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ORIGINAL RESEARCH

A Menon-type identity derived using Cohen-Ramanujan sum

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Abstract Menon's identity is a classical identity involving gcd sums and the Euler totient function ϕ. We derived the Menon-type identity $\sum\limits_{\substack{m=1\\(m,n^s)_s=1}}^{n^s}(m-1,n^s)_s=\Phi_s(n^s)\tau_s(n^s)$ in [*Czechoslovak Math. J., 72(1):165-176 (2022)*] where Φ_s denotes the Klee's function and $(a,b)_s$ denotes a a generalization of the gcd function. Here we give an alternate method to derive this identity using the properties of the Cohen-Ramanujan sum defined by E. Cohen.

Keywords Menon's identity · Euler totient function · Generalized gcd · Jordan totient function · Klee's function · Ramanujan sum · Cohen-Ramanujan sum

Mathematics Subject Classification 11A07 · 11A25 · 20D60 · 20D99

1 Introduction

The classical Menon's identity which originally appeared in [10] is a gcd sum turning out to be equal to a product of the Euler totient function ϕ and the number of divisors function τ. If (m,n) denotes the gcd of m and n, the identity is precisely the following:

$$\sum_{\substack{m=1\\(m,n)=1}}^{n}(m-1,n)=\phi(n)\tau(n). \tag{1}$$

It has been generalized and extended by many authors, see, for example, the papers [8] [13], [14], and [17]. For a positive integer s, and integers a,b, not both zero, the largest l^s (where $l\in\mathbb{N}$) dividing both a and b denoted by $(a,b)_s$ is called the *generalized gcd* of a and b (following E. Cohen [4]). When $s=1$, this will be equal to

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the usual gcd of a and b. For positive integers s and n, the *Klee's function* $\Phi_s(n)$ is defined to give the cardinality of the set $\{m \in \mathbb{N} : 1 \leq m \leq n, (m,n)_s = 1\}$.

Li and Kim gave a generalization of the Menon's identity (1) in [8]. Generalizing it further, these authors proved in [1] that if $m_1, m_2, \cdots, m_k, b_1, b_2, \cdots, b_r, n, s \in \mathbb{N}$ and $a_1, a_2, \cdots, a_k \in \mathbb{Z}$ are such that $(a_i, n^s)_s = 1$ for $i = 1, 2, \cdots, k$, then

$$\sum_{\substack{1\leq m_1,m_2,\ldots,m_k\leq n^s\\(m_1,n^s)_s=1\\(m_2,n^s)_s=1\\\cdots\\(m_k,n^s)_s=1\\1\leq b_1,b_2,\ldots,b_r\leq n^s}} (m_1-a_1, m_2-a_2, \ldots, m_k-a_k, b_1, b_2, \cdots, b_r, n^s)_s$$

$$= \Phi_s(n^s)^k \sum_{d^s|n^s} \frac{(d^s)^r}{\Phi_s(\frac{n^s}{d^s})^{k-1}}. \tag{2}$$

As a corollary to this, we also derived the following Menon-type identity [1, Corollary 3.1]:

$$\sum_{\substack{m=1\\(m,n^s)_s=1}}^{n^s} (m-1, n^s)_s = \Phi_s(n^s)\tau_s(n^s). \tag{3}$$

The *Ramanujan sum* denoted by $c_r(n)$ is defined as the sum $c_r(n) := \sum\limits_{\substack{m=1\\(m,r)=1}}^{r} e^{\frac{2\pi imn}{r}}$ where $r \in \mathbb{N}$ and $n \in \mathbb{Z}$. This exponential sum appeared for the first time in [11] has been generalized in many ways. E. Cohen gave the generalization

$$c_r^s(n) = \sum_{\substack{(h,r^s)_s=1\\h=1}}^{r^s} e^{\frac{2\pi inh}{r^s}}. \tag{4}$$

in [2]. This sum will be henceforth called as the *Cohen-Ramanujan sum*. When $s = 1$, Cohen-Ramanujan sum reduces to the Ramanujan sum. Some other important properties of this sum are given in the next section.

We derived the identity (2) using elementary number theory techniques. Menon's identity (1) was also proved originally using elementary number theory techniques. In [15], L. Tóth gave a proof of the Menon's identity using the concept of Ramanujan sum. This was surprising as most of the other Menon-type identities were derived using techniques from elementary number theory, group theory or some sort of character theory. In this paper, we derive the Menon-type identity (3) using the properties of the generalized gcd function and the Cohen-Ramanujan sum. We believe that the properties of the Ramanujan sum and its generalizations can most probably be applied to derive more interesting identities similar to the already derived generalizations of Menon's identity. This paper underlines the significance of the concept of Ramanujan sum which was originally employed by Srinivasa Ramanujan and later by many other authors (see, for example, [6], [5], and [9]) for finding Fourier-series like expansions of certain arithmetical functions. Also, we expect this paper to further motivate future attempts in using the Ramanujan sum in seemingly different contexts.

