is convergent if parameters α and ω satisfy one of the following conditions:
- *(d) $\omega > \omega_4$, and $\alpha > \alpha_1$;*
- *(e) $\omega < \omega_2$, and $\alpha < \alpha_1$.*

Proof It follows from Theorem 3.1 that the spectral radius $\rho(S_\omega(\alpha, V))$ of the iteration matrix of the LPMQHSS method satisfies

$$\rho(S_\omega(\alpha, V)) \le \frac{\lambda_{\max}}{\alpha + \lambda_{\max}} \left(\frac{\sqrt{\alpha^2 + \mu_{\min}^2}}{\mu_{\min}} + \alpha\omega\eta \right).$$

To get $\rho(S_\omega(\alpha, V)) < 1$, it is enough to have $\frac{\lambda_{\max}}{\alpha+\lambda_{\max}}(\frac{\sqrt{\alpha^2+\mu_{\min}^2}}{\mu_{\min}} + \alpha\omega\eta) < 1$. By simplifying this inequality, we have

$$\alpha(\lambda_{\max}^2 - \mu_{\min}^2) + \alpha\omega\eta\mu_{\min}^2\lambda_{\max}(2 - \omega\eta\lambda_{\max}) < 2\mu_{\min}^2\lambda_{\max}(1 - \omega\eta\lambda_{\max}). \tag{3.4}$$

(1) If $\lambda_{\max} \le \mu_{\min}$, it can be divided into three situations:

(a) If $2 - \omega\eta\lambda_{\max} < 0$, this is $\omega > \frac{2}{\eta\lambda_{\max}}$, then it follows from (3.4) that

$$\alpha > \frac{2\mu_{\min}^2\lambda_{\max}(1 - \omega\eta\lambda_{\max})}{\lambda_{\max}^2 - \mu_{\min}^2 + \omega\eta\mu_{\min}^2\lambda_{\max}(2 - \omega\eta\lambda_{\max})}.$$

(b) If $2 - \omega\eta\lambda_{\max} \ge 0$, this is $\omega \le \frac{2}{\eta\lambda_{\max}}$, we consider the following two cases:

(b1) When $1 - \omega\eta\lambda_{\max} > 0$, ie., $\omega < \frac{1}{\eta\lambda_{\max}}$.

- If $\lambda_{\max}^2-\mu_{\min}^2+\omega\eta\mu_{\min}^2\lambda_{\max}(2-\omega\eta\lambda_{\max})\geq 0$, solving the quadratic inequality about ω yields
$$\frac{\mu_{\min}-\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}\leq\omega\leq\frac{\mu_{\min}+\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}.$$
Due to the fact that $\frac{1}{\eta\lambda_{\max}}<\frac{2}{\eta\lambda_{\max}}$ and $\frac{1}{\eta\lambda_{\max}}<\frac{\mu_{\min}+\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}$, then
$$\frac{\mu_{\min}-\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}\leq\omega<\frac{1}{\eta\lambda_{\max}}.$$
In this case, solution of Inequality (3.4) is
$$\alpha<\frac{2\mu_{\min}^2\lambda_{\max}(1-\omega\eta\lambda_{\max})}{\lambda_{\max}^2-\mu_{\min}^2+\omega\eta\mu_{\min}^2\lambda_{\max}(2-\omega\eta\lambda_{\max})}.$$
- If $\lambda_{\max}^2-\mu_{\min}^2+\omega\eta\mu_{\min}^2\lambda_{\max}(2-\omega\eta\lambda_{\max})<0$, we can get
$$\omega<\frac{\mu_{\min}-\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}\quad\text{or}\quad\omega>\frac{\mu_{\min}+\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}.$$
Because of $\lambda_{\max}\leq\mu_{\min}$, we have $\frac{\mu_{\min}-\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}\geq 0$ and $\frac{\mu_{\min}-\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}\leq\frac{1}{\eta\lambda_{\max}}$, then
$$0\leq\omega<\frac{\mu_{\min}-\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}.$$
Thus, if $0\leq\omega<\frac{\mu_{\min}-\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}$, then Inequality (3.4) holds for any $\alpha>0$.

(b2) $1-\omega\eta\lambda_{\max}\leq 0$, this is $\frac{1}{\eta\lambda_{\max}}\leq\omega\leq\frac{2}{\eta\lambda_{\max}}$.
- If $\lambda_{\max}^2-\mu_{\min}^2+\omega\eta\mu_{\min}^2\lambda_{\max}(2-\omega\eta\lambda_{\max})\geq 0$, there do not exist parameters α and ω satisfying the conditions to make $\rho(S_\omega(\alpha,V))<1$.
- If $\lambda_{\max}^2-\mu_{\min}^2+\omega\eta\mu_{\min}^2\lambda_{\max}(2-\omega\eta\lambda_{\max})<0$, we can get
$$\omega<\frac{\mu_{\min}-\lambda_{\max}}{\eta\mu_{\min_{\lambda_{\max}}}}\quad\text{or}\quad\omega>\frac{\mu_{\min}+\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}.$$
Due to the fact that $\frac{\mu_{\min}-\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}<\frac{1}{\eta\lambda_{\max}}<\frac{\mu_{\min}+\lambda_{\max}}{\eta\mu_{\min}\lambda_{\max}}<\frac{2}{\eta\lambda_{\max}}$, one has
$$\frac{\mu_{min}+\lambda_{max}}{\eta\mu_{min}\lambda_{max}}<\omega\leq\frac{2}{\eta\lambda_{max}}.$$
In this case, it follows from (3.4) that
$$\alpha>\frac{2\mu_{\min}^2\lambda_{\max}(1-\omega\eta\lambda_{\max})}{\lambda_{\max}^2-\mu_{\min}^2+\omega\eta\mu_{\min}^2\lambda_{\max}(2-\omega\eta\lambda_{\max})}.$$

