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ORIGINAL RESEARCH

Robust numerical scheme for 2D fractional integro-differential equations of Volterra type

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Abstract This article provides a numerical study of two-dimensional Volterra integro-differential equations involving fractional derivatives in the Caputo sense of order α, γ $(0 < \alpha, \gamma < 1)$. First, we establish a sufficient condition for the existence and uniqueness of the solution using the Banach fixed point theorem. Due to the limitation of finding the exact analytical solution, we derive and analyze an efficient numerical scheme to approximate the solution. The proposed scheme uses the L1 technique to discretize the differential components, whereas a composite trapezoidal rule is used to approximate the double integral. The convergence analysis and error estimation are carried out. It is shown that the proposed scheme converges with an optimal convergence rate of $\min\{2-\alpha, 2-\gamma\}$ for sufficiently smooth initial data. In addition, we apply the proposed difference scheme to solve the semilinear problem. The well-known Newton's linearization technique is used to deal with semilinearity. Finally, a couple of numerical experiments are conducted to support our theoretical findings and validate the proposed scheme.

Keywords 2D-Volterra integro-differential equations · Caputo derivative · L1 scheme · Composite trapezoidal rule · Convergence analysis · Newton's linearization

Mathematics Subject Classification 45D05 · 35R09 · 65R20 · 26A33

1 Introduction

Fractional calculus is a generalization of integration and differentiation to arbitrary order. It plays a significant role in the mathematical modeling of engineering and physical phenomena; for details one may refer to [4, 14, 24] and references therein. Mathematicians and physicists have shown a great interest in fractional integral equations (FIEs) and fractional integro-differential equations (FIDEs), which are extensively used in engineering and scientific fields such as in glass-forming processes, nonlinear oscillation of earthquakes, tumor-immune cell competition, epidemiology, hereditary properties of various materials, fluid-dynamic traffic model, signal processing, etc. [8, 26–28]. As a result, the development of solution techniques for FIEs/FIDEs has become an essential area in applied mathematics. A significant amount of literature has been published to find the approximate solutions including: Adomian decomposition and homotopy perturbation method [20], hybrid Bernstein method [9], wavelet method [12, 16], collocation method [13, 23], Galerkin method [15], finite difference method [6, 25].

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In this paper, we consider the following two-dimensional fractional order Volterra integro-differential equations (2D-FVIDEs) defined on a rectangular domain $\Omega := (0,1] \times (0,1]$

$$\begin{cases} \left(D_x^{\alpha} + D_y^{\gamma} + a(x,y)\right)w(x,y) + \mu \int_0^y \int_0^x K(x,y,r,s)w(r,s)drds = f(x,y) \text{ in } \Omega, \\ w(0,y) = \phi(y) \quad \text{for } y \in [0,1], \\ w(x,0) = \psi(x) \quad \text{for } x \in (0,1], \end{cases} \tag{1}$$

where the parameter μ is a real constant. Assume $a : \Omega \to \mathbb{R}$ and $K(>0) : \Omega \times \Omega \to \mathbb{R}$ being sufficiently smooth functions. ϕ and ψ are given continuous functions in their respective domains. D_x^{α} and D_y^{γ} indicate the fractional partial differential operator with respect to x and y, respectively.

In recent years, the approximate analytical and numerical methods in resolving the two-dimensional FIDEs have gained considerable importance: Ahsan *et al.* in [2] obtained the approximate solutions of 2D-FVIDEs using the optimal homotopy asymptotic method. Khajehnasiri in [10], solves two-dimensional Volterra-Fredholm IDEs using the triangular function. In [18], Mojahedfar *et al.* applied the Legendre wavelets method to solve a class of 2D-FIDEs. Using operational matrices of fractional integration based on the Haar wavelets, Khajehnasiri *et al.* in [11] found the numerical solution for a nonlinear 2D-FVIDEs system. Asgari and Ezzati [3] proposed 2D Bernstein polynomials operational matrix for the approximation of 2D-FIEs. Nemati *et al.* [19] solved a class of nonlinear 2D Volterra integral equations employing Legendre polynomials. In [1], Aghazadeh and Khajehnasiri introduced an operational matrix using two-dimensional block-pulse functions to treat nonlinear 2D-FVIDEs. Rahimi *et al.* [22] developed the operational Tau method for the Fredholm integral equations and also provided error estimation.

Approximating the solution of 2D integro-differential equations involving fractional differentiation using an efficient finite difference technique is yet to be discovered. The objective of this article is to establish a difference scheme of optimal accuracy for obtaining the approximate solutions to the problem (1). The composite trapezoidal rule approximates the double integral, and the L1 technique is used to discretize the differential part. Additionally, the proposed scheme is extended to solve the corresponding semilinear problem. This paper is organized as follows: Section 2 introduces a few basic definitions and the existence of a unique solution to the considered system. The proposed scheme is derived in Sect. 3. Section 4 deals with the convergence analysis and error estimation. Section 5 extends the proposed scheme to solve the semilinear problem. The efficiency and applicability of the proposed scheme are validated through a couple of numerical examples in Sect. 6. Finally, some concluding remarks are drawn in Sect. 7.

Notation: On a domain Q, $\mathcal{C}^k(Q)$ represents the continuously differentiable functions to k^{th} order. The discrete maximum norm $\| \varphi \| = \max_j |\varphi_j|$, and C is a positive constant.

2 Preliminaries

The following definitions will be useful for this work. For more details one may refer [5,21].

Definition 1 The Riemann-Liouville fractional integral of $\varphi(x) \in \mathcal{C}[a,b]$ of order $\widetilde{\alpha} \in \mathbb{R}^+$ is defined as:

$$J_x^{\widetilde{\alpha}}\varphi(x) = \frac{1}{\Gamma(\widetilde{\alpha})}\int_a^x (x-s)^{\widetilde{\alpha}-1}\varphi(s)\,ds.$$

$\Gamma(\cdot)$ being Euler's gamma function.

Definition 2 The Caputo fractional derivative of $\varphi(x) \in \mathcal{C}^m[a,b]$ of order $\widetilde{\alpha} \in \mathbb{R}^+$, $m-1 < \widetilde{\alpha} \le m$ is defined as:

$$D_x^{\widetilde{\alpha}}\varphi(x) = J_x^{m-\widetilde{\alpha}}\varphi^{(m)}(x) = \begin{cases} \dfrac{1}{\Gamma(m-\widetilde{\alpha})}\displaystyle\int_a^x \frac{\varphi^{(m)}(s)}{(x-s)^{1-m+\widetilde{\alpha}}}\,ds, & \text{if } m-1 < \widetilde{\alpha} < m, \\ \varphi^{(m)}(x), & \text{if } \widetilde{\alpha} = m. \end{cases}$$

Definition 3 Let $m-1 < \widetilde{\alpha} \le m, m \in \mathbb{N}$. The Caputo partial fractional derivative of order $\widetilde{\alpha} \in \mathbb{R}^+$ of the function $\varphi(x,y)$ such that $\partial_x^m \varphi(x,y) \in \mathcal{C}[a,b]$, is defined by:

$$D_x^{\widetilde{\alpha}}\varphi(x,y) = J_x^{m-\widetilde{\alpha}}\partial_x^m\varphi(x,y) = \begin{cases} \dfrac{1}{\Gamma(m-\widetilde{\alpha})}\displaystyle\int_a^x \frac{\partial_t^m\varphi(t,y)}{(x-t)^{\widetilde{\alpha}-m+1}}\,dt, & \text{if } m-1 < \widetilde{\alpha} < m, \\ \partial_x^m\varphi(x,y), & \text{if } \widetilde{\alpha} = m, \end{cases}$$

where ∂_x denotes the partial derivative with respect to x.

