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ORIGINAL RESEARCH

Fundamental property of $2 \times n$ row Suslin matrices

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Abstract The study of unimodular rows over a commutative ring involves the utilization of Suslin matrices. These matrices possess a Fundamental Property, which plays a crucial role in establishing connections between the group generated by special Suslin matrices and the special orthogonal group. Moreover, the Fundamental Property serves as a means to establish a link between Suslin matrices and Spin groups. In this work, we focus on $2 \times n$ row Suslin matrices, defining them and exploring their properties. We provide a proof for the Fundamental Property specific to $2 \times n$ row Suslin matrices.

Keywords $2 \times n$ rows · Suslin matrices

Mathematics Subject Classification 13C10 · 15A63 · 11E88 · 19A13

1 Introduction

In [1, §5], A.A. Suslin introduced a matrix called a Suslin matrix denoted as $S_r(v, w)$. These matrices are associated with pairs of vectors (v, w) in $R^{2(r+1)}$, and their determinants for $r \geq 1$, are given by $\langle v, w\rangle^{2^{r-1}}$, where R is a commutative ring with unity. Suslin matrices are defined recursively as follows: For $v = (a_0, v_1)$ and $w = (b_0, w_1)$ in $M_{1\times(r+1)}(R)$, where $a_0, b_0 \in R$, $v_1, w_1 \in M_{1\times r}(R)$ and $M_{1\times(r+1)}(R)$ denotes the set of $1 \times (r + 1)$ matrices with entries in R, set $S_0(v, w) = a_0$ and define $S_r(v, w)$ as the matrix:

$$S_r(v, w) = \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1, w_1) \\ -S_{r-1}(w_1, v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix},$$

where the superscript T stands for the transpose of a matrix.

A Suslin matrix $S_r(v, w)$ is called special if $\langle v, w\rangle = 1$ and the group generated by these matrices is referred to as "special unimodular vector group" denoted as $SUm_r(R)$. A.A. Suslin showed that special Suslin matrices can be used to find completion of the unimoduar row $(a_0, a_1, a_2^2, \ldots, a_r^r)$. The study of Suslin matrices was further advanced by Jose-Rao in [2–4], where they proved the Key Lemma and the Fundamental Property. They also developed the commutator calculus for the elementary subgroup $EUm_r(R)$ of $SUm_r(R)$. The Fundamental Property plays a crucial role in defining an action of Suslin matrices on the Suslin space $\mathbb{S}_r$, which is a free

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R-module with rank $2(r+1)$ having basis elements $S_r(e_i, 0)$ and $S_r(0, e_i)$, where e_i represents the i-th unit coordinate vector with $1 \leq i \leq r+1$. This action establishes a connection between a quotient of $SUm_r(R)$ and a quotient of the special orthogonal group $SO_{2(r+1)}(R)$.

In [5], Vineeth Chintala established a link between Suslin matrices and Clifford algebra of hyperbolic form $H(R^n)(R-$ Module $R^n \times R^n)$ equipped with the quadratic form $q(v, w) = v.w^T$. By this approach, he proved Fundamental Property of Suslin matrices. This link, along with the Fundamental Property, established a connection between Spin group

$$Spin_{2n}(R) = \{x \in Cl_0(R) \mid xx^* = 1 \text{ and } xH(R^n)x^{-1} = H(R^n)\}$$

and the matrix group

$$G_{n-1} = \left\{g \in GL_{2^{n-1}}(R) \mid gSg^* \text{ is Suslin matrix, for all Suslin matrix } S\right\}.$$

In this paper, we define $2 \times n$ row Suslin matrices and denote them as $S_n^2(V, W)$ for the vectors $V, W \in M_{2\times n}(R)$, where $M_{2\times n}(R)$ represents the set of $2 \times n$ matrices with entries in commutative ring R. We analyze their properties and explore their connections with the vectors on which they are defined. We provide a proof of the Fundamental Property satisfied by $2 \times n$ row Suslin matrices. We call this property as Fundamental Property since it plays an crucial role in further advancement of theory in context of $2 \times n$ row Suslin matrices. The Fundamental Property is useful in understanding $2 \times n$ right invertible matrices and their orbits.

2 Preliminaries

In this section, we revisit and review some of the properties of Suslin matrices $S_r(v, w)$ from [3] to [1], to understand their characteristics.