2 Notations and basic results

Notations, definitions, and properties most frequently used across this paper are the following. A postive integer a is said to be s-power free or s-free if no l^s where $l \in \mathbb{N}$ divides a. The Euler totient function ϕ has several generalizations of which three are relevant to us. They are the Jordan totient function, Cohen totient function and the Klee's function. The Jordan totient function $J_s(n)$ [7, pp 95-97] defined for positive integers s and n gives the number of ordered sets of s elements from a complete residue system (mod n) such that the greatest common divisor of each set is prime to n. The Cohen totient function ϕ_s [4] is defined as follows. If $(a,b)_s = 1$, then a and b are said to be relatively s-prime. The subset N of a complete residue system M (mod n^s) consisting of all elements of M that are relatively s-prime to n^s is called an s-reduced residue system (mod n). Cohen totient function $\phi_s(n)$ gives the number of elements of an s-reduced residue system (mod n).

Some of the important properties of the Klee's function Φ_s are listed in [1, Section 2]. Most notably, it has a product formula

$$\Phi_s(n) = n \prod_{\substack{p^s \mid n \\ p \text{ prime}}} \left(1 - \frac{1}{p^s}\right). \tag{5}$$

In [4, Theorem 3], Cohen showed that $\phi_s(n) = n^s \prod_{\substack{p^s \mid n \\ p \text{ prime}}} (1 - \frac{1}{p^s})$. So $\phi_s(n) = \Phi_s(n^s)$. Further J_s and ϕ_s are related by the identity $J_s(n) = \phi_s(n)$ [4, Theorem 5]. Note also that $\Phi_1 = J_1 = \phi_1 = \phi$.

The Cohen-Ramanujan sum $c_r^s(n)$ defined in equation (4) has the following properties.

CRS1 $c_r^s(n) = \sum_{\substack{d \mid r \\ d^s \mid n}} \mu(r/d) d^s$ where μ is the Möbius function [2].

CRS2 $c_r^s(n) = \dfrac{\phi_s(r)\mu(d)}{\phi_s(d)} = \dfrac{\Phi_s(r^s)\mu(d)}{\Phi_s(d^s)}$ where $d^s = \dfrac{r^s}{(n, r^s)_s}$ [3].

CRS3 $c_r^s(n) = c_r^s(-n)$.

CRS4 $c_r^s(1) = \mu(r)$.

3 Main results

We will derive the Menon-type identity (3) in this section. To begin with, we prove three lemmas which will be used in the proof of the identity.

Lemma 3.1 *For positive integers n, s, we have* $\dfrac{n^s}{\Phi_s(n^s)} = \sum_{d \mid n} \dfrac{(\mu(d))^2}{\Phi_s(d^s)}$.

Proof Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Then

$$\begin{aligned}
\frac{n^s}{\Phi_s(n^s)} &= \frac{1}{\prod_{\substack{p^s \mid n^s \\ p \text{ prime}}} \left(1 - \frac{1}{p^s}\right)} \\
&= \prod_{\substack{p^s \mid n^s \\ p \text{ prime}}} (1 + \frac{1}{p^s - 1}) \\
&= \left(1 + \frac{1}{p_1^s - 1}\right)\left(1 + \frac{1}{p_2^s - 1}\right) \cdots \left(1 + \frac{1}{p_k^s - 1}\right) \\
&= \prod_{i=1}^{k} \left(1 + \frac{(\mu(p_i))^2}{\Phi_s(p_i^s)}\right) \\
&= \sum_{d \mid n} \frac{(\mu(d))^2}{\Phi_s(d^s)}.
\end{aligned}$$

□

An element in $\mathbb{N}^2$ will be denoted by $(a; b)$ to avoid any confusion with the notation of gcd of a and b. The next result generalizes [16, Proposition 1].