(2) If $\lambda_{\max}>\mu_{\min}$, it can be divided into three situations:
(c) If $2-\omega\eta\lambda_{\max}<0$, this is $\omega>\frac{2}{\eta\lambda_{\max}}$, we consider the following two cases:
(c1) If $\lambda_{\max}^2-\mu_{\min}^2+\omega\eta\mu_{\min}^2\lambda_{\max}(2-\omega\eta\lambda_{\max})\geq 0$, there do not exist parameters α and ω satisfying the conditions to make $\rho(S_\omega(\alpha,V))<1$.

(c2) If $\lambda_{\max}^2-\mu_{\min}^2+\omega\eta\mu_{\min}^2\lambda_{\max}(2-\omega\eta\lambda_{\max})<0$, we can get
$$\omega>\frac{\mu_{\min}+\lambda_{\max}}{\eta\lambda_{\max}\mu_{\min}}.$$
Owing to the fact that $\frac{\mu_{\min}+\lambda_{\max}}{\eta\lambda_{\max}\mu_{\min}}>\frac{2}{\eta\lambda_{\max}}$, then for $\omega>\frac{\mu_{\min}+\lambda_{\max}}{\eta\lambda_{\max}\mu_{\min}}$, we have
$$\alpha>\frac{2\mu_{\min}^2\lambda_{\max}(1-\omega\eta\lambda_{\max})}{\lambda_{\max}^2-\mu_{\min}^2+\omega\eta\mu_{\min}^2\lambda_{max}(2-\omega\eta\lambda_{\max})}.$$

(d) If $2-\omega\eta\lambda_{\max} \geq 0$, two different cases are considered:
(d1) If $1-\omega\eta\lambda_{\max} > 0$, this is $\omega < \frac{1}{\eta\lambda_{\max}}$, then solving Inequality (3.4) leads to

$$\alpha < \frac{2\mu_{\min}^2\lambda_{\max}(1-\omega\eta\lambda_{\max})}{\lambda_{\max}^2-\mu_{\min}^2+\omega\eta\mu_{\min}^2\lambda_{\max}(2-\omega\eta\lambda_{\max})}.$$

(d2) $1-\omega\eta\lambda_{\max} \leq 0$, there do not exist parameters α and ω such that $\rho(S_\omega(\alpha, V)) < 1$.

The results of Theorem 3.2 can be obtained by sorting out the above proof results. □

Remark 3.1 Keep in mind that the maximum eigenvalue λ_{max} of the matrix $V^{-1}H(\omega, V)$ is related to ω, then the upper and lower bounds of the convergent intervals of the parameters α and ω involved in the LPMQHSS method in Theorem 3.2 depend on the parameter ω. Thus, the convergence properties of the LPMQHSS method are conservative. The detailed discussions about the parameters of the LPMQHSS method will be a valuable topic for our future study.

Next, Theorem 3.3 gives the convergence properties of the LPMQHSS method with $V=T$.

Theorem 3.3 *Let $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$ be a SPSD matrix, $T \in \mathbb{R}^{n\times n}$ be a SPD matrix. Let $\tau_{\max}$ be the largest eigenvalue of the matrix $T^{-1}W$. If $0 < \tau_{\max} < 1$, $\alpha > \frac{\omega\tau_{\max}}{1-\tau_{\max}-\omega(1+\tau_{\max})}$ and $0 \leq \omega < \frac{1-\tau_{\max}}{1+\tau_{\max}}$, then the LPMQHSS iteration method with $V = T$ is convergent, i.e., $\rho(S_\omega(\alpha, T)) < 1$.*

Proof From (2.7), the iteration matrix of the LPMQHSS method with $V=T$ is as follows

$$S_\omega(\alpha, T) = (1+\mathrm{i}\alpha)(\alpha T + H_\omega)^{-1}H_\omega + \alpha\omega(\alpha T + H_\omega)^{-1}W,$$