Definition 4 The two parameters $\beta, \zeta > 0$ Mittag-Leffler function is defined by:

$$E_{\beta,\zeta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \zeta)}, \qquad z \in \mathbb{C}.$$

Definition 5 Suppose (X, d) be a complete metric space, then each contraction mapping $\varphi : X \to X$ has a unique fixed point x in X *i.e.,* $\varphi x = x$.

Theorem 1 *If* $\dfrac{\Gamma(2-\gamma)\big[(\alpha+1)\,\zeta + \mu\eta\big] + (\alpha+1)}{\Gamma(2+\alpha)\Gamma(2-\gamma)} < 1$, *then the model problem (1) has unique solution in* Ω.

Proof Banach's fixed point theorem is used to prove the theorem. Without any loss of generality, we assume that $\alpha \geq \gamma$. Applying J_x^{α} to (1), we have

$$w(x, y) = \Psi(w(x, y)) \quad \text{for all } (x, y) \in \Omega,$$

where

$$\begin{aligned}\Psi(w(x, y)) = \phi(y) &+ J_x^{\alpha} f(x, y) - J_x^{\alpha}\big\{D_y^{\gamma}\, w(x, y)\big\} - J_x^{\alpha}\big\{a(x, y)\, w(x, y)\big\}\\ &- \mu J_x^{\alpha}\Big\{\int_0^y \int_0^x K(x, y, r, s)\, w(r, s)\, dr\, ds\Big\}.\end{aligned}$$

Our aim is to show that Ψ is a contraction mapping. For any $w_1(x, y),\ w_2(x, y) \in \mathcal{C}(\Omega, \mathbb{R})$, we have

$$\begin{aligned}&\big\|\, \Psi(w_1(x, y)) - \Psi(w_2(x, y))\,\big\|\\
&\leq \big\|\, J_x^{\alpha}\big[a(x, y)\,\{w_1(x, y) - w_2(x, y)\}\big]\big\| + \big\|\, J_x^{\alpha}\big[D_y^{\gamma}\,\{w_1(x, y) - w_2(x, y)\}\big]\big\|\\
&+ \mu\Big\| J_x^{\alpha}\Big\{\int_0^y \int_0^x K(x, y, r, s)\,\big(w_1(r, s) - w_2(r, s)\big)\, dr\, ds\Big\}\Big\|\\
&\leq \frac{1}{\Gamma(\alpha)}\Big\| \int_0^x (x-t)^{\alpha-1}\big|\, a(t, y)\big|\,\big|\, w_1(t, y) - w_2(t, y)\big|\, dt\,\Big\|\\
&+ \frac{1}{\Gamma(\alpha)}\Big\| \int_0^x (x-t)^{\alpha-1}\Big[\frac{1}{\Gamma(1-\gamma)} \int_0^y (y-s)^{-\gamma}\,\Big|\,\frac{\partial}{\partial s}\big\{w_1(t, s) - w_2(t, s)\big\}\Big| ds\Big]\, dt\,\Big\|\\
&+ \frac{\mu}{\Gamma(\alpha)}\Big\| \int_0^x (x-t)^{\alpha-1}\Big\{\int_0^y \int_0^t \big|K(t, y, r, s)\big|\,\Big|\big(w_1(r, s) - w_2(r, s)\big)\Big|\, dr\, ds\Big\}\, dt\Big\|.\end{aligned}$$

Since the kernel function $K(x, y, r, s)$ and the coefficient $a(x, y)$ are bounded functions, there exist real constants η and ζ such that $\big|K(x, y, r, s)\big| \leq \eta$ and $\big|a(x, y)\big| \leq \zeta$. So, we get

$$\begin{aligned}&\big\|\, \Psi(w_1(x, y)) - \Psi(w_2(x, y))\,\big\|\\
&\leq \frac{\zeta}{\Gamma(\alpha)}\Big\|\, w_1(x, y) - w_2(x, y)\Big\|\,\Big|\int_0^x (x-t)^{\alpha-1}\, dt\,\Big|\\
&+ \frac{1}{\Gamma(\alpha)\Gamma(1-\gamma)}\Big\|\, w_1(x, y) - w_2(x, y)\Big\|\,\Big|\int_0^x (x-t)^{\alpha-1}\Big\{\int_0^y \frac{ds}{(y-s)^{\gamma}}\Big\}\, dt\,\Big|\\
&+ \frac{\mu\eta}{\Gamma(\alpha)}\Big\|\, w_1(x, y) - w_2(x, y)\Big\|\,\Big|\int_0^x (x-t)^{\alpha-1}\Big\{\int_0^y \int_0^t dr\, ds\Big\} dt\Big|\\
&\leq \frac{\Gamma(2-\gamma)\big[(\alpha+1)\,\zeta + \mu\eta\big] + (\alpha+1)}{\Gamma(2+\alpha)\Gamma(2-\gamma)}\,\Big\|\, w_1(x, y) - w_2(x, y)\Big\|\\
&\leq \big\|\, w_1(x, y) - w_2(x, y)\big\|.\end{aligned}$$

Hence, Ψ is a contraction mapping. Since $\big(\mathcal{C}(\Omega),\ \|\cdot\|\big)$ is a Banach space, Ψ has a unique fixed point which is the solution of the problem (1). □

3 The discretization

The model problem (1) will be discretized by the L1 scheme and a composite trapezoidal rule. Set M_x, M_y be natural numbers. Consider a uniform mesh $x_j = jh_x$ for $j = 0(1)M_x$ with step size $h_x = M_x^{-1}$ and $y_i = ih_y$ for $i = 0(1)M_y$ with step size $h_y = M_y^{-1}$. Denote W_j^i be the approximation of u computed at $(x_j, y_i) \in \overline{\Omega}_{M_x}^{M_y}$.

