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ORIGINAL RESEARCH

A sufficient and necessary condition for one dimensional boundary blow-up problem with p-Laplace operator

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Abstract The existence, uniqueness and asymptotic behavior for one dimensional boundary blow-up problem with p-Laplace operator has been investigated. This problem arises in many fields, such as in the theory of automorphic functions and Riemann surfaces of constant negative curvature, in the study of the electric potential in a glowing hollow metal body, etc. Our result shows that the boundary blow-up solution exists provided that the Keller-Osserman condition holds true and the absorb terms satisfy a divergent condition in some neighbourhood of zero. By symmetry method, comparison theorem and the regularly varying theory, we also investigate the uniqueness of the boundary blow-up solutions, these results can be seen as the beneficial supplement to the corresponding results of elliptic equations.

Keywords Keller-Osserman conjecture · Boundary blow-up problem · Symmetry method

1 Introduction

Boundary blow-up or large solution problem is a hot topic in the theory and application of differential equations. The problem originated from the study of the theory of Riemann surfaces with negative curvature constants and the theory of automatic-functions, and appeared in many fields, such as stable constrained stochastic control, superdiffusion processes in stochastic processes, differential geometry, the concentrated wealth effect in ecology, the potential in hot hollow metal shells, and high speed diffusion in chemical reactions. In view of the above findings, many mathematicians pay their attention to the research of boundary blow-up problems, which makes the great progress in the theory of boundary blow-up.

The classical boundary blow-up problem is

$$\begin{cases} \Delta u(x) = f(u(x)), x \in \Omega, \\ u|_{\partial\Omega} = +\infty, \end{cases} \tag{1.1}$$

where Ω is a boundary domain in $R^N (N \geq 2)$. Here $u|_{\partial\Omega} = +\infty$ means that when $d(x, \partial\Omega) \to 0, u(x) \to +\infty$.

In 1916, Bieberbach first studied this type of problem (see [1]) and in his paper, $f(u) = e^u$. Later Keller [2] and Osserman [3] obtained an important result:

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Theorem A *Assume that $f(0) = 0$, $f'(t)$ is continuous and $f'(t) \geq 0$ for all $t \geq 0$, then (1.1) has a solution if and only if*

$$\int_1^{+\infty} \left(\int_0^t f(s)ds \right)^{-1/2} dt < +\infty. \tag{1.2}$$

The condition (1.2) plays an important role in the studying of boundary blow-up problem, it is usually called Keller-Osserman condition, also see [4–10] and [11]. Now, we might wonder whether the following result is true.

Keller-Osserman conjecture *When $N = 1$, the conclusion of Theorem* A *is also true.*

For this conjecture, V. Anuradha [12] etc. considered the following autonomous two point boundary value problem

$$\begin{cases} -u''(t) = \lambda f(u(t)); \ 0 < t < 1, \\ \lim_{t \to 0^+} u(t) = \lim_{t \to 1^-} u(t) = +\infty. \end{cases} \tag{1.3}$$

Using the quadrature method, they gave some necessary and sufficient conditions for the existence of nonnegative solutions. As they said, "The gap between the class of functions that satisfy the necessary condition but not the sufficient condition is quite small". Also they gave some examples to illustrate their main results. Here is one of their main conclusions.

Theorem B *Assume that there exists $a > 1$ such that*

$$\limsup_{u \to \infty} \frac{-f(u)}{u^a} = \infty, \tag{1.4}$$

then there exist solutions to (1.3) for some $\lambda > 0$.

Wang [13] used the method proposed by Anuradha and generalized his results, and he obtained a more suitable condition

$$\liminf_{u \to \infty} \frac{-f(u)}{u(\ln u)^3} = L, \ (0 < L \leq \infty) \tag{1.5}$$

for the existence of (1.3).

The gap between the class of functions that satisfy the necessary condition but not the sufficient condition becomes smaller than that of Anuradha's, but there also exists a distance to Keller-Osserman conjecture. Later, Zhang [14] also considered this problem, and improved this condition to

$$\liminf_{u \to \infty} \frac{-f(u)}{u(\ln u)^\alpha} = L, (0 < L \leq \infty), \ \alpha \leq 2. \tag{1.6}$$

It is easy to see that the conditions (1.4), (1.5) and (1.6) all satisfy Keller-Osserman condition (1.2). For the Keller-Osserman conjecture, in [15], we gave a complete answer.