where $H_\omega = W + \omega T$. Therefore, we have

$$\begin{aligned}
\rho(S_\omega(\alpha, T)) &= \rho((1+\mathrm{i}\alpha)(\alpha T + H_\omega)^{-1}H_\omega + \alpha\omega(\alpha T + H_\omega)^{-1}W)\\
&= \rho((1+\mathrm{i}\alpha)T^{-\frac{1}{2}}(\alpha I + T^{-\frac{1}{2}}H_\omega T^{-\frac{1}{2}})^{-1}T^{-\frac{1}{2}}H_\omega T^{-\frac{1}{2}}T^{\frac{1}{2}}\\
&\quad + \alpha\omega T^{-\frac{1}{2}}(\alpha I + T^{-\frac{1}{2}}H_\omega T^{-\frac{1}{2}})^{-1}T^{-\frac{1}{2}}WT^{-\frac{1}{2}}T^{\frac{1}{2}})\\
&= \rho((1+\mathrm{i}\alpha)(\alpha I + T^{-\frac{1}{2}}H_\omega T^{-\frac{1}{2}})^{-1}T^{-\frac{1}{2}}H_\omega T^{-\frac{1}{2}} + \alpha\omega(\alpha I + T^{-\frac{1}{2}}H_\omega T^{-\frac{1}{2}})^{-1}T^{-\frac{1}{2}}WT^{-\frac{1}{2}})\\
&\leq \sqrt{\alpha^2+1}\left\|(\alpha I + T^{-\frac{1}{2}}H_\omega T^{-\frac{1}{2}})^{-1}T^{-\frac{1}{2}}H_\omega T^{-\frac{1}{2}}\right\|_2 + \alpha\omega\left\|(\alpha I + T^{-\frac{1}{2}}H_\omega T^{-\frac{1}{2}})^{-1}T^{-\frac{1}{2}}WT^{-\frac{1}{2}}\right\|_2\\
&= \sqrt{\alpha^2+1}\left\|(\alpha I + \omega I + T^{-\frac{1}{2}}WT^{-\frac{1}{2}})^{-1}(\omega I + T^{-\frac{1}{2}}WT^{-\frac{1}{2}})\right\|_2\\
&\quad + \alpha\omega\left\|(\alpha I + \omega I + T^{-\frac{1}{2}}WT^{-\frac{1}{2}})^{-1}T^{-\frac{1}{2}}WT^{-\frac{1}{2}}\right\|_2\\
&= \sqrt{\alpha^2+1}\max_{\tau_j\in sp(T^{-1}W)}\frac{\omega+\tau_j}{\alpha+\omega+\tau_j} + \max_{\tau_j\in sp(T^{-1}W)}\frac{\alpha\omega\tau_j}{\alpha+\omega+\tau_j}\\
&\leq \sqrt{\alpha^2+1}(\max_{\tau_j\in sp(T^{-1}W)}\frac{\omega+\tau_j}{\alpha+\omega+\tau_j} + \max_{\tau_j\in sp(T^{-1}W)}\frac{\omega\tau_j}{\alpha+\omega+\tau_j}).
\end{aligned}$$

Let $h(x) = \frac{\omega+x}{\alpha+\omega+x}$ and $s(x) = \frac{\omega x}{\alpha+\omega+x}$ for $\alpha > 0$, $\omega \geq 0$. Taking the derivative of the two functions with respect to x leads to

$$\frac{dh(x)}{dx} = \frac{\alpha}{(\alpha+\omega+x)^2} > 0 \ \text{ and } \ \frac{ds(x)}{dx} = \frac{\alpha\omega+\omega^2}{(\alpha+\omega+x)^2} > 0.$$

From the monotonicity of the two functions, we have

$$\max_{\tau_j\in sp(T^{-1}W)}\frac{\omega+\tau_j}{\alpha+\omega+\tau_j} = \frac{\omega+\tau_{\max}}{\alpha+\omega+\tau_{\max}}$$

and

$$\max_{\tau_j\in sp(T^{-1}W)}\frac{\omega\tau_j}{\alpha+\omega+\tau_j} = \frac{\omega\tau_{\max}}{\alpha+\omega+\tau_{\max}}.$$

So we can get

$$\begin{aligned}\rho(S_\omega(\alpha, T)) &\le \sqrt{\alpha^2+1}(\max_{\tau_j \in sp(T^{-1}W)} \frac{\omega+\tau_j}{\alpha+\omega+\tau_j} + \max_{\tau_j \in sp(T^{-1}W)} \frac{\omega\tau_j}{\alpha+\omega+\tau_j})\\ &= \sqrt{\alpha^2+1}\frac{\omega+\tau_{\max}+\omega\tau_{\max}}{\alpha+\omega+\tau_{\max}}\\ &\le (\alpha+1)\frac{\omega+\tau_{\max}+\omega\tau_{\max}}{\alpha+\omega+\tau_{\max}}.\end{aligned}$$

To get $\rho(S_\omega(\alpha, T)) < 1$, it is enough to have

$$(\alpha+1)\frac{\omega+\tau_{\max}+\omega\tau_{\max}}{\alpha+\omega+\tau_{\max}} < 1.$$

By simplifying the above inequality, we have

$$\alpha(1-\omega-\tau_{\max}-\omega\tau_{\max}) > \omega\tau_{\max} \iff \alpha[(1-\tau_{\max})-\omega(1+\tau_{\max})] > \omega\tau_{\max}.$$

(a) If $0 \le \tau_{\max} < 1$ and $0 \le \omega < \frac{1-\tau_{\max}}{1+\tau_{\max}}$, then $(1-\tau_{\max})-\omega(1+\tau_{\max}) > 0$ holds. Solving the above inequality results in $\alpha > \frac{\omega\tau\max}{1-\tau_{\max}-\omega(1+\tau_{\max})}$.

(b) If $\tau_{\max} \ge 1$, one has $(1-\tau_{\max})-\omega(1+\tau_{\max}) < 0$, then there do not exist parameters α and ω such that $\alpha[(1-\tau_{\max})-\omega(1+\tau_{\max})] > \omega\tau_{\max}$.