Caputo's fractional derivative $D_x^{\alpha} w(x, y)$ at $(x_j, y_i) \in \overline{\Omega}_{M_x}^{M_y}$ for $j = 1(1)M_x$ and $i = 0(1)M_y$ can be expressed as follows:

$$D_x^{\alpha} w(x_j, y_i) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{s=x_k}^{x_{k+1}} (x_j - s)^{-\alpha} \frac{\partial w}{\partial s}(s, y_i) ds. \tag{2}$$

The L1 approximation [17] of (2) is

$$\begin{aligned} D_{M_x}^{\alpha} w(x_j, y_i) &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} h_x^{-1} \big[w(x_{k+1}, y_i) - w(x_k, y_i)\big] \int_{s=x_k}^{x_{k+1}} (x_j - s)^{-\alpha} ds \\ &= \mu_x \sum_{k=0}^{j-1} \big[w(x_{k+1}, y_i) - w(x_k, y_i)\big] b_{j-k}, \end{aligned} \tag{3}$$

where $\mu_x = \dfrac{h_x^{-\alpha}}{\Gamma(2-\alpha)}$ and $b_p = p^{1-\alpha} - (l-p)^{1-\alpha}$ for $p = 1(1)M_x$. The truncation error $\varepsilon_{i,j}^{(1)} = (D_x^{\alpha} - D_{M_x}^{\alpha}) w(x_j, y_i)$. Similarly, the L1 approximation of $D_y^{\gamma} w(x, y)$ at $(x_j, y_i) \in \overline{\Omega}_{M_x}^{M_y}$ for $i = 1(1)M_y$ and $j = 0(1)M_x$ is

$$D_{M_y}^{\gamma} w(x_j, y_i) = \mu_y \sum_{k=0}^{i-1} \big[w(x_j, y_{k+1}) - w(x_j, y_k)\big] d_{i-k}, \tag{4}$$

where $\mu_y = \dfrac{h_y^{-\gamma}}{\Gamma(2-\gamma)}$ and $d_p = b_p$ for $p = 1(1)M_y$. The truncation error $\varepsilon_{i,j}^{(2)} = (D_y^{\gamma} - D_{M_y}^{\gamma}) w(x_j, y_i)$. The integral for $j = 1(1)M_x$ and $i = 1(1)M_y$ can be written as

$$\begin{aligned} I_x^y w(x_j, y_i) &= \int_0^{y_i} \int_0^{x_j} K(x_j, y_i, r, s)\, w(r, s)\, dr\, ds \\ &= \sum_{p=0}^{i-1} \sum_{k=0}^{j-1} \int_{y_p}^{y_{p+1}} \int_{x_k}^{x_{k+1}} K(x_j, y_i, r, s)\, w(r, s)\, dr\, ds. \end{aligned}$$

Applying the composite trapezoidal rule, we get

$$\begin{aligned} I_{M_x}^{M_y} w(x_j, y_i) = \frac{h_x h_y}{4} \sum_{p=0}^{i-1} \sum_{k=0}^{j-1} \Big\{ & K(x_j, y_i, x_k, y_p)\, w(x_k, y_p) \\ & + K(x_j, y_i, x_{k+1}, y_p)\, w(x_{k+1}, y_p) + K(x_j, y_i, x_k, y_{p+1})\, w(x_k, y_{p+1}) \\ & + K(x_j, y_i, x_{k+1}, y_{p+1})\, w(x_{k+1}, y_{p+1}) \Big\}. \end{aligned} \tag{5}$$

The remainder $\varepsilon_{i,j}^{(3)} = \big(I_x^y - I_{M_x}^{M_y}\big) w(x_j, y_i)$. Using (3) and (5), the problem (1) is transformed as

$$\begin{cases} D_{M_x}^{\alpha} w(x_j, y_i) + D_{M_y}^{\gamma} w(x_j, y_i) + a(x_j, y_i)\, w(x_j, y_i) + \mu I_{M_x}^{M_y} w(x_j, y_i) \\ = f(x_j, y_i) + r_j^i \quad \text{for } j = 1(1)M_x;\ i = 1(1)M_y, \\ w(x_0, y_i) = \phi(y_i) \quad \text{for } i = 0(1)M_y, \\ w(x_j, y_0) = \psi(x_j) \quad \text{for } j = 1(1)M_x, \end{cases} \tag{6}$$

where $r_j^i = \varepsilon_{i,j}^{(1)} + \varepsilon_{i,j}^{(2)} + \varepsilon_{i,j}^{(3)}$ be the total truncation error. Neglecting r_j^i, the transformed problem (6) reduces to

$$\begin{cases} \mu_x \sum_{k=0}^{j-1} \big(W_{k+1}^i - W_k^i\big) b_{j-k} + \mu_y \sum_{k=0}^{i-1} \big(W_j^{k+1} - W_j^k\big) d_{i-k} + a_j^i W_j^i + \dfrac{\mu h_x h_y}{4} \times \\ \sum_{p=0}^{i-1} \sum_{k=0}^{j-1} \Big\{ K(x_j, y_i, x_k, y_p) W_k^p + K(x_j, y_i, x_{k+1}, y_p) W_{k+1}^p + K(x_j, y_i, x_k, y_{p+1}) W_k^{p+1} \\ + K(x_j, y_i, x_{k+1}, y_{p+1}) W_{k+1}^{p+1} \Big\} = f_j^i \text{ for } j = 1(1)M_x;\ i = 1(1)M_y, \\ W_0^i = \phi(y_i) \quad \text{for } i = 0(1)M_y, \\ W_j^0 = \psi(x_j) \quad \text{for } j = 1(1)M_x, \end{cases} \tag{7}$$

where $a_j^i = a(x_j, y_i)$, a similar notation is used for $f(x, y)$. It is possible to express the solution to the above scheme in the explicit form as:

$$\begin{aligned} &\Big\{ \hat{b}_1 + \hat{d}_1 + a_j^i + \frac{\mu h_x h_y}{4} K(x_j, y_i, x_j, y_i) \Big\} W_j^i = f_j^i + \hat{b}_1 W_{j-1}^i + \sum_{k=0}^{j-2} \big(W_k^i - W_{k+1}^i\big) \hat{b}_{j-k} \\ &\quad + \hat{d}_1 W_j^{i-1} + \sum_{k=0}^{i-2} \big(W_j^k - W_j^{k+1}\big) \hat{d}_{i-k} - \frac{\mu h_x h_y}{4} \Bigg[\sum_{p=0}^{i-1} \sum_{k=0}^{j-2} K(x_j, y_i, x_{k+1}, y_{p+1}) W_{k+1}^{p+1} \\ &\quad + \sum_{p=0}^{i-2} K(x_j, y_i, x_j, y_{p+1}) W_j^{p+1} \Bigg] - \frac{\mu h_x h_y}{4} \sum_{p=0}^{i-1} \sum_{k=0}^{j-1} \Big\{ K(x_j, y_i, x_k, y_p) W_k^p \\ &\quad + K(x_j, y_i, x_{k+1}, y_p) W_{k+1}^p + K(x_j, y_i, x_k, y_{p+1}) W_k^{p+1} \Big\} \text{ for } j = 1(1)M_x; i = 1(1)M_y, \end{aligned} \tag{8}$$

where $\hat{b}_p = \dfrac{b_p h_x^{-\alpha}}{\Gamma(2-\alpha)}$ for $p = 1(1)M_x$ and $\hat{d}_q = \dfrac{d_q h_x^{-\gamma}}{\Gamma(2-\gamma)}$ for $q = 1(1)M_y$.

4 Error analysis

This section provides the convergence analysis and the order of accuracy of the proposed scheme (7).