Lemma 2.1 [1, Lemma 1.2] *Let R be a commutative ring and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in M_{r\times r}(R)$. If $\alpha_1\alpha_2 = \alpha_2\alpha_1$, then*

$$\det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \det(\beta_2\alpha_1 - \beta_1\alpha_2).$$

Notation 2.2 *If $v = (a_0, \dots, a_r)$, $w = (b_0, \dots, b_r) \in M_{1\times(r+1)}(R)$, then we denote $\langle v, w\rangle = a_0b_0 + \cdots + a_rb_r = v.w^T$.*

Lemma 2.3 [3, Lemma 3.1]
Let $v, w, s, t \in M_{1\times(r+1)}(R)$, $v = (a_0, a_1, \dots, a_r)$, $w = (b_0, b_1, \dots, b_r)$. Then

$$\begin{aligned} S_r(v, w) + S_r(w, v)^T &= \{a_0 + b_0\}I_{2^r}, \\ S_r(s, t)S_r(w, v)^T + S_r(v, w)S_r(t, s)^T &= \{\langle s, w\rangle + \langle v, t\rangle\}I_{2^r}, \\ S_r(w, v)^T S_r(s, t) + S_r(t, s)^T S_r(v, w) &= \{\langle s, w\rangle + \langle v, t\rangle\}I_{2^r}. \end{aligned}$$

Lemma 2.4 [1, Lemma 5.1]
For $r \geq 1$, and $v, w \in M_{1\times(r+1)}(R)$, the Suslin matrix $S_r(v, w)$ has the following properties:

(i) $S_r(v, w)S_r(w, v)^T = \langle v, w\rangle I_{2^r} = S_r(w, v)^T S_r(v, w)$.
(ii) $\det S_r(v, w) = \langle v, w\rangle^{2^{r-1}}$.

3 The $2 \times n$ row Suslin matrix

In this section, we will present the concept of $2 \times n$ row Suslin matrices and explore their properties.

Notation 3.1 *We denote $2 \times n$ row by $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in M_{2\times n}(R)$, where $v_1 = (a_1, \dots, a_n)$, $v_2 = (b_1, \dots, b_n) \in M_{1\times n}(R)$.*

Definition 3.2 For $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in M_{2\times n}(R)$, we define the $2 \times n$ row Suslin matrix by,

$$S_n^2(V, W) = S_{2n-1}((v_1, v_2), (w_1, w_2)).$$

Following are some properties of $2 \times n$ row Suslin matrices:

Lemma 3.3 *Let* $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *and* $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in M_{2\times n}(R)$. *Then,*

(i) $S_n^2(V, W)S_n^2(W, V)^T = S_n^2(W, V)^T S_n^2(V, W) = (\langle v_1, w_1\rangle + \langle v_2, w_2\rangle)I_{2^{2n-1}}$.

(ii) $\det S_n^2(V, W) = (\langle v_1, w_1\rangle + \langle v_2, w_2\rangle)^{2^{2n-2}}$.

Proof Let $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_1, \ldots, a_n \\ b_1, \ldots, b_n \end{pmatrix}$ and $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} c_1, \ldots, c_n \\ d_1, \ldots, d_n \end{pmatrix}$. We first prove (i) by induction on n. When $n = 1$, $V = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $W = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$. Thus by definition,

$$\begin{aligned} S_1^2(V, W)S_1^2(W, V)^T &= S_1^2\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}\right) S_1^2\left(\begin{pmatrix} c_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\right)^T \\ &= S_1((a_1, b_1), (c_1, d_1))S_1((c_1, d_1), (a_1, b_1))^T \\ &= \begin{pmatrix} a_1 & b_1 \\ -d_1 & c_1 \end{pmatrix}\begin{pmatrix} c_1 & -b_1 \\ d_1 & a_1 \end{pmatrix} \\ &= \begin{pmatrix} a_1c_1 + b_1d_1 & 0 \\ 0 & d_1b_1 + a_1c_1 \end{pmatrix} \\ &= (\langle v_1, w_1\rangle + \langle v_2, w_2\rangle)I_2. \end{aligned}$$

