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ORIGINAL RESEARCH

Bounded solutions in anisotropic degenerate parabolic problems with a singular term

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Abstract In this paper, we study the existence of bounded solutions for a nonlinear anisotropic parabolic equation with degenerate coercivity and a singular term on the right-hand side. The model problem considered is as follows

$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i=1}^{N} D_i \left(\frac{u^{p_i-1}(1+Du)^{-1} Du + |Du|^{p_i-2} Du}{(1+|u|)^{\theta}} \right) = \frac{f}{u^{\gamma}} & \text{in } Q, \\ u(x,0) = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $N \geq 2$, $T > 0$, $2 \leq p_i < N$ for every $i = 1, \ldots, N$, $\theta, \gamma \geq 0$, $0 \leq f \in L^m(Q)$ with $m > \frac{N}{\overline{p}} + 1$ ($\overline{p}$ defined in (2.1)) and $Q = \Omega \times (0, T)$. The main idea in the proof is based on Stampacchia's lemma, which allows us to obtain a priori estimates by making a suitable choice of a test function.

Keywords Anisotropic parabolic equation · Singular non-linearity · Non coercive operator · Bounded solutions · Fixed point theorem

Mathematics Subject Classification 35K65 · 35K55 · 35B45 · 35B65

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$), Q be the cylinder $\Omega \times (0, T)$ ($T > 0$), and Γ be the lateral surface $\partial\Omega \times (0, T)$. We consider the following singular anisotropic parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} + Bu = g(u) f & \text{in } Q, \\ u(x,0) = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

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where

$$Bu = -\sum_{i=1}^{N} D_i\big[b_i(x,t,u,Du)\big],$$

and $f \geq 0$ is a function belong to $L^m(\Omega)$ with $m > \frac{N}{\overline{p}} + 1$. Here, we assume that $b_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ for $i = 1, \ldots, N$ are Carathéodory functions. For almost every (x,t) in Q, for every $\sigma \in \mathbb{R}$, for all $\xi, \eta \in \mathbb{R}^N$, and for any $i = 1, \ldots, N$, the following conditions hold

$$b_i(x,t,\sigma,\xi)\cdot\xi_i \geq \frac{\alpha|\xi_i|^{p_i}}{(1+|\sigma|)^{\theta}}, \tag{1.2}$$

$$0 \leq \theta < \min\left\{\overline{p} - 1 + \frac{\overline{p}}{N} + \gamma\left(1+\frac{\overline{p}}{N}\right), \overline{p} - 2 + \frac{\overline{p}^2}{N}\right\}, \tag{1.3}$$

$$|b_i(x,t,\sigma,\xi_i)| \leq \beta\left(k(x,t) + |\sigma|^{\overline{p}} + \sum_{j=1}^{N}|\xi_j|^{p_j}\right)^{1-\frac{1}{p_i}}, \tag{1.4}$$

and

$$\big[b_i(x,t,\sigma,\xi) - b_i(x,t,\sigma,\xi')\big]\cdot(\xi_i - \xi_i') > 0, \quad \xi_i \neq \xi_i', \tag{1.5}$$

where α, β are strictly positive real numbers and k is a given positive function in $L^1(Q)$.

Finally, the singular term $g : [0,\infty) \to [0,\infty]$ is continuous, bounded outside the origin, and may potentially approach infinity at the origin. Additionally, the following properties hold true

$$\exists\, c > 0,\ \exists\, \gamma \in (0, \overline{p} - 1) \quad \text{such that} \quad g(s) \leq \frac{c}{s^{\gamma}} \quad \forall s \in (0,\infty). \tag{1.6}$$

Nowadays, anisotropic equations are prevalent in various mathematical models. They are particularly necessary for studying physical fields like electro-rheological and thermo-rheological liquids [1]. Additionally, they find applications in biology, as demonstrated by Bendahmane et al. [2], where they serve as a model for describing the spread of epidemic diseases in heterogeneous environments. There are other modern applications of anisotropic problems in the thermistor phenomenon (see [11]).

Below, we provide brief descriptions of several papers that are directly relevant to this work. To simplify the presentation for the reader, we will divide this description into two parts: the isotropic case and the anisotropic case.

1.1 Analysis of the isotropic scenario: $p_i = p$

The case where $p = 2$ and $\gamma = 0$ was studied in Ref. [9]. In Ref. [20], the authors investigated the existence and regularity of problem (1.7), which is described as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u)\widehat{a}(x,t,u,Du)) = f & \text{in } Q, \\ u = 0 & \text{on } \Gamma, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.7}$$

where $f \in L^m(Q)$ with $m > 1$, $2 \leq p < N$, and $a : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies, for almost every $(x,t) \in Q$ and all $s \in \mathbb{R}$, the following inequality

$$\frac{\alpha}{(1+|s|)^{\theta}} \leq a(x,t,s) \leq \beta,$$

where $0 \leq \theta < p - 1 + \frac{p}{N}$. The constants α and β are positive constants. The regularity for entropy solutions to problem (1.7) was obtained in Ref. [12]. If $f = -\operatorname{div}(g)$, with $0 \leq \theta < p - 1$, where $g \in (L^r(0,T; L^q(\Omega)))^N$ under certain conditions on r and q, existence and regularity results for problem (1.7) have been obtained in Refs. [7] and [30].

In [5], the authors have addressed problem (1.1) in the coercive case with $\gamma \neq 0$. They have proven the existence and regularity of solutions to the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = \dfrac{f}{u^\gamma} & \text{in } Q, \\ u(x,0) = u_0(x) & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

where $\gamma > 0$, $p \geq 2$, $f > 0$, $f \in L^m(Q)$ with $m \geq 1$, and $u_0 \in L^\infty(\Omega)$. For the existence and regularity results regarding the anisotropic elliptic case of problem (1.1), we recommend referring to Refs. [22,31].

Moreover, the authors in [14] study the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,Du)) = fg(u) & \text{in } Q, \\ u = 0 & \text{on } \Gamma, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases}$$

where f is a given function belonging to some Lebesgue space $L^m(Q)$ with $m \geq 1$, the singular term g satisfies inequality (1.6) with $0 < \gamma < 1$, and $a : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function that satisfies, for almost every $(x,t) \in Q$ and all $\sigma \in \mathbb{R}$, $\xi \in \mathbb{R}^N$

$$a(x,t,\sigma,\xi).\xi \geq \frac{\alpha|\xi|^p}{(1+|\sigma|)^\theta},$$

where

$$0 \leq \theta < p - 1 + \frac{p}{N} + \gamma\left(1 + \frac{p}{N}\right).$$

Finally, in Ref. [21], the authors studied the existence and regularity of the following singular model problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,Du)) + |u|^{s-1}u = fg(u) & \text{in } Q, \\ u = 0 & \text{on } \Gamma, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases}$$

here, $f \in L^m(Q)$ with $m \geq 1$, $a : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function that satisfies, for almost every $(x,t) \in Q$ and all $\sigma \in \mathbb{R}$, $\xi \in \mathbb{R}^N$

$$a(x,t,\sigma,\xi).\xi \geq \frac{\alpha|\xi|^p}{(1+|\sigma|)^{\theta(p-1)}} \quad \text{with} \quad 0 \leq \theta < 1,$$

and the singular term g satisfies inequality (1.6) with $0 < \gamma < 1$.

1.2 Analysis of anisotropic situations

In the case of uniform ellipticity and non-singularity (i.e., $\theta = 0$ and $\gamma = 0$), the existence results for problem (1.1) have been proven in Refs. [17] and [18] when $f \in L^m(Q)$ or when f is a bounded Radon measure on Q. If $\gamma = 0$, the existence and regularity of entropy solutions for problem (1.1) with lower order terms and variable exponents were investigated in [19]. The problem (1.7) was further generalized in Refs. [13,29], where the authors studied the following singular problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{N} D_i\left(a_i(x,t,u,Du)\right) = f & \text{in } Q, \\ u(x,0) = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

Here, $f \in L^m(Q)$ with $m \geq 1$, $a_i : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ for $i = 1, \ldots, N$ are Carathéodory functions that satisfy inequalities (1.3) and (1.5). Moreover, for almost every $(x, t) \in Q$, for all $s \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^N$, the following inequality holds

$$\sum_{i=1}^{N} a_i(x, t, s, \xi)\xi_i \geq \alpha \sum_{i=1}^{N} \frac{|\xi_i|^{p_i}}{(1 + |s|)^{\theta(p_i-1)}},$$

where $p_i \in (1, +\infty)$ for every $i = 1, \ldots, N$, and $0 \leq \theta < 1 + \frac{\overline{p}}{N(\overline{p}-1)}$. In Ref. [29], the authors primarily focused on studying the existence and regularity of solutions u with respect to the parameter m. However, for the regularity results of weak solutions to the anisotropic elliptic case of problem (1.1), one can refer to the recent work in Ref. [10].

1.3 The dilemmas posed by problem (1.1)

Our main objective in this article is to investigate the results of [20] regarding the framework of the non-coercive operator $B(u)$. However, we face several difficulties in achieving this goal. Firstly, while the differential operator B is well-defined between $\mathbb{L}^{\overrightarrow{p}}\left(0, T; X_0^{1,\overrightarrow{p}}(\Omega)\right)$ and its dual $\left(\mathbb{L}^{\overrightarrow{p}}\left(0, T; X_0^{1,\overrightarrow{p}}(\Omega)\right)\right)^*$, it fails to be coercive on $\mathbb{L}^{\overrightarrow{p}}\left(0, T; X_0^{1,\overrightarrow{p}}(\Omega)\right)$ when u is unbounded. Due to this lack of coercivity, the classical theory for elliptic operators acting between spaces in duality (as described in [27]) cannot be applied in general, even if the datum f is regular (see [16,25]).

Secondly, it is important to note that (1.1) can have a singular term on the right-hand side. In other words, the solution u is required to be zero on the boundary of the domain, but simultaneously, the right-hand side of (1.1) could become unbounded.

Finally, the tools and techniques available for the isotropic case cannot be extended to the anisotropic setting, which further complicates the analysis in this article.