To sum up, Theorem 3.3 can be obtained. □

3.2 Properties of the DLPMQHSS method

Theorem 3.4 *Let $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$ be a SPSD matrix, $T \in \mathbb{R}^{n\times n}$ be a SPD matrix. Let $\tau_{\max}$ be the largest eigenvalue of the matrix $T^{-1}W$. If $0 < \tau_{\max} \le 1$, for $0 \le \omega \le \frac{-\tau_{\max}^2+\sqrt{-\tau_{\max}^6+\tau_{\max}^2+1}}{\tau_{\max}^3+\tau_{\max}}$, the DLPMQHSS iteration method approximates to the unique solution of the complex symmetric linear equations* (1.1) *for any initial vector.*

Proof From (2.13), we can get the iteration matrix of the DLPMQHSS method of the form

$$L(\omega) = \mathrm{i}T^{-1}WT^{-1}(\omega W + \mathrm{i}H_\omega), \tag{3.5}$$

where $H_\omega = W + \omega T$. Then

$$\begin{aligned}\rho(L(\omega)) &= \rho(\mathrm{i}T^{-1}WT^{-1}(\omega W + \mathrm{i}H_\omega))\\ &= \rho(T^{-1}WT^{-1}(\omega W + \mathrm{i}W + \mathrm{i}\omega T))\\ &= \rho(T^{-\frac{1}{2}}WT^{-\frac{1}{2}}(\omega T^{-\frac{1}{2}}WT^{-\frac{1}{2}} + \mathrm{i}T^{-\frac{1}{2}}WT^{-\frac{1}{2}} + \mathrm{i}\omega I))\\ &\le \left\|T^{-\frac{1}{2}}WT^{-\frac{1}{2}}(\omega T^{-\frac{1}{2}}WT^{-\frac{1}{2}} + \mathrm{i}T^{-\frac{1}{2}}WT^{-\frac{1}{2}} + \mathrm{i}\omega I)\right\|_2\\ &= \max_{\tau_j \in sp(T^{-1}W)} \tau_j\sqrt{\tau_j^2 + (\omega+\tau_j)^2}\\ &= \tau_{\max}\sqrt{\omega^2\tau_{\max}^2 + (\omega+\tau_{\max})^2}\\ &\triangleq \sigma(\omega).\end{aligned}$$

In order to make the upper bound $\sigma(\omega)$ of the spectral radius of the iteration matrix less than 1, one has $\tau_{\max}\sqrt{\omega^2\tau_{\max}^2 + (\omega+\tau_{\max})^2} < 1$, this is $(\tau_{\max}^4 + \tau_{\max}^2)\omega^2 + 2\omega\tau_{\max}^3 + \tau_{\max}^4 - 1 < 0$. In order to consider the sign of the left function of the above inequality, we take into account the equation

$$(\tau_{\max}^4 + \tau_{\max}^2)\omega^2 + 2\omega\tau_{\max}^3 + \tau_{\max}^4 - 1 = 0, \tag{3.6}$$

Table 1 Maximum eigenvalues of the matrix $T^{-1}W$ for Example 4.1

m	64	128	256	512	1024
$\tau_{\max}(T^{-1}W)$	3.38×10^{-4}	1.33×10^{-3}	5.28×10^{-3}	-	-

which is regarded as a quadratic equation about ω, and the discriminant of the root of the equation is

$$\triangle_\omega = 4\tau_{\max}^6 - 4(\tau_{\max}^4 + \tau_{\max}^2)(\tau_{\max}^4 - 1) = 4\tau_{\max}^2(-\tau_{\max}^6 + \tau_{\max}^2 + 1).$$

If $\triangle_\omega \le 0$, we know that there does not exit parameter ω making $(\tau_{\max}^4 + \tau_{\max}^2)\omega^2 + 2\omega\tau_{\max}^3 + \tau_{\max}^4 - 1 < 0$. If $\triangle_\omega > 0$, from which one can derive $0 < \tau_{\max} < 1.1510$. Then the solution of Equation (3.6) is

$$\omega_1 = \frac{-\tau_{\max}^2 - \sqrt{-\tau_{\max}^6 + \tau_{\max}^2 + 1}}{\tau_{\max}^3 + \tau_{\max}}, \ \omega_2 = \frac{-\tau_{\max}^2 + \sqrt{-\tau_{\max}^6 + \tau_{\max}^2 + 1}}{\tau_{\max}^3 + \tau_{\max}}.$$

(a) If $0 < \tau_{\max} \le 1$, we have $\omega_2 = \frac{-\tau_{\max}^2 + \sqrt{-\tau_{\max}^6 + \tau_{\max}^2 + 1}}{\tau_{\max}^3 + \tau_{\max}} \ge 0$, then $0 \le \omega \le \frac{-\tau_{\max}^2 + \sqrt{-\tau_{\max}^6 + \tau_{\max}^2 + 1}}{\tau_{\max}^3 + \tau_{\max}}$.

