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ORIGINAL RESEARCH

Gallai–Ramsey numbers for rainbow trees and monochromatic double stars or books

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Abstract Given two non-empty graphs G, H and a positive integer k, the Gallai-Ramsey number $\mathrm{gr}_k(G : H)$ is defined as the minimum positive integer N such that for all $n \geq N$, every k-edge-coloring of K_n contains either a rainbow subgraph G or a monochromatic subgraph H. In this paper, we get some exact values or bounds of Gallai–Ramsey numbers for rainbow $K_{1,3}$, P_5 or P_4^+ and monochromatic double stars or books.

Keywords Ramsey theory · Gallai–Ramsey number · Double star · Book

Mathematics Subject Classification 05D10 · 05C35 · 05C55

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G, respectively. A *k-edge-coloring* of G is a function $c : E(G) \to \{1, 2, \dots, k\}$, where $\{1, 2, \dots, k\}$ is a set of colors. When the integer k is small, we usually use red, blue, green and other specific colors to represent these colors. A k-edge-coloring of a graph is *exact* if all colors are used at least once. In this paper, we consider only exact edge-colorings of graphs. An edge-colored graph is called *rainbow* if all its edges have distinct colors and *monochromatic* if all its edges have the same color.

For an edge-colored graph G, we use $G[U]$ to denote the subgraph induced by a subset U of $V(G)$. Let G^i denote the subgraph of G induced by all i-color edges in G. The *degree*, $\deg_G(v)$ or $\deg(v)$ for short, of a vertex v of G is the number of edges incident to it. The *minimum degree* of G, $\delta(G)$, is the smallest of the degrees of vertices in G and the *maximum degree*, $\Delta(G)$, of G is the largest of the degrees of the vertices in G. For a color i, let $\Delta^i(G)$ denote the the *maximum i-color degree* of G, that is, the maximum degree of the subgraph G^i of G. The *complement* of a graph G, denoted by $\overline{G}$, is a graph with vertex set $V(G)$, and any two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G

Let G and H be two graphs, the *join* of G and H, denoted by $G + H$, is to insert all the edges between $V(G)$ and $V(H)$. In other words, $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{vu | \text{for all} v \in V(G) \text{and} u \in V(H)\}$. The *double star* $S_{m,n}$ for integers $n \geq m \geq 1$ is the graph obtained from $K_{1,n}$ and $K_{1,m}$ by adding an edge between their centers, this added edge is called the center edge of $S_{m,n}$. The center of $K_{1,n}$ is called *n-center* in $S_{m,n}$ and the center of $K_{1,m}$ is called *m-center* in $S_{m,n}$. The *book* B_n is the graph that n triangles share only one edge, that is $B_n = \overline{K_n} + K_2$. For more notation and terminology not defined here, we refer to [2].

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1.1 Ramsey number

Since the original paper of Ramsey [15] was published in 1930, Ramsey theory has become a hot topic in discrete mathematics. Given k graphs $G_1, G_2, \ldots, G_k$, the k-color Ramsey number $R(G_1, G_2, \ldots, G_k)$ is defined as the minimum positive integer n such that every k-edge-coloring of K_n contains a monochromatic subgraph G_i with color i, where $1 \le i \le n$. If $G_1 = G_2 = \cdots = G_k = G$, then we simply write the Ramsey number as $R_k(G)$. If $k = 2$ and $G_1 = G_2 = G$, then we simply write the Ramsey number as $R(G)$. At present, the exact value of Ramsey number of only a few graphs has been determined.

In 1972, Harary in [9] gave the exact value of Ramsey numbers involving stars.

Theorem 1 [9] *$R(K_{1,n}, K_{1,m}) = n + m - \varepsilon$, where $\varepsilon = 1$ if both n and m are even and $\varepsilon = 0$ otherwise.*

In 1978, Rousseau and Sheehan in [10] gave the result of star-book Ramsey numbers.

Theorem 2 [10] *Let an integer $k = \lfloor \frac{m-1}{n} \rfloor$. If $kn \le m - 1 \le \left(k + \frac{2}{(k+1)(k+2)}\right) n$, then*

$$R(K_{1,n}, B_m) = \max\left((k+2)n, \left\lfloor \frac{(k+3)\left((k+2)(m-1) + 2(k+1)n\right)}{(k+1)(k+4)} \right\rfloor\right) + 1.$$

In this paper, we will use the following corollary. For more results on Ramsey numbers, we refer to [14].

Corollary 1 *For an integer $n \ge 3$, we have*

$$R(K_{1,n}, B_n) = 3n - 1 \quad \text{and} \quad R(K_{1,n-1}, B_n) = 3n - 2.$$

1.2 Gallai–Ramsey number

An edge-colored complete graph without rainbow triangles has very special structure. In 1967, Gallai in [6] first examined this structure under the guise of transitive orientations of graphs and it can also be traced back to the paper [3]. For this reason, an edge-coloring of a complete graph containing no rainbow triangles is called a *Gallai coloring*. Gallai's result was restated in [8] by the terminology of graphs. For the following statement, a nontrivial partition is a partition with at least two parts.

Theorem 3 [3,6,8] *In any edge-coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition of the vertices (called a Gallai partition) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

In 2010, Faudree, Gould, Jacobson and Magnant in [4] defined *Gallai–Ramsey number* $\mathrm{gr}_k(G : H)$. For more recent results about Gallai–Ramsey numbers, we refer to [13].

Definition 1 [4] Given two non-empty graphs G, H and a positive integer k, define the Gallai–Ramsey number $\mathrm{gr}_k(G : H)$ to be the minimum integer N such that for all $n \ge N$, every k-edge-coloring of K_n contains either a rainbow subgraph G or a monochromatic subgraph H.

Noticing that Gallai–Ramsey numbers consider only edge-colorings of complete graphs. So, according to the definitions of Ramsey number and Gallai–Ramsey number, we have

$$\mathrm{gr}_k(G : H) \le R_k(H) < \infty.$$

Additionally, if $2 \le k \le |E(G)| - 1$, then it is clear that there is no rainbow subgraph G in any k-edge-colored complete graph. Therefore, in this case, we have

$$\mathrm{gr}_k(G : H) = R_k(H).$$

In the study of k-edge-colorings, in addition to "exact k-edge-coloring", another definition is the so-called "at most k-edge-coloring", which means that the actual number of colors used does not exceed k, and it is allowed to be less than k. In [11], Li, Besse, Magnant, Wang and Watts gave a conjecture about the Gallai-Ramsey number for rainbow P_5 under the at most k-edge-coloring rule.