To address these difficulties, we will employ approximation techniques to overcome the issues related to the degenerate coercivity of the operator term and the singularity of the right-hand side (as stated in problems (3.3)). By employing truncation methods, we will ensure that the operator term remains bounded, and we will handle the singularity on the right-hand side appropriately. Furthermore, we will utilize Schauder's theorem to establish the existence of a bounded finite energy solution u_n to these problems. This theorem guarantees the existence of a solution within a certain function space under suitable conditions, providing a valuable tool for our analysis.

2 Fundamentals and main result statement

In this section, we provide the necessary preliminary information and state the main result of our study.

Let Ω be a bounded smooth open subset of $\mathbb{R}^N$, and let $\overrightarrow{p} = (p_1, p_2, ..., p_N) \in \mathbb{R}^N$ with $p_i \geq 1$ for all $i = 1, \ldots, N$. The anisotropic Sobolev spaces can be defined as follows

$$\begin{cases} W^{1,p_i}(\Omega) = \left\{v \in W^{1,1}(\Omega) : \ D_i v \in L^{p_i}(\Omega)\right\}, \\ W_0^{1,p_i}(\Omega) = \left\{v \in W_0^{1,1}(\Omega) : \ D_i v \in L^{p_i}(\Omega)\right\}, \end{cases}$$

where $D_i u = \frac{\partial u}{\partial x_i}$ represents the partial derivative of u with respect to x_i. The anisotropic space $W_0^{1,p_i}(\Omega)$ can also be defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p_i}(\Omega)$ with respect to the norm

$$\|v\|_{W_0^{1,p_i}(\Omega)} = \sum_{i=1}^{N} \|D_i v\|_{L^{p_i}(\Omega)}.$$

which makes it a Banach space. Its dual is denoted by $\left(W_0^{1,p_i}(\Omega)\right)'$.

Let $\overrightarrow{r} = (r_1, r_2, \ldots, r_N) \in \mathbb{R}^N$ with $r_i > 1$ for all $i = 1, \ldots, N$, we define the spaces as follows

$$X_0^{1,\overrightarrow{p}}(\Omega) = \bigcap_{i=1}^{N} W_0^{1,p_i}(\Omega),$$

$$\mathbb{L}^{\overrightarrow{r}}\big(0,T;X_0^{1,\overrightarrow{p}}(\Omega)\big)=\bigcap_{i=1}^{N}L^{r_i}\big(0,T;W_0^{1,p_i}(\Omega)\big),$$

where the norms are given by

$$\|u\|_{X_0^{1,\overrightarrow{p}}(\Omega)}=\sum_{i=1}^{N}\|D_iu\|_{L^{p_i}(\Omega)},$$

$$\|u\|_{\mathbb{L}^{\overrightarrow{r}}(0,T;X_0^{1,\overrightarrow{p}}(\Omega))}=\sum_{i=1}^{N}\|u\|_{L^{r_i}(0,T;W_0^{1,p_i}(\Omega))}.$$

Now, we recall the following theorem on anisotropic Sobolev embeddings.

Lemma 2.1 [28] *Let $v\in W_0^{1,p_i}(\Omega)$, and define*

$$\overline{p}^*=\frac{N\overline{p}}{N-\overline{p}},\quad \overline{p}=\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}\right)^{-1}. \tag{2.1}$$

If $1<\overline{p}<N$, then there exists a positive constant C, depending only on Ω, such that

$$\|v\|_{L^r(\Omega)}\le C\prod_{i=1}^{N}\|D_iv\|_{L^{p_i}(\Omega)}^{\frac{1}{N}},\quad \forall r\in[1,\overline{p}^*], \tag{2.2}$$

$$\|v\|_{L^{\overline{p}^*}(\Omega)}^{p_N}\le C\sum_{i=1}^{N}\|D_iv\|_{L^{p_i}(\Omega)}^{p_i}. \tag{2.3}$$

Next, we present the embedding result for the anisotropic Sobolev space (refer to [15,23,28] for more details).

Lemma 2.2 *Suppose $p_i\ge 1$ for $i=1,\dots,N$. If $v\in\bigcap_{i=1}^{N}W^{1,p_i}(Q)$ and $\overline{p}<N$, then*

$$\|v\|_{L^r(Q)}\le C\prod_{i=1}^{N}\big(\|u\|_{L^{p_i}(Q)}+\|D_iv\|_{L^{p_i}(Q)}\big)^{\frac{1}{N}},\quad \forall r\in[1,\overline{p}^*], \tag{2.4}$$

where the constant C depends on N and p_i. Furthermore, if $\overline{p}\ge N$, the inequality (2.4) *holds for all $r\ge 1$, and C depends on r and $|Q|$.*

The following lemma is useful when proving the boundedness of the solution u_n of problem (3.3).

Lemma 2.3 [3,27] *Let $\Psi:\mathbb{R}_+\to\mathbb{R}_+$ be a non-increasing function such that*

$$\Psi(h)\le\frac{M_1k^{\eta\rho}}{(h-k)^{\eta}}[\Psi(k)]^{\delta},\quad \forall h>k\ge k_0.$$

where M_1,η,k_0,δ,ρ are positive real numbers with $\delta>1$ and $\rho\in[0,1)$. Then there exists $k>0$ such that $\Psi(k)=0$.

We begin by recalling the well-known Gagliardo-Nirenberg interpolation inequality.

Lemma 2.4 [24] *Let v be a function in $W_0^{1,1}(\Omega)\cap L^{\rho}(\Omega)$, with $q\ge 1$ and $\rho\ge 1$. Then there exists a positive constant M_1, depending on N, q, and ρ, such that*

$$\|v\|_{L^q(\Omega)}\le M_1\|\nabla v\|_{(L^1(\Omega))^N}^{\widehat{\theta}}\|v\|_{L^{\rho}(\Omega)}^{1-\widehat{\theta}}, \tag{2.5}$$

for every θ and q satisfying

$$0\le\widehat{\theta}\le 1,\quad 1\le q<+\infty,\quad \frac{1}{q}=\widehat{\theta}\left(1-\frac{1}{N}\right)+\frac{1-\widehat{\theta}}{\rho}. \tag{2.6}$$

An immediate consequence of the previous lemma is the following embedding result:

Lemma 2.5 [6] *Let $v \in L^q(0,T;W_0^{1,h}(\Omega)) \cap L^\infty(0,T;L^\varrho(\Omega))$, where $h, \varrho \geq 1$. Then v belongs to $L^{q_0}(Q)$, where $q_0 = q\frac{N+\varrho}{N}$, and there exists a positive constant M_2 depending only on N, q, and ϱ such that*

$$\int_Q |v|^{q_0}\, dx\, dt \leq M_2 \|v\|^{\frac{\rho q}{N}}_{L^\infty(0,T;L^\rho(\Omega))} \int_Q |\nabla v|^q\, dx\, dt. \tag{2.7}$$

Let $k > 0$ be a given constant. We will make use of the truncation function T_k defined as $T_k(s) = \max\{-k, \min\{s,k\}\}$, and its complement function $G_k(s) = s - T_k(s)$. Additionally, we will utilize the following function:

$$V_\lambda(s) = \begin{cases} 1, & \text{if } s \leq \lambda, \\ \frac{2\lambda - s}{\lambda}, & \text{if } \lambda < s < 2\lambda, \\ 0, & \text{if } s \geq 2\lambda. \end{cases} \tag{2.8}$$

We define the notion of a weak solution to problem (1.1) as follows:

Definition 2.6 A function u is a weak solution of problem (1.1) if $u \in L^1(0,T;W_0^{1,1}(\Omega))$, $b_i(x,t,u,Du) \in (L^1(Q))^N$ for all $i = 1,\ldots,N$, $g(u)f \in L^1_{\text{loc}}(Q)$, $u \geq 0$ almost everywhere in Q, and the equality

$$-\int_Q u\frac{\partial \varphi}{\partial t} dxdt + \sum_{i=1}^N \int_Q b_i(x,t,u,Du).D_i\varphi dxdt = \int_Q g(u) f\varphi dxdt, \tag{2.9}$$

holds for every test function $\varphi \in C^\infty([0,T] \times \overline{\Omega})$ such that $\varphi = 0$ in a neighborhood of $\partial\Omega \times (0,T)$ and $\varphi = 0$ on $\Omega \times T$.

The main result of this work is as follows.

Theorem 2.7 *Assuming that hypotheses* (1.2), (1.5), *and* (1.6) *hold true, if the datum f is a nonnegative function belonging to $L^m(Q)$ with $m > \frac{N}{p} + 1$, then there exists a function*

$$u \in \bigcap_{i=1}^N L^{p_i}\left(0,T;W_0^{1,p_i}(\Omega)\right) \cap L^\infty(Q)$$

that is a solution of problem (1.1) *in the sense of Definition 2.6.*

Remark 1 To obtain the a priori uniform L^∞-estimates for the solutions of approximate equations, we apply Lemma 6.1 in [27], which is stated as Lemma 2.3 below. However, it should be noted that Lemma 2.3 requires the assumption $p_i \geq 2$ for every $i = 1,\ldots,N$.

Remark 2 In the isotropic case (i.e., $p_i = p = 2$ and $\gamma = 0$), the result of Theorem 2.7 coincides with the classical boundedness results for parabolic equations with degenerate coercivity and non-singularity, as shown in [9, Theorem 1.1.].