Similarly one can prove $S_1^2(W, V)^T S_1^2(V, W) = (\langle v_1, w_1\rangle + \langle v_2, w_2\rangle)I_2$. Therefore the result is true for $n = 1$. Now for $n > 1$, let $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_1, v_1' \\ v_2 \end{pmatrix}$ and $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} c_1, w_1' \\ w_2 \end{pmatrix}$, where $v_1' = (a_2, \ldots, a_n)$, $w_1' = (c_2, \ldots, c_n)$. For $I = I_{2^{2n-2}}$, by definition,

$$\begin{aligned} &S_n^2(V, W)S_n^2(W, V)^T \\ &= S_{2n-1}((v_1, v_2), (w_1, w_2))S_{2n-1}((w_1, w_2), (v_1, v_2))^T \\ &= \begin{pmatrix} a_1 I & S_{2n-2}((v_1', v_2), (w_1', w_2)) \\ -S_{2n-2}((w_1', w_2), (v_1', v_2))^T & c_1 I \end{pmatrix} \\ &\quad \times \begin{pmatrix} c_1 I & -S_{2n-2}((v_1', v_2), (w_1', w_2)) \\ S_{2n-2}((w_1', w_2), (v_1', v_2))^T & a_1 I \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{aligned}$$

where,

$$\begin{aligned} \alpha &= a_1c_1 I + S_{2n-2}((v_1', v_2), (w_1', w_2))S_{2n-2}(w_1', w_2), (v_1', v_2))^T, \\ \beta &= -a_1 S_{2n-2}((v_1', v_2), (w_1', w_2)) + a_1 S_{2n-2}((v_1', v_2), (w_1', w_2)) \\ &= 0, \\ \gamma &= -c_1 S_{2n-2}((w_1', w_2), (v_1', v_2))^T + c_1 S_{2n-2}((w_1', w_2), (v_1', v_2))^T \\ &= 0, \\ \delta &= c_1a_1 I + S_{2n-2}((w_1', w_2), (v_1', v_2))^T S_{2n-2}((v_1', v_2), (w_1', w_2)). \end{aligned}$$

By Lemma 2.4,

$$\begin{aligned} \alpha &= (a_1c_1 + \langle v_1', w_1'\rangle + \langle v_2, w_2\rangle)I \\ &= (\langle v_1, w_1\rangle + \langle v_2, w_2\rangle)I, \\ \delta &= (c_1a_1 + \langle v_1', w_1'\rangle + \langle v_2, w_2\rangle)I \\ &= (\langle v_1, w_1\rangle + \langle v_2, w_2\rangle)I. \end{aligned}$$

Hence, $S_n^2(V,W)S_n^2(W,V)^T = (\langle v_1,w_1\rangle + \langle v_2,w_2\rangle)I_{2^{2n-1}}$. Similarly one can prove $S_n^2(W,V)^T S_n^2(V,W) = (\langle v_1,w_1\rangle + \langle v_2,w_2\rangle)I_{2^{2n-1}}$.
We now prove (ii). Note that

$$\det S_n^2(V,W) = \begin{vmatrix} a_1 I & S_{2n-2}((v_1',v_2),(w_1',w_2)) \\ -S_{2n-2}((w_1',w_2),(v_1',v_2))^T & c_1 I \end{vmatrix}$$

Since $(a_1 I)S_{2n-2}((v_1',v_2),(w_1',w_2)) = S_{2n-2}((v_1',v_2),(w_1',w_2))(a_1 I)$, by Lemmas 2.1 and 2.4, $\det S_n^2(V,W)$

$$\begin{aligned} &= \det\,(c_1 a_1 I + S_{2n-2}((w_1',w_2),(v_1',v_2))^T S_{2n-2}((v_1',v_2),(w_1',w_2))) \\ &= (\langle v_1,w_1\rangle + \langle v_2,w_2\rangle)^{2^{2n-2}}. \end{aligned}$$

This proves (ii). □

Lemma 3.4 *Let* $V = \begin{pmatrix} a_1,\ldots,a_n \\ b_1,\ldots,b_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $W = \begin{pmatrix} c_1,\ldots,c_n \\ d_1,\ldots,d_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in M_{2\times n}(R)$. *Then,*

(i) $S_n^2(V,W)^T = S_n^2(V',W')$, *where* $V' = \begin{pmatrix} a_1,-c_2,\ldots,-c_n \\ -d_1,-d_2,\ldots,-d_n \end{pmatrix}$ *and* $W' = \begin{pmatrix} c_1,-a_2,\ldots,-a_n \\ -b_1,-b_2,\ldots,-b_n \end{pmatrix} \in M_{2\times n}(R)$.

(ii) $\operatorname{adj} S_n^2(V,W) = (\langle v_1,w_1\rangle + \langle v_2,w_2\rangle)^{2^{2n-2}-1} S_n^2(W,V)^T$.