(b) If $1 < \tau_{\max} < 1.1510$, $\omega_2 = \frac{-\tau_{\max}^2 + \sqrt{-\tau_{\max}^6 + \tau_{\max}^2 + 1}}{\tau_{\max}^3 + \tau_{\max}} < 0$ holds, then there does not exist ω such that $(\tau_{\max}^4 + \tau_{\max}^2)\omega^2 + 2\omega\tau_{\max}^3 + \tau_{\max}^4 - 1 < 0$.

□

4 Numerical experiments

We present two examples to verify the the effectiveness of the LPMQHSS and DLPMQHSS methods. We compare the LPMQHSS, DLPMQHSS, PMQUSS, CRI, DSS and PMHSS iteration methods in detail in terms of both the number of iteration steps (denoted by "IT"), the elapsed CPU time (denoted by "CPU") in seconds and relative residual (denoted by "RES").

In addition, the operating environment for this numerical experiment is MATLAB2018b on an 1.19Hz, 8.00GB RAM win10 and Intel(R) Core(TM) i5-1035G1 CPU@1.00 for laptop. In actual computations, the initial vector is adopted as $x^{(0)} = \mathbf{0}$, and in the following experiments we choose $V = T$. All iterations are terminated once the current relative residual satisfies

$$\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} < 1.00 \times 10^{-6}, \quad r^{(k)} = b - Ax^{(k)},$$

or the number of iteration steps exceeds $k_{\max} = 10m$ (m is the problem size of the following numerical examples). In the following numerical experiments, we set the maximum value of α to be 1000.00. Moreover, for the LPMQHSS and DLPMQHSS iteration methods we select the matrix T as the quantity matrix, and it follows from the iterative schemes (2.3) and (2.11) that selecting such a quantity matrix can make our methods needing less computation.

Example 4.1 [10, 20] Consider the following complex Helmholtz equation

$$-\triangle u + \sigma_1 u + \mathrm{i}\sigma_2 u = f$$

where $\sigma_1 = 1.00$, $\sigma_2 = 1.00 \times 10^8$. For the above equations, we use the five point difference scheme in the case that the grid size is $h = \frac{1}{m+1}$, and obtain an equation similar to (1.1)

$$(M + \sigma_1 I + \mathrm{i}\sigma_2 I)x = b,$$

where M is an block-tridiagonal matrix of order $n(n = m^2)$, $M = I \otimes H + H \otimes I$, $H = h^2 \cdot tridiag(-1.00, 2.00, -1.00) \in \mathbb{R}^{m \times m}$. We define the vector on the right side of the equation as $b = (1.00 + \mathrm{i})A\mathbf{1}$, $\mathbf{1}$ means that all elements in the vector are 1.00.

Table 2 The parameter ranges that minimize the iterations number for Example 4.1

		m				
		64	128	256	512	1024
PMHSS [30]	α	[0.91,1.10]	[0.91,1.10]	[0.91,1.10]	[0.91,1.10]	[0.91,1.10]
CRI [17]	α	[0.04,31.21]	[0.11,9.41]	[0.41,2.46]	[0.23,4.51]	[0.37,2.76]
DSS [10]	α	[0.04,31.21]	[0.11,9.41]	[0.41,2.46]	[0.23,4.51]	[0.37,2.76]
PMQHSS [3]	α	[0.96,1.02]	[0.97,1.02]	[0.97,1.02]	[0.97,1.02]	[0.97,1.02]
	ω	[0.89,1.10]	[0.89,1.10]	[0.89,1.10]	[0.89,1.10]	[0.89,1.10]
LPMQHSS	α	[0.04,1000.00]	[0.11,1000.00]	[0.38,1000.00]	[0.22,1000.00]	[1.15,1000.00]
	ω	1.00×10^{-8}	1.00×10^{-8}	1.00×10^{-8}	1.00×10^{-8}	1.00×10^{-8}
DLPMQHSS	ω	[0.00,0.09]	[0.00,0.03]	[0.00,0.01]	[0.00,0.84]	[0.00,0.22]

Table 3 Numerical results for different iteration methods for Example 4.1

		m				
		64	128	256	512	1024
PMHSS [30]	α	0.99	1.04	0.99	1.05	0.92
	IT	41	41	41	41	41
	CPU	0.04	0.23	1.24	5.77	26.05
	RES	9.54E-7	9.61E-7	9.54E-7	9.64E-7	9.86E-7
CRI [17]	α	26.01	8.38	2.38	4.50	0.98
	IT	3	3	3	4	5
	CPU	0.01	0.06	0.31	1.62	8.12
	RES	6.96E-7	7.98E-7	9.54E-7	9.88E-7	1.94E-7
DSS [10]	α	18.01	7.03	2.03	0.23	1.62
	IT	3	3	3	4	5
	CPU	0.01	0.06	0.32	1.73	8.60
	RES	3.35E-7	5.69E-7	7.75E-7	9.04E-7	2.94E-7
PMQHSS [3]	α	0.97	1.01	1.02	1.02	0.99
	ω	0.92	0.99	1.07	0.89	0.94
	IT	21	21	21	21	21
	CPU	0.02	0.09	0.50	2.29	9.77
	RES	9.63E-7	9.55E-7	9.60E-7	1.00E-6	9.61E-7
LPMQHSS	α	2.00	2.00	2.00	2.00	2.00
	ω	1.00×10^{-8}	1.00×10^{-8}	1.00×10^{-8}	1.00×10^{-8}	1.00×10^{-8}
	IT	3	3	3	4	4
	CPU	0.01	0.05	0.21	1.31	4.46
	RES	1.29E-9	1.39E-8	1.53E-7	1.45E-8	6.28E-7
DLPMQHSS	ω	0.03	0.02	0.01	0.13	0.15
	IT	2	2	2	3	3
	CPU	0.00	0.01	0.03	0.14	0.56
	RES	3.23E-7	6.03E-7	9.40E-7	2.60E-7	5.01E-7