Lemma 2 *If $w(x, y) \in \mathcal{C}^2(\overline{\Omega})$ be the solution of (1), then*

$$\begin{aligned} |\varepsilon_{i,j}^{(1)}| &\le C h_x^{2-\alpha} \quad \textit{for } j = 1(1)M_x;\ i = 0(1)M_y, \\ |\varepsilon_{i,j}^{(2)}| &\le C h_y^{2-\gamma} \quad \textit{for } i = 1(1)M_y;\ j = 0(1)M_x. \end{aligned}$$

Proof We have $\big|\varepsilon_{i,j}^{(1)}\big| = \big|(D_x^\alpha - D_{M_x}^\alpha)\, w(x_j, \cdot)\big|$

$$\begin{aligned} &= \left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{x_k}^{x_{k+1}} (x_j - s)^{-\alpha} \Big[\frac{\partial w}{\partial s}(s, \cdot) - \frac{w(x_{k+1}, \cdot) - w(x_k, \cdot)}{h_x} \Big] ds \right| \\ &\le C \left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{x_k}^{x_{k+1}} \frac{x_j + x_{j-1} - 2s}{(x_j - s)^\alpha}\, ds + \mathcal{O}(h_x^2) \right|. \end{aligned}$$

From [17], we get $\left| \dfrac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{x_k}^{x_{k+1}} \dfrac{x_j + x_{j-1} - 2s}{(x_j - s)^\alpha}\, ds \right| \le 2 h_x^{2-\alpha}$. Therefore, $|\varepsilon_{i,j}^{(1)}| \le C h_x^{2-\alpha}$. Following the similar procedure, one can easily proof $|\varepsilon_{i,j}^{(2)}| \le C h_y^{2-\gamma}$. Hence completes the proof. □

Lemma 3 *Suppose $w(x, y) \in \mathcal{C}^2(\overline{\Omega})$, then the remainder of the trapezoidal formula (5) satisfies*

$$\big|\varepsilon_{i,j}^{(3)}\big| \le C(h_x^2 + h_y^2) \quad \textit{for } j = 1(1)M_x;\ i = 1(1)M_y.$$

Proof Using the Taylor series expansion, we have

$$\begin{aligned}\left|\varepsilon^{(3)}_{i,j}\right| &= \left|(I^y_x - I^{M_y}_{M_x})\, w(x_j, y_i)\right| \\ &= \Bigg|\sum_{p=0}^{i-1}\sum_{k=0}^{j-1}\int_{y_p}^{y_{p+1}}\int_{x_k}^{x_{k+1}} \frac{1}{2}\Big[(r - x_{k+\frac{1}{2}})^2\, \psi_{rr}(\zeta_r, \zeta_s) + (s - y_{p+\frac{1}{2}})^2\, \psi_{ss}(\eta_r, \eta_s) \\ &\quad + 2(r - x_{k+\frac{1}{2}})(s - y_{p+\frac{1}{2}})\, \psi_{rs}(\tau_r, \tau_s)\Big] dr\, ds\Bigg|,\end{aligned}$$

for some $x_k < \zeta_r, \eta_r, \tau_r < x_{k+1},\ y_p < \zeta_s, \eta_s, \tau_s < y_{p+1}$, where $\psi(x, y) = K(x_j, y_i, x, y)w(x, y) \in \mathcal{C}^2(\overline{\Omega})$, since K is smooth function and $w \in \mathcal{C}^2(\overline{\Omega})$. Hence, there is a real constant L such that the second order partial derivatives and mixed derivative of ψ are bounded in absolute value by L and $x_{k+\frac{1}{2}} = \frac{1}{2}(x_k + x_{k+1}),\ y_{p+\frac{1}{2}} = \frac{1}{2}(y_p + y_{p+1})$. With the help of triangle inequality, get

$$\begin{aligned}\left|\varepsilon^{(3)}_{i,j}\right| &\le \sum_{p=0}^{i-1}\sum_{k=0}^{j-1}\int_{y_p}^{y_{p+1}}\int_{x_k}^{x_{k+1}} \frac{1}{2}\Big[(r - x_{k+\frac{1}{2}})^2 \left|\psi_{rr}(\zeta_r, \zeta_s)\right| + (s - y_{p+\frac{1}{2}})^2 \left|\psi_{ss}(\eta_r, \eta_s)\right| \\ &\quad + 2(r - x_{k+\frac{1}{2}})(s - y_{p+\frac{1}{2}}) \left|\psi_{rs}(\tau_r, \tau_s)\right|\Big] dr\, ds \\ &\le \frac{L}{2}\sum_{p=0}^{i-1}\sum_{k=0}^{j-1}\Bigg[(y_{p+1} - y_p)\int_{x_k}^{x_{k+1}} (r - x_{k+\frac{1}{2}})^2\, dr + (x_{k+1} - x_k)\int_{y_p}^{y_{p+1}} (s - y_{p+\frac{1}{2}})^2\, ds \\ &\quad + 2\int_{y_p}^{y_{p+1}}\int_{x_k}^{x_{k+1}} (r - x_{k+\frac{1}{2}})(s - y_{p+\frac{1}{2}})\, dr ds\Bigg] \\ &= \frac{L}{6}\sum_{p=0}^{i-1}\sum_{k=0}^{j-1}\Big[h_y \frac{h_x^3}{4} + h_x \frac{h_y^3}{4}\Big] \\ &\le \frac{L}{24}(h_y h_x^3 + h_x h_y^3) M_x M_y \le C(h_x^2 + h_y^2),\end{aligned}$$

which is our desired bound. □

Lemma 4 *For any mesh function* $\{W^i_j\}_{j=0}^{M_x}$ *with* $W^i_0 = 0$, *we have*

$$|W^i_j| \ \le \max_{k=1,2,\ldots,j}\left\{\frac{h_x^\alpha\, \Gamma(2-\alpha)}{b_k}\, D^\alpha_{M_x}|W^i_k|\right\} \ \textit{for}\ j = 1(1)M_x.$$

Proof Suppose $\max_{k=1,2,\ldots,j}|W^i_k| = |W^i_m|$ for some $m \in \{1, 2, \ldots, j\}$. Since $W^i_0 = 0$, by using (3) without remainder term, we get

$$D^\alpha_{M_x}|W^i_m| \ = \frac{h_x^{-\alpha}}{\Gamma(2-\alpha)}\Bigg[|W^i_m| - \sum_{k=1}^{m-1}\big(b_k - b_{k+1}\big)\,|W^i_{m-k}|\Bigg].$$

Now, $b_1 = 1$ and $b_k > b_{k+1}$ for all $k \ge 1$ implies that

$$D^\alpha_{M_x}|W^i_m| \ \ge \frac{h_x^{-\alpha}}{\Gamma(2-\alpha)}\Bigg[|W^i_m| - \sum_{k=1}^{m-1}\big(b_k - b_{k+1}\big)\,|W^i_m|\Bigg] \ge \frac{h_x^{-\alpha}}{\Gamma(2-\alpha)} b_m |W^i_m|.$$

Therefore, $|W^i_m| \ \le \dfrac{h_x^\alpha\, \Gamma(2-\alpha)}{b_m}\, D^\alpha_{M_x}|W^i_m|$. □

Denote the absolute error $e_j^i = w(x_j, y_i) - W_j^i$ at (x_j, y_i) for $j = 1(1)M_x;\ i = 1(1)M_y$. Using (1) and (6), yields the error equation

$$\begin{cases} D_{M_x}^{\alpha}\, e_j^i + D_{M_y}^{\gamma}\, e_j^i + a_j^i\, e_j^i + \mu I_{M_x}^{M_y} e_j^i = r_j^i \quad \text{for } j = 1(1)M_x;\ i = 1(1)M_y, \\ e_0^i = 0 \quad \text{for } i = 0(1)M_y, \\ e_j^0 = 0 \quad \text{for } j = 1(1)M_x. \end{cases} \tag{9}$$

Theorem 5 *The solution of (9) satisfies the following stability results*

$$\|E^{(i)}\| \le C\|R^{(i)}\|,$$

where $E^{(i)} = (e_0^i, e_1^i, \dots, e_{M_x}^i)^T$ *and* $R^{(i)} = (r_0^i, r_1^i, \dots, r_{M_x}^i)^T$.