3 Solving the approximation problem

Let $f_n \in C_0^\infty(\overline{Q})$ (for example $f_n = T_n(f)$), such that

$$\|f_n\|_{L^m(Q)} \leq \|f\|_{L^m(Q)} \quad \forall n, \tag{3.1}$$

$$f_n \to f \quad \text{strongly in } L^m(Q). \tag{3.2}$$

Moreover, let us define $g(0) = \lim_{s\to 0} g(s)$ (see [14,31]). We set

$$g_n(s) = \begin{cases} T_n(g(s)) & \text{for } s > 0, \\ \min\{n, g(0)\} & \text{otherwise.} \end{cases}$$

We consider the following approximate problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \displaystyle\sum_{i=1}^{N} D_i\left(b_i(x,t,T_n(u_n),Du_n)\right) = g_n(u_n) f_n \text{ in } Q, \\ u_n(x,0) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \Omega, \\ u_n = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \Gamma. \end{cases} \tag{3.3}$$

Lemma 3.1 *Let b_i be Carathéodory functions satisfying* (1.2)–(1.5) *for every $i = 1, \dots, N$, and g satisfies* (1.6). *Then there exists a weak solution u_n of* (3.3) *such that*

$$\begin{cases} u_n \in \displaystyle\bigcap_{i=1}^{N} L^{p_i}\left(0,T; W_0^{1,p_i}(\Omega)\right) \cap C([0,T]; L^2(\Omega)) \ , \\ \dfrac{\partial u_n}{\partial t} \in \displaystyle\sum_{i=1}^{N} L^{p_i'}\left(0,T; (W_0^{1,p_i}(\Omega))'\right) + L^1(Q) \qquad , \end{cases}$$

and it satisfies the weak formulation

$$\begin{aligned} &\int_Q \frac{\partial u_n}{\partial t}\varphi dxdt + \sum_{i=1}^{N}\int_Q b_i(x,t,T_n(u_n),Du_n).D_i\varphi dxdt \\ &= \int_Q g_n(u_n) f_n \varphi dxdt. \end{aligned} \tag{3.4}$$

for every fixed $n \in \mathbb{N}$ and for every $\varphi \in \bigcap_{i=1}^{N} L^{p_i}\left(0,T; W_0^{1,p_i}(\Omega)\right) \cap L^\infty(Q)$.

Proof Let $n \in \mathbb{N}$ and $v \in L^{\overline{p}^*}(Q)$ be fixed. Consider the following nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial w}{\partial t} - \displaystyle\sum_{i=1}^{N} D_i\left(b_i(x,t,T_n(w),Dw)\right) = g_n(v) f_n \text{ in } Q, \\ w(x,0) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \Omega, \\ w = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \Gamma. \end{cases} \tag{3.5}$$

It is clear that problem (3.5) has a unique solution w such that

$$\begin{cases} w \in \displaystyle\bigcap_{i=1}^{N} L^{p_i}\left(0,T; W_0^{1,p_i}(\Omega)\right) \cap L^\infty(Q), \\ \dfrac{\partial w}{\partial t} \in \displaystyle\sum_{i=1}^{N} L^{p_i'}\left(0,T; (W_0^{1,p_i}(\Omega))'\right) + L^1(Q), \end{cases}$$

since the right-hand side in (3.5) belongs to $L^\infty(Q)$ (see, for instance, [16]), also see [17,18]. In particular, the map $P : L^{\overline{p}}(Q) \to L^{\overline{p}}(Q)$ defined as $P(v) = w$ is well-defined for every $i = 1, \dots, N$. Moreover, since $g_n(v) f_n$ is bounded, we have that $w \in L^\infty(Q)$, and there exists a positive constant C_∞, independent of v and w (but possibly depending on n), such that

$$\|w\|_{L^\infty(Q)} \le C_\infty. \tag{3.6}$$

Choosing w as a test function in (3.5), we obtain the following equation

$$\begin{aligned} &\frac{1}{2}\int_\Omega |w(T)|^2 dx + \sum_{i=1}^{N}\int_Q b_i(x,t,T_n(w),Dw).D_i w dxdt \\ &= \int_Q g_n(v) f_n w dxdt. \end{aligned} \tag{3.7}$$

By excluding a positive term on the left-hand side of (3.7), and employing (1.2) and (1.6), we can conclude

$$\alpha \sum_{i=1}^{N} \int_0^T \int_\Omega \frac{|D_i w|^{p_i}}{(1+|T_n(w)|)^\theta} dxdt \leq n^{\gamma+1} \int_Q |w| dxdt.$$

Using Hölder's inequality with exponent $\overline{p}^*$ and the inequality $(1+T_n(w))^\theta \leq (1+n)^\theta$, we get

$$\sum_{i=1}^{N} \int_Q |D_i w|^{p_i} dxdt \leq \frac{C_1}{\alpha} n^{\gamma+1} (1+n)^\theta |Q|^{\frac{1}{(\overline{p}^*)'}} \left(\int_Q |w|^{\overline{p}^*} dxdt \right)^{\frac{1}{\overline{p}^*}}.$$

The previous inequality and (2.3) yield the following estimate

$$\left(\int_Q |w|^{\overline{p}^*} dxdt \right)^{\frac{p_N}{\overline{p}^*}} \leq \frac{C_1}{\alpha} n^{\gamma+1} (1+n)^\theta |Q|^{\frac{1}{(\overline{p}^*)'}} \left(\int_Q |w|^{\overline{p}^*} dxdt \right)^{\frac{1}{\overline{p}^*}}.$$

Since $p_N > 1$, then there exists a positive constant $C(n, |Q|)$ independent of v and w, such that

$$\|w\|_{L^{\overline{p}^*}(Q)} \leq C(n, |Q|).$$

By the previous inequality and the fact that $\overline{p} \leq \overline{p}^*$, we have

$$\|w\|_{L^{\overline{p}}(Q)} \leq C_1 C(n, |Q|) = R(n, |Q|). \tag{3.8}$$

Thus, (3.8) implies that the ball B in $L^{\overline{p}}(Q)$ of radius $R(n, |Q|)$ is invariant for the map P. We are going to prove that P is continuous. Let $v_k k$ be a bounded sequence in B. By (3.8), there exists a subsequence of vkk, still denoted by vk, and a measurable function v belonging to $L^{\overline{p}}(Q)$ such that the following convergence holds

$$v_k \to v \quad \text{strongly in } L^{\overline{p}}(Q). \tag{3.9}$$

So, $v_k \longrightarrow v$ almost everywhere in Q as $k \longrightarrow \infty$, hence

$$g_n(v_k) f_n \to g_n(v) f_n \quad \text{a.e. in } Q,$$

as $k \longrightarrow \infty$. Consequently, since for all $k \in \mathbb{N}$

$$g_n(v_k) f_n \leq n f_n \in L^\infty(Q) \subset L^{\overline{p}}(Q), \quad \text{a.e. in } Q,$$

by the dominated convergence theorem, one has

$$g_n(v_k) f_n \to g_n(v) f_n \quad \text{strongly in } L^{\overline{p}}(Q).$$

Hence, by uniqueness, one deduces that $w_k = P(v_k)$ converges to $w = P(v)$ in $L^{\overline{p}}(Q)$. This gives the continuity of P.

On the other hand, one can choose w_k as a test function in the weak formulation of the problem solved by w_k, which is the following

$$\begin{cases} \dfrac{\partial w_k}{\partial t} - \displaystyle\sum_{i=1}^{N} D_i \left(b_i(x,t,T_n(w_k), Dw_k) \right) = g_n(v_k) f_n & \text{in } Q, \\ w_k(x,0) = 0 & \text{on } \Omega, \\ w_k = 0 & \text{on } \Gamma, \end{cases}$$

we obtain, for all $i = 1, \ldots, N$

$$\int_0^T \|D_i w_k\|^{p_i}_{L^{p_i}(\Omega)} dt \leq C \int_0^T \left(\int_\Omega |w_k|^{\overline{p}^*} dx \right)^{\frac{1}{\overline{p}^*}} dt. \tag{3.10}$$

Since the ball in $L^{\overline{p}}(Q)$ is invariant under P, we have w_k belonging to B. From equations (3.8) and (3.10), we can deduce that w_k is bounded in $\bigcap_{i=1}^{N} L^{p_i}\big((0,T;W_0^{1,p_i}(\Omega)\big)$, and $w_k \in L^{\overline{p}}(Q)$. This is because there exists $j \in i = 1,\dots,N$ such that $p_j \geq \overline{p}$. The growth assumption on b_i implies the following for all $i = 1,\dots,N$

$$\begin{aligned}
&\int_Q |b_i(x,t,w_k,D_i w_k)|^{p_i'} dxdt \\
&\leq C\beta \int_Q \left[|k(x,t)| + |w_k|^{\overline{p}} + \sum_{j=1}^{N} |D_j w_k|^{p_j}\right]^{(1-\frac{1}{p_i})p_i'} dxdt \\
&\leq C\Big[\|k\|^{p_i'}_{L^{p_i'}(Q)} + \|w_k\|^{\overline{p}}_{L^{\overline{p}}(Q)} + \|w_k\|^{p_i}_{L^{p_i}(0,T;W_0^{1,p_i}(\Omega))}\Big] \\
&< +\infty.
\end{aligned}$$

Therefore, $\frac{\partial w_k}{\partial t}$ is bounded in $\sum_{i=1}^{N} L^{p_i'}\left(0,T;(W_0^{1,p_i}(\Omega))'\right) \cap L^1(Q)$ for all $i = 1,\dots,N$. As mentioned in [26], this implies that w_k is relatively strongly compact in $L^1(Q)$. Thus, there exists a subsequence of w_k (still denoted by w_k), and a measurable function w belonging to $L^1(Q)$ such that

$$w_k \to w \quad \text{a.e in } L^1(Q). \tag{3.11}$$

Using equations (3.10) and (3.11), along with the Lebesgue theorem, we can conclude that w_k converges strongly to w in $L^{p_i}(Q)$. This implies that the operator P is compact. By Schauder's fixed point theorem, there exists u_n such that

$$u_n \in \bigcap_{i=1}^{N} L^{p_i}\left(0,T;W_0^{1,p_i}(\Omega)\right) \cap C([0,T];L^2(\Omega)),$$

and

$$\frac{\partial u_n}{\partial t} \in \sum_{i=1}^{N} L^{p_i'}\left(0,T;(W_0^{1,p_i}(\Omega))'\right) \cap L^1(Q),$$

and $u_n = P(u_n)$ for every fixed n.

Moreover, taking $\varphi = u_n^-$ in (3.3), we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u_n^-(T)|^2 dx + \sum_{i=1}^{N}\int_Q b_i(x,t,T_n(u_n),Du_n).D_i u_n^- dxdt \\
&= \int_Q g_n(u_n) f_n u_n^- dxdt.
\end{aligned}$$

Using (1.2), omitting the non-negative terms, and recalling that $g_n(u_n) f_n \geq 0$, we can deduce the following inequality

$$\frac{\alpha}{(1+n)^\theta}\sum_{i=1}^{N}\int_Q |D_i u_n^-|^{p_i} dxdt \leq \int_Q g_n(u_n) f_n u_n^- dxdt \leq 0.$$

From this inequality, we can conclude that $\|u_n^-\|_{L^{p_i}(0,T;W_0^{1,p_i}(\Omega))} = 0$, which implies that $u_n \geq 0$ almost everywhere in Q. □

4 A priori estimate in the L^∞ norm

In this section, we will use the symbol C to represent various positive constants (not depending on n) whose values may vary from line to line. These constants will only depend on the given data.