Conjecture 1 For any graph H with no isolated vertices, we have

$$\mathrm{gr}_k(P_5 : H) = R_3(H).$$

For more recent results about Gallai–Ramsey numbers, we refer to the monograph [13] and the survey paper [5].

1.3 Structural theorems under rainbow-tree-free colorings

Thomason and Wagner in [17] obtained the following results.

Theorem 4 [17] *For an integer $n \geq 4$, let K_n be edge-colored complete graph so that it contains no rainbow P_4. Then one of the following statements holds.*

(i) *At most two colors are used;*
(ii) *$n = 4$ and three colors are used, each color forming a perfect matching.*

Theorem 5 [17] *For positive integers k and n, if K_n is a k-edge-colored complete graph without rainbow subgraph P_5, then after renumbering the colors, one of the following statements holds.*

(i) *$k \leq 3$ or $n \leq 4$;*
(ii) *There exists a partition $(V_1, V_2, \dots, V_k)$ of $V(K_n)$ such that for each i, all the edges connecting two vertices in V_i are colored by either 1 or i (so V_1 forms a monochromatic clique), and all the edges between V_i and V_j with $i \neq j$ are colored by 1;*
(iii) *$K_n - v$ is monochromatic for some vertex v;*
(iv) *There are three vertices a, b and c such that the edges ab, bc and ac have color 2, 3 and 4, respectively, some edges incident with a have color 3, and all the other edges have color 1;*
(v) *There are four vertices a, b, c and d such that the edges ab, ac, ad, bc and bd have color 2, 3, 4, 4 and 3, respectively, the edge cd has color 1 or 2, and all the other edges have color 1;*
(vi) *$n = 5$, $V(K_n) = \{a, b, c, d, e\}$, the edges ad, ae and bc have color 1, the edges bd, be and ac have color 2, the edges cd, ce and ab have color 3, and the edge de has color 4.*

In order to facilitate the description of the following structural theorems, we give some edge-colored complete graphs with special structures.

Definition 2 [1] For an integer $n \geq 4$, let $G_1(n)$ be a 3-edge-colored K_n that satisfies the following conditions: The vertices of K_n are partitioned into three pairwise disjoint sets V_1, V_2 and V_3 such that for $1 \leq i \leq 3$ (with indices modulo 3), all the edges between V_i and V_{i+1} have color i, and all the edges within V_{i+1} have color i or $i+1$.

In the $G_1(n)$ coloring structure, one of V_1, V_2 and V_3 is allowed to be empty, but at least two of them are non-empty (Otherwise at most only two colors can appear).

Definition 3 [1] For an integer $n \geq 4$, let $G_2(n)$ be a 4-edge-colored K_n in which there is exactly one edge, say xy, having color 2. Every edge from x to all the other vertices except y has color 3, and every edge from y to all the other vertices except x have color 4. All the edges not incident to vertices x, y have color 1.

In the $G_2(n)$ coloring structure, K_n contains no rainbow subgraph P_4^+ but contains a rainbow subgraph $K_{1,3}$ and a rainbow subgraph P_5 (if $n \geq 5$).

Definition 4 [1] For an integer $n \geq 4$, let $G_3(n)$ be a 4-edge-colored K_n in which there exists a rainbow subgraph K_3 having colors 1, 2 and 3, say $V(K_3) = \{a, b, c\}$, the edge ab has color 1, the edge bc has color 2 and the edge ac has color 3. Let every edge incident with at most one vertex in the rainbow subgraph K_3 have color 4.

In the $G_3(n)$ coloring structure, K_n contains no rainbow subgraphs P_4^+ and P_5, but contains a rainbow subgraph $K_{1,3}$. If the edge-coloring structure of a complete graph is the same as $G_i(n)$, $i \in \{1, 2, 3\}$, then we write it as $K_n = G_i(n)$.

The *local k-coloring* of a graph G refers to the edge coloring of G, satisfying that the colors of the edges incident to each vertex of G are at most k. In [7], Gyárfás, Lehel, Schelp and Tuza gave the coloring structure of a local 2-colored complete graph K_n with k colors. Let A_{ij} be a vertex subset of complete graph K_n, and each edge of the induced subgraph by A_{ij} has either color i or color j. Then there are only two types of coloring structures of the local 2-colored complete graph K_n with k colors. One structure is $k = 3$ and there exists a partition of $V(K_n)$, denoted as (A_{12}, A_{13}, A_{23}). The other structure is $k \geq 3$ and there exists a dominant color, which may be assumed to be color 1, and the vertex set of K_n has a partition, denoted as $(A_{12}, A_{13}, \dots, A_{1k})$. In [1], Bass, Magnant, Ozeki and Pyron studied the edge-colored complete graphs without rainbow $K_{1,3}$ from structural perspectives.

Theorem 6 [1,7] *For positive integers k and n, if K_n is a k-edge-colored complete graph without rainbow subgraph $K_{1,3}$, then after renumbering the colors, one of the following statements holds.*

(*i*) $k \leq 2$ *or* $n \leq 3$*;*
(*ii*) $k = 3$ *and* $K_n = G_1(n)$*;*
(*iii*) $k \geq 4$ *and Item* (*ii*) *in Theorem 5 holds.*

Next we give two types of edge-colored complete graphs without rainbow P_4^+, where P_4^+ is the graph consisting of a P_4 with one extra pendent edge incident with an inner vertex of P_4.