Table 1 shows the maximum eigenvalues of the matrix $T^{-1}W$ in Examples 4.1 for the different m. It can be seen from Table 1 that when $m < 512$, the maximum eigenvalue of the matrix $T^{-1}W$ satisfies the condition of Theorem 3.4, which shows that the DLPMQHSS iteration method for solving Example 4.1 is undoubtedly convergent. When the matrix order is quite high, the amount of eigenvalue calculation exceeds the memory of computer, so there are no results in Table 1 for $m \geq 512$. Here, we use the symbol "-" to indicate this case. In other words, we do not know whether the maximum eigenvalue of the matrix $T^{-1}W$ satisfies the condition in Theorem 3.4 for $m = 512$ and 1024 or not, but it can be seen from the results shown in Table 3 that the DLPMQHSS iteration method is still convergent when $m \geq 512$. The reason for this phenomenon is that the convergence condition in Theorem 3.4 is a suffcient condition of the convergence of the DLPMQHSS method, not a suffcient and necessary condition.

In Table 2, we dispaly the ranges of parameters with minimize IT when the PMQHSS, PMHSS, DLPMQHSS, DSS and CRI methods are used to solve Example 4.1. As is pointed in [8,14,24] that ω should be chosen reasonably small so that the matrix $I - \mathrm{i}\omega T$ could be well-conditioned. Hence, we fix ω to be 1.00×10^{-8} in the LPMQHSS iteration method. From Table 2 we can see that when $\omega = 1.00 \times 10^{-8}$ is taken in the LPMQHSS iteration method, the value ranges of α minimizing IT are very wide.

Table 4 Maximum eigenvalue of the matrix $T^{-1}W$ for Example 4.2

m	30	40	50	60
$\tau_{\max}(T^{-1}W)$	1.20×10^{-5}	1.20×10^{-5}	1.20×10^{-5}	1.20×10^{-5}

Table 5 The parameter ranges that minimize the iterations number for Example 4.2

		m			
		30	40	50	60
PMHSS [30]	α	[0.91,1.10]	[0.91,1.10]	[0.91,1.10]	[0.91,1.10]
CRI [17]	α	[0.70,1.43]	[0.51,1.97]	[0.43,2.37]	[0.37,2.70]
DSS [10]	α	[0.70,1.43]	[0.51,1.97]	[0.43,2.37]	[0.37,2.70]
PMQHSS [3]	α	[0.97,1.02]	[0.97,1.02]	[0.97,1.02]	[0.97,1.02]
	ω	[0.89,1.10]	[0.89,1.10]	[0.89,1.10]	[0.89,1.10]
LPMQHSS	α	[0.54,1000.00]	[0.45,1000.00]	[0.39,1000.00]	[0.35,1000.00]
	ω	1.00×10^{-8}	1.00×10^{-8}	1.00×10^{-8}	1.00×10^{-8}
DLPMQHSS	α	[0.00,2.13]	[0.00,2.48]	[0.00,2.79]	[0.00,3.07]

In Table 3, for the PMQHSS, PMHSS, CRI, DSS and DLPMQHSS methods we choose the parameters from the parameters ranges in Table 2, which make CPU time least. And the parameters, IT, CPU time and RES are shown in Table 3. It follows from Table 2 that the ranges of α involved in the LPMQHSS method include 2 when ω is taken as 1.00×10^{-8}. Therefore, in Table 3, we display the IT, CPU time and RES of the LPMQHSS iteration method when $\alpha = 2.00$. From Table 3, we can see that the DLPMQHSS iteration method is superior to other existing ones in IT and CPU time. In addition, the LPMQHSS iteration method is more efficient than the PMHSS and PMQHSS ones in terms of IT and CPU time, and better than the CRI and DSS ones with respect to CPU time. If both α and ω involved in the LPMQHSS iteration method are chosen as the optimal values, it will have better results.

Example 4.2 [22] Let's consider the following 3-D Helmholtz equation

$$\begin{cases} -\triangle u - \tau_1 u + \mathrm{i}\tau_2 u = f, \ in\ \Omega \\ u = g, \ on\ \partial\Omega \end{cases}.$$

We set $\tau_2 = 1.00 \times 10^6$ and $\tau_1 = 4.00$. For the above three-dimensional Helmholtz equation, process it under the grid size of $h = \frac{1}{m+1.00}$ is used. The matrix $W = H \otimes I \otimes I + I \otimes H \otimes I + I \otimes I \otimes H - \tau_1^2 h^2 (I \otimes I \otimes I)$ and $T = \tau_2(I \otimes I \otimes I)$, where $H = tridiag(-1.00, 2.00, -1.00)$ and I represents the identity matrix of order m. We define the vector on the right side of the equation as $b = (1.00 + \mathrm{i})A\mathbf{1}$, $\mathbf{1}$ means that all elements in the vector are 1.00.