Recently, A. Mohammed et al studied the following p-Laplace equations with weight function (see [16])

$$\begin{gathered} \left[r^\alpha \left(u'\right)^{p-1} \right]' = r^{\alpha+\beta} g(r) f(u), \\ u'(0) = 0, \ u(r) \to +\infty \text{ as } r \to R^-, \end{gathered} \tag{1.7}$$

where

$$\alpha \geq 0, \beta \geq 0, p > 1, \ r > 0, \ 0 < r < R.$$

For f and g, they assumed that

(f-1) $f(0) = 0$, $f(s) > 0$, $\int_0^1 \left(\int_0^s f(t)dt \right)^{-1/p} ds = +\infty$, $f'(s) \geq 0$ for $s > 0$.

(f-2)

$$\int_1^{+\infty} (F(t))^{-1/p} dt < +\infty, \ F(t) = \int_0^t f(s)ds. \tag{1.8}$$

(g-1) $(r^\beta g(r))' \geq 0$ and $g^{1/p}(r) \in L^1(0, R)$.

(g-2)

$$\liminf_{r \to R^-} g(r)(R - r)^p > 0.$$

After carefully investigations, they obtained

Theorem C *Let f satisfy (1.8) and (f-1), g satisfies (g-1), then problem (1.7) has a solution.*

Theorem D *Let $g(r) > 0,\ f(t) > 0$ is increasing, $f(t)t^{1-p}$ decreasing and such that for some $0 < \gamma < p-1$, $lim_{t\to+\infty} f(t)t^{-\gamma} = A$, with $0 < A < +\infty$. If $u(r)$ and $v(r)$ are two positive solutions of (1.7), then*

$$\lim_{r\to R^-} \frac{u(r)}{v(r)} = 1.$$

For the study on p-Laplace equations, one can also refer to [17–23] and so on.

In this paper, for simplicity, we want to investigate the following p-Laplace boundary blow-up problem

$$\begin{cases} \left[|u'(t)|^{p-2}u'(t)\right]' = f(u(t));\ 0 < t < 1, \\ \lim_{t\to0^+} u(t) = \lim_{t\to1^-} u(t) = +\infty, \end{cases} \tag{1.9}$$

where $f(t) \geq 0,\ f'(t)$ is continuous and $f'(t) > 0$ for all $t > 0,\ p > 2$ is a constant. Our aim is to generalize the Keller-Osserman conjecture to p-Laplace problems and to generalize Theorem C and D. We should point out that the methods we used here can be easily applied to problem (1.7).

We organized the paper as follows. In section 2, we gave some preliminaries about the Comparison Theorem for p-Laplace equations, the definition of regularly varying functions and some lemmas. In section 3, under the Keller-Osserman condition, we gave a detailed proof of the existence of positive solution for (1.9). In section 4, by using the properties of regularly varying functions, we established the theorem of asymptotic stability and the uniqueness theorem of solutions for (1.9). In section 5, we gave an example to illustrate the feasibility of our main results and took a further look at what could be done in the future.

2 Preliminaries

First we introduce the definition of regularly varying which can be found in [24].

Definition 2.1 A positive measurable function R defined on $[D, +\infty)$, for some $D > 0$, is called regularly varying (at infinity) with index $q \in \mathbb{R}$, written $R \in \mathbb{R}_q$, if for all $\xi > 0$,

$$\lim_{u\to\infty} R(\xi u)/R(u) = \xi^q.$$

When the index of regular variation q is zero, we say that the function is slowly varying.

Remark 2.1 a. Any function $R \in \mathbb{R}_q$ can be written in terms of a slowly varying function. Indeed, set $R(u) = u^q L(u)$, then L varies slowly.

b. For any $m > 0$, $u^m L(u) \to +\infty$, $u^{-m}L(u) \to 0$ as $u \to +\infty$.