Proof Let $I = I_{2^{2n-2}}$, $v_1' = (a_2,\ldots,a_n)$, $w_1' = (c_2,\ldots,c_n) \in M_{1\times(n-1)}(R)$. By definition,

$$\begin{aligned} S_n^2(V,W)^T &= S_{2n-1}((v_1,v_2),(w_1,w_2))^T \\ &= \begin{pmatrix} a_1 I & -S_{2n-2}((w_1',w_2),(v_1',v_2)) \\ S_{2n-2}((v_1',v_2),(w_1',w_2))^T & c_1 I \end{pmatrix} \\ &= S_{2n-1}((a_1,-c_2,\ldots,-c_n,-w_2),(c_1,-a_2,\ldots,-a_n,-v_2)) \\ &= S_n^2(V',W'), \end{aligned}$$

where $V' = \begin{pmatrix} a_1,-c_2,\ldots,-c_n \\ -d_1,-d_2,\ldots,-d_n \end{pmatrix}$, $W' = \begin{pmatrix} c_1,-a_2,\ldots,-a_n \\ -b_1,-b_2,\ldots,-b_n \end{pmatrix}$. This proves (i). Note that

$$(\operatorname{adj} S_n^2(V,W))S_n^2(V,W) = \det S_n^2(V,W) I_{2^{2n-1}},$$

and hence

$$(\operatorname{adj} S_n^2(V,W) S_n^2(V,W))S_n^2(W,V)^T = (\det S_n^2(V,W))S_n^2(W,V)^T.$$

Therefore by Lemma 3.3,

$$(\operatorname{adj} S_n^2(V,W))(\langle v_1,w_1\rangle + \langle v_2,w_2\rangle) = (\langle v_1,w_1\rangle + \langle v_2,w_2\rangle)^{2^{2n-2}} S_n^2(W,V)^T.$$

If $(\langle v_1,w_1\rangle + \langle v_2,w_2\rangle)$ is not a zero divisor in R then we are done as

$$\operatorname{adj} S_n^2(V,W) = (\langle v_1,w_1\rangle + \langle v_2,w_2\rangle)^{2^{2n-2}-1} S_n^2(W,V)^T \tag{1}$$

Now if $(\langle v_1,w_1\rangle + \langle v_2,w_2\rangle)$ is a zero divisor in R then let $v_1(x) = (a_1+x,a_2,\ldots,a_n)$, $w_1(x) = (c_1+x,c_2,\ldots,c_n) \in M_{1\times n}(R[x])$ and $V(x) = \begin{pmatrix} v_1(x) \\ v_2 \end{pmatrix}$, $W(x) = \begin{pmatrix} w_1(x) \\ w_2 \end{pmatrix} \in M_{2\times n}(R[x])$. Thus $(\langle v_1(x),w_1(x)\rangle + \langle v_2,w_2\rangle)$ is a unitary polynomial and hence a non zero divisor in $R[x]$. Therefore by Eq. (1),

$$\operatorname{adj} S_n^2(V(x),W(x)) = (\langle v_1(x),w_1(x)\rangle + \langle v_2,w_2\rangle)^{2^{2n-2}-1} S_n^2(W(x),V(x))^T.$$

Setting $x=0$, since $v_1(0) = v_1$, $w_1(0) = w_1$, $V(0) = V$ and $W(0) = W$, we get the result. This proves (ii). □

Lemma 3.5 *Let* $V = \begin{pmatrix} a_1, \ldots, a_n \\ b_1, \ldots, b_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $W = \begin{pmatrix} c_1, \ldots, c_n \\ d_1, \ldots, d_n \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ *and* $T = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in M_{2\times n}(R)$. *Then for* $r = 2n-1$,

$$S_n^2(V, W) + S_n^2(W, V)^T = \{a_1 + c_1\} I_{2^r},$$
$$S_n^2(S, T) S_n^2(W, V)^T + S_n^2(V, W) S_n^2(T, S)^T = \{\langle s_1, w_1\rangle + \langle s_2, w_2\rangle + \langle v_1, t_1\rangle + \langle v_2, t_2\rangle\} I_{2^r}$$
$$S_n^2(W, V)^T S_n^2(S, T) + S_n^2(T, S)^T S_n^2(V, W) = \{\langle s_1, w_1\rangle + \langle s_2, w_2\rangle + \langle v_1, t_1\rangle + \langle v_2, t_2\rangle\} I_{2^r}.$$