Theorem 7 [1,16] *For positive integers k and n, if K_n is a k-edge-colored complete graph without rainbow subgraph P_4^+, then after renumbering the colors, one of the following holds.*

(*i*) $k \leq 3$ *or* $n \leq 4$*;*
(*ii*) $k = 4$ *and* $K_n \in \{G_2(n), G_3(n)\}$*;*
(*iii*) $k \geq 4$ *and* K_n *contains no rainbow* $K_{1,3}$*. In particular, Item* (*ii*) *in Theorem 5 holds.*

Although the Gallai–Ramsey number was first studied for rainbow triangles, research on other rainbow subgraphs has gradually been carried out in recent years. There are many results on Gallai-Ramsey numbers involving rainbow $K_{1,3}$, P_5 or P_4^+. In [12], Li, Wang and Liu got some exact values and bounds for $\mathrm{gr}_k(P_5 : K_t)$, and got the structural theorems for complete bipartite graphs without rainbow subgraphs P_4 and P_5. In [18], Wei, He, Mao and Zhou obtained some exact values and bounds of Gallai–Ramsey for rainbow P_5 and monochromatic fans or wheels. In [19], Zhou, Li, Mao and Wei studied Gallai–Ramsey numbers for rainbow $K_{1,3}$, P_5 or P_4^+ and monochromatic small path union stars. In [20], Zou, Wang, Lai and Mao derived results for $\mathrm{gr}_k(P_5 : H)$ $(k \geq 3)$, where H is a general or special graph.

This paper is organized as follows: In next section, we will give some propositions and lemmas. In Section 3, we determine some exact values or bounds of Gallai–Ramsey numbers for rainbow $K_{1,3}$, P_5 or P_4^+ and monochromatic $S_{m,n}$ or B_n. In the last section, some related open problems are proposed.

2 Preliminaries

Li, Wang and Liu in [12] determined the sharp bound of k such that any k-edge-colored K_n always has a rainbow subgraph P_5. Bass, Magnant, Ozeki and Pyron in [1] obtained the sharp bound of k such that any k-edge-colored K_n always has a rainbow subgraph $K_{1,3}$ by studying anti-Ramsey numbers.

Proposition 8 [12] *For integers $n \geq 5$ and k with $n + 1 \leq k \leq \binom{n}{2}$, there is always a rainbow subgraph P_5 under any k-edge-colored K_n.*

Proposition 9 [1] *For integers $n \geq 4$ and k with $\lceil \frac{n+3}{2} \rceil \leq k \leq \binom{n}{2}$, there is always a rainbow subgraph $K_{1,3}$ under any k-edge-colored K_n.*

The following two lemmas are very important in this paper. Using these lemmas, we can avoid a lot of repetitive work in proving the upper bound of Gallai-Ramsey number involving rainbow P_5 or P_4^+.

Lemma 1 *Let H be a graph such that H is a subgraph of $\overline{K_2} + K_{|V(H)|-2}$ and H is also a subgraph of $\overline{K_4} + K_{|V(H)|-3}$. For every 4-edge-colored K_N $(N \geq |V(H)| + 1 > 5)$, if Theorem 5 (iii), (iv) and (v) hold, then there exists a monochromatic subgraph H.*

Proof We can easily observe that there is a monochromatic subgraph H when Theorem 5 (*iii*) holds. Assume that Theorem 5 (*iv*) holds. Noticing that $\{a, b, c, v_1, \ldots, v_{|V(H)|-2}\} \subseteq V(K_N)$ and the coloring of all edges incident with $\{a, b, c\}$ as Theorem 5 (*iv*), hence there is a monochromatic subgraph H with color 1. Next, we assume that Theorem 5 (*v*) holds. Noticing that $\{a, b, c, d, v_1, \ldots, v_{|V(H)|-3}\} \subseteq V(K_N)$ and the coloring of all edges incident with $\{a, b, c, d\}$ as Theorem 5 (*v*), hence there is a monochromatic subgraph H with color 1. □

Lemma 2 *Let H be a graph such that H is a subgraph of $\overline{K_3} + K_{|V(H)|-1}$. For every 4-edge-colored K_N $(N \geq |V(H)| + 2)$, if Theorem 7 (ii) holds, then there exists a monochromatic subgraph H.*

Proof Assume that Theorem 7 (*ii*) holds. If $K_N = G_2(N)$, then $K_N - x - y$ is monochromatic with color 1, and hence there is a monochromatic subgraph H. Next, we assume that $K_N = G_3(N)$. Noticing that $\{a, b, c, v_1, \ldots, v_{|V(H)|-1}\} \subseteq V(K_N)$ and the coloring of all edges incident with $\{a, b, c\}$ as the definition of $G_3(N)$, hence there is a monochromatic H with color 4. □

In 2022, Zhou, Li, Mao and Wei in [19] gave some general results between $\mathrm{gr}_k(K_{1,3}:H)$, $\mathrm{gr}_k(P_5:H)$ and $\mathrm{gr}_k(P_4^+:H)$ $(k\geq 4)$. We correct a small flaw in Lemmas 4 and 5 given in [19], which is that the lack of condition $\mathrm{gr}_k(K_{1,3}:H)\geq 5$ can lead to errors. Noticing that if $k\geq 5$ and $\mathrm{gr}_k(K_{1,3}:H)=4$, then $\mathrm{gr}_k(P_5:H)>4$ and $\mathrm{gr}_k(P_4^+:H)>4$. This is because for any k-edge-colored K_4 with $5\leq k\leq 6$, there is no rainbow subgraph P_5 or P_4^+, and also no monochromatic subgraph H (except for the trivial case where $H=K_2$ or $H=2K_2$).

Lemma 3 [19] *For an integer $k\geq 4$, we have*

$$\mathrm{gr}_k(P_5:H)\geq \mathrm{gr}_k(K_{1,3}:H) \text{ and } \mathrm{gr}_k(P_4^+:H)\geq \mathrm{gr}_k(K_{1,3}:H).$$

Lemma 4 [19] *For integers $k\geq 5$ and $\mathrm{gr}_k(K_{1,3}:H)\geq 5$, we have*

$$\mathrm{gr}_k(P_5:H)=\begin{cases}\max\left\{|V(H)|+1,\mathrm{gr}_k(K_{1,3}:H)\right\}, & 5\leq k\leq |V(H)|;\\ \mathrm{gr}_k(K_{1,3}:H), & k\geq |V(H)|+1\geq 5.\end{cases}$$

Lemma 5 [19] *For integers $k\geq 5$ and $\mathrm{gr}_k(K_{1,3}:H)\geq 5$, we have*

$$\mathrm{gr}_k(P_4^+:H)=\mathrm{gr}_k(K_{1,3}:H).$$

In 2023, Zou, Wang, Lai and Mao in [20] provided results on the Gallai–Ramsey number for rainbow P_5.