For equations (1.9), we can easily obtain

Lemma 2.1 *If $u(t)$ is a solution of (1.9), then $u(1-t)$ is also a solution of (1.9).*

The proof of this lemma is trivial, we omit it here.

Lemma 2.2 *If $u(t)$ is a solution of (1.9), then there exists only one point $t_0 \in (0, 1)$ such that $u'(t_0) = 0$, in fact, $t_0 = \frac{1}{2}$.*

The proof of this lemma can be obtained by the generalized Rolle Theorem. $t_0 = \frac{1}{2}$ can be obtained easily by Lemma 2.1.

Lemma 2.3 *If $u(t) \in C^2[\frac{1}{2}, 1)$ is a solution of the following problem*

$$\begin{cases} \left[|u'(t)|^{p-2}u'(t)\right]' = f(u(t));\ \frac{1}{2} \leq t < 1, \\ u'\left(\frac{1}{2}\right) = 0,\ u(1) = +\infty, \end{cases} \tag{2.1}$$

then

$$w(t) = \begin{cases} u(1-t),\ t \in \left(0, \frac{1}{2}\right); \\ u(t),\ t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{2.2}$$

is a solution of (1.9), conversely, the conclusion also holds true.

Proof The proof is trivial, we omit it. □

Lemma 2.4 *Assume that the Keller-Osserman condition holds true. If $f(0) > 0$, then the function*

$$g(a) = \int_a^{+\infty} \left(\int_a^t f(s)ds \right)^{-1/p} dt \tag{2.3}$$

is well-defined on the interval $[0, +\infty)$ and satisfies

$$\lim_{a \to +\infty} g(a) = 0.$$

If $f(0) = 0$ and $\int_0^b [f(t)t]^{-1/p} dt = +\infty (b > 0)$, then the function $g(a)$ is well-defined on the interval $(0, +\infty)$ and satisfies

$$\lim_{a \to 0^+} g(a) = +\infty, \quad \lim_{a \to +\infty} g(a) = 0.$$

Proof Notice that $f(s)$ is increasing, thus

$$f(a)(t-a) \le \int_a^t f(s)ds \le f(t)(t-a), \text{ for } t \ge a.$$

Therefore

$$[f(t)(t-a)]^{-1/p} \le \left(\int_a^t f(s)ds \right)^{-1/p} \le [f(a)(t-a)]^{-1/p},$$

set

$$g(a) = I(a,b) + J(a,b),$$

where

$$I(a,b) = \int_a^b \left(\int_a^t f(s)ds \right)^{-1/p} dt, \ J(a,b) = \int_b^{+\infty} \left(\int_a^t f(s)ds \right)^{-1/p} dt, \ b > a.$$

Notice that $\left(\int_a^t f(s)ds \right)^{-1/p} \in C^1[b, +\infty)$, $\left(\int_a^t f(s)ds \right)^{-1/p} > 0$ for any $t \ge b > a$, we know that $\int_1^{+\infty} \left(\int_0^t f(s)ds \right)^{-1/p} dt$ and $\int_b^{+\infty} \left(\int_0^t f(s)ds \right)^{-1/p} dt (b > 0)$ have the same convergence, also notice that

$$\lim_{t \to +\infty} \frac{\left(\int_0^t f(s)ds \right)^{-1/p}}{\left(\int_a^t f(s)ds \right)^{-1/p}} = 1,$$

thus from the limit form of comparative criterion for convergence of infinite generalized integrals, we know $\int_b^{+\infty} \left(\int_0^t f(s)ds \right)^{-1/p} dt$ and $\int_b^{+\infty} \left(\int_a^t f(s)ds \right)^{-1/p} dt$ have the same convergence, therefore by (1.8), we obtain that $J(a,b) < +\infty$ for any $b > a$.

If $f(0) > 0$, then

$$I(a,b) \le \int_a^b [f(a)(t-a)]^{-1/p} dt = (f(a))^{-1/p} \frac{p}{p-1} (b-a)^{\frac{p-1}{p}} < +\infty, \tag{2.4}$$

which means that the function $g(a)$ is well-defined on the interval $[0, +\infty)$.

