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ORIGINAL RESEARCH

Nonlinear weighted elliptic problem with variable exponents and L^1 data

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Abstract In this paper, we prove the existence of weak solutions for a class of nonlinear weighted elliptic equations in Ω with $p(x)$ growth conditions and integrable data. The functional setting involves Lebesgue-Sobolev spaces with variable exponents. Our results are generalizations of the corresponding results in the constant exponent case in L. Boccardo et al (Boll. Unione Mat. Ital. 15, No. 4, 503-514 (2022)) and some results given in D. Arcoya et al (Journal of Functional Analysis, 268(5), 1153-1166 (2015)).

Keywords Elliptic problem · Degenerate coercivity · Sobolev weights

Mathematics Subject Classification 35A01 · 35J60 · 35J70

1 Introduction

The first problem considered in this work is the nonlinear weighted elliptic problem having a degenerate coercivity:

$$\begin{cases} -\text{div}\left(a(x)\frac{\widehat{a}(x,\nabla u)}{(1+|u|)^{\varrho(x)}}\right) = f(x) \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N > 2)$ with boundary denoted by $\partial\Omega$, $\varrho(x) \geq 0$. The variable exponents $p(.) : \mathbb{R}^N \to (1, \infty)$ are continuous functions and f belongs to $L^{p_*(x)}(\Omega)$ where

$$p_*(x) = (p^*(x))' = \frac{Np(x)}{Np(x) - N + p(x)}.$$

The weight function $a(x)$ belongs to $W^{1,p(x)}(\Omega)$ such that for some $\lambda \in \mathbb{R}_+^*$,

$$a(x) \geq \lambda. \tag{1.1}$$

Here, suppose that $\widehat{a}(x, \xi)$ be a function verifies the following conditions.

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($\widehat{a_1}$) The function $\widehat{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and there exist a positive constant $\beta > 1$ such that

$$\widehat{a}(x,\xi)\xi \geq \beta|\xi|^{p(x)}, \quad \text{a.e. } x \in \Omega \text{ and } \forall \xi \in \mathbb{R}^N.$$

($\widehat{a_2}$) There exist a positive constant α such that

$$|\widehat{a}(x,\xi)| \leq \alpha\Big(g(x) + h(x)|\xi|^{p(x)}\Big)^{1-\frac{1}{p(x)}}, \quad \text{a.e. } x \in \Omega \text{ and } \forall \xi \in \mathbb{R}^N,$$

where $g \in L^1(\Omega)$, $h \in L^\infty$ are a given positive functions.

($\widehat{a_3}$) For every $\xi, \xi' \in \mathbb{R}^N$ and for almost every $x \in \Omega$, we have that

$$(\widehat{a}(x,\xi) - \widehat{a}(x,\xi'))(\xi - \xi') > 0 \quad \text{with } \xi \neq \xi'.$$

Next we will study existence of solutions for the problem

$$\begin{cases} -\text{div}\left(a(x)\frac{\widehat{a}(x,\nabla u)}{(1+|u|)^{\varrho(x)}}\right) + e(x)|u|^{p(x)-2}u = f(x) \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{P_1}$$

with the function $0 \leq e(x) \in L^1(\Omega)$. Even if $f \in L^1(\Omega)$, the assumption

$$\text{there exists} \quad k_0 > 0 \quad \text{such that} \quad |f(x)| \leq k_0.e(x). \tag{1.2}$$

implies the existence of a weak solution u belonging to $W_0^{1,p(x)}(\Omega)$ and to $L^\infty(\Omega)$.

Finally we deal the existence of weak solutions to **non degenerate problem** ($i.e.$, $\varrho(x) = 0$) having lower order terms with natural growth

$$\begin{cases} -\text{div}\,(a(x)\widehat{a}(x,\nabla u)) + e(x)|u|^{p(x)-2}u = \pi(x)F(x,u,\nabla u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_2}$$

With the function $\pi(x) \in L^1(\Omega)$ such that

$$\forall x \in \Omega, |\pi(x)| \leq a(x), \tag{1.3}$$

and we also suppose that the term $F(x,s,\xi)$ is a Carathéodory function (that is, measurable with respect to x for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) for almost every $x \in \Omega$) satisfying for some positive constant τ,

$$|F(x,s,\xi)| \leq \tau|\xi|^{p(x)}, \tag{1.4}$$

for a.e.$x \in \Omega$ and for every $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$.