Lemma 4.1 *Suppose that the assumptions of Theorem* 2.7 *are satisfied, and let u_n be a solution of* (3.4)*. Then, there exists a positive constant C such that*

$$\|u_n\|_{L^\infty(Q)} \leq C, \tag{4.1}$$

$$\|u_n\|_{\mathbb{L}^{\overrightarrow{p}}(0,T;X_0^{1,\overrightarrow{p}}(\Omega))} \leq C. \tag{4.2}$$

Proof For all $\tau \in (0, T]$, we take

$$\varphi(u_n) = [(1+u_n)^{\overline{p}-1} - 1]G_k'(u_n)\chi_{(0,\tau)}, \quad \forall i = 1, \ldots, N,$$

as a test function in equation (3.4), where $\chi_{(0,\tau)}$ denotes the characteristic function of the interval $(0, \tau)$ within the interval $(0, T]$ and $G_k(u_n) = u_n - T_k(u_n)$ for all $k > 0$. By using (1.2),(1.6) and the fact that

$$\begin{aligned} D_i\varphi(u_n) &= \left[(\overline{p}-1)(1+u_n)^{\overline{p}-2}G_k'(u_n) + [(1+u_n)^{\overline{p}-1} - 1]G_k''(u_n)\right] D_i u_n \chi_{(0,\tau)} \\ &= (\overline{p}-1)(1+u_n)^{\overline{p}-2}G_k'(u_n) D_i u_n \chi_{(0,\tau)}, \end{aligned}$$

we obtain

$$\begin{aligned} &\int_\Omega \Phi(u_n)dx + \alpha(p^- - 1)\sum_{i=1}^N \int_0^\tau \int_\Omega \frac{|D_i u_n|^{p_i}}{(1+u_n)^\theta}(1+u_n)^{\overline{p}-2}G_k'(u_n)dxdt \\ &\leq \int_0^\tau \int_\Omega f_n(1+u_n)^{\overline{p}-1-\gamma}G_k'(u_n)dxdt, \end{aligned} \tag{4.3}$$

where

$$\Phi(u_n) = \int_0^{u_n} \left((1+s)^{\overline{p}-1} - 1\right) G_k'(s)ds.$$

Using the fact that $(1+y)^\delta \geq 1 + y^\delta$ for all $b \geq 0$ and all $\delta > 1$, we obtain

$$\begin{aligned} \Phi(u_n) &\geq \int_0^{u_n} s^{\overline{p}-1}G_k'(s)ds \\ &= \int_0^{u_n} \left(\frac{1}{\overline{p}}s^{\overline{p}}G_k'(s)\right)' ds \\ &\geq \frac{1}{\overline{p}}u_n^{\overline{p}}G_k'(u_n) \\ &\geq \frac{1}{\overline{p}}G_k(u_n)^{\overline{p}}G_k'(u_n). \end{aligned} \tag{4.4}$$

this occurs because, when $u_n \geq 0$ almost everywhere in Ω, we can observe that $u_n \geq G_k(u_n) \geq 0$. Let $A_{k,n}(t) = \{x \in \Omega : u_n(x,t) > k\}$ for $t \in (0, T)$. By using (4.3), (4.4), (3.1), Hölder's inequality, and the fact that $1 + u_n \leq 2(k + G_k(u_n))$ for $k \geq 1$ and that $G_k'(u_n) = 0$ on the set $\{x \in \Omega : u_n(x,t) \leq k\}$, we obtain the following inequality for all $i = 1, \ldots, N$

$$\begin{aligned} &\int_{A_{k,n}(\tau)} G_k(u_n(\tau))^{\overline{p}}dx + \int_0^\tau \int_{A_{k,n}(t)} \frac{|D_i u_n|^{p_i}}{(1+u_n)^{\theta-\overline{p}+2}}dxdt \\ &\leq C_1\|f\|_{L^m(Q)} \left(\int_0^T \int_{A_{k,n}(t)} (k + G_k(u_n))^{(\overline{p}-1-\gamma)m'}dxdt\right)^{\frac{1}{m'}}. \end{aligned}$$

Then,

$$\sup_{\tau\in[0,T]}\int_{A_{k,n}(\tau)} G_k(u_n(\tau))^{\overline{p}}dx + \sup_{\tau\in[0,T]}\int_0^\tau\int_{A_{k,n}(t)}\frac{|D_iu_n|^{p_i}}{(1+u_n)^{\theta-\overline{p}+2}}dxdt$$
$$\leq C_1\|f\|_{L^m(Q)}\left(\int_0^T\int_{A_{k,n}(t)}(k+G_k(u_n))^{(\overline{p}-1-\gamma)m'}dxdt\right)^{\frac{1}{m'}}.$$

Hence, for all $i=1,\dots,N$, we have

$$\|G_k(u_n)\|^{\overline{p}}_{L^\infty(0,T;L^{\overline{p}}(A_{k,n}))}$$
$$\leq C_1\|f\|_{L^m(Q)}\left(\int_0^T\int_{A_{k,n}(t)}(k+G_k(u_n))^{(\overline{p}-1-\gamma)m'}dxdt\right)^{\frac{1}{m'}}, \tag{4.5}$$

and

$$\int_0^T\int_{A_{k,n}(t)}\frac{|D_iu_n|^{p_i}}{(1+u_n)^{\theta-\overline{p}+2}}dxdt$$
$$\leq C_1\|f\|_{L^m(Q)}\left(\int_0^T\int_{A_{k,n}(t)}(k+G_k(u_n))^{(\overline{p}-1-\gamma)m'}dxdt\right)^{\frac{1}{m'}}. \tag{4.6}$$

The proof is divided into two cases.

Case 1: Suppose that

$$\overline{p}-2<\theta<\min\left\{\overline{p}-1+\frac{\overline{p}}{N}+\gamma\left(1+\frac{\overline{p}}{N}\right),\overline{p}-2+\frac{\overline{p}^2}{N}\right\}. \tag{4.7}$$

For all $1<\sigma_i<p_i$, we can write

$$\int_Q|D_iG_k(u_n)|^{\sigma_i}dxdt=\int_Q\frac{|D_iG_k(u_n)|^{\sigma_i}}{(1+u_n)^{\frac{(\theta-\overline{p}+2)\sigma_i}{p_i}}}(1+u_n)^{\frac{(\theta-\overline{p}+2)\sigma_i}{p_i}}dxdt.$$

Using (4.6) and Hölder's inequality, we have

$$\int_Q|D_iG_k(u_n)|^{\sigma_i}dxdt$$
$$\leq\left(\int_0^T\int_{A_{k,n}(t)}\frac{|D_iG_k(u_n)|^{p_i}}{(1+u_n)^{\theta-\overline{p}+2}}dxdt\right)^{\frac{\sigma_i}{p_i}}$$
$$\times\left(\int_0^T\int_{A_{k,n}(t)}(1+u_n)^{\frac{(\theta-\overline{p}+2)\sigma_i}{p_i-\sigma_i}}dxdt\right)^{\frac{p_i-\sigma_i}{p_i}}$$
$$\leq C_2\left(\int_0^T\int_{A_{k,n}(t)}(k+G_k(u_n)^{m'(\overline{p}-1-\gamma)}dxdt\right)^{\frac{\sigma_i}{p_im'}}$$
$$\times\left(\int_0^T\int_{A_{k,n}(t)}(k+G_k(u_n))^{\frac{(\theta-\overline{p}+2)\sigma_i}{p_i-\sigma_i}}dxdt\right)^{\frac{p_i-\sigma_i}{p_i}}. \tag{4.8}$$

Now let's consider $\sigma_i=\lambda p_i$, where $\lambda\in[0,1)$ such that,

$$\frac{(\theta-\overline{p}+2)\sigma_i}{p_i-\sigma_i}=\frac{(\theta-\overline{p}+2)\lambda}{1-\lambda}=\frac{N+\overline{p}}{N}\overline{\sigma}, \tag{4.9}$$

so that

$$\overline{\sigma} = \frac{\overline{p}(\overline{p}+N)+N(\overline{p}-2-\theta)}{N+\overline{p}}, \tag{4.10}$$

$$\overline{\sigma}^* = \frac{N[\overline{p}(N+\overline{p})+N(\overline{p}-2-\theta)]}{N^2-\overline{p}^2-N(\overline{p}-2-\theta)}. \tag{4.11}$$

Putting $\mu = \frac{N+\overline{p}}{N}\overline{\sigma}$, we get

$$\mu = \frac{\overline{p}(\overline{p}+N)+N(\overline{p}-2-\theta)}{N} \tag{4.12}$$

Taking the $\frac{N\sigma_i}{\overline{\sigma}}$-th root of each side of inequality (4.8), and utilizing (4.9) and (4.12), we arrive at the following inequality

$$\begin{aligned}
&\left(\int_Q |D_i G_k(u_n)|^{\sigma_i} dxdt\right)^{\frac{\overline{\sigma}}{N\sigma_i}} \\
&\leq C_3 \left(\int_0^T \int_{A_{k,n}(t)} (k+G_k(u_n)^{m'(\overline{p}-1-\gamma)} dxdt\right)^{\frac{\overline{\sigma}}{Np_i m'}} \\
&\quad \times \left(\int_0^T \int_{A_{k,n}(t)} (k+G_k(u_n))^{\mu} dxdt\right)^{(1-\lambda)\frac{\overline{\sigma}}{N\sigma_i}},
\end{aligned}$$

this implies that

$$\begin{aligned}
&\prod_{i=1}^N \left(\int_Q |D_i G_k(u_n)|^{\sigma_i} dxdt\right)^{\frac{\overline{\sigma}}{N\sigma_i}} \\
&\leq C_4 \left(\int_0^T \int_{A_{k,n}(t)} (k+G_k(u_n)^{m'(\overline{p}-1-\gamma)} dxdt\right)^{\frac{\overline{\sigma}}{\overline{p}m'}} \\
&\quad \times \left(\int_0^T \int_{A_{k,n}(t)} (k+G_k(u_n))^{\mu} dxdt\right)^{\frac{\overline{p}-\overline{\sigma}}{\overline{p}}}.
\end{aligned} \tag{4.13}$$