Theorem 10 [20] *For a graph H and an integer $k\geq 5$, we have*

$$\mathrm{gr}_k(P_5:H)=\begin{cases}\max\left\{\lceil\frac{1+\sqrt{1+8k}}{2}\rceil,5\right\}, & k\geq |V(H)|+1;\\ |V(H)|+1, & k=|V(H)| \text{ and } H \text{ is not a complete graph};\\ (|V(H)|-1)^2+1, & k=|V(H)| \text{ and } H \text{ is a complete graph}.\end{cases}$$

Based on the idea of Zou, Wang, Lai and Mao [20], the number of colors does not exceed the total number of edges in an edge-colored graph. So by solving the equation

$$k\leq\binom{n}{2}=|E(K_n)|,$$

we obtain $n\geq\frac{1+\sqrt{1+8k}}{2}$. Therefore, the following lemma is directly given.

Lemma 6 *For integer $k\geq 3$ and any graph H, we have*

$$\mathrm{gr}_k(K_{1,3}:H)\geq\left\lceil\frac{1+\sqrt{1+8k}}{2}\right\rceil.$$

When the number of colors $k\geq 4$, we know from Theorem 6 (iii) (i.e., Theorem 5 (ii)) that if a k-edge-colored complete graph does not contain a rainbow subgraph $K_{1,3}$, then there is only one coloring structure. Conversely, if the coloring structure of a k-edge-colored complete graph satisfies what is described in Theorem 6 (iii), then the complete graph does not contain a rainbow subgraph $K_{1,3}$. In order to describe the edge-coloring structure of lower bounds in the following section more concisely, we construct a family of k-edge-colored complete graphs based on the coloring structure given in Theorem 6 (iii). Therefore, every k-edge-colored complete graph described in Definition 5 does not contain a rainbow subgraph $K_{1,3}$.

Definition 5 Let integer $k\geq 4$ and $[K_{t_1},K_{t_2},\ldots,K_{t_{k-1}}]$ be a k-edge-colored complete graph obtained from $k-1$ vertex-disjoint complete graphs $K_{t_1},K_{t_2},\ldots,K_{t_{k-1}}$ such that all the edges of K_{t_i} are colored by $i+1$ for each $1\leq i\leq k-1$ and all the edges between K_{t_i} and K_{t_j} are colored by 1 for any two integers $1\leq i<j\leq k-1$.

3 Main results

3.1 Rainbow $K_{1,3}$

In this subsection, we give the Gallai-Ramsey numbers for monochromatic double stars or books involving rainbow $K_{1,3}$.

Theorem 11 *Let integer $k \geq 4$. If H is a subgraph of balanced complete $(k-1)$-partite graph $K_{2,2,\ldots,2}$, then*

$$\mathrm{gr}_k(K_{1,3} : H) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil.$$

Proof The lower bound is obtained by Lemma 6, and we only need to prove the upper bound. Consider any k-edge-coloring of K_N $(N \geq N_k)$. Noticing that $N_k < 2k-2$ for $k \geq 4$. If $N_k \leq N \leq 2k-3$, then it follows from Proposition 9 that there is always a rainbow subgraph $K_{1,3}$, the result thus follows. Next we assume that $N \geq 2k-2$. Suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph H. It follows from the fact $k \geq 4$ that Theorem 6 (i) and (ii) do not hold. If Theorem 6 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2, 3, \ldots, k\}$. Since H is a subgraph of balanced complete $(k-1)$-partite graph $K_{2,2,\ldots,2}$, it follows that there is a monochromatic subgraph H, a contradiction. The result thus follows. □

For convenience of description, a vertex x *contributes* to n-center (or m-center) if x is a pendent vertex and adjacent to n-center (or m-center) in subgraph $S_{m,n}$. For a vertex subset V' of $V(G)$, the *remaining* vertices in V' are those that are neither center vertices of $S_{m,n}$ nor contribute to a center vertex of $S_{m,n}$. For $k=3$, we have the following results.

Theorem 12 *For integers $n \geq m \geq 1$, we have*

$$\mathrm{gr}_3(K_{1,3} : S_{m,n}) = 2n + m + 2.$$

Proof Let G_1 and G_2 be monochromatic copies of K_n with colors 3 and 2, respectively. Let G_3 be a monochromatic copy of K_{m+1} with color 3. We construct K_{2n+m+1} by making G_1, G_2, G_3 and inserting all edges between these copies such that the edges from G_1 to G_2 are colored by 1, and the edges from G_2 to G_3 are colored by 2, and the edges from G_1 to G_3 are colored by 3. One can easily check that there is neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $S_{m,n}$ under such a coloring of K_{2n+m+1}, and so $\mathrm{gr}_3(K_{1,3} : S_{m,n}) \geq 2n+m+2$.

Consider any 3-edge-coloring of K_N $(N \geq 2n+m+2)$ and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $S_{m,n}$. By Theorem 6 (ii), there is a partition (V_1, V_2, V_3) of $V(K_N)$ such that $K_N = G_1(N)$ when $k=3$. Without loss of generality, we assume that $|V_1| \geq |V_2| \geq |V_3|$. If $|V_1| \geq n+1$ and $|V_2| \geq m+1$, then there is a monochromatic subgraph $S_{m,n}$ with n-center in V_2 and m-center in V_1, a contradiction.

Suppose that $|V_1| \geq n+1$ and $|V_2| \leq m$. If $\Delta^1(G[V_1]) \geq m - |V_2| + 1$, then we choose an m-center in V_1 and an n-center in V_2, and then we choose $|V_2|-1$ vertices from V_2 to contribute to the m-center and $m-|V_2|+1$ vertices from V_1 to contribute to the m-center. So the number of remaining vertices in V_1 is at least

$$\begin{aligned} |V_1| - (m - |V_2| + 1) - 1 &= 2n + m + 2 - (|V_2| + |V_3|) - (m - |V_2| + 1) - 1 \\ &= 2n - |V_3| \geq n. \end{aligned}$$

Therefore, at least n vertices in V_1 can be contributed to n-center. Hence, there is a monochromatic subgraph $S_{m,n}$ with color 1, a contradiction. So $\Delta^1(G[V_1]) \leq m - |V_2|$ and similarly $\Delta^3(G[V_1]) \leq m - |V_3|$, but it contradicts with the fact $\Delta^1(G[V_1]) + \Delta^3(G[V_1]) \geq \Delta(G[V_1]) = |V_1| - 1 \geq 2n + m + 1 - (|V_2| + |V_3|)$.