If $f(0) = 0$, then

$$I(0,b) \ge \int_0^b [f(t)t]^{-1/p} dt = +\infty, \tag{2.5}$$

which means that the function $g(a)$ is well-defined on the interval $(0, +\infty)$ ((2.4) holds for any $a > 0$) and $\lim_{a \to 0^+} g(a) = +\infty$.

Noticing the condition (1.8) and that $\left(\int_0^t f(s)ds \right)^{-1/p}$ is monotone decreasing, from [25], we can obtain

$$\lim_{t \to +\infty} t \left(\int_0^t f(s)ds \right)^{-1/p} = 0,$$

hence

$$\lim_{t\to+\infty} t^p \left(\int_0^t f(s)ds\right)^{-1} = 0,$$

thus

$$\lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} = +\infty,$$

by (2.4), we have

$$\lim_{a\to+\infty} I(a, 2a) = 0.$$

Note that

$$J(a, b) = \int_b^{+\infty} \left(\int_a^t f(s)ds\right)^{-1/p} dt < +\infty,$$

then

$$\lim_{a\to+\infty} J(a, 2a) = 0,$$

therefore

$$\lim_{a\to+\infty} g(a) = 0.$$

□

Lemma 2.5 *(Comparison Theorem). Let* $u(t) \in C^2[\frac{1}{2}, 1)$ *such that*

$$\left[|u'(t)|^{p-2}u'(t)\right]' \le 0,\ t \in \left[\frac{1}{2}, 1\right), \tag{2.6}$$

$$u'(\frac{1}{2}) \le 0,\ u(1) \ge 0, \tag{2.7}$$

then $u(t) \ge 0$ *for all* $t \in [\frac{1}{2}, 1)$.

Proof Notice that $\left[|u'(t)|^{p-2}u'(t)\right]' \le 0$ implies that the function $|u'(t)|^{p-2}u'(t)$ is monotonic decreasing in $[\frac{1}{2}, 1)$, then $|u'(t)|^{p-2}u'(t) \le |u'(\frac{1}{2})|^{p-2}u'(\frac{1}{2}) \le 0$ for $t \in [\frac{1}{2}, 1)$, thus $u'(t) \le 0$ for $t \in [\frac{1}{2}, 1)$, therefore, $u(t) \ge u(1) \ge 0$ for all $t \in [\frac{1}{2}, 1)$. □

Remark 2.1 In Lemma 2.5, if we replace $\frac{1}{2}$ and 1 by any other real numbers a and b ($a < b$), then the conclusion is also true.

Lemma 2.6 *(Comparison Theorem). Assume that* $f(t) \ge 0$, $f'(t)$ *is continuous and* $f'(t) > 0$ *for all* $t > 0$. *Let* $u(t), v(t) \in C^2[\frac{1}{2}, 1)$ *such that*

$$\left[|u'(t)|^{p-2}u'(t)\right]' \le f(u(t)),\ t \in \left[\frac{1}{2}, 1\right), \tag{2.8}$$

$$\left[|v'(t)|^{p-2}v'(t)\right]' \ge f(v(t)),\ t \in \left[\frac{1}{2}, 1\right),$$

$$u'\left(\frac{1}{2}\right) \le v'\left(\frac{1}{2}\right),\ u(1) \ge v(1), \tag{2.9}$$

then $u(t) \ge v(t)$ *for all* $t \in [\frac{1}{2}, 1]$.

Proof Assume that it is false, then there exist some $t_0,\ t_1 \in [\frac{1}{2}, 1]$ such that $u(t) \le v(t)$ for $t \in [t_0, t_1]$ and $u'(t_1) \ge v'(t_1), u(t_1) = v(t_1), u(t_0) < v(t_0), u'(t_0) \le v'(t_0)$. Let $w(t) = u(t) - v(t)$, $t \in [t_0, t_1]$, then $w(t) \le 0$ and $w(t_0) < 0$, (2.8) and (2.9) imply that

$$\left[|u'(t)|^{p-2}u'(t)\right]' - \left[|v'(t)|^{p-2}v'(t)\right]' \le f(u(t)) - f(v(t)) \le 0 \text{ for } t \in [t_0, t_1]. \tag{2.10}$$