Proof Let $I = I_{2^{2n-2}}$, $s_1 = (x_1, \ldots, x_n)$, $s_2 = (y_1, \ldots, y_n)$, $t_1 = (z_1, \ldots, z_n)$ and $t_2 = (w_1, \ldots, w_n)$ in $M_{1\times n}(R)$. Let $v_1' = (a_2, \ldots, a_n)$, $w_1' = (c_2, \ldots, c_n) \in M_{1\times(n-1)}(R)$ By definition,
$S_n^2(V, W) + S_n^2(W, V)^T$

$$\begin{aligned}
&= S_{2n-1}((v_1, v_2), (w_1, w_2)) + S_{2n-1}((w_1, w_2), (v_1, v_2))^T \\
&= \begin{pmatrix} a_1 I & S_{2n-2}((v_1', v_2), (w_1', w_2)) \\ -S_{2n-2}((w_1', w_2), (v_1', v_2))^T & c_1 I \end{pmatrix} \\
&\quad + \begin{pmatrix} c_1 I & -S_{2n-2}((v_1', v_2), (w_1', w_2)) \\ S_{2n-2}((w_1', w_2), (v_1', v_2))^T & a_1 I \end{pmatrix} \\
&= (a_1 + c_1) I_{2^{2n-1}}
\end{aligned}$$

This proves the first equality. Now by Lemma 3.3,
$S_n^2(V + S, W + T) S_n^2(W + T, V + S)^T$

$$\begin{aligned}
&= \{\langle (v_1 + s_1), (w_1 + t_1)\rangle + \langle (v_2 + s_2), (w_2 + t_2)\rangle\} I_{2^{2n-1}} \\
&= \{\langle v_1, w_1\rangle + \langle v_1, t_1\rangle + \langle s_1, w_1\rangle + \langle s_1, t_1\rangle \\
&\quad + \langle v_2, w_2\rangle + \langle v_2, t_2\rangle + \langle s_2, w_2\rangle + \langle s_2, t_2\rangle\} I_{2^{2n-1}}.
\end{aligned}$$

By definition,

$$\begin{aligned}
S_n^2(V + S, W + T) &= S_{2n-1}((v_1 + s_1, v_2 + s_2), (w_1 + t_1, w_2 + t_2)) \\
&= S_{2n-1}((v_1, v_2), (w_1, w_2)) + S_{2n-1}((s_1, s_2), (t_1, t_2)) \\
&= S_n^2(V, W) + S_n^2(S, T).
\end{aligned}$$

Similarly $S_n^2(W + T, V + S)^T = S_n^2(W, V)^T + S_n^2(T, S)^T$. Therefore,
$S_n^2(V + S, W + T) S_n^2(W + T, V + S)^T$

$$\begin{aligned}
&= S_n^2(V, W) S_n^2(W, V)^T + S_n^2(V, W) S_n^2(T, S)^T + S_n^2(S, T) S_n^2(W, V)^T \\
&\quad + S_n^2(S, T) S_n^2(T, S)^T \\
&= (\langle v_1, w_1\rangle + \langle v_2, w_2\rangle) I_{2^{2n-1}} + S_n^2(V, W) S_n^2(T, S)^T + S_n^2(S, T) S_n^2(W, V)^T \\
&\quad + (\langle s_1, t_1\rangle + \langle s_2, t_2\rangle) I_{2^{2n-1}}.
\end{aligned}$$

Now equating the two equations we get,
$S_n^2(S, T) S_n^2(W, V)^T + S_n^2(V, W) S_n^2(T, S)^T$

$$= \{\langle s_1, w_1\rangle + \langle s_2, w_2\rangle + \langle v_1, t_1\rangle + \langle v_2, t_2\rangle\}\, I_{2^{2n-1}}.$$

This proves second equality. Similarly one can prove the third equation. □

4 The fundamental property for 2 × n row Suslin matrices

In this section we prove Fundamental Property of $2 \times n$ row Suslin matrices.

Theorem 4.1 (Fundamental Property of $2 \times n$ Row Suslin Matrices)
Let $V, W, S, T \in M_{2\times n}(R)$ be given by

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \quad T = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}.$$

Then we have,

$$S_n^2(S,T)S_n^2(V,W)S_n^2(S,T) = S_n^2(V',W')$$
$$S_n^2(T,S)S_n^2(W,V)S_n^2(T,S) = S_n^2(W',V')$$

for some $V' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}$, $W' = \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} \in M_{2\times n}(R)$ with

$$\langle (v_1', v_2'), (w_1', w_2') \rangle = \langle (s_1, s_2), (t_1, t_2) \rangle^2 \langle (v_1, v_2), (w_1, w_2) \rangle.$$