Table 4 shows that for Example 4.2, the maximum eigenvalue of the matrix $T^{-1}W$ satisfies the condition $0.00 < \tau_{\max} \leq 1.00$ of Theorem 3.4 for all m. Therefore, the DLPMQHSS iteration method can converge to the unique solution of Equations (1.1) when solving Example 4.2.

Table 5 shows the value ranges of parameters that lead to the least IT when PMQHSS, PMHSS, CRI, DSS, DLPMQHSS methods are used to process Example 4.2. In Table 5, the parameter ω in the LPMQHSS iteration method is selected as $\omega = 1.00 \times 10^{-8}$, and α is chosen as the one that leads to the least IT when $\omega = 1.00 \times 10^{-8}$.

Table 6 lists the optimal parameter values, IT, CPU time and RES when PMQHSS, PMHSS, LPMQHSS, DLPMQHSS, DSS and CRI methods are used to solve Example 4.2. The optimal values of parameters involved in the PMQHSS, PMHSS, DLPMQHSS, DSS and CRI methods are the ones resulting in the least CPU time according to the ranges of the parameters given in Table 5. For the LPMQHSS iteration method, we select $\omega = 1.00 \times 10^{-8}$ and $\alpha = 1.00$. From Table 6, we can see that the LPMQHSS and DLPMQHSS iteration methods are more efficient than the PMHSS and PMQHSS ones in terms of IT and CPU time. Although the IT of the LPMQHSS and DLPMQHSS methods are roughly the same as those of the CRI and DSS ones, CPU time of the LPMQHSS and DLPMQHSS methods is less than that of the CRI and DSS ones. In addition, Table 6 displays the experimental results of the LPMQHSS method with the parameters $\alpha = 1.00$ and $\omega = 1.00 \times 10^{-8}$. And if the two parameters are selected as the optimal ones, the numerical behavior of the LPMQHSS method may be better.

Table 6 Numerical results for different iteration methods for Example 4.2

		m			
		30	40	50	60
PMHSS [30]	α	1.03	0.97	0.98	0.92
	IT	41	41	41	41
	CPU	2.66	13.80	55.26	187.50
	RES	9.58E-7	9.58E-7	9.56E-7	9.87E-7
CRI [17]	α	0.98	0.62	1.87	1.89
	IT	2	2	2	2
	CPU	1.22	12.16	57.08	224.13
	RES	9.39E-7	8.99E-7	8.61E-7	7.87E-7
DSS [10]	α	0.95	0.97	0.57	1.88
	IT	2	2	2	2
	CPU	1.23	12.07	56.57	222.95
	RES	9.40E-7	8.06E-7	8.32E-7	7.85E-7
PMQHSS [3]	α	1.00	0.97	0.97	0.97
	ω	0.93	0.90	0.89	0.89
	IT	21	21	21	21
	CPU	1.19	8.13	33.78	123.43
	RES	9.66E-7	9.69E-7	9.74E-7	9.74E-7
LPMQHSS	α	1.00	1.00	1.00	1.00
	ω	1.00×10^{-8}	1.00×10^{-8}	1.00×10^{-8}	1.00×10^{-8}
	IT	2	2	2	2
	CPU	0.58	5.58	25.99	111.61
	RES	6.69E-7	5.75E-7	5.11E-7	4.64E-7
DLPMQHSS	ω	2.00	0.82	2.23	2.99
	IT	2	2	2	2
	CPU	0.01	0.03	0.06	0.11
	RES	9.39E-7	3.30E-7	7.98E-7	9.73E-7

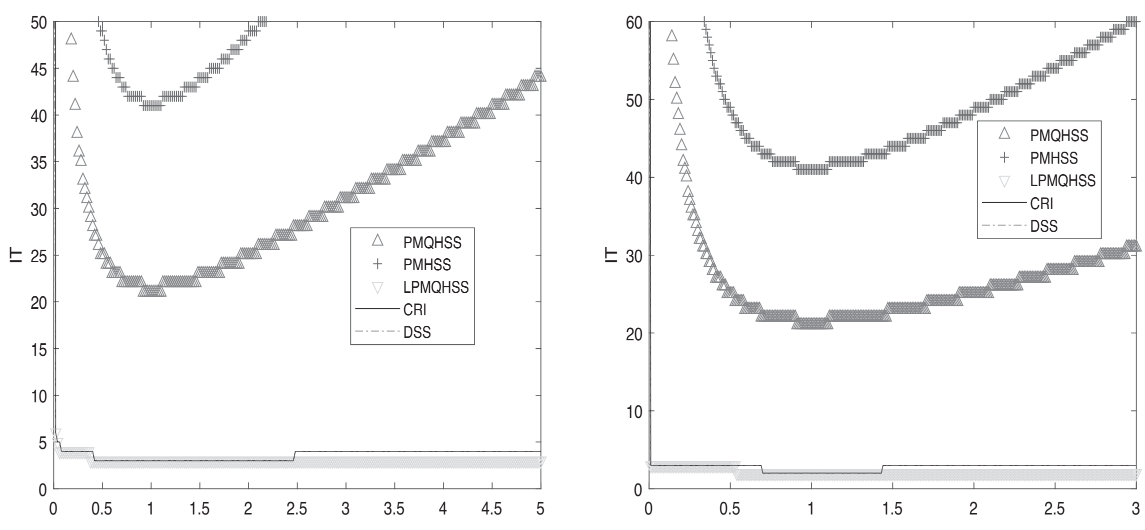

Fig. 1 Iterations number (IT) of the above methods depends on the change of parameter α (Example 4.1 (left $m = 256$), Example 4.2 (right $m = 30$): $\omega = 1.00$ in the PMQHSS, $\omega = 1.00 \times 10^{-8}$ in the LPMQHSS)