Suppose that $|V_1| \leq n$. If $\Delta^1(G[V_2]) \geq n - |V_1| + 1$, then we choose an m-center in V_1 and an n-center in V_2, and then we choose $|V_1|-1$ vertices from V_1 to contribute to the n-center and $n-|V_1|+1$ vertices from V_2 to contribute to the n-center. So the number of remaining vertices in V_2 is at least

$$\begin{aligned} |V_2| - (n - |V_1| + 1) - 1 &= |V_1| + |V_2| - n - 2 \\ &\geq \frac{2}{3}(2n + m + 2) - n - 2 \\ &= \frac{1}{3}n + \frac{2}{3}m - \frac{2}{3} \geq m - \frac{2}{3}. \end{aligned}$$

Noticing that $|V_2| - (n - |V_1| + 1) - 1$ is an integer, thus the number of remaining vertices in V_2 is at least m. Next we choose m vertices from V_2 to contribute to the m-center, there is a monochromatic subgraph $S_{m,n}$ with color 1, a contradiction. Similarly, if $\Delta^3(G[V_3]) \geq n - |V_1| + 1$, then we can also find a monochromatic subgraph $S_{m,n}$ with color 3. Therefore, $\Delta^1(G[V_2]) \leq n - |V_1|$ and $\Delta^3(G[V_3]) \leq n - |V_1|$, which implies $\Delta^2(G[V_2]) \geq |V_2| - 1 - (n - |V_1|) = |V_1| + |V_2| - n - 1$ and $\Delta^2(G[V_3]) \geq |V_3| - 1 - (n - |V_1|) = |V_1| + |V_3| - n - 1$. Recall that the edges from $G[V_2]$ to $G[V_3]$ are colored by 2. Noticing that $|V_1| + |V_3| - n - 1 - (m + n - |V_2| + 1) = |V_1| + |V_2| + |V_3| - (2n + m + 2) \geq 0$. Next we choose an m-center in V_2 and an n-center in V_3, and then we choose $|V_2| - 1$ vertices from V_2 to contribute to the n-center and $n - |V_2| + 1$ vertices from V_3 to contribute to the n-center and m vertices from V_3 to contribute to the m-center. Therefore, there is a monochromatic subgraph $S_{m,n}$ with color 2, a contradiction. The result thus follows. □

For $k \geq 4$, we completely solve and give the exact values of the Gallai–Ramsey numbers for rainbow $K_{1,3}$ and monochromatic double stars.

Theorem 13 *For integers $k \geq 4$ and $n \geq m \geq 1$, we have*

$$\mathrm{gr}_k(K_{1,3} : S_{m,n}) = \begin{cases} \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & n + m \leq 2k - 4; \\ n + m + 2, & n + m \geq 2k - 3 \text{ and } n \leq (m+1)(k-2) - 1; \\ \binom{k-1}{k-2}(n-a) + a + 1, & n \geq (m+1)(k-2) \text{ and } n \equiv a \pmod{k-2} \\ & \text{where } 0 \leq a \leq k-3. \end{cases}$$

Proof Assume that $n + m \leq 2k - 4$. It follows from Theorem 11 that $\mathrm{gr}_k(K_{1,3} : S_{m,n}) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$. Next, we distinguish the following two cases to prove this theorem.

Case 1 $n + m \geq 2k - 3$ *and* $n \leq (m+1)(k-2) - 1$.

Since $n + m \geq 2k - 3$, it follows from $\mathrm{gr}_k(K_{1,3} : S_{m,n}) \geq |V(S_{m,n})| = n + m + 2$ that the lower bound of the theorem holds. Consider any k-edge-coloring of K_N $(N \geq n + m + 2)$ and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $S_{m,n}$. It follows from the fact $k \geq 4$ that Theorem 6 (i) and (ii) do not hold. If Theorem 6 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2, 3, \ldots, k\}$ and $\sum_{i=1}^{k} |V_i| \geq n + m + 2$. Without loss of generality, we assume that $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 2$. If $|V_{k-1}| \geq |V_k| \geq n + 1$, then there is a monochromatic subgraph $S_{m,n}$ with the center edge between V_{k-1} and V_k, a contradiction. If $|V_{k-1}| \geq n + 1$ and $|V_k| \leq n$, then $|V_{k-2}| \geq |V_{k-1}| \geq n + 1$, and hence there is a monochromatic subgraph $S_{m,n}$ with the center edge between V_{k-2} and V_{k-1}, a contradiction.

Next we assume that $|V_{k-1}| \leq n$ and $|V_k| \leq n$. If $|V_{k-1}| \leq n$ and $|V_k| \leq m$, then it follows from $N - (|V_{k-1}| + |V_k|) \geq (n - |V_{k-1}| + 1) + (m - |V_k| + 1)$ that there is a monochromatic subgraph $S_{m,n}$ with the center edge between V_{k-1} and V_k, a contradiction. If $|V_{k-1}| \leq n$ and $m + 1 \leq |V_k| \leq n$, then $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq m + 1$. Since $n \leq (m+1)(k-2) - 1$, it follows from $\bigcup_{i=2}^{k-1} |V_i| \leq (m+1)(k-2)$ that there is a monochromatic subgraph $S_{m,n}$ with the center edge between V_{k-1} and V_k, a contradiction. The result thus follows.

Case 2 $n \geq (m+1)(k-2)$ *and* $n \equiv a \pmod{k-2}$ *where* $0 \leq a \leq k-3$.

It follows from $n \equiv a \pmod{k-2}$ and $n \geq (m+1)(k-2)$ that $\frac{n-a}{k-2}$ is an integer and $\frac{n-a}{k-2} \geq m + 1$. Let $t_1 = \frac{n-a}{k-2} + a$ and $t_i = \frac{n-a}{k-2}$ for each $2 \leq i \leq k-1$. Then $K_{(\frac{k-1}{k-2})(n-a)+a} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph. One can easily check that there is neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $S_{m,n}$, and so $\mathrm{gr}_k(K_{1,3} : S_{m,n}) \geq (\frac{k-1}{k-2})(n-a) + a + 1$.

Consider any k-edge-coloring of K_N $(N \geq (\frac{k-1}{k-2})(n-a) + a + 1)$ and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph $S_{m,n}$. It follows from the fact $k \geq 4$ that Theorem 6 (i) and (ii) do not hold. If Theorem 6 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2, 3, \ldots, k\}$ and $\sum_{i=1}^{k} |V_i| \geq (\frac{k-1}{k-2})(n-a) + a + 1$. Without loss of generality, we assume that $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 2$.