Proof Let $v_1 = (v_{11}, \overline{v_1})$, $w_1 = (w_{11}, \overline{w_1})$, $s_1 = (s_{11}, \overline{s_1})$, $t_1 = (t_{11}, \overline{t_1}) \in M_{1\times n}(R)$, where $\overline{v_1} = (v_{12}, \ldots, v_{1n})$, $\overline{w_1} = (w_{12}, \ldots, w_{1n})$, $\overline{s_1} = (s_{12}, \ldots, s_{1n})$, and $\overline{t_1} = (t_{12}, \ldots, t_{1n}) \in M_{1\times(n-1)}(R)$. Then for $r = 2n-2$ and $I = I_{2^r}$ by definition,
$S_n^2(S,T)S_n^2(V,W)S_n^2(S,T)$

$$\begin{aligned}
&= S_{2n-1}((s_1,s_2),(t_1,t_2))S_{2n-1}((v_1,v_2),(w_1,w_2))S_{2n-1}((s_1,s_2),(t_1,t_2)) \\
&= \begin{pmatrix} s_{11}I & S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) \\ -S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T & t_{11}I \end{pmatrix} \\
&\quad \times \begin{pmatrix} v_{11}I & S_r((\overline{v_1},v_2),(\overline{w_1},w_2)) \\ -S_r((\overline{w_1},w_2),(\overline{v_1},v_2))^T & w_{11}I \end{pmatrix} \\
&\quad \times \begin{pmatrix} s_{11}I & S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) \\ -S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T & t_{11}I \end{pmatrix} \\
&= \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}.
\end{aligned}$$

where,

$$\begin{aligned}
\gamma_{11} &= s_{11}^2 v_{11} I - s_{11}\{S_r((\overline{s_1},s_2),(\overline{t_1},t_2))S_r((\overline{w_1},w_2),(\overline{v_1},v_2))^T \\
&\quad + S_r((\overline{v_1},v_2),(\overline{w_1},w_2))S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T\} \\
&\quad - w_{11}\{S_r((\overline{s_1},s_2),(\overline{t_1},t_2))S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T\}, \\
\gamma_{12} &= (s_{11}v_{11} + t_{11}w_{11})S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) + t_{11}s_{11}S_r((\overline{v_1},v_2),(\overline{w_1},w_2)) \\
&\quad - S_r((\overline{s_1},s_2),(\overline{t_1},t_2))S_r((\overline{w_1},w_2),(\overline{v_1},v_2))^T S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) \\
\gamma_{21} &= -(s_{11}v_{11} + t_{11}w_{11})S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T - s_{11}t_{11}S_r((\overline{w_1},w_2),(\overline{v_1},v_2))^T \\
&\quad + S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T S_r((\overline{v_1},v_2),(\overline{w_1},w_2))S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T \\
\gamma_{22} &= t_{11}^2 w_{11} I - t_{11}\{S_r((\overline{w_1},w_2),(\overline{v_1},v_2))^T S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) \\
&\quad + S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T S_r((\overline{v_1},v_2),(\overline{w_1},w_2))\} \\
&\quad - v_{11}\{S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T S_r((\overline{s_1},s_2),(\overline{t_1},t_2))\}.
\end{aligned}$$

By Lemma 2.3,
$S_r((\overline{s_1},s_2),(\overline{t_1},t_2))S_r((\overline{w_1},w_2),(\overline{v_1},v_2))^T + S_r((\overline{v_1},v_2),(\overline{w_1},w_2))S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T$

$$\begin{aligned}
&= \big(\langle (\overline{s_1},s_2),(\overline{w_1},w_2)\rangle + \langle (\overline{v_1},v_2),(\overline{t_1},t_2)\rangle\big)\, I \\
&= \big(\langle \overline{s_1},\overline{w_1}\rangle + \langle s_2,w_2\rangle + \langle \overline{t_1},\overline{v_1}\rangle + \langle t_2,v_2\rangle\big)\, I.
\end{aligned}$$

And by Lemma 2.4,

$$S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) = \big(\langle\overline{t_1},\overline{s_1}\rangle + \langle t_2,s_2\rangle\big) I.$$