Proof Multiplying (9) by e_j^i, then using (3), one can reach at

$$\begin{aligned} &2|e_j^i|^2 + \zeta|e_j^i|^2 + \frac{\mu h_x h_y}{4}\sum_{p=0}^{i-1}\sum_{k=0}^{j-1}\Big\{K(x_j, y_i, x_k, y_p)\,|e_k^p||e_j^i| + K(x_j, y_i, x_{k+1}, y_p)\,|e_{k+1}^p||e_j^i| \\ &\quad + K(x_j, y_i, x_k, y_{p+1})\,|e_k^{p+1}||e_j^i| + K(x_j, y_i, x_{k+1}, y_{p+1})\,|e_{k+1}^{p+1}||e_j^i|\Big\} \\ &\le |r_j^i|\,|e_j^i| + \sum_{k=1}^{j-1}\big(b_k - b_{k+1}\big)\,|e_{j-k}^i|\,|e_j^i| + \sum_{k=1}^{i-1}\big(d_k - d_{k+1}\big)\,|e_j^{i-k}|\,|e_j^i|. \end{aligned}$$

Since $K(x, y, r, s) > 0$, we have

$$2|e_j^i|^2 \le |r_j^i|\,|e_j^i| + \sum_{k=1}^{j-1}\big(b_k - b_{k+1}\big)\,|e_{j-k}^i|\,|e_j^i| + \sum_{k=1}^{i-1}\big(d_k - d_{k+1}\big)\,|e_j^{i-k}|\,|e_j^i|.$$

Dividing both sides by $|e_j^i|$, we get $D_{M_x}^{\alpha}|e_j^i| + D_{M_y}^{\gamma}|e_j^i| \le |r_j^i|$. Since $b_1 = 1$ and $b_p > b_{p+1}$ for all $p \ge 1$, we have

$$(1-\alpha)p^{-\alpha} \le b_p \le (1-\alpha)(p-1)^{-\alpha} \quad \text{for } p \ge 2.$$

Using Lemma 4, we obtain

$$\begin{aligned} |e_j^i| &\le \max_{k=1(1)j}\left\{\frac{h_x^{\alpha}\,\Gamma(2-\alpha)}{b_k}\,|r_k^i|\right\} + \max_{k=1(1)i}\left\{\frac{h_y^{\gamma}\,\Gamma(2-\gamma)}{d_k}\,|r_j^k|\right\} \\ &\le \left[\frac{h_x^{\alpha}\,\Gamma(2-\alpha)}{(1-\alpha)M_x^{-\alpha}} + \frac{h_y^{\alpha}\,\Gamma(2-\gamma)}{(1-\gamma)M_y^{-\gamma}}\right]\max_{k=1(1)j}\max_{p=1(1)i}\left\{|r_k^p|\right\} \le C\max_{k=1(1)j}\max_{p=1(1)i}\left\{|r_k^p|\right\}, \end{aligned}$$

where the constant C is independent of step sizes $h_x,\ h_y$. This proves the theorem. □

Theorem 6 *If* $w(x_j, y_i)$ *be the true solution and* W_j^i *be the approximate solution of the problem (1), then we have*

$$\max_{0\le j\le M_x}\max_{0\le i\le M_y}\left|w(x_j, y_i) - W_j^i\right| \le C\big(h_x^{2-\alpha} + h_y^{2-\gamma}\big).$$

Proof Lemma 2 and Lemma 3 show that

$$|r_j^i| \le |\varepsilon_{i,j}^{(1)}| + |\varepsilon_{i,j}^{(2)}| + |\varepsilon_{i,j}^{(3)}| \le C\big(h_x^{2-\alpha} + h_y^{2-\gamma} + h_x^2 + h_y^2\big).$$

Using Theorem 5, we obtain

$$\|E^{(i)}\| \le C\|R^{(i)}\| \le C\big(h_x^{2-\alpha} + h_y^{2-\gamma} + h_x^2 + h_y^2\big) \le C\big(h_x^{2-\alpha} + h_y^{2-\gamma}\big).$$

This proves the theorem. □

Remark 1 The order of convergence of the proposed scheme is $\min\{2-\alpha,\ 2-\gamma\}$.

5 Semilinear equation

Consider the following two-dimensional semilinear fractional order Volterra integro-differential equation (2D-SFVIDE):

$$\begin{cases} D_x^{\alpha} w(x,y) + D_y^{\gamma} w(x,y) + a(x,y)w(x,y) + \mu \int_0^y \int_0^x K(x,y,r,s)w(r,s)drds \\ = g(x,y,w) \ \text{ in } \Omega, \\ w(0,y) = \phi(y) \quad \text{for } y \in [0,1], \\ w(x,0) = \psi(x) \quad \text{for } x \in (0,1], \end{cases} \tag{10}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. Using the Newton linearization technique the semilinear problem (10) is transform to an approximate linearized problem. Let, $w^{(0)} = w^{(0)}(x,y)$ be the initial guess. Then, for any $k > 0$, $g(x,y,w^{(k)}(x,y))$ can expand with respect to $w^{(k-1)}(x,y))$ as the following infinite series

$$g(x,y,w^{(k)}(x,y)) \approx g(x,y,w^{(k-1)}(x,y)) + \left[w^{(k)}(x,y) - w^{(k-1)}(x,y)\right]\partial_w g(x,y,w^{(k-1)}(x,y)) + \cdots \tag{11}$$

Substituting (11) in (10), we get the following linearized FVIDE

$$D_x^{\alpha} w^{(k)}(x,y) + D_y^{\gamma} w^{(k)}(x,y) + A(x,y)w^{(k)}(x,y) + \mu \int_0^y \int_0^x K(x,y,r,s)w^{(k)}(r,s)drds = G(x,y), \tag{12}$$

where

$$\begin{cases} A(x,y) = a(x,y) - \partial_w g(x,y,w^{(k-1)}(x,y)), \\ G(x,y) = g(x,y,w^{(k-1)}(x,y)) - w^{(k-1)}(x,y)\partial_w g(x,y,w^{(k-1)}(x,y)). \end{cases}$$

Therefore, the linearized problem corresponding to the linear problem (10) becomes

$$\begin{cases} D_x^{\alpha}\, w^{(k)}(x,y) + D_y^{\gamma}\, w^{(k)}(x,y) + A(x,y)\, w^{(k)}(x,y) \\ +\mu \int_0^y \int_0^x K(x,y,r,s)\, w^{(k)}(r,s)\, dr\, ds = G(x,y) \ \text{ in } \Omega, \\ w^{(k)}(0,y) = \phi(y) \quad \text{for } y \in [0,1], \\ w^{(k)}(x,0) = \psi(x) \quad \text{for } x \in (0,1]. \end{cases} \tag{13}$$