Noticing that $m + 1 \leq \left\lfloor \frac{n}{k-2} \right\rfloor = \frac{n-a}{k-2} \leq \frac{n-a}{k-2} + a$. If $2 \leq |V_k| \leq m$, then it follows from $N - (|V_{k-1}| + |V_k|) \geq (n - |V_{k-1}| + 1) + (m - |V_k| + 1)$ that there is a monochromatic subgraph $S_{m,n}$ with the center edge between V_{k-1} and V_k, a contradiction. If $m + 1 \leq |V_k| \leq \frac{n-a}{k-2}$, then $|V(K_N)| - |V_k| \geq n + 1$, and hence there is a monochromatic subgraph $S_{m,n}$ with the center edge between V_{k-1} and V_k, a contradiction. If $|V_k| \geq \frac{n-a}{k-2} + 1$, then $\bigcup_{i=2}^{k-1} |V_i| \geq n - a + k - 2 \geq n + 1$, and hence there is a monochromatic subgraph $S_{m,n}$ with the center edge between V_{k-1} and V_k, a contradiction. The result thus follows. □

The following two lemmas will help us simplify the following proof details and avoid repetitive statements.

Lemma 7 *For an integer $n \geq 2$, we have*

$$\mathrm{gr}_3(K_{1,3} : B_n) \geq \begin{cases} 7, & n = 2; \\ 10, & n = 3; \\ 3n, & n \geq 4. \end{cases}$$

Proof First, we consider the case of $n = 2$. Let G_1 and G_2 be monochromatic edge-colored complete graphs of order 3 with colors 2 and 3, respectively. We construct K_6 from G_1 and G_2 by adding the edges such that the edges between G_1 and G_2 are colored by 1. One can easily check that there is neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph B_2 under such a coloring of K_6, and so $\mathrm{gr}_3(K_{1,3} : B_2) \geq 7$. Then, we consider the case of $n = 3$. Let G_1 be a copy of K_5 with colors 1 and 3 such that the induced subgraph of color 1 is a C_5 and the induced subgraph of color 3 is a C_5. Let G_2 and G_3 be monochromatic copies of K_2 with color 2. We construct K_9 by making G_1, G_2, G_3 and inserting all edges between these copies such that the edges from G_1 to G_2 are colored by 1, and the edges from G_2 to G_3 are colored by 2, and the edges from G_1 to G_3 are colored by 3. One can easily check that there is neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph B_3 under such a coloring of K_9, and so $\mathrm{gr}_3(K_{1,3} : B_3) \geq 10$.

Next, we consider the case of $n \geq 4$. Let G_1 be a copy of $K_{R(K_{1,n-1},B_n)-1}$ with colors 1 and 3 such that the induced subgraph of color 1 with maximum degree less than $n-1$ and the induced subgraph of color 3 without subgraph B_n. Let G_2 be a monochromatic copy of K_2 with color 2. Recall that $R(K_{1,n-1}, B_n) = 3n - 2$. We construct K_{3n-1} by making G_1, G_2 and inserting all edges between these copies such that the edges from G_1 to G_2 are colored by 1. One can easily check that there is neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph B_n under such a coloring of K_{3n-1}, and so $\mathrm{gr}_3(K_{1,3} : B_n) \geq 3n$. □

Lemma 8 *For every 3-edge-coloring of K_N ($N \geq 2n + 3$), if K_N contains no rainbow subgraph $K_{1,3}$, then there is a partition (V_1, V_2, V_3) of $V(K_N)$ such that $K_N = G_1(N)$. Furthermore, if $|V_1| \geq |V_2| \geq |V_3|$ and $|V_2| \geq n$ or $|V_2| = 1$, then there exists a monochromatic subgraph B_n.*

Proof Consider any 3-edge-coloring of K_N ($N \geq 2n + 3$). Since K_N contains no rainbow subgraph $K_{1,3}$, it follows from Theorem 6 (ii) that there is a partition (V_1, V_2, V_3) of $V(K_N)$ such that if $k = 3$, then $K_N = G_1(N)$.

Assume that $|V_2| \geq n$. Since $|V_1| \geq |V_2| \geq n$ and the edges between V_1 and V_2 are colored by 1, it follows from that the color of all edges in the induced subgraph of V_i is not color 1 for each $i \in \{1, 2\}$. Therefore, the induced subgraph of V_1 is a monochromatic complete graph with color 3 and the induced subgraph of V_2 is a monochromatic complete graph with color 2. If $|V_1| \geq n + 2$, then there is a monochromatic subgraph B_n with color 3. If $|V_1| = n + 1$, then $|V_3| = |V(K_N)| - (|V_1| + |V_2|) \geq 1$, there is a monochromatic subgraph B_n with color 3. If $|V_1| = |V_2| = n$, then $|V_3| = |V(K_N)| - (|V_1| + |V_2|) \geq 3$, there is a monochromatic subgraph B_n with color 3.

Assume that $|V_2| = 1$. Since each color is used at least once, it follows that $|V_2| = |V_3| = 1$, and then $|V_1| = |V(K_N)| - (|V_2| + |V_3|) \geq 2n + 1$. Noticing that $R(K_{1,n}) \leq 2n$, there is a monochromatic subgraph $K_{1,n}$ with color 1 or 3 in the induced subgraph of V_1, and so a monochromatic subgraph B_n in K_N. The result thus follows. □

According to the above two lemmas, we can get the following theorem directly. For $n \geq 4$, the exact value of $\mathrm{gr}_3(K_{1,3} : B_n)$ may be related to the classical Ramsey number.

Theorem 14 $\mathrm{gr}_3(K_{1,3} : B_2) = 7$ *and* $\mathrm{gr}_3(K_{1,3} : B_3) = 10$.