Therefore,

$$\begin{aligned}
\gamma_{11} &= (s_{11}^2 v_{11} - s_{11}\{\langle\overline{s_1},\overline{w_1}\rangle + \langle s_2,w_2\rangle + \langle\overline{t_1},\overline{v_1}\rangle + \langle t_2,v_2\rangle\} \\
&\quad - w_{11}\{\langle\overline{t_1},\overline{s_1}\rangle + \langle t_2,s_2\rangle\})I. \\
&= v'_{11} I \quad \text{(say)} \\
\gamma_{12} &= (s_{11}v_{11} + t_{11}w_{11})S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) + t_{11}s_{11}S_r((\overline{v_1},v_2),(\overline{w_1},w_2)) \\
&\quad - S_r((\overline{s_1},s_2),(\overline{t_1},t_2))\{(\langle\overline{w_1},\overline{s_1}\rangle + \langle w_2,s_2\rangle + \langle\overline{v_1},\overline{t_1}\rangle + \langle v_2,t_2\rangle)I \\
&\quad - S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T S_r((\overline{v_1},v_2),(\overline{w_1},w_2))\} \\
&= (s_{11}v_{11} + t_{11}w_{11})S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) + t_{11}s_{11}S_r((\overline{v_1},v_2),(\overline{w_1},w_2)) \\
&\quad - \big(\langle\overline{w_1},\overline{s_1}\rangle + \langle w_2,s_2\rangle + \langle\overline{v_1},\overline{t_1}\rangle + \langle v_2,t_2\rangle\big) S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) \\
&\quad + S_r((\overline{s_1},s_2),(\overline{t_1},t_2))S_r((\overline{t_1},t_2),(\overline{s_1},s_2))^T S_r((\overline{v_1},v_2),(\overline{w_1},w_2)) \\
&= (s_{11}v_{11} + t_{11}w_{11})S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) \\
&\quad - \big(\langle\overline{w_1},\overline{s_1}\rangle + \langle w_2,s_2\rangle + \langle\overline{v_1},\overline{t_1}\rangle + \langle v_2,t_2\rangle\big) S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) \\
&\quad + t_{11}s_{11}S_r((\overline{v_1},v_2),(\overline{w_1},w_2)) + (\langle\overline{s_1},\overline{t_1}\rangle + \langle s_2,t_2\rangle)S_r((\overline{v_1},v_2),(\overline{w_1},w_2)) \\
&= \big(s_{11}v_{11} + t_{11}w_{11} - \langle\overline{w_1},\overline{s_1}\rangle - \langle w_2,s_2\rangle - \langle\overline{v_1},\overline{t_1}\rangle - \langle v_2,t_2\rangle\big) \\
&\quad \times S_r((\overline{s_1},s_2),(\overline{t_1},t_2)) + (\langle s_1,t_1\rangle + \langle s_2,t_2\rangle)S_r((\overline{v_1},v_2),(\overline{w_1},w_2)) \\
&= S_r((x\overline{s_1} + y\overline{v_1}, xs_2 + yv_2),(x\overline{t_1} + y\overline{w_1}, xt_2 + yw_2))
\end{aligned}$$

where,

$$\begin{aligned}
x &= s_{11}v_{11} + t_{11}w_{11} - \langle\overline{w_1},\overline{s_1}\rangle - \langle w_2,s_2\rangle - \langle\overline{v_1},\overline{t_1}\rangle - \langle v_2,t_2\rangle, \\
y &= \langle s_1,t_1\rangle + \langle s_2,t_2\rangle.
\end{aligned}$$

Similarly by Lemmas 2.3 and 2.4,

$$\begin{aligned}
\gamma_{21} &= -S_r((x\overline{t_1} + y\overline{w_1}, xt_2 + yw_2),(x\overline{s_1} + y\overline{v_1}, xs_2 + yv_2))^T, \quad \text{and} \\
\gamma_{22} &= w'_{11} I \quad \text{(say)}.
\end{aligned}$$

Then for $v'_1 = (v'_{11}, x\overline{s_1} + y\overline{v_1})$, $v'_2 = (xs_2 + yv_2)$, $w'_1 = (w'_{11}, x\overline{t_1} + y\overline{w_1})$, $w'_2 = (xt_2 + yw_2) \in M_{1\times n}(R)$, we have

$$\begin{aligned}
S_n^2(S,T)S_n^2(V,W)S_n^2(S,T) &= S_{2n-1}((v'_1,v'_2),(w'_1,w'_2)) \\
&= S_n^2(V',W'),
\end{aligned}$$

where, $V' = \begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix}$, $W' = \begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} \in M_{2\times n}(R)$.