Since $m > \frac{N}{\overline{p}} + 1$ and $\overline{\sigma} > \overline{p} - 1 - \gamma$, we can conclude that

$$\frac{\overline{\sigma}(N+\overline{p})}{m'N(\overline{p}-1-\gamma)} = \frac{\overline{\sigma}}{\overline{p}-1-\gamma} \times \frac{N+\overline{p}}{m'N} > 1.$$

By applying Hölder's inequality, we can derive the following inequality

$$\begin{aligned}
&\int_0^T \int_{A_{k,n}(t)} (k+G_k(u_n))^{m'(\overline{p}-1-\gamma)} dxdt \\
&\leq \left(\int_0^T \int_{A_{k,n}(t)} (k+G_k(u_n))^{\mu}\right)^{\frac{m'N(\overline{p}-1-\gamma)}{(N+\overline{p})\overline{\sigma}}} \\
&\quad \times \left(\int_0^T |A_{k,n}(t)| dt\right)^{1-\frac{Nm'(\overline{p}-1-\gamma)}{(N+\overline{p})\overline{\sigma}}}.
\end{aligned} \tag{4.14}$$

Therefore, using (4.13) and (4.14), we can write for all $k \geq 1$

$$\prod_{i=1}^N \left(\int_Q |D_i G_k(u_n)|^{\sigma_i} dxdt\right)^{\frac{\overline{\sigma}}{N\sigma_i}}$$

$$\leq C_5 \left(\int_0^T \int_{A_{k,n}(t)} (k + G_k(u_n))^{\mu} dx dt \right)^{\frac{N(\overline{p}-1-\gamma)+(\overline{p}-\overline{\sigma})(N+\overline{p})}{\overline{p}(N+\overline{p})}} \times \left(\int_0^T |A_{k,n}(t)| dt \right)^{\frac{\overline{\sigma}}{\overline{p}m'} - \frac{N(\overline{p}-1-\gamma)}{(N+\overline{p})\overline{p}}}. \tag{4.15}$$

Now, observe that

$$\theta < \overline{p} - 2 + \frac{\overline{p}^2}{N} \Leftrightarrow \overline{p} < \mu < \overline{\sigma}^*$$

Using the interpolation inequality, we obtain

$$\|G_k(u_n)(t,.)\|_{L^{\mu}(A_{k,n})} \leq \|G_k(u_n)(t,.)\|^{1-\tau}_{L^{\overline{p}}(A_{k,n})} \times \|G_k(u_n)(t,.)\|^{\tau}_{L^{\overline{\sigma}^*}(A_{k,n})}, \tag{4.16}$$

where

$$\tau = \frac{(\overline{p}-\mu)\overline{\sigma}^*}{(\overline{p}-\overline{\sigma}^*)\mu}. \tag{4.17}$$

By equations (4.10), (4.11), (4.12), and (4.17), we can observe the following relationships

$$\tau = \frac{N}{N+\overline{p}}, \quad \tau\mu = \overline{\sigma} \quad \text{and} \quad (1-\tau)\mu = \frac{\overline{p}}{N}\overline{\sigma}. \tag{4.18}$$

Using (4.16), (4.14) and (4.18), the result is

$$\begin{aligned}
&\int_0^T \|G_k(u_n)\|^{\mu}_{L^{\mu}(A_{k,n})} dt \\
&\leq \|G_k(u_n)\|^{\frac{\overline{p}}{N}\overline{\sigma}}_{L^{\infty}(0,T;L^{\overline{p}}(A_{k,n}))} \int_0^T \|G_k(u_n)\|^{\overline{\sigma}}_{L^{\overline{\sigma}^*}(A_{k,n})} dx \\
&\leq C_6 \left(\int_0^T \int_{A_{k,n}(t)} (k + G_k(u_n))^{\mu} dx dt \right)^{\frac{\overline{p}-1-\gamma}{N+\overline{p}}} \\
&\quad \times \left(\int_0^T |A_{k,n}(t)| dt \right)^{\frac{\overline{\sigma}}{Nm'} - \frac{\overline{p}-1-\gamma}{N+\overline{p}}} \int_0^T \|G_k(u_n)\|^{\overline{\sigma}}_{L^{\overline{\sigma}^*}(A_{k,n})} dx.
\end{aligned} \tag{4.19}$$

We apply the anisotropic Sobolev inequality with $r = \overline{\sigma}^*$, denoted as (2.4), to obtain

$$\int_0^T \|G_k(u_n)\|^{\overline{\sigma}}_{L^{\overline{\sigma}^*}(A_{k,n})} dt \leq \int_0^T \prod_{i=1}^N \left(\int_{A_{k,n}(t)} |D_i G_k(u_n)|^{\sigma_i} dx \right)^{\frac{\overline{\sigma}}{N\sigma_i}} dt.$$

Since $\sum_{i=1}^N \frac{\overline{\sigma}}{N\sigma_i} = 1$, applying the Hölder's inequality and (4.15), we have

$$\begin{aligned}
\int_0^T \|G_k(u_n)\|^{\overline{\sigma}}_{L^{\overline{\sigma}^*}(A_{k,n})} dt &\leq \prod_{i=1}^N \left(\int_Q |D_i G_k(u_n)|^{\sigma_i} dx dt \right)^{\frac{\overline{\sigma}}{N\sigma_i}} \\
&\leq C_5 \left(\int_0^T \int_{A_{k,n}(t)} (k + G_k(u_n))^{\mu} dx dt \right)^{\frac{N(\overline{p}-1-\gamma)+(\overline{p}-\overline{\sigma})(N+\overline{p})}{\overline{p}(N+\overline{p})}} \\
&\quad \times \left(\int_0^T |A_{k,n}(t)| dt \right)^{\frac{\overline{\sigma}}{\overline{p}m'} - \frac{N(\overline{p}-1-\gamma)}{(N+\overline{p})\overline{p}}}.
\end{aligned} \tag{4.20}$$

Based on inequalities (4.19) and (4.20), we can deduce that

$$\int_0^T \int_{A_{k,n}} G_k(u_n)^\mu dxdt \le C_7 \left(\int_0^T \int_{A_{k,n}(t)} (k + G_k(u_n))^\mu dxdt \right)^{\frac{N(\overline{p}-1-\gamma)}{\overline{p}(N+\overline{p})} + \frac{\overline{p}-\overline{\sigma}}{\overline{p}}} \times \left(\int_0^T \int_{A_{k,n}(t)} (k + G_k(u_n))^\mu dxdt \right)^{\frac{\overline{p}-1-\gamma}{N+\overline{p}}} \times \left(\int_0^T |A_{k,n}(t)| dt \right)^{\frac{(N+\overline{p})\overline{\sigma}}{\overline{p}Nm'} - \frac{\overline{p}-1-\gamma}{\overline{p}}}.$$

Hence,

$$\mathbf{A}_{nk} \le C_8 \mathbf{A}_{nk}^{\frac{2\overline{p}-1-\gamma-\overline{\sigma}}{\overline{p}}} (\Psi_n(k))^{\frac{(N+\overline{p})\overline{\sigma}}{\overline{p}Nm'} - \frac{\overline{p}-1-\gamma}{\overline{p}}} + C_9 k^{\frac{(N+\overline{p})\overline{\sigma}}{N}\left(\frac{2\overline{p}-1-\gamma-\overline{\sigma}}{\overline{p}}\right)} (\Psi_n(k))^{\frac{(N+\overline{p})\overline{\sigma}}{\overline{p}Nm'} + \frac{\overline{p}-\overline{\sigma}}{\overline{p}}}, \tag{4.21}$$

where

$$\Psi_n(k) = \int_0^T |A_{k,n}(t)| dt, \quad \mathbf{A}_{nk} = \int_0^T \int_{A_{k,n}(t)} G_k(u_n)^{\frac{(N+p)\sigma}{N}} dxdt.$$

Thus, according to Young's inequality, for all $\varepsilon > 0$,

$$\mathbf{A}_{nk}^{\frac{2\overline{p}-1-\gamma-\overline{\sigma}}{\overline{p}}} (\Psi_n(k))^{\frac{(N+\overline{p})\overline{\sigma}}{\overline{p}Nm'} - \frac{\overline{p}-1-\gamma}{\overline{p}}} \le \varepsilon \mathbf{A}_{nk} + C(\varepsilon) \Psi_n(k)^{\left(\frac{(N+\overline{p})\overline{\sigma}}{Nm'} - (\overline{p}-1-\gamma)\right)\frac{1}{\overline{\sigma}+1+\gamma-\overline{p}}}. \tag{4.22}$$

Applying (4.21) to (4.22) and taking $\varepsilon = \frac{1}{2C_8}$, we get

$$\mathbf{A}_{nk} \le C_{10} \Psi_n(k)^{\left(\frac{(N+\overline{p})\overline{\sigma}}{Nm'} - (\overline{p}-1-\gamma)\right)\frac{1}{\overline{\sigma}+1+\gamma-\overline{p}}} + C_{10} k^{\frac{(N+\overline{p})\overline{\sigma}}{N}\left(\frac{2\overline{p}-1-\gamma-\overline{\sigma}}{\overline{p}}\right)} (\Psi_n(k))^{\frac{(N+\overline{p})\overline{\sigma}}{\overline{p}Nm'} + \frac{\overline{p}-\overline{\sigma}}{\overline{p}}}. \tag{4.23}$$

Since $m > \frac{N}{\overline{p}} + 1$, we can conclude that

$$\left(\frac{(N+\overline{p})\overline{\sigma}}{Nm'} - (\overline{p}-1-\gamma)\right)\frac{1}{\overline{\sigma}+1+\gamma-\overline{p}} > \frac{(N+\overline{p})\overline{\sigma}}{\overline{p}Nm'} + \frac{\overline{p}-\overline{\sigma}}{\overline{p}} > 1.$$