Lemma 3.2 *Let $\psi : \mathbb{N}^2 \to \mathbb{C}$ be an arbitrary function and k, n, s positive integers. Then we have*

$$\sum_{k=1}^{n^s} \psi(k; n^s)(k, n^s)_s = \sum_{d^s \mid n^s} \Phi_s(d^s) \sum_{j=1}^{\frac{n^s}{d^s}} \psi(d^s j; n^s).$$

Proof We have $n^s = \sum_{d|n} J_s(d)$ [12, Section V.3] and $J_s(n) = \Phi_s(n^s)$. Hence $n^s = \sum_{d|n} \Phi_s(d^s) = \sum_{d^s|n^s} \Phi_s(d^s)$. So

$$\begin{aligned}\sum_{k=1}^{n^s} \psi(k; n^s)(k, n^s)_s &= \sum_{k=1}^{n^s} \psi(k; n^s) \sum_{d^s|(k,n^s)_s} \Phi_s(d^s)\\ &= \sum_{k=1}^{n^s} \psi(k; n^s) \sum_{\substack{d^s|k\\ d^s|n^s}} \Phi_s(d^s).\end{aligned}$$

Since $d^s \mid k$ and $1 \leq k \leq n^s$, we may write $k = d^s j$ where $1 \leq j \leq \frac{n^s}{d^s}$ and hence we get the first part of the lemma $\sum_{k=1}^{n^s} \psi(k; n^s)(k, n^s)_s = \sum_{d^s|n^s} \Phi_s(d^s) \sum_{j=1}^{\frac{n^s}{d^s}} \psi(d^s j; n^s)$.

□

Lemma 3.3 *For n, s, j positive integers, we have that*

$$\sum_{b=1}^{n^s} (b, n^s)_s e^{\frac{2\pi i b j}{n^s}} = \sum_{d^s|(j,n^s)_s} d^s \Phi_s\left(\frac{n^s}{d^s}\right).$$

Proof Define $\psi(b; n) = e^{\frac{2\pi i b j}{n}}$ so that

$$\begin{aligned}\sum_{b=1}^{n^s} (b, n^s)_s e^{\frac{2\pi i b j}{n^s}} &= \sum_{b=1}^{n^s} (b, n^s)_s \psi(b; n^s)\\ &= \sum_{d^s|n^s} \Phi_s(d^s) \sum_{k=1}^{\frac{n^s}{d^s}} \psi\left(k, \frac{n^s}{d^s}\right)\\ &= \sum_{d^s|n^s} \Phi_s(d^s) \sum_{k=1}^{\frac{n^s}{d^s}} e^{\left(\frac{2\pi i k j}{\frac{n^s}{d^s}}\right)}.\end{aligned}$$

Note that $\frac{1}{d^s} \sum_{k=1}^{d^s} e^{\frac{2\pi i k n}{d^s}} = \begin{cases} 1 \text{ if } d^s \mid n \\ 0 \text{ otherwise .}\end{cases}$

So

$$\begin{aligned}\sum_{b=1}^{n^s} (b, n^s)_s e^{\frac{2\pi i b j}{n^s}} &= \sum_{\substack{d^s|n^s\\ \frac{n^s}{d^s}|j}} \Phi_s(d^s) \frac{n^s}{d^s}\\ &= \sum_{\substack{d^s|n^s\\ d^s|j}} \Phi_s\left(\frac{n^s}{d^s}\right) d^s\\ &= \sum_{d^s|(j,n^s)_s} \Phi_s\left(\frac{n^s}{d^s}\right) d^s.\end{aligned}$$

□

Now we are ready to derive identity (3) which we restate below.

Theorem 3.4 *For n, s positive integers, we have that*

$$\sum_{\substack{a=1\\(a,n^s)_s=1}}^{n^s}(a-1,n^s)_s=\Phi_s(n^s)\tau_s(n^s).$$

Proof

$$\begin{aligned}\sum_{\substack{a=1\\(a,n^s)_s=1}}^{n^s}(a-1,n^s)_s &= \sum_{\substack{a,b=1\\(a,n^s)_s=1\\a-1\equiv b(\mathrm{mod}\ n^s)}}^{n^s}(b,n^s)_s\\ &= \sum_{\substack{a,b=1\\(a,n^s)_s=1}}^{n^s}\ \sum_{a-1\equiv b(\mathrm{mod}\ n^s)}(b,n^s)_s\\ &= \sum_{\substack{a,b=1\\(a,n^s)_s=1}}^{n^s}(b,n^s)_s\frac{1}{n^s}\sum_{j=1}^{n^s}e^{\frac{2\pi ij(1-a+b)}{n^s}}.\end{aligned}$$