Similarly one can prove the other statement. For the inner product relation, we have by Fundamental Property,

$$S_n^2(V',W') = S_n^2(S,T)S_n^2(V,W)S_n^2(S,T),$$

$$S_n^2(W',V')^T = S_n^2(T,S)^T S_n^2(W,V)^T S_n^2(T,S)^T.$$

By Lemma 3.3,
$(\langle v'_1,w'_1\rangle + \langle v'_2,w'_2\rangle)I_{2^{2n-1}}$

$$\begin{aligned}
&= S_n^2(V',W')S_n^2(W',V')^T \\
&= S_n^2(S,T)S_n^2(V,W)S_n^2(S,T)S_n^2(T,S)^T S_n^2(W,V)^T S_n^2(T,S)^T \\
&= (\langle s_1,t_1\rangle + \langle s_2,t_2\rangle)^2(\langle v_1,w_1\rangle + \langle v_2,w_2\rangle)I_{2^{2n-1}}
\end{aligned}$$

Comparing we get,

$$\langle v_1', w_1' \rangle + \langle v_2', w_2' \rangle = (\langle s_1, t_1 \rangle + \langle s_2, t_2 \rangle)^2 (\langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle).$$

or,

$$\langle (v_1', v_2'), (w_1', w_2') \rangle = \langle (s_1, s_2), (t_1, t_2) \rangle^2 \langle (v_1, v_2), (w_1, w_2) \rangle.$$

□

Corollary 4.2 *Let* $V, W, S, T \in M_{2\times n}(R)$ *be given by* $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$, $T = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$. *Then we have*

$$S_n^2(T, S)^T S_n^2(V, W) S_n^2(T, S)^T = S_n^2(V', W')$$

for some $V' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}$, $W' = \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} \in M_{2\times n}(R)$ *with*

$$\langle (v_1', v_2'), (w_1', w_2') \rangle = \langle (s_1, s_2), (t_1, t_2) \rangle^2 \langle (v_1, v_2), (w_1, w_2) \rangle.$$

Proof Let $T = \begin{pmatrix} t_{11}, t_{12}, \ldots, t_{1n} \\ t_{21}, t_{22}, \ldots, t_{2n} \end{pmatrix}$ and $S = \begin{pmatrix} s_{11}, s_{12}, \ldots, s_{1n} \\ s_{21}, s_{22}, \ldots, s_{2n} \end{pmatrix}$. Then by Lemma 3.4, $S_n^2(T, S)^T = S_n^2(T', S')$, where $T' = \begin{pmatrix} t_{11}, -s_{12}, \ldots, -s_{1n} \\ -s_{21}, -s_{22}, \ldots, -s_{2n} \end{pmatrix} = \begin{pmatrix} t_1' \\ t_2' \end{pmatrix}$ and $S' = \begin{pmatrix} s_{11}, -t_{12}, \ldots, -t_{1n} \\ -t_{21}, -t_{22}, \ldots, -t_{2n} \end{pmatrix} = \begin{pmatrix} s_1' \\ s_2' \end{pmatrix}$. Note that

$$\langle (t_1', t_2'), (s_1', s_2') \rangle = \langle (t_1, t_2), (s_1, s_2) \rangle = \langle (s_1, s_2), (t_1, t_2) \rangle.$$

Now the result follows from Theorem 4.1. □

5 Future applications

The Fundamental Property, played a vital role in connecting special Suslin matrices $S_r(v, w)$ with orthogonal maps for unimodular rows. In this paper, we proved the Fundamental Property for $2 \times n$ row Suslin matrices, thereby laying the foundation for further exploration. We anticipate extending our research by introducing an anti-involution and a reflection map, which are expected to establish a link between the $2 \times n$ row Suslin matrix and an orthogonal transformation. Once this connection is established, the group of $2 \times n$ row Suslin matrices can be associated with special orthogonal groups which are well known object.

Another application of Fundamental Property is to establish a connection between $2 \times n$ row Suslin matrices and Clifford algebras of certain hyperbolic form. This connection is further to be explored.

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Declarations

Conflict of Interest The authors have no conflict of interest to declare that are relevant to this article.

References

1. Suslin, A.A.; *On Stably Free Modules (Russian)*, Mat. Sb. (N.S.) 102(144) (1977), no. 4, 537-550.
2. Jose, Selby; Rao, Ravi A.; *A local global principle for the elementary unimodular vector group*. Commutative Algebra and Algebraic Geometry (Banglore, India, 2003), Contemp. Math., vol. 390, Amer. Math. Soc., Providence, RI, 2005, pp. 119-125.
3. Jose, Selby; Rao, Ravi A.; *A Structure Theorem for the Elementary Unimodular Vector group*. Trans. Amer. Math. Soc. **358** (2006), no. 7, 3097–3112.
4. Jose, Selby; Rao, Ravi A.; *A Fundamental Property of Suslin matrices*, Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology (2010), 5, 407-436.
5. Chintala, Vineeth.; *On Suslin matrices and their connection to spin groups*. Doc. Math. **20** (2015), 531-550