Which shows that the function $|u'(t)|^{p-2}u'(t) - |v'(t)|^{p-2}v'(t)$ is monotonic decreasing in $[t_0, t_1]$, thus

$$|u'(t)|^{p-2}u'(t) - |v'(t)|^{p-2}v'(t) \ge |u'(t_1)|^{p-2}u'(t_1) - |v'(t_1)|^{p-2}v'(t_1). \tag{2.11}$$

We claim that

$$|u'(t_1)|^{p-2}u'(t_1) - |v'(t_1)|^{p-2}v'(t_1) \geq 0. \tag{2.12}$$

In the following, we divide the proof of this claim into three cases according to the signature of $u'(t_1)$ and $v'(t_1)$.

Case 1. $u'(t_1) \geq 0$, $v'(t_1) \leq 0$, in this case, the conclusion is obvious.

Case 2. $u'(t_1) \geq 0$, $v'(t_1) \geq 0$, notice that $u'(t_1) \geq v'(t_1)$ and $|u'(t_1)|^{p-2}u'(t_1) - |v'(t_1)|^{p-2}v'(t_1) = \left[u'(t_1)\right]^{p-1} - \left[v'(t_1)\right]^{p-1}$, therefore, $|u'(t_1)|^{p-2}u'(t_1) - |v'(t_1)|^{p-2}v'(t_1) \geq 0$.

Case 3. $u'(t_1) \leq 0$, notice that $u'(t_1) \geq v'(t_1)$, in this case, $v'(t_1) \leq 0$, thus $|u'(t_1)|^{p-2}u'(t_1) - |v'(t_1)|^{p-2}v'(t_1) = -\left[\left[-u'(t_1)\right]^{p-1} - \left[-v'(t_1)\right]^{p-1}\right] \geq 0$.

From (2.11) and (2.12), we have

$$|u'(t)|^{p-2}u'(t) - |v'(t)|^{p-2}v'(t) \geq 0 \text{ for } t \in [t_0, t_1]. \tag{2.13}$$

Which implies that $u'(t) \geq v'(t)$.

By (2.10) and (2.13), we get $|u'(t)|^{p-2}u'(t) - |v'(t)|^{p-2}v'(t) \equiv 0$ for $t \in [t_0, t_1]$, thus $w(t) \equiv 0$ for $t \in [t_0, t_1]$, which is a contradiction. □

Remark 2.2 In Lemma 2.6, if we replace $\frac{1}{2}$ and 1 by any other real numbers a and $b (a < b)$, then the conclusion is also true.

Corollary 2.1 *Assume that* $f(0) = 0$, $f'(t)$ *is continuous and* $f'(t) \geq 0$ *for all* $t \geq 0$. *Let* $u(t) \in C^2[\frac{1}{2}, 1)$ *such that*

$$\left[|u'(t)|^{p-2}u'(t)\right]' \leq f(u(t)),\ t \in \left[\frac{1}{2}, 1\right),$$
$$u'\left(\frac{1}{2}\right) \leq 0,\ u(1) \geq 0,$$

then $u(t) \geq 0$ *for all* $t \in [\frac{1}{2}, 1]$.

Proof In Lemma 2.6, let $v(t) \equiv 0$, then we can immediately get the conclusion. □

Remark 2.3 One can easily see that the result of Corollary 2.1 implies that Lemma 2.5 hold.

Lemma 2.7 *If* $f(0) = 0$, $f'(t)$ *is continuous and* $f'(t) \geq 0$ *for all* $t \geq 0$. *Moreover, assume that the Keller-Osserman condition (1.8) holds true, then the equation*

$$\begin{cases} \left[|u'(t)|^{p-2}u'(t)\right]' = f(u(t));\ \frac{1}{2} \leq t < 1, \\ u'\left(\frac{1}{2}\right) = 0,\ u(1) = n, \end{cases} \tag{2.14}$$

has only one positive solution, denoted as $u_n(t)$, $u_n(t)$ *is increasing in* n.

Proof The proof can be easily obtained by Lemma 2.6, we omit it here. □

3 Existence

The existence of positive solutions for the equation (1.9) can be listed as follows.