The stopping criterion for the successive iteration is

$$\max_{(x_j,y_i)\in\Omega_{M_x}^{M_y}} \left|w^{(k)}(x_j,y_i) - u^{(k-1)}(x_j,y_i)\right| \le TOL.$$

6 Numerical experiments

The following two numerical examples demonstrate the applicability of the proposed scheme. Evaluate the maximum error as:

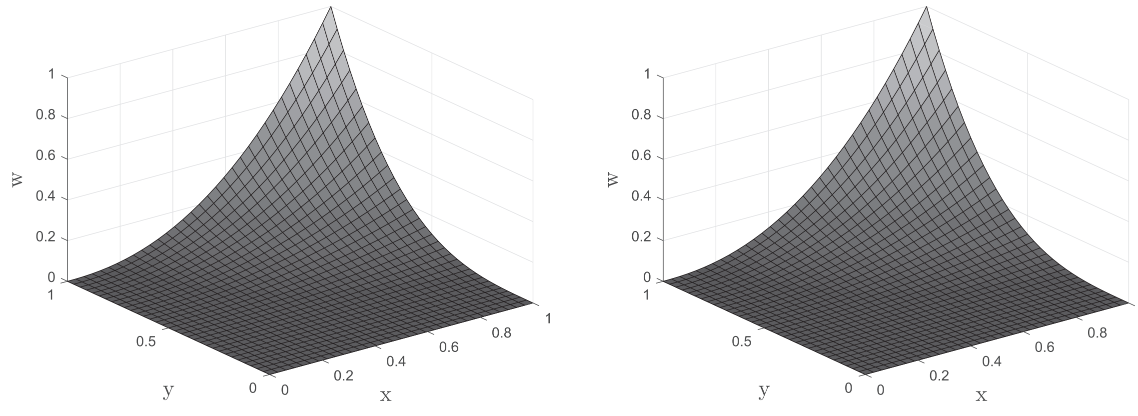
$$\Sigma_{M_x}^{M_y} = \max_{0\le j\le M_x} \max_{0\le i\le M_y} \left|w(x_j,y_i) - W_j^i\right|.$$

The order of convergence as:

$$\varrho_{M_x}^{M_y} = \log_2\left(\frac{\Sigma_{M_x}^{M_y}}{\Sigma_{2M_x}^{2M_y}}\right).$$

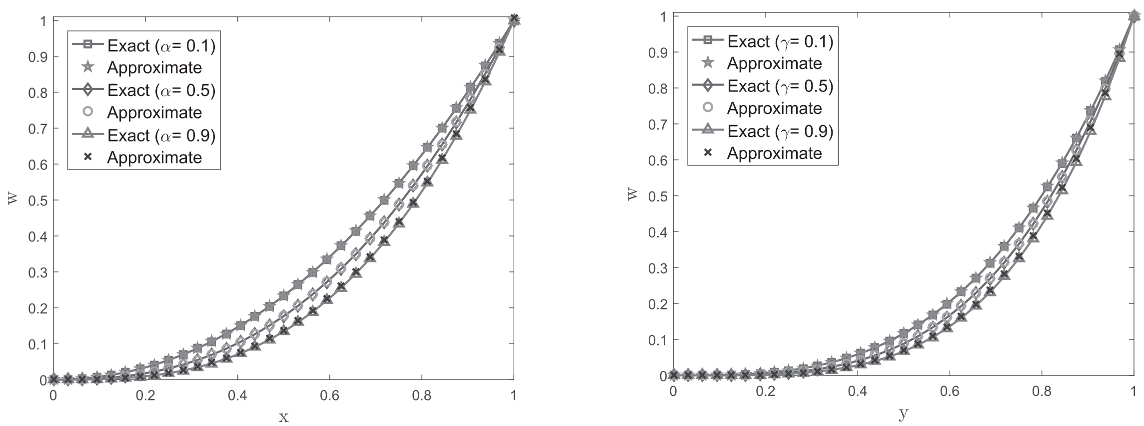
Example 1 Consider the test problem:

$$\begin{cases} D_x^{\alpha}\, w(x,y) + D_y^{\gamma}\, w(x,y) + e^{x+y}\, w(x,y) - \int_0^y \int_0^x (y+s)\, w(r,s)\, dr\, ds = f(x,y), \\ \text{for } (x,y) \in (0,1] \times (0,1], \\ w(0,y) = 0 \quad \text{for } y \in [0,1], \\ w(x,0) = 0 \quad \text{for } x \in (0,1]. \end{cases}$$



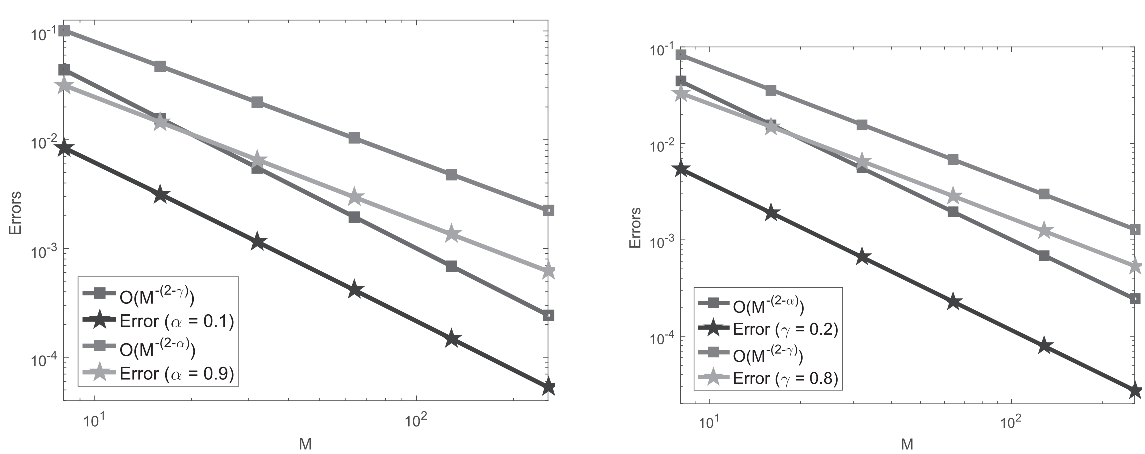
(a) Exact (b) Numerical

Fig. 1 Solution with $\alpha = 0.3,\ \gamma = 0.2$ and $M_x = M_y = 32$ for Example 1



(a) $\gamma = 0.5,\ y = 1$ (b) $\alpha = 0.5,\ x = 1$

Fig. 2 Comparisons between exact and approximate solutions with $M_x = M_y = 32$ for Example 1



(a) $\gamma = 0.5$ (b) $\alpha = 0.5$

Fig. 3 Log-log plots of $\Sigma_{M_x}^{M_y}$ for Example 1

Algorithm 1 Algorithm for the proposed scheme

1. Set α, γ, M_x, M_y.
2. Construct uniform mesh for Ω as $\{x_j = j/M_x\}_{j=0}^{M_x}$ and $\{y_i = i/M_y\}_{i=0}^{M_y}$.
3. Compute b_p for $p = 1, 2, \ldots, M_x$ and d_p for $p = 1, 2, \ldots, M_y$.
4. Repeat the following steps for each $j = 1, 2, \ldots, M_x$ and $i = 1, 2, \ldots, M_y$
   I. Construct algebraic equation (8) for W_j^i substituting (3), (4) and (5) in (1).
   II. Solve implicit form (8) for the approximate solution W_j^i.