We observe that $|\Psi_n(k)| \le T|\Omega|$, for $k \ge 1$. Therefore, we can state that

$$\mathbf{A}_{nk} \le C_{11} k^{\frac{(N+\overline{p})\overline{\sigma}}{N}\left(\frac{2\overline{p}-1-\gamma-\overline{\sigma}}{\overline{p}}\right)} (\Psi_n(k))^{\frac{(N+\overline{p})\overline{\sigma}}{\overline{p}Nm'} + \frac{\overline{p}-\overline{\sigma}}{\overline{p}}} \tag{4.24}$$

Recalling the property of $G_k(s)$, if $h > k$, we have $|G_k(u_n)| > h - k$ on $A_{h,n}(t)$, and $A_{h,n}(t) \subset A_{k,n}(t)$. Considering $\frac{2\overline{p}-1-\gamma-\overline{\sigma}}{\overline{p}} < 1$, we can express (4.24) as follows

$$\Psi_n(h) \le \frac{C_{11} k^{\frac{(N+\overline{p})\overline{\sigma}}{N}\left(\frac{2\overline{p}-1-\gamma-\overline{\sigma}}{\overline{p}}\right)} (\Psi_n(k))^{\frac{(N+\overline{p})\overline{\sigma}}{\overline{p}Nm'} + \frac{\overline{p}-\overline{\sigma}}{\overline{p}}}}{(h-k)^{\frac{(N+\overline{p})\overline{\sigma}}{N}}}, \quad \forall h > k \ge 1. \tag{4.25}$$

By applying Lemma 2.3 to the values of ρ, η, and δ given as

$$\rho = \frac{2\overline{p}-1-\gamma-\overline{\sigma}}{\overline{p}}, \quad \eta = \frac{(N+\overline{p})\overline{\sigma}}{N}, \quad \text{and} \quad \delta = \frac{(N+\overline{p})\overline{\sigma}}{\overline{p}Nm'} + \frac{\overline{p}-\overline{\sigma}}{\overline{p}},$$

we can conclude that there exists a positive constant k^* such that $\Psi_n(k^*) = 0$. Since it is known that $|\Psi_n(k)| \leq T|\Omega|$ (as stated in the proof of Lemma A.1 in [23]), there exists a positive constant ω independent of n such that $k^* \leq \omega$. Therefore, we have

$$\Psi_n(\omega) = 0. \tag{4.26}$$

Hence, from (4.26), we obtain (4.1).

Case 2: Suppose that $\theta \in [0, \overline{p} - 2]$. Using (4.6), we can write

$$\begin{aligned}
&\int_Q |D_i G_k(u_n)|^{p_i} dxdt \\
&\leq \int_0^T \int_{A_{k,n}(t)} |D_i u_n|^{p_i} (1 + |u_n|)^{\overline{p}-\theta-2} dxdt \\
&\leq C \left(\int_0^T \int_{A_{k,n}(t)} (k + |G_k(u_n)|)^{(\overline{p}-1-\gamma)m'} dxdt \right)^{\frac{1}{m'}},
\end{aligned} \tag{4.27}$$

For each $i = 1, \ldots, N$, we take the $p_i N$-th root of both sides of inequality (4.27), resulting in

$$\begin{aligned}
&\prod_{i=1}^N \left(\int_Q |D_i G_k(u_n)|^{p_i} dxdt \right)^{\frac{1}{Np_i}} \\
&\leq C \left(\int_0^T \int_{A_{k,n}(t)} (k + G_k(u_n))^{(\overline{p}-1-\gamma)m'} dxdt \right)^{\frac{1}{\overline{p}m'}},
\end{aligned}$$

Applying Lemma 2.2 with $r = \overline{p}^*$ (where $\overline{p} < N$) to the left-hand side of the above inequality and using Hölder's inequality with exponent $\frac{\overline{p}N}{(\overline{p}-1-\gamma)m'(N-\overline{p})} > 1$ (since $m > \frac{N}{\overline{p}} + 1 > \frac{N}{\overline{p}}$ and $\gamma \geq 0$) on the right-hand side, we obtain

$$\begin{aligned}
\int_0^T \int_{A_{k,n}} G_k(u_n)^{\overline{p}^*} dxdt \leq C &\left(\int_0^T \int_{A_{k,n}(t)} (k + G_k(u_n))^{\overline{p}^*} dxdt \right)^{\frac{\overline{p}-1-\gamma}{\overline{p}}} \\
&\times \left(\int_0^T |A_{k,n}(t)| dt \right)^{\frac{\overline{p}^*}{m'\overline{p}} - \frac{\overline{p}-1-\gamma}{\overline{p}}}.
\end{aligned}$$

Thus, we can establish the following inequality

$$\mathbf{B}_{nk} \leq C \mathbf{B}_{nk}^{\frac{\overline{p}-1-\gamma}{\overline{p}}} (\Psi_n(k))^{\frac{\overline{p}^*}{\overline{p}m'} - \frac{\overline{p}-1-\gamma}{\overline{p}}} + C k^{\frac{\overline{p}-1-\gamma}{\overline{p}}} (\Psi_n(k))^{\frac{\overline{p}^*}{\overline{p}m'}}, \tag{4.28}$$

where

$$\mathbf{B}_{nk} = \int_0^T \int_{A_{k,n}(t)} G_k(u_n)^{\overline{p}^*} dxdt.$$

For any positive value of ε, Young's inequality implies that

$$\mathbf{B}_{nk}^{\frac{\overline{p}-1-\gamma}{\overline{p}}} (\Psi_n(k))^{\frac{\overline{p}^*}{\overline{p}m'} - \frac{\overline{p}-1-\gamma}{\overline{p}}} \leq \varepsilon \mathbf{B}_{nk} + C(\varepsilon) \Psi_n(k)^{\frac{\overline{p}^* - (\overline{p}-1-\gamma)m'}{m'(1+\gamma)}}. \tag{4.29}$$

Using (4.28) and (4.29), and choosing $\varepsilon = \frac{1}{2C}$, we get

$$\mathbf{B}_{nk} \leq C \Psi_n(k)^{\frac{\overline{p}^* - (\overline{p}-1-\gamma)m'}{m'(1+\gamma)}} + C k^{\frac{\overline{p}-1-\gamma}{\overline{p}}} (\Psi_n(k))^{\frac{\overline{p}^*}{\overline{p}m'}}.$$

The condition $m > \frac{N}{\overline{p}} + 1 > \frac{N}{\overline{p}}$ gives rise to the inequality

$$\frac{\overline{p}^* - (\overline{p} - 1 - \gamma)m'}{m'(1+\gamma)} > \frac{\overline{p}^*}{\overline{p}m'}.$$

Hence,

$$\mathbf{B}_{nk} \leq Ck^{\frac{\overline{p}-1-\gamma}{\overline{p}}} (\Psi_n(k))^{\frac{\overline{p}^*}{\overline{p}m'}}. \tag{4.30}$$

Taking into account the property of $G_k(s)$, when $h > k$, we observe that $|G_k(u_n)| > h - k$ on $A_{h,n}(t)$, and note that $A_{h,n}(t)$ is a subset of $A_{k,n}(t)$. With reference to equation (4.30), and considering the condition $\overline{p}^* > 1$, we can conclude that

$$\Psi_n(h) \leq \frac{Ck^{\overline{p}^*\frac{\overline{p}-1-\gamma}{\overline{p}\,\overline{p}^*}} (\Psi_n(k))^{\frac{\overline{p}^*}{\overline{p}m'}}}{(h-k)^{\overline{p}^*}}, \quad \forall h > k \geq 1. \tag{4.31}$$

Hence, by employing Lemma 2.3 in combination with

$$\rho = \frac{\overline{p} - 1 - \gamma}{\overline{p}\ \overline{p}^*} < 1, \quad \eta = \overline{p}^* > 0 \quad \text{and} \quad \delta = \frac{\overline{p}^*}{\overline{p}m'} > 1,$$

As a result, we obtain $\Psi_n(\omega) = 0$, which concludes the proof of (4.1).

By considering u_n as a test function for problem (3.4), and using (1.2) and (3.1), we obtain the following

$$\frac{1}{2}\int_\Omega u_n(T)^2 dx + \alpha \sum_{i=1}^{N} \int_Q \frac{|D_i u_n|^{p_i}}{(1+u_n)^\theta} dxdt \leq \int_Q f_n u_n^{1-\gamma} dxdt. \tag{4.32}$$

By discarding the nonnegative first term in (4.32), and using the boundedness of the sequence $\{u_n\}_n$ in $L^\infty(Q)$, we obtain

$$\begin{aligned} \int_Q \frac{|D_i u_n|^{p_i}}{(1+u_n)^\theta} dxdt &\leq \frac{1}{\alpha} \|f\|_{L^m(Q)} |Q|^{\frac{1}{m'}} \|u_n^{1-\gamma}\|_{L^\infty(Q)} \\ &\leq C \quad \forall i = 1, \ldots, N. \end{aligned} \tag{4.33}$$

Given that the sequence $\{u_n\}_n$ is bounded in $L^\infty(Q)$ and based on inequality (4.33), we can conclude

$$\begin{aligned} \int_Q |D_i u_n|^{p_i} dxdt &= \int_Q \frac{|D_i u_n|^{p_i}}{(1+u_n)^\theta} (1+u_n)^\theta dxdt \\ &\leq \left(1 + \|u_n\|_{L^\infty(Q)}\right) \int_Q \frac{|D_i u_n|^{p_i}}{(1+u_n)^\theta} dxdt \\ &\leq C \quad \forall i = 1, \ldots, N. \end{aligned}$$

Therefore, we can deduce estimate (4.2). □

Lemma 4.2 *Suppose that the assumptions of Theorem* 2.7 *are satisfied, and let u_n be a solution of* (3.4)*. Then there exists a positive constant C such that the following estimate holds*

$$\left\| \frac{\partial u_n}{\partial t} \right\|_{L^r(0,T;W^{-1,r}(\Omega)) + L^1(Q)} \leq C. \tag{4.34}$$

where

$$1 < r < \min_{i=\overline{1,N}} \left\{ \frac{N + \overline{p}}{\overline{p}m} \frac{p_i}{p_i - 1} \right\}. \tag{4.35}$$

Proof For all $n \in \mathbb{N}$, if u_n is a solution of problem (3.4), then we have the expression

$$\frac{\partial u_n}{\partial t} = \sum_{i=1}^{N} D_i(b_i(x,t,T_n(u_n),Du_n)) + g_n(u_n)f_n,$$