To arrive at the last line above, we used the fact that $\frac{1}{d^s}\sum\limits_{k=1}^{d^s}e^{\frac{2\pi ikn}{d^s}}=1$ if $d^s|n$. Continuing with the computation, we get

$$\begin{aligned}\sum_{\substack{a=1\\(a,n^s)_s=1}}^{n^s}(a-1,n^s)_s &= \frac{1}{n^s}\sum_{j(\mathrm{mod}\ n^s)}e^{\frac{2\pi ij}{n^s}}\sum_{b(\mathrm{mod}\ n^s)}(b,n^s)_se^{\frac{2\pi ibj}{n^s}}\sum_{\substack{a=1\\(a,n^s)_s=1}}^{n^s}e^{\frac{2\pi ia(-j)}{n^s}}\\ &= \frac{1}{n^s}\sum_{j(\mathrm{mod}\ n^s)}e^{\frac{2\pi ij}{n^s}}\sum_{b(\mathrm{mod}\ n^s)}(b,n^s)_se^{\frac{2\pi ibj}{n^s}}c_n^s(j).\end{aligned}$$

Now let $(j,n^s)_s=\delta^s\Rightarrow\delta^s\mid n^s,\delta^s\mid j\Rightarrow\delta^s\mid n^s,\ j=t\delta^s$ for some integer t. Also $(\frac{j}{\delta^s},\frac{n^s}{\delta^s})_s=1$. That is $(t,\frac{n^s}{\delta^s})_s=1$.

Since $\frac{n^s}{(j,n^s)_s}=\frac{n^s}{\delta^s}=(\frac{n}{\delta})^s$, by property CRS2 of the Cohen-Ramanujan sum and Lemma 3.3, we get

$$\begin{aligned}&\sum_{\substack{a=1\\(a,n^s)_s=1}}^{n^s}(a-1,n^s)_s\\ &= \frac{1}{n^s}\sum_{\substack{t=1\\(t,\frac{n^s}{\delta^s})_s=1\\\delta^s|n^s}}^{\frac{n^s}{\delta^s}}e^{\frac{2\pi it\delta^s}{n^s}}\sum_{d^s|\delta^s}d^s\Phi_s\left(\frac{n^s}{d^s}\right)\frac{\Phi_s(n^s)\mu(\frac{n}{\delta})}{\Phi_s(\frac{n^s}{\delta^s})}\\ &= \frac{\Phi_s(n^s)}{n^s}\sum_{\delta^s|n^s}\frac{\mu(\frac{n}{\delta})}{\Phi_s(\frac{n^s}{\delta^s})}\sum_{d^s|\delta^s}d^s\Phi_s\left(\frac{n^s}{d^s}\right)\sum_{\substack{t=1\\(t,\frac{n^s}{\delta^s})_s=1}}^{\frac{n^s}{\delta^s}}e^{\frac{2\pi it\delta^s}{n^s}}\\ &= \frac{\Phi_s(n^s)}{n^s}\sum_{\delta^s|n^s}\frac{\mu(\frac{n}{\delta})}{\Phi_s(\frac{n^s}{\delta^s})}\sum_{d^s|\delta^s}d^s\Phi_s\left(\frac{n^s}{d^s}\right)c^s_{\frac{n}{\delta}}(1)\\ &= \frac{\Phi_s(n^s)}{n^s}\sum_{\delta^s|n^s}\frac{(\mu(\frac{n}{\delta}))^2}{\Phi_s(\frac{n^s}{\delta^s})}\sum_{d^s|\delta^s}d^s\Phi_s\left(\frac{n^s}{d^s}\right)\ \text{since}\ \mu(\frac{n}{\delta})=c^s_{\frac{n}{\delta}}(1)\qquad\text{by CRS4.}\end{aligned}$$

Now $d^s \mid \delta^s, \delta^s \mid n^s \Rightarrow n^s = \delta^s h^s = d^s k^s h^s$ so that

$$\begin{aligned}
\sum_{\substack{a=1 \\ (a,n^s)_s=1}}^{n^s} (a-1, n^s)_s &= \frac{\Phi_s(n^s)}{n^s} \sum_{n^s=d^s k^s h^s} d^s \Phi_s(k^s h^s) \frac{(\mu(h))^2}{\Phi_s(h^s)} \\
&= \frac{\Phi_s(n^s)}{n^s} \sum_{n^s=d^s m^s} d^s \Phi_s(m^s) \sum_{m=kh} \frac{(\mu(h))^2}{\Phi_s(h^s)} \\
&= \frac{\Phi_s(n^s)}{n^s} \sum_{n^s=d^s m^s} d^s \Phi_s(m^s) \frac{m^s}{\Phi_s(m^s)} \quad \text{by Lemma 3.1} \\
&= \frac{\Phi_s(n^s)}{n^s} \sum_{n^s=d^s m^s} d^s m^s \\
&= \frac{\Phi_s(n^s)}{n^s} \sum_{d^s \mid n^s} n^s \\
&= \Phi_s(n^s)\tau_s(n^s)
\end{aligned}$$

which is what we claimed. □

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Declarations

Competing Interests The authors declare that they have no competing interests.

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