Table 1 *$\Sigma_{M_x}^{M_y}$ and $\varrho_{M_x}^{M_y}$ with $\gamma = 0.5$ for Example* 1

$\alpha/(M_x = M_y)$	8	16	32	64	128	256
0.1	8.466E-03	3.150E-03	1.150E-03	4.149E-04	1.486E-04	5.300E-05
	1.426	1.454	1.471	1.481	1.487	
0.3	9.187E-03	3.384E-03	1.224E-03	4.381E-04	1.558E-04	5.515E-05
	1.441	1.467	1.482	1.492	1.498	
0.5	1.146E-02	4.262E-03	1.558E-03	5.629E-04	2.020E-04	7.214E-05
	1.427	1.452	1.468	1.479	1.486	
0.7	1.747E-02	6.963E-03	2.744E-03	1.076E-03	4.211E-04	1.649E-04
	1.327	1.343	1.351	1.353	1.353	
0.9	3.181E-02	1.445E-02	6.546E-03	2.969E-03	1.352E-03	6.178E-04
	1.138	1.143	1.140	1.135	1.130	

Table 2 *$\Sigma_{M_x}^{M_y}$ and $\varrho_{M_x}^{M_y}$ with $\alpha = 0.5$ for Example* 1

$\alpha/(M_x = M_y)$	8	16	32	64	128	256
0.2	5.430E-03	1.900E-03	6.606E-04	2.290E-04	7.935E-05	2.751E-05
	1.515	1.524	1.528	1.529	1.528	
0.4	8.480E-03	3.035E-03	1.069E-03	3.733E-04	1.295E-04	4.476E-05
	1.483	1.505	1.519	1.527	1.533	
0.5	1.146E-02	4.262E-03	1.558E-03	5.629E-04	2.020E-04	7.214E-05
	1.427	1.452	1.468	1.479	1.485	
0.6	1.605E-02	6.297E-03	2.427E-03	9.252E-04	3.506E-04	1.323E-04
	1.350	1.376	1.391	1.400	1.405	
0.8	3.325E-02	1.486E-02	6.532E-03	2.848E-03	1.236E-03	5.359E-04
	1.162	1.186	1.198	1.204	1.206	

Table 3 *Comparison between exact and approximate solution with $\alpha = 0.2$ and $\gamma = 0.7$ for Example* 1

$(x, y)/(M_x = M_y)$	Exact	16	32	64	128	256
(0.25, 0.25)	0.00028	0.00032	0.00030	0.00029	0.00028	0.00028
(0.50, 0.50)	0.01675	0.01753	0.01708	0.01689	0.01680	0.01677
(0.75, 0.75)	0.18317	0.18677	0.18469	0.18380	0.18343	0.18328
(0.94, 0.94)	0.68333	0.69068	0.68640	0.68459	0.68385	0.68354

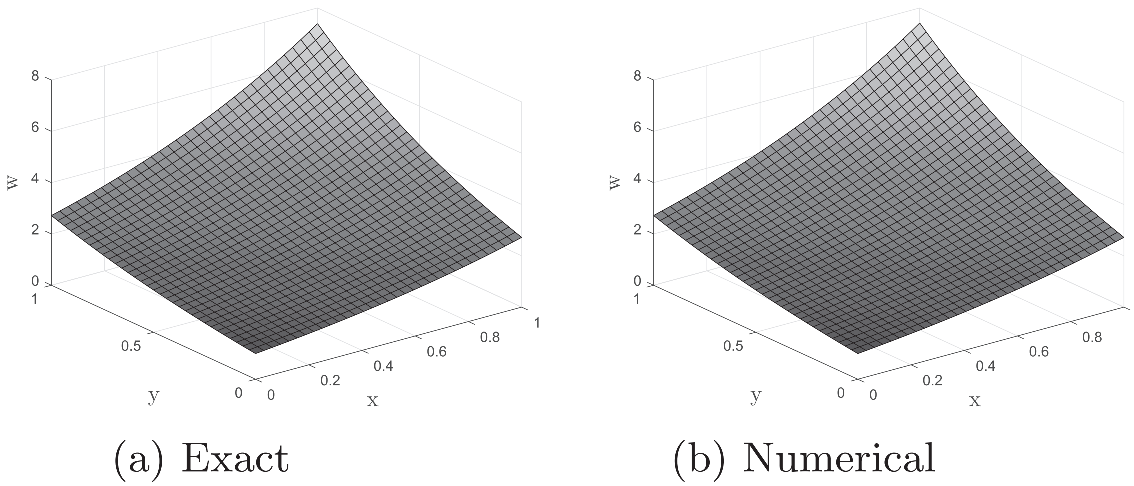


Fig. 4 Solution with $\alpha = 0.2,\ \gamma = 0.8$ and $M_x = M_y = 32$ for Example 2

The exact solution is $w(x, y) = x^{2+\alpha} y^{3+\gamma}$ for suitable choice of the source function $f(x, y)$. Figure 1 displays the exact solution and numerical solution with $\alpha = 0.3,\ \gamma = 0.2$ for Example 1. Figure 2 shows the comparisons between exact and approximate solutions for different values of α and γ. The log-log plot in Figure 3 of the computational error shows that the rate of convergence of the proposed scheme is sharp. Tables 1 and 2 depicted $\Sigma_{M_x}^{M_y}$ and $\varrho_{M_x}^{M_y}$ with fixed γ and fixed α, respectively for Example 1. From this data, it is evident that the approximate solution converges to the exact solution (See Table 3).

Example 2 Consider the following model:

$$\begin{cases} D_x^{\alpha}\, w(x, y) + D_y^{\gamma}\, w(x, y) + (1 - xy)w(x, y) + \int_0^y \int_0^x (xs + yr)\, w(r, s)\, dr\, ds = g(x, y, w) \\ \text{for } (x, y) \in (0, 1] \times (0, 1], \\ w(0, y) = e^y \quad \text{for } y \in [0, 1], \\ w(x, 0) = e^x \quad \text{for } x \in (0, 1], \end{cases}$$

where the semilinear term $g(x, y, w(x, y))$ is given by

$$g(x, y, w) = \frac{w^3}{1 + w^2} + f(x, y).$$

We choose $g(x, y)$, so that the exact solution is $w(x, y) = e^{x+y}$. Figure 4 shows the exact and approximate solutions with $\alpha = 0.2,\ \gamma = 0.8$ for Example 2. Figure 5 displays the comparisons between exact and approximate solutions for different values of x and y with $\alpha = 0.9,\ \gamma = 0.1$. The log-log plot is given in Fig. 6. The maximum error and convergence rate are displayed in Tables 4 and 5 with fixed γ and fixed α, respectively, illustrating that the convergence rate is sharp.