In [3], L.Boccardo et al. have studied the existence of the solutions to problem (P) where the data f in $L^{p_*}(\Omega)$ and $p(x) = p,\ \varrho(x) = 0$. The problem (P_1) was also considered in [3], when $p(x) = p,\ \varrho(x) = 0$ and f in $L^1(\Omega)$. In the case of $a(x) \equiv 1$, $p(x) = 2$ and $\varrho(x) = 0$ the existence of the solutions to problem (P_2) have been intensively studied by many authors, we refer the reader to the bibliography [1] and references therein. And so, in this work, I would like to apply the ideas and results mentioned in [3] to the case of nonlinear weighted elliptic equations with variable exponent. The method used to prove the existence of solutions $u \in W_0^{1,p(x)}(\Omega)$ for the problem (P) is the same as used in [3].

The aim of this paper is to extend the results in [1,3] to the case of degenerate elliptic equations and establish the existence of weak solutions for the problems (P), (P_1) and non-degenerate problem (P_2).

The main difficulty associated with this strategy. In view of the assumption that the weight function $a(x) \in W^{1,p(x)}(\Omega)$, the difficulty is the fact that the differential operator $Au = -\text{div}\left(a(x)\frac{\widehat{a}(x,\nabla u)}{(1+|u|)^{\varrho(x)}}\right)$ is NOT coercive on the space $W_0^{1,p(\cdot)}(\Omega)$ due to the fact that if u is very large, $\frac{1}{(1+|u|)^{\varrho(x)}}$ tends to zero. To overcome this difficulty, we will proceed by approximation by means of truncatures in $\frac{1}{(1+|u|)^{\varrho(x)}}$ to get a coercive differential operator.

This paper is organized as follows: in Section 2 we recall some basic notations of the weighted Sobolev spaces with variable exponents, and we provide a technical lemma. Next, in Section 3, we prove the existence of solutions to problem (P). Finally, in Section 4, we give the proof of existence of solutions to problems (P_1) and (P_2).

2 Mathematical preliminaries and a technical lemma

In the following, we will revisit some facts on variable exponent spaces $L^{p(\cdot)}(\Omega)$. We refer to ([4], [11]) and the references therein, for further properties of variable exponent Lebesgue-Sobolev spaces.

Hereinafter, we set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1, \text{ for any } x \text{ in } \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we denote

$$p^+ = \max_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx,$$

is finite. The expression

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\big\{\lambda > 0 : \rho_{p(\cdot)}\big(\frac{u}{\lambda}\big) \le 1\big\},$$

defines a norm on $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|u\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \le p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\big|\int_\Omega uv\, dx\big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \le 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

holds. We define also the Banach space

$$W^{1,p(\cdot)}(\Omega) = \big\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\big\},$$

which is equipped with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|u\|_{1,p(\cdot)})$ is a Banach space. Next, we define $W_0^{1,p(\cdot)}(\Omega)$ the Sobolev space with zero boundary values by

$$W_0^{1,p(\cdot)}(\Omega) = \big\{u \in W^{1,p(\cdot)}(\Omega) : u = 0 \text{ on } \partial\Omega\big\},$$

endowed with the norm $\|\cdot\|_{1,p(\cdot)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that $1 < p^- \le p^+ < +\infty$. For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C_+(\overline{\Omega})$, the Poincaré inequality holds

$$\|u\|_{p(\cdot)} \le C\|\nabla u\|_{p(\cdot)}, \tag{2.1}$$

for some $C > 0$ which depends on Ω and $p(\cdot)$. Therefore, $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms. Given a real positive number k and $s \in \mathbb{R}$ we will define the functions $T_k(s) = \max(-k, \min(s, k))$.

As in [3], we have the following result.

Lemma 2.1 *Lets K_n be a sequence of functions in $L^1(\Omega)$, $p(x) > 1$ and let a_n be a sequence of functions in $W^{1,p(x)}(\Omega)$ such that:*

1. *the sequence $\{K_n\}_n$ is bounded functions which converges to K in $L^1(\Omega)$;*
2. *the sequence $\{a_n\}_n$ is bounded in $W^{1,p(x)}(\Omega)$;*

3. there exists a weak solution v_n in $W_0^{1,p(x)}(\Omega)$ of

$$\begin{cases} -div\left(a_n(x)\frac{\widehat{a}(x,\nabla v_n)}{(1+|T_n(v_n)|)^{\varrho(x)}}\right) = K_n, \quad in\ \Omega \\ \qquad\qquad\qquad\qquad\qquad v_n = 0,\ on\ \partial\Omega. \end{cases} \tag{*}$$

4. The sequence $(T_k(v_n))_n$ is bounded in $W_0^{1,p(x)}(\Omega)$;

Then, we have

$$\int_\Omega \frac{\widehat{