Proof From Lemmas 7 and 8, we can directly obtain $\mathrm{gr}_3(K_{1,3} : B_2) = 7$. Next, we consider the Gallai-Ramsey number $\mathrm{gr}_3(K_{1,3} : B_3)$. It follows from Lemma 7 that $\mathrm{gr}_3(K_{1,3} : B_3) \geq 10$. Consider any 3-edge-coloring of K_N ($N \geq 10$) and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph B_3. By Theorem 6 (ii), there is a partition (V_1, V_2, V_3) of $V(K_N)$ such that if $k = 3$, then $K_N = G_1(N)$. Without loss of generality, we assume that $|V_1| \geq |V_2| \geq |V_3|$. It follows from Lemma 8 that we only need to consider the case of $|V_2| = 2$. If $|V_2| = 2$ and $1 \leq |V_3| \leq 2$, then $|V_1| = |V(K_N)| - (|V_2| + |V_3|) \geq 6$. It follows from Theorem 1($R(K_{1,3}) = 6$) that there is a monochromatic subgraph B_3, a contradiction. If $|V_2| = 2$ and $|V_3| = 0$, then $|V_1| = |V(K_N)| - |V_2| \geq 8$. It follows from Corollary 1($R(K_{1,3}, B_3) = 8$) that there is a monochromatic subgraph B_3, a contradiction. □

For $k \geq 4$, we completely solve and give the exact values of the Gallai–Ramsey numbers for rainbow $K_{1,3}$ and monochromatic books.

Theorem 15 *For an integer $k \geq 4$, we have*

$$\mathrm{gr}_k(K_{1,3} : B_n) = \begin{cases} \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & 1 \leq n \leq 2k-6; \\ \binom{k-1}{k-3}(n-1-a)+a+1, & n \geq 2k-5 \text{ and } n-1 \equiv a \pmod{k-3} \\ & \text{where } 0 \leq a \leq k-4. \end{cases}$$

Proof Assume that $1 \leq n \leq 2k-6$, it follows from Theorem 11 that $\mathrm{gr}_k(K_{1,3} : B_n) = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$. Next, we consider the case of $n \geq 2k-5$ and $n-1 \equiv a \pmod{k-3}$ where $0 \leq a \leq k-4$. It follows from $n-1 \equiv a \pmod{k-3}$ that $\frac{n-1-a}{k-3}$ is an integer and $\frac{n-1-a}{k-3} \geq 2$. Let $t_1 = \frac{n-1-a}{k-3} + a$ and $t_i = \frac{n-1-a}{k-3}$ for each $2 \leq i \leq k-1$. Then $K_{(\frac{k-1}{k-3})(n-1-a)+a} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph. One can easily check that there is neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph B_n, and so $\mathrm{gr}_k(K_{1,3} : B_n) \geq (\frac{k-1}{k-3})(n-1-a)+a+1$.

Consider any k-edge-coloring of K_N ($N \geq (\frac{k-1}{k-3})(n-1-a)+a+1$) and suppose to the contrary that K_N contains neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph B_n. It follows from the fact $k \geq 4$ that Theorem 6 (i) and (ii) do not hold. If Theorem 6 (iii) holds, then $|V_i| \geq 2$ for each $i \in \{2, 3, \ldots, k\}$ and $\sum_{i=1}^{k} |V_i| \geq (\frac{k-1}{k-3})(n-1-a)+a+1$. Without loss of generality, we assume that $|V_2| \geq |V_3| \geq \ldots \geq |V_k| \geq 2$.

If $2 \leq |V_{k-1}| \leq \frac{n-1-a}{k-3}$, then $4 \leq |V_{k-1}| + |V_k| \leq \frac{2(n-1-a)}{k-3}$ and $|V(K_N)| - (|V_{k-1}| + |V_k|) \geq (\frac{k-3}{k-3})(n-3-a)+a-1 = n$, and hence there is a monochromatic subgraph B_n, a contradiction. If $|V_{k-1}| \geq \frac{n-1-a}{k-3} + 1$, then $\bigcup_{i=2}^{k-2} |V_i| \geq n-1-a+k-3 \geq n-1-a+a+1 = n$, and hence there is a monochromatic subgraph B_n, a contradiction. The result thus follows. □

3.2 Rainbow P_5

In this subsection, we give the Gallai-Ramsey numbers for monochromatic double stars or books involving rainbow P_5. For $k = 4$, according to Lemma 3 and Theorem 13, we have the following corollary.

Corollary 2 *For integers $n \geq m \geq 1$, we have*

$$\mathrm{gr}_4(P_5 : S_{m,n}) \geq \mathrm{gr}_4(K_{1,3} : S_{m,n}) = \begin{cases} 4, & n \leq 4-m; \\ n+m+2, & 5-m \leq n \leq 2m+1; \\ \frac{3}{2}n+1, & \text{even } n \geq 2m+2; \\ \frac{3}{2}n+\frac{1}{2}, & \text{odd } n \geq 2m+3. \end{cases}$$

Similarly, for $k = 4$, according to Lemma 3 and Theorem 15, we have the following corollary.

Corollary 3 *For an integer $n \geq 1$, we have*

$$\mathrm{gr}_4(P_5 : B_n) \geq \mathrm{gr}_4(K_{1,3} : B_n) = \begin{cases} 4, & 1 \leq n \leq 2; \\ 3n-2, & n \geq 3. \end{cases}$$

Lemma 9 *Let integer $k \geq 4$. For a graph H with at least k vertices, we have*

$$\mathrm{gr}_k(P_5 : H) \geq |V(H)| + 1.$$

Proof Let G_1 be a monochromatic copy of $K_{|V(H)|-1}$ with color 1 and G_2 be a copy of K_1. We construct a $K_{|V(H)|}$ by making use of G_1, G_2 by inserting all edges between these copies such that the edges from G_1 to G_2 are colored by $2, 3, \ldots, k$. One can easily check that there is neither a rainbow subgraph P_5 nor a monochromatic subgraph H under such a k-edge-coloring of $K_{|V(H)|}$, and so $\mathrm{gr}_k(P_5 : H) \geq |V(H)| + 1$. □

Since $|V(S_{m,n})| = n+m+2$, $S_{m,n}$ is a subgraph of $\overline{K_2} + K_{n+m}$ and $\overline{K_4} + K_{n+m-1}$, it follows from Corollary 2, Lemmas 1 and 9 that we have the following corollary directly.