Theorem 3.1 *If* $f(0) = 0$ *and* $\int_0^b [f(t)t]^{-1/p}\,dt = +\infty$ *(for* $b > 0$*), then the equation (1.9) has a positive solution if and only if the Keller-Osserman condition (1.8) holds true.*

Proof Sufficiency. Conversely, suppose that there is no solution. Given any positive real number a, notice that (2.1) has a positive solution $u(t)$ such that

$$\begin{cases} \left[|u'(t)|^{p-2}u'(t)\right]' = f(u(t));\ \frac{1}{2} \leq t < 1, \\ u'\left(\frac{1}{2}\right) = 0, u(\frac{1}{2}) = a, \end{cases} \tag{3.1}$$

its maximal existence interval is $[\frac{1}{2}, r_a)$, where r_a satisfies $\lim_{t\to r_a^-} u(t) = +\infty$, $\frac{1}{2} < r_a$ and $r_a \neq 1$. Notice that (3.1) implies

$$u'(t) = \left(\frac{p}{p-1}\int_a^{u(t)} f(s)ds\right)^{1/p} \quad \text{for } t \in \left[\frac{1}{2}, r_a\right), \tag{3.2}$$

here we use the inequality $u'(t) \geq 0$ for $t \in [\frac{1}{2}, r_a)$. From (3.2), we have

$$\int_a^{+\infty}\left(\int_a^t f(s)ds\right)^{-1/p} dt = \left(\frac{p}{p-1}\right)^{1/p}\left(r_a - \frac{1}{2}\right). \tag{3.3}$$

By Lemma 2.4 and the intermediate value theorem, we know that there exists $a_0 > 0$ such that

$$g(a_0) = \frac{1}{2}\left(\frac{p}{p-1}\right)^{1/p},$$

then (3.3) implies that $r_{a_0} = 1$, which is a contradiction.

Necessity. Assume that

$$\int_1^{+\infty}\left(\int_0^t f(s)ds\right)^{-1/p} dt = +\infty,$$

then for any $a > 0$, we have $g(a) = +\infty$, by (3.3), we know that there is no solution for (2.1), which is a contradiction. □

Theorem 3.2 *If $f(0) > 0$, then the equation (1.9) has a positive solution if and only if the Keller-Osserman condition (1.8) and $\max_{a\geq 0} g(a) \geq \frac{1}{2}\left(\frac{p}{p-1}\right)^{1/p}$ hold true, here $g(a)$ is defined in (2.3).*

Proof From Lemmas 2.1-2.4 and the proof of Theorem 3.1, we can easily get the conclusion. □

4 Asymptotical stability

In this section, we want to study the asymptotical behavior of large solutions near the boundary and the uniqueness of the solution. Firstly, we have

Theorem 4.1 *Assume that $f(0) = 0$, $\int_0^b [f(t)t]^{-1/p} dt = +\infty$ and $f(t) \in \mathbb{R}_q$, $q > p-1$. Then the solution $u(t)$ of (1.9) satisfies*

$$\lim_{t\to\partial I}\frac{u(t)}{\phi(\delta(t))} = 1, \tag{4.1}$$

where

$$\delta(t) = dist(t, \partial I),\ I = [0,1],\ \phi(s) = \psi^{-1}(s),\ F(t) = \int_0^t f(s)ds \text{ and } \psi(s) = \int_s^{+\infty}\frac{1}{\left(\frac{p}{p-1}F(t)\right)^{1/p}}dt.$$

That is

$$\lim_{t\to 0^+}\frac{u(t)}{\phi(t)} = 1 \text{ and } \lim_{t\to 1^-}\frac{u(t)}{\phi(1-t)} = 1.$$