Furthermore, to better show the advantages of the proposed LPMQHSS method, Fig. 1 shows IT of the PMQHSS, PMHSS, DSS, LPMQHSS and CRI methods versus the parameter α for Examples 4.1 and 4.2. For the PMQHSS and LPMQHSS iteration methods, we take ω as 1.00 and 1.00×10^{-8}, respectively. It can be seen from Fig. 1 that the minimum number of iterations of the DSS, LPMQHSS and CRI iteration methods are the same and less than that of the PMHSS and PMQHSS ones. In addition, for the parameter α, the LPMQHSS iteration method is more stable than the CRI and DSS ones. In Fig. 1, we can see that iterations number of the CRI and DSS are the same with respect to parameter α.

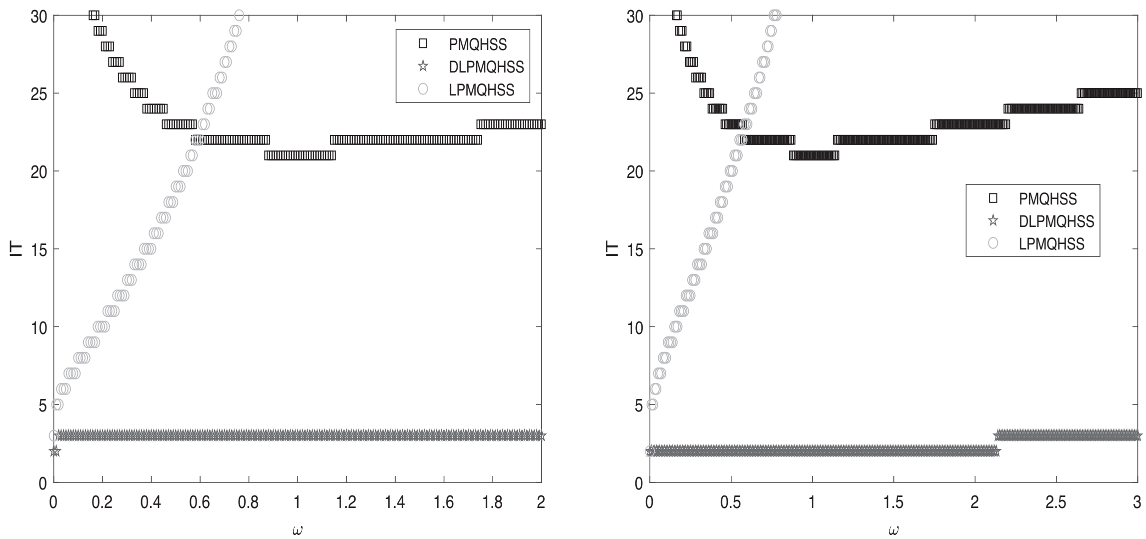

Fig. 2 Iterations number (IT) of the PMQHSS, LPMQHSS, DLPMQHSS methods depends on the change of parameter ω (Example 4.1 (left $m = 256$): $\alpha = 1.00$ in the PMQHSS, $\alpha = 2.00$ in the LPMQHSS; Example 4.2 (right $m = 30$): $\alpha = 1.00$ of the PMQHSS, LPMQHSS)

In Fig. 2, we compare IT of the PMQHSS, LPMQHSS and DLPMQHSS iteration methods with respect to the parameter ω. It can be observed from Fig. 2 that the DLPMQHSS iteration method is not sensitive to the parameter ω, and IT of the LPMQHSS iteration method gradually increases with the increase of ω, and the IT of the PMHSS iteration method first decreases and then increases with the increase of the parameter ω. If the parameter ω is selected appropriately, the IT of the DLPMQHSS and LPMQHSS iteration methods can be much less than that of the PMQHSS one.

In addition, the LPMHSS iteration method is not convergent when solving Examples 4.1 and 4.2. Therefore, we do not list numerical results of the LPMHSS method in the above tables and figures.

5 Concluding remarks

For the large sparse complex symmetric linear equations (1.1), we present the LPMQHSS and DLPMQHSS iteration methods based on PMQHSS method. Theoretically, we discuss the convergence conditions of the LPMQHSS and DLPMQHSS methods, and study the convergence properties of the LPMQHSS methods with $V = T$. The results of numerical examples show that the LPMQHSS and DLPMQHSS iteration methods are more effective than the PMHSS, CRI, DSS and PMQHSS ones. In order to more intuitively observe the sensitivity of the PMQHSS, PMHSS, CRI, DSS, LPMQHSS and DLPMQHSS methods on parameters, we give diagrams of curves of iterations number of these methods varying with respect to the parameters.

Declarations

Conflict of interest The authors declare that they have no competing interests.

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