Given that $g_n(u_n)f_n$ is bounded in $L^1(Q)$, our objective is to demonstrate that

$$v_n = \sum_{i=1}^{N} D_i(b_i(x,t,T_n(u_n),Du_n))$$

belongs to a bounded set in $L^r(0,T;W^{-1,r}(\Omega)) + L^1(Q)$. To be precise, we define $v_n(t) = v_n$ for $t \in (0,T)$. By using the condition (1.4) and applying the Hölder inequality, we obtain

$$\begin{aligned}
&\|v_n\|_{W^{-1,r}(\Omega)} \\
&= \sup_{\substack{\varphi \in W_0^{1,r'}(\Omega) \\ \|\varphi\|_{W_0^{1,r'}(\Omega)} \le 1}} \left| \int_\Omega \sum_{i=1}^{N} b_i(x,t,T_n(u_n),Du_n) D_i\varphi dx \right| \\
&\le \beta \sup_{\substack{\varphi \in W_0^{1,r'}(\Omega) \\ \|\varphi\|_{W_0^{1,r'}(\Omega)} \le 1}} \sum_{i=1}^{N} \int_\Omega \left(k + |u_n|^{\overline{p}} + \sum_{j=1}^{N} |D_j u_n|^{p_j} \right)^{1-\frac{1}{p_i}} |D_i\varphi| dx \\
&\le C \sup_{\substack{\varphi \in W_0^{1,r'}(\Omega) \\ \|\varphi\|_{W_0^{1,r'}(\Omega)} \le 1}} \sum_{i=1}^{N} \|D_i\varphi\|_{L^{r'}(\Omega)} \\
&\quad \times \left[\sum_{i=1}^{N} \int_\Omega \left(k + |u_n|^{\overline{p}} + \sum_{j=1}^{N} |D_j u_n|^{p_j} \right)^{\left(1-\frac{1}{p_i}\right)r} \right]^{\frac{1}{r}},
\end{aligned}$$

since $\varphi \in W_0^{1,r'}(\Omega)$, we conclude that

$$\begin{aligned}
&\int_0^T \|v_n\|_{W^{-1,r}(\Omega)}^r dt \\
&\le C \int_0^T \left[\sum_{i=1}^{N} \int_\Omega \left(k + |u_n|^{\overline{p}} + \sum_{j=1}^{N} |D_j u_n|^{p_j} \right)^{\left(1-\frac{1}{p_i}\right)r} \right] dt. \qquad (4.36)
\end{aligned}$$

Thanks to (4.35), we obtain

$$\left(1 - \frac{1}{p_i}\right) r < \frac{N+\overline{p}}{\overline{p}m} \quad \forall\, i = 1, \ldots, N,$$

there exists ϵ such that for all $i = 1, \ldots, N$

$$\left(1 - \frac{1}{p_i}\right) r < \epsilon < \frac{N+\overline{p}}{\overline{p}m}.$$

Since $m > \frac{N}{\overline{p}} + 1$ we obtain that

$$\frac{N+\overline{p}}{\overline{p}m} < 1.$$

Therefore $\epsilon \in (0, 1)$ and for every $i = 1, \ldots, N$

$$1 \leq \epsilon p_i < \frac{N + \overline{p}}{\overline{p}m} p_i \leq p_i, \tag{4.37}$$

$$\left(1 - \frac{1}{p_i}\right) \frac{r}{\epsilon} < 1. \tag{4.38}$$

By Hölder inequality (4.36), (4.38) and the imbedding $W_0^{1,p_i}(\Omega) \hookrightarrow L^{\overline{p}}(\Omega)$, and using the inequality

$$\left(\sum_{i=1}^{N} a_i\right)^{\sigma} \leq \sum_{i=1}^{N} a_i^{\sigma}, \qquad \forall \sigma \in [0, 1],$$

we obtain

$$\begin{aligned}
&\int_0^T \|v_n\|^r_{W^{-1,r}(\Omega)} dt \\
&\leq C \sum_{i=1}^{N} \left[\int_Q \left(k^{\epsilon} + |u_n|^{\epsilon \overline{p}} + \sum_{j=1}^{N} |D_j u_n|^{\epsilon p_j} \right) dxdt \right]^{\left(1 - \frac{1}{p_i}\right)\frac{r}{\epsilon}} \\
&\leq C \sum_{i=1}^{N} \left[\int_Q \left(k^{\epsilon} + |D_i u_n|^{\epsilon p_i} + \sum_{j=1}^{N} |D_j u_n|^{\epsilon p_j} \right) dxdt \right]^{\left(1 - \frac{1}{p_i}\right)\frac{r}{\epsilon}}.
\end{aligned} \tag{4.39}$$

Using (4.37), (4.38) and the fact that $\epsilon < 1$, we get

$$\int_0^T \|v_n\|^r_{W^{-1,r}(\Omega)} dt \leq C \sum_{i=1}^{N} \int_Q \left(k^{\epsilon} + C|D_i u_n|^{p_i}\right) dxdt + C.$$

Since $k \in L^{p_i}(Q)$, from Lemma 4.1, we find (4.34). This completes the proof of Lemma 4.2. □

5 Proof of the main result

By Lemma 4.1, the sequence $\{u_n\}_n$ remains in a bounded set of $\mathbb{L}^{\overrightarrow{p}}\left(0, T; X_0^{1,\overrightarrow{p}}(\Omega)\right)$. Additionally, from Lemma 4.2, the sequence $\left\{\frac{\partial u_n}{\partial t}\right\}_n$ remains in a bounded set of the space

$$L^1(0, T; (W^{1,r'}(\Omega))') + L^1(Q) \hookrightarrow L^1\left(0, T; W^{-1,s}(\Omega)\right) + L^1(Q),$$

for all $s < \min\left\{\frac{N}{N-1}, r\right\}$. Then, by Lemma 4.2, the sequence $\left\{\frac{\partial u_n}{\partial t}\right\}_n$ remains in a bounded set of the space

$$L^1\left(0, T; W^{-1,s}(\Omega)\right) + L^1(Q).$$

Now, using Corollary 4 in [26], we deduce that $\{u_n\}_n$ is relatively compact in $L^1(Q)$. This implies that we can extract a sub-sequence (denoted again by $\{u_n\}_n$) such that

$$u_n \to u \quad \text{strongly in} \quad L^1(Q), \tag{5.1}$$

$$u_n \to u \quad \text{a.e. on} \quad Q. \tag{5.2}$$

To proceed with the proof, we require the following lemma.

Lemma 5.1 [17] *For every $k > 0$, there exists a function θ_k such that for any $\varepsilon > 0$, the following inequality holds*

$$\limsup_n \sum_{i=1}^{N} \int_{\{|u_n - u^k| \leq \varepsilon\}} b_i(x, t, T_n(u_n), Du_n)(D_i u_n - D_i u^k) dxdt \leq \theta_k(\varepsilon),$$

where $\lim \theta_k(\varepsilon) = 0$, $u^k = \phi_k(u)$, *with ϕ_k is family of functions satisfy* (*see* [17])

(i) ϕ_k *is a twice differentiable function,* ϕ_k', ϕ_k'' *are bounded on* $\mathbb{R}$
(ii) $\phi_k(\sigma) = \sigma$ *if* $|\sigma| \leq k$, *and* $\phi_k'(\sigma) = 0$ *if* $|\sigma| \geq k + \frac{1}{k}$, $0 < \phi_k' < 1$ *on the set* $(k, k + \frac{1}{k}) \cup (-(k + \frac{1}{k}), -k)$.

Now, by adapting the approach of Lemma 4.11 in [19] and using Lemma 5.1, there exists a subsequence (still denoted $\{u_n\}_n$) such that for any $i = 1, \ldots, N$

$$(b_i(x, t, u_n, Du_n) - b_i(x, t, u, Du))(D_i u_n - D_i u) \to 0 \text{ a.e. } (x, t) \in Q.$$

By employing a technique similar to the one used in [8], we have

$$D_i u_n \to D_i u \quad \text{a.e. in } \ Q \ \ \forall i = 1, \ldots, N. \tag{5.3}$$

As the functions b_i are Carathéodory functions, and from (5.2) and (5.3), we can deduce that

$$b_i(x, t, u_n, Du_n) \to b_i(x, t, u, Du) \quad \text{a.e. in } \ Q \ \ \forall i = 1, \ldots, N. \tag{5.4}$$

By combining (1.4) and (5.4), it follows that

$$b_i(x, t, u_n, Du_n) \rightharpoonup b_i(x, t, u, Du) \quad \text{weakly in } \ L^{p_i'}(Q) \ \ \forall i = 1, \ldots, N. \tag{5.5}$$

We begin by proving an important lemma that will be useful in the proof of Theorem 2.7.