Proof Notice that $f(t) \in \mathbb{R}_q$, $q > p-1$ implies that (1.8) holds, then from Theorem 3.1, we know that the solution of (1.9) exist, by Lemma 2.3 and Corollary 2.1, any solution $u(t)$ satisfies $u(t) \geq 0$. Then (1.9) implies that

$$u'(t) = \left(\frac{p}{p-1}\int_a^{u(t)} f(s)ds\right)^{1/p}. \tag{4.2}$$

From (4.2), we have

$$u'(t) = \left(\frac{p}{p-1}\right)^{1/p}\left(\int_0^{u(t)} f(s)ds - \int_0^a f(s)ds\right)^{1/p},$$

thus

$$\frac{u'(t)}{\left(\frac{p}{p-1}\int_0^{u(t)} f(s)ds\right)^{1/p}} = \left(1 - \frac{\int_0^{u(\frac{1}{2})} f(s)ds}{\int_0^{u(t)} f(s)ds}\right)^{1/p}, \tag{4.3}$$

for any boundary blow-up solution $u(t)$ of (1.9), we can easily obtain

$$\lim_{t\to 1^-} \frac{\int_0^{u(\frac{1}{2})} f(s)ds}{\int_0^{u(t)} f(s)ds} = 0.$$

Then (4.3) implies that

$$\psi(u(t)) = \int_t^1 (1+o(1))dt = (1-t)(1+o(1)),$$

therefore

$$u(t) = \phi\left(\beta(t)(1-t)\right),$$

where $\beta(t)$ satisfies

$$\lim_{t\to 1^-} \beta(t) = 1 \text{ and } \beta(t) < 1. \tag{4.4}$$

In order to prove (4.1), we only need to prove

$$\lim_{t\to 1^-} \frac{\phi\left(\beta(t)(1-t)\right)}{\phi(1-t)} = 1. \tag{4.5}$$

From (4.4), we have for any $\varepsilon > 0$, there exists some $\delta < 1/2$ such that

$$1-\varepsilon < \beta(t) < 1 \text{ for } 1-\delta < t < 1.$$

Notice that

$$\phi'(s) < 0,$$

thus

$$\phi(1-t) \le \phi\left(\beta(t)(1-t)\right) \le \phi\left((1-t)(1-\varepsilon)\right),$$

which implies that

$$1 \le \frac{\phi\left(\beta(t)(1-t)\right)}{\phi(1-t)} \le \frac{\phi\left((1-t)(1-\varepsilon)\right)}{\phi(1-t)}. \tag{4.6}$$

In the following, we prove that $\phi(t) \in \mathbb{R}_{2/(1-q)}$ (at zero) when $f(t) \in \mathbb{R}_q$. In fact, If $f(t) \in \mathbb{R}_q$, then

$$\lim_{t\to+\infty} \frac{F(\xi t)}{F(t)} = \lim_{t\to+\infty} \frac{\xi F'(\xi t)}{F'(t)} = \lim_{t\to+\infty} \frac{\xi f(\xi t)}{f(t)} = \xi^{q+1},$$

which implies that

$$\lim_{t\to+\infty} \frac{\psi(\xi t)}{\psi(t)} = \lim_{t\to+\infty} \frac{\xi\psi'(\xi t)}{\psi'(t)} = \lim_{t\to+\infty} \frac{\xi\left(\frac{p}{p-1}F(t)\right)^{1/p}}{\left(\frac{p}{p-1}F(\xi t)\right)^{1/p}} = \xi^{1-\frac{q+1}{p}},$$

therefore

$$\lim_{t\to+\infty} \frac{\phi\left[\xi^{\left(1-\frac{q+1}{p}\right)}\psi(t)\right]}{t} = \xi,$$

thus

$$\lim_{t\to 0^+} \frac{\phi(\xi t)}{\phi(t)} = \lim_{y\to+\infty} \frac{\phi(\xi\psi(y))}{y} = \lim_{y\to+\infty} \frac{\phi(\sigma^{\left(1-\frac{q+1}{p}\right)}\psi(y))}{y} = \sigma = \xi^{-\frac{p}{q-(p-1)}}.$$

So

$$\lim_{t\to 1^-} \frac{\phi\left((1-t)(1-\varepsilon)\right)}{\phi(1-t)} = (1-\varepsilon)^{-\frac{p}{q-(p-1)}},$$

letting $\varepsilon \to 0^+$, and then by (4.6), (4.5) holds true. $\square$

Theorem 4.2 *Assume that $f(0)=0$, $\frac{f(u)}{u^{p-1}}$ is increasing on $(0,+\infty)$ and $f(t)\in\mathbb{R}_q$, $q>p-1$. Then the equation (1.9) has only one positive solution.*