Corollary 4 *For integers $n \geq m \geq 1$, we have*

$$\begin{aligned} \mathrm{gr}_4(P_5 : S_{m,n}) &= \max\left\{n+m+3, \mathrm{gr}_4(K_{1,3} : S_{m,n})\right\} \\ &= \begin{cases} n+m+3, & 2 \leq n \leq 2m+3; \\ \frac{3}{2}n+1, & \text{even } n \geq 2m+4; \\ \frac{3}{2}n+\frac{1}{2}, & \text{odd } n \geq 2m+5. \end{cases} \end{aligned}$$

Since $|V(B_n)| = n+2$, B_n is a subgraph of $\overline{K_2}+K_n$ and $\overline{K_4}+K_{n-1}$, it follows from Corollary 3, Lemmas 1 and 9 that we have the following corollary directly.

Corollary 5 *For an integer $n \geq 3$, we have*

$$\mathrm{gr}_4(P_5 : B_n) = \max\left\{n+3, \mathrm{gr}_4(K_{1,3} : B_n)\right\} = 3n-2.$$

3.3 Rainbow P_4^+

In this subsection, we give the Gallai–Ramsey numbers for monochromatic double stars or books involving rainbow P_4^+. For $k = 4$, according to Lemma 3 and Theorem 13, we have the following corollary.

Corollary 6 *For integers $n \geq m \geq 1$, we have*

$$\mathrm{gr}_4(P_4^+ : S_{m,n}) \geq \mathrm{gr}_4(K_{1,3} : S_{m,n}) = \begin{cases} 4, & n \leq 4-m; \\ n+m+2, & 5-m \leq n \leq 2m+1; \\ \frac{3}{2}n+1, & \text{even } n \geq 2m+2; \\ \frac{3}{2}n+\frac{1}{2}, & \text{odd } n \geq 2m+3. \end{cases}$$

Similarly, for $k = 4$, according to Lemma 3 and Theorem 15, we have the following corollary.

Corollary 7 *For an integer $n \geq 1$, we have*

$$\mathrm{gr}_4(P_4^+ : B_n) \geq \mathrm{gr}_4(K_{1,3} : B_n) = \begin{cases} 4, & 1 \leq n \leq 2; \\ 3n-2, & n \geq 3. \end{cases}$$

Lemma 10 *For a graph H except a star, we have*

$$\mathrm{gr}_4(P_4^+ : H) \geq |V(H)| + 2.$$

Proof Let $K_{|V(H)|+1} = G_2(|V(H)|+1)$. It follows from Theorem 7 (ii) that there is neither a rainbow subgraph P_4^+ nor a monochromatic subgraph H under such a 4-edge-coloring of H, and so $\mathrm{gr}_4(P_4^+ : H) \geq |V(H)| + 2$. □

Since $|V(S_{m,n})| = n+m+2$, $S_{m,n}$ is a subgraph of $\overline{K_3}+K_{n+m+1}$, it follows from Corollary 6, Lemmas 2 and 10 that we have the following corollary directly.

Corollary 8 *For integers $n \geq m \geq 1$, we have*

$$\mathrm{gr}_4(P_4^+ : S_{m,n}) = \max\left\{n+m+4, \mathrm{gr}_4(K_{1,3} : S_{m,n})\right\} = \begin{cases} n+m+4, & 1 \leq n \leq 2m+5; \\ \frac{3}{2}n+1, & \text{even } n \geq 2m+6; \\ \frac{3}{2}n+\frac{1}{2}, & \text{odd } n \geq 2m+7. \end{cases}$$

Since $|V(B_n)| = n+2$, B_n is a subgraph of $\overline{K_3}+K_{n+1}$, it follows from Corollary 7, Lemmas 2 and 10 that we have the following corollary directly.

Corollary 9 *For an integer $n \geq 1$, we have*

$$\mathrm{gr}_4(P_4^+ : B_n) = \max\left\{n+4, \mathrm{gr}_4(K_{1,3} : B_n)\right\} = \begin{cases} n+4, & 1 \leq n \leq 2; \\ 3n-2, & n \geq 3. \end{cases}$$

At the end of this section, in order to supplement Lemma 5, we directly give the following corollary according to Lemma 11 and Proposotion 9.

Lemma 11 *Gallai-Ramsey number* $\mathrm{gr}_k(K_{1,3} : H) = 4$ *if and only if* $4 \le k \le 6$ *and* H *is a subgraph of balanced complete* $(k-1)$*-partite graph* $K_{2,2,\ldots,2}$.

Proof If $4 \le k \le 6$ and H is a subgraph of balanced complete $(k-1)$-partite graph $K_{2,2,\ldots,2}$, then it follows from Theorem 11 that $\mathrm{gr}_k(K_{1,3} : H) = 4$. If $\mathrm{gr}_k(K_{1,3} : H) = 4$, then it follows from the definition of exact k-edge coloring and Proposition 9 that $4 \le k \le 6$. Suppose H is not a subgraph of balanced complete $(k-1)$-partite graph $K_{2,2,\ldots,2}$. For $4 \le k \le 6$, let $t_i = 2$ for each $1 \le i \le k-1$. Then $K_{2(k-1)} = [K_{t_1}, K_{t_2}, \ldots, K_{t_{k-1}}]$ is a k-edge-colored complete graph. One can easily check that there is neither a rainbow subgraph $K_{1,3}$ nor a monochromatic subgraph H, and so $\mathrm{gr}_k(K_{1,3} : H) \ge 2(k-1)+1 \ge 7$, which contradicts with $\mathrm{gr}_k(K_{1,3} : H) = 4$. The result thus follows. □

Corollary 10 *Let integer* $5 \le k \le 6$. *If* $\mathrm{gr}_k(K_{1,3} : H) = 4$, *then*

$$\mathrm{gr}_k(P_4^+ : H) = 5.$$

4 Conclusion

Gallai–Ramsey number involving rainbow $K_{1,3}$ plays a very significant role in Gallai-Ramsey number involving rainbow P_5 or P_4^+. When the number of colors $k \ge 5$, if we know $\mathrm{gr}_k(K_{1,3} : H)$, then we can directly get $\mathrm{gr}_k(P_5 : H)$ and $\mathrm{gr}_k(P_4^+ : H)$ from Lemmas 4, 5 or Corollary 10. Essentially, this special property is determined by Theorems 5, 6 and 7. Although we have obtained some results, there are still some unsolved problems. We end this section with an open problem.

Problem 1 *For an integer* $n \ge 4$, *determine the exact value of* $\mathrm{gr}_3(K_{1,3} : B_n)$.

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Declarations

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