Lemma 5.2 *Let* u_n *be a weak solution of* (3.4). *Then, for all* $\mathbb{L}^{\overrightarrow{p}}(0, T; X_0^{1,\overrightarrow{p}}(\Omega)) \cap L^\infty(Q)$, *we have*

$$\lim_{n \to +\infty} \int_Q g_n(u_n) f_n \varphi dx dt = \int_Q g(u) f \varphi dx dt. \tag{5.6}$$

Proof Considering $\mathbb{L}^{\overrightarrow{p}}(0, T; X_0^{1,\overrightarrow{p}}(\Omega)) \cap L^\infty(Q)$ as a test function in (3.4), we obtain

$$\begin{aligned} &\int_Q \frac{\partial u_n}{\partial t} \varphi dx dt + \sum_{i=1}^N \int_Q b_i(x, t, T_n(u_n), Du_n).D_i \varphi dx dt \\ &\quad = \int_Q g_n(u_n) f_n \varphi dx dt. \end{aligned} \tag{5.7}$$

If $g(0) < +\infty$, we can conclude that (5.6) holds true. Now, let's consider the case where $g(0) = +\infty$. Suppose φ is a non-negative function in $\mathbb{L}^{\overrightarrow{p}}(0, T; X_0^{1,\overrightarrow{p}}(\Omega)) \cap L^\infty(Q)$ as a test function in the weak formulation (5.7). By using(1.4) and Young's inequality, we get

$$\begin{aligned} &\int_Q g_n(u_n) f_n \varphi dx dt \\ &\leq \int_Q |u_n| \left| \frac{\partial \varphi}{\partial t} \right| dx dt \\ &\quad + \beta \sum_{i=1}^N \int_Q \left(k + |u_n|^{\overline{p}} + \sum_{j=1}^N |D_j u_n|^{p_j} \right)^{1 - \frac{1}{p_i}} |D_i \varphi| dx dt \\ &\leq \frac{1}{p^-} \int_Q |u_n|^{p_i} dx dt + \frac{1}{(p^-)'} \int_Q \left| \frac{\partial \varphi}{\partial t} \right|^{p_i'} dx dt + \beta \frac{1}{p^-} \sum_{i=1}^N \int_Q |D_i \varphi|^{p_i} dx dt \\ &\quad + \beta \frac{1}{(p^-)'} \sum_{i=1}^N \int_Q \left(|k| + |u_n|^{\overline{p}} + \sum_{j=1}^N |D_j u_n|^{p_j} \right) dx dt \\ &\leq C \left(\|u_n\|_{L^{p_i}(Q)}^{p_i} + \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^{p_i'}(Q)}^{p_i'} + \|k\|_{L^{p_i}(Q)}^{p_i} \right) \\ &\quad + C \left(\|u_n\|_{\mathbb{L}^{\overrightarrow{p}}(0,T;X_0^{1,\overrightarrow{p}}(\Omega))} + \|\varphi\|_{\mathbb{L}^{\overrightarrow{p}}(0,T;X_0^{1,\overrightarrow{p}}(\Omega))} \right) \end{aligned}$$

$$\leq C. \tag{5.8}$$

Therefore, (5.8) implies that the sequence $f_n g_n(u_n)\varphi_n$ is bounded in $L^1(Q)$. By taking the limit as $n \to +\infty$ in (5.8), we can apply Fatou's lemma to obtain

$$\int_Q fg(u)\varphi dxdt \leq C, \tag{5.9}$$

where C does not depend on n. Hence, we have $fg(u)\varphi \in L^1(Q)$ for all nonnegative $\mathbb{L}^{\overrightarrow{p}}(0,T; X_0^{1,\overrightarrow{p}}(\Omega)) \cap L^\infty(Q)$. This implies that

$$\int_{\{u=0\}} g(u) f\varphi dxdt = g(0)\int_{\{u=0\}} f\varphi dxdt < +\infty,$$

so that

$$\int_{\{u=0\}} f\varphi dxdt = 0, \quad \text{for all} \quad 0 \leq \varphi \in C_c^1([0,T)\times\Omega).$$

so that so that, $f\varphi = 0$ a.e. on $\{u=0\}$ for all $\mathbb{L}^{\overrightarrow{p}}(0,T; X_0^{1,\overrightarrow{p}}(\Omega)) \cap L^\infty(Q)$. Using the fact that $f \in L^m(Q) \subset L^1(Q)$, we can deduce (as shown in [4])

$$\{u=0\} \subset \{f=0\}. \tag{5.10}$$

For any fixed $\lambda > 0$, we can express

$$\begin{aligned}\int_Q f_n g_n(u_n)\varphi dxdt &= \int_{Q\cap\{u_n>\lambda\}} f_n g_n(u_n)\varphi dxdt \\ &\quad + \int_{Q\cap\{u_n\leq\lambda\}} f_n g_n(u_n)\varphi dxdt \\ &= \mathbf{I}^1_{n,\lambda} + \mathbf{I}^2_{n,\lambda}.\end{aligned} \tag{5.11}$$

For $\mathbf{I}^1_{n,\lambda}$, we can observe that

$$0 \leq g_n(u_n) f_n \chi_{\{u_n>\lambda\}}\varphi \leq \sup_{s\in[\lambda,+\infty)} [g(s)] f\varphi \in L^1(Q). \tag{5.12}$$

Using the Lebesgue's dominated convergence theorem and the fact that $\chi_{u_n>\lambda}$ converges to $\chi_{u\geq\lambda}$ almost everywhere in Ω, we have

$$\lim_{n\to\infty} \mathbf{I}^1_{n,\lambda} = \int_{Q\cap\{u\geq\lambda\}} g(u) f\varphi dxdt.$$

Since $g(u) f\varphi \in L^1(Q)$, applying the Lebesgue's dominated convergence theorem with respect to λ, we conclude that

$$\lim_{\lambda\to 0^+}\lim_{n\to\infty} \mathbf{I}^1_{n,\lambda} = \int_{Q\cap\{u\geq 0\}} g(u) f\varphi dxdt = \int_Q g(u) f\varphi dxdt.$$

By (5.10), it follows that

$$\lim_{\lambda\to 0^+}\lim_{n\to\infty} \mathbf{I}^1_{n,\lambda} = \int_{Q\cap\{u>0\}} g(u) f\varphi dxdt = \int_Q g(u) f\varphi dxdt. \tag{5.13}$$

To eliminate $\mathbf{I}^2_{n,\lambda}$, we consider $V_\lambda(u_n)\varphi$ as the test function in (3.4), where V_λ is defined in (2.8). By doing so, we obtain the following expression

$$\int_Q \frac{\partial u_n}{\partial t} V_\lambda(u_n)\varphi dxdt + \sum_{i=1}^N \int_Q b_i(x,t,T_n(u_n), Du_n) D_i(V_\lambda(u_n)\varphi) dxdt$$

$$= \int_Q f_n g_n(u_n) V_\lambda(u_n) \varphi dxdt. \tag{5.14}$$

Using integration by parties and definition of V_λ, we have

$$\int_Q \frac{\partial u_n}{\partial t} V_\lambda(u_n) \varphi dxdt = -\int_Q \Phi(u_n) \frac{\partial \varphi}{\partial t} dxdt, \tag{5.15}$$

where

$$\Phi(s) = \int_0^s V_\lambda(y) dy.$$

Using (1.2), $V'_\lambda(u_n) = -\frac{1}{\lambda}$ and the fact that

$$D_i(V_\lambda(u_n)\varphi) = D_i\varphi V_\lambda(u_n) + V'_\lambda(u_n)\varphi D_i u_n,$$

we get,

$$\begin{aligned} &\sum_{i=1}^N \int_Q b_i(x,t,T_n(u_n), Du_n) D_i(V_\lambda(u_n)\varphi) dxdt \\ &\quad \geq \sum_{i=1}^N \int_Q b_i(x,t,T_n(u_n), Du_n) V_\lambda(u_n). D_i\varphi dxdt. \end{aligned} \tag{5.16}$$

On the other hand, we have

$$\begin{aligned} \int_Q f_n g_n(u_n) V_\lambda(u_n) \varphi dxdt &= \int_{Q\cap\{u_n\leq\lambda\}} f_n g_n(u_n) V_\lambda(u_n) \varphi dxdt \\ &\quad + \int_{Q\cap\{\lambda<u_n<2\lambda\}} f_n g_n(u_n) V_\lambda(u_n) \varphi dxdt \\ &\leq \int_{Q\cap\{u_n\leq\lambda\}} f_n g_n(u_n) V_\lambda(u_n) \varphi dxdt. \end{aligned} \tag{5.17}$$

Combining (5.14)–(5.16) and (5.17), we obtain

$$\begin{aligned} &\int_{Q\cap\{u_n\leq\lambda\}} f_n g_n(u_n) V_\lambda(u_n) \varphi dxdt \\ &\leq \sum_{i=1}^N \int_Q b_i(x,t,T_n(u_n), Du_n) V_\lambda(u_n). D_i\varphi dxdt - \int_Q \Phi(u_n)\varphi_t dxdt. \end{aligned}$$

Using the boundedness of V_λ and the continuity of Φ, we can deduce that as n tends to infinity, $\Phi(u_n)\frac{\partial\varphi}{\partial t} \to \Phi(u)\frac{\partial\varphi}{\partial t}$ strongly in $L^1(Q)$. Additionally, we have

$$b_i(x,t,T_n(u_n), Du_n) V_\lambda(u_n) \rightharpoonup b_i(x,t,u,Du) V_\lambda(u) \quad \text{weakly in } L^{p'_i}(Q),$$

for all $i = 1, \ldots, N$. This implies that

$$\begin{aligned} &\limsup_{n\to+\infty} \int_{Q\cap\{u_n\leq\lambda\}} f_n g_n(u_n) V_\lambda(u_n) \varphi dxdt \\ &\leq -\int_{\{u=0\}} \Phi(u) \frac{\partial\varphi}{\partial t} dxdt + \sum_{i=1}^N \int_{\{u=0\}} b_i(x,t,u,Du) V_\lambda(u). D_i\varphi dxdt, \end{aligned}$$

then,

$$\lim_{\lambda\to 0} \lim_{n\to\infty} \mathbf{I}^2_{n,\lambda} = 0. \tag{5.18}$$

By (5.11), (5.13) and (5.18), we deduce that for all nonnegative $\mathbb{L}^{\overrightarrow{p}}(0,T;X_0^{1,\overrightarrow{p}}(\Omega))\cap L^\infty(Q)$, the following holds

$$\lim_{n\to\infty}\int_Q f_n g_n(u_n)\varphi dxdt=\int_Q fg(u)\varphi dxdt. \tag{5.19}$$

Moreover, by decomposing any $\varphi=\varphi^+-\varphi^-$ with $\varphi^+=\max\{\varphi,0\}$ and $\varphi^-=-\min\{\varphi,0\}$, we deduce that (5.19) holds for every $\varphi\in\mathbb{L}^{\overrightarrow{p}}(0,T;X_0^{1,\overrightarrow{p}}(\Omega))\cap L^\infty(Q)$. This concludes the proof of (5.6). □

Let $n\to+\infty$ in (5.7), by (5.1), (5.5) and (5.6) we get

$$\int_Q\frac{\partial u}{\partial t}\varphi dxdt+\sum_{i=1}^N\int_Q b_i(x,t,u,Du).D_i\varphi dxdt=\int_Q g(u)f\varphi dxdt,$$

for every $\varphi\in\mathbb{L}^{\overrightarrow{p}}(0,T;X_0^{1,\overrightarrow{p}}(\Omega))\cap L^\infty(Q)$.

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Conflict of interest The authors declares that he has no conflict of interest.

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