Proof Notice that $\frac{f(u)}{u^{p-1}}$ is increasing on $(0,+\infty)$ implies $\int_0^b [f(t)t]^{-1/p}\,dt=+\infty$, then Theorem 4.1 holds true. Let $u(t)$, $v(t)$ be the solutions of (1.9), by (4.1), we have

$$\lim_{t\to 1^-}\frac{u(t)}{v(t)}=1,$$

then for any $\varepsilon>0$, there exists $\delta>0$ such that

$$(1-\varepsilon)v(t)\le u(t)\le(1+\varepsilon)v(t)\text{ for }t\in(1-\delta,1),$$

set $\overline{u}(t)=(1+\varepsilon)v(t)$, $t\in[\frac{1}{2},1)$, then

$$\overline{u}'\left(\frac{1}{2}\right)=u'\left(\frac{1}{2}\right)=0\text{ and }u(t)\le\overline{u}(t)\text{ for }t\in(1-\delta,1). \tag{4.7}$$

We claim that for any $t\in[\frac{1}{2},1)$, $u(t)\le\overline{u}(t)$. Notice that

$$\left[|\overline{u}'(t)|^{p-2}\overline{u}'(t)\right]'=(1+\varepsilon)^{p-1}\left[|v'(t)|^{p-2}v'(t)\right]'=(1+\varepsilon)^{p-1}f(v(t))\le f(\overline{u}(t)).$$

The last inequality holds true provided that $\frac{f(u)}{u^{p-1}}$ is increasing on $(0,+\infty)$, from (4.7) and Lemma 2.6, the above claim is obvious. Similarly, we can prove that for any $t\in[\frac{1}{2},1)$, $u(t)\ge\underline{u}(t)$, where $\underline{u}(t)=(1-\varepsilon)\,v(t)$. Passing to the limit $\varepsilon\to 0^+$, we conclude that $u(t)\equiv v(t)$. □

Remark 4.1 From the proofs of Theorem 4.1 and Theorem 4.2, we can easily obtain that the condition "$\frac{f(u)}{u^{p-1}}$ is increasing on $(0,+\infty)$ and $f(t)\in\mathbb{R}_q$, $q>p-1$" in Theorem 4.2 can be replaced by

(H) There exists $q>p-1$ such that for any $\varepsilon>0$ sufficiently small, the function f satisfies

$$f[(1+\varepsilon)s]\ge(1+\varepsilon)^q f(s)\text{ and }f[(1-\varepsilon)s]\le(1-\varepsilon)^q f(s),\ s\in(0,+\infty).$$

Thus we have

Corollary 4.1 *Assume that $f(0)=0$ and (H) hold true, then the equation (1.9) has only one positive solution.*

5 Discussions

From the proof of Theorem 3.1, one can easily see that the condition $\int_0^b [f(t)t]^{-1/p}\,dt=+\infty$ can be replaced by

$$\int_0^b\left[\int_0^t f(s)ds\right]^{-1/p}dt=+\infty. \tag{5.1}$$

So Theorem 3.1 can be generalized to:

Corollary 5.1 *If $f(0)=0$, then the equation (1.9) has a positive solution if and only if the Keller-Osserman condition (1.8) and (5.1) hold true.*

Remark 5.1 One can easily see that Corollary 5.1 generalized Theorem C.

Example By Corollary 5.1, we can easily obtain that the problem

$$\begin{cases}\left[|u'(t)|^{p-2}u'(t)\right]'=u^q(t);\ 0<t<1,\\ \lim_{t\to 0^+}u(t)=\lim_{t\to 1^-}u(t)=+\infty,\end{cases}$$

has a positive solution if and only if $q>p-1$. In fact, it is unique.

For $f(u)=u^q$ and $f(u)=e^u$, we can easily see that the conditions (1.8) and (5.1), in fact, are equivalent to each other, thus we have the following conjecture.

Conjecture *If $f(0)=0$, then the equation (1.9) has a positive solution if and only if the Keller-Osserman condition (1.8) holds true.*

We leave it for further investigations.

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