7 Conclusion

In this work, an efficient finite difference scheme is developed and analyze to solve the two-dimensional fractional Volterra integro-differential equations. The classical L1 technique is employed to discretize the Caputo fractional derivatives and the double integral is approximated with the help of the composite trapezoidal rule. The convergence analysis and the error estimation ensure that the proposed scheme achieves a $\min\{2-\alpha, 2-\gamma\}$ order of accuracy. In addition, we extend the proposed scheme to solve the corresponding semilinear problem. The results of the theoretical and numerical experiments confirm the efficiency and applicability of the proposed scheme. Tables and graphs show that the proposed scheme converges uniformly.

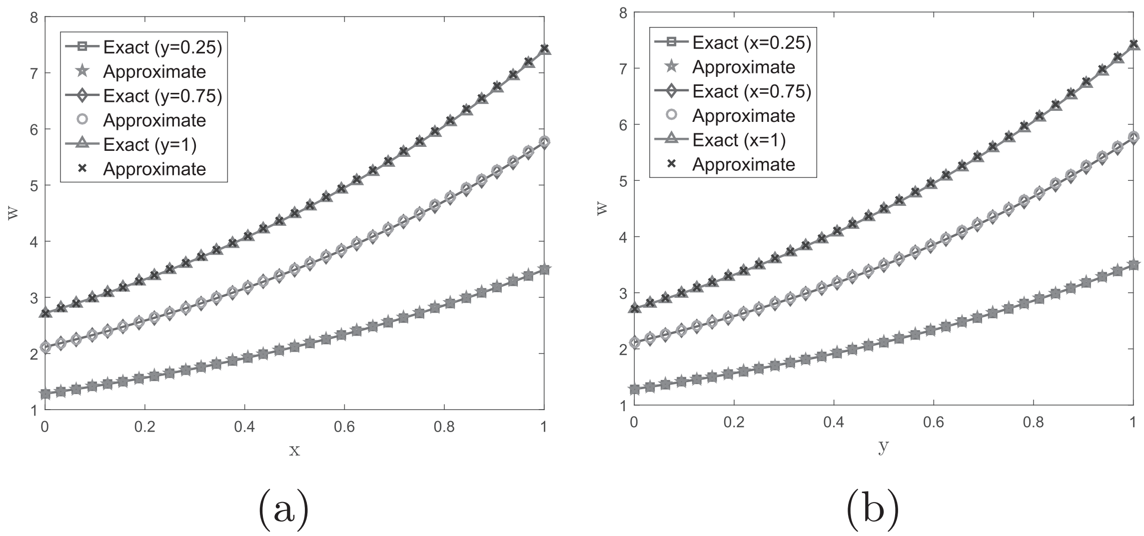


Fig. 5 Comparisons between exact and approximate solutions with $\alpha = 0.9$, $\gamma = 0.1$, and $M_x = M_y = 32$ for Example 2

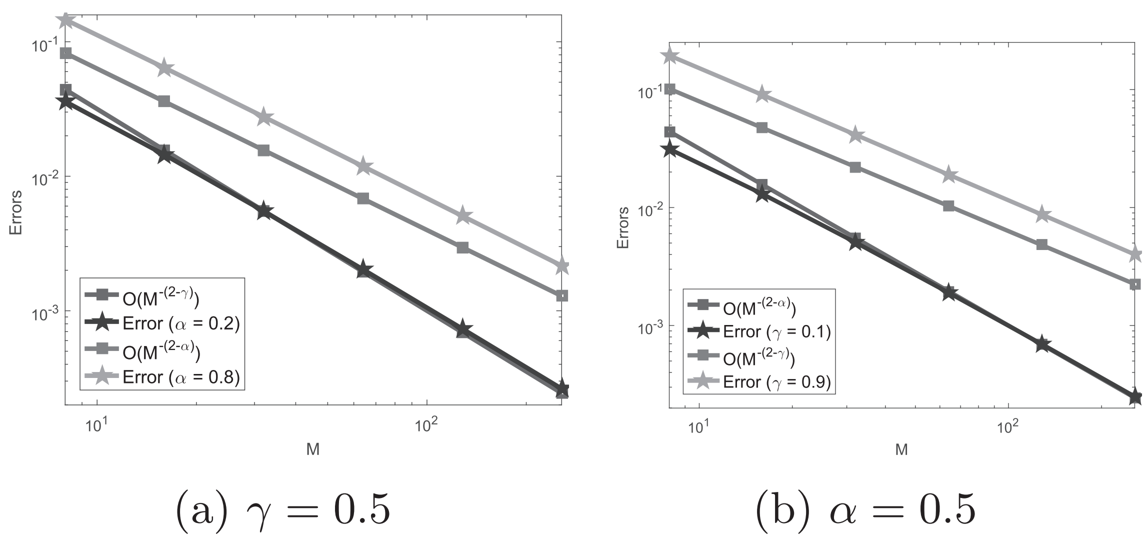


Fig. 6 Log-log plots of $\Sigma_{M_x}^{M_y}$ for Example 2

Table 4 $\Sigma_{M_x}^{M_y}$ *and* $\varrho_{M_x}^{M_y}$ *with* $\gamma = 0.5$ *for Example* 2

$\alpha/(M_x = M_y)$	8	16	32	64	128	256
0.2	3.598E-02	1.446E-02	5.496E-03	2.027E-03	7.343E-04	2.633E-04
	1.315	1.396	1.439	1.465	1.480	
0.4	5.223E-02	2.028E-02	7.547E-03	2.741E-03	9.805E-04	3.479E-04
	1.365	1.426	1.461	1.483	1.495	
0.5	6.571E-02	2.571E-02	9.692E-03	3.575E-03	1.300E-03	4.657E-04
	1.354	1.408	1.439	1.459	1.481	
0.6	8.455E-02	3.392E-02	1.319E-02	5.039E-03	1.905E-03	7.174E-04
	1.318	1.363	1.388	1.403	1.409	
0.8	1.462E-01	6.408E-02	2.765E-02	1.185E-02	5.068E-03	2.164E-03
	1.190	1.213	1.222	1.226	1.228	

Table 5 $\Sigma_{M_x}^{M_y}$ *and* $\varrho_{M_x}^{M_y}$ *with* $\alpha = 0.5$ *for Example* 2

$\alpha/(M_x = M_y)$	8	16	32	64	128	256
0.1	3.135E-02	1.303E-02	5.063E-03	1.898E-03	6.964E-04	2.497E-04
	1.266	1.364	1.416	1.446	1.480	
0.3	4.267E-02	1.673E-02	6.255E-03	2.277E-03	8.156E-04	2.878E-04
	1.351	1.420	1.458	1.481	1.503	
0.5	6.571E-02	2.571E-02	9.692E-03	3.575E-03	1.300E-03	4.651E-04
	1.354	1.408	1.439	1.459	1.483	
0.7	1.106E-01	4.613E-02	1.880E-02	7.566E-03	3.026E-03	1.204E-03
	1.261	1.295	1.313	1.322	1.329	
0.9	1.946E-01	9.013E-02	4.142E-02	1.900E-02	8.729E-03	4.002E-03
	1.110	1.122	1.124	1.122	1.125	

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Data availability statements The data sets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflicts of interest The authors declare that there is no conflicts of interest.

Ethical approval N/A

Informed consent On behalf of the authors, Dr. Jugal Mohapatra shall be communicating the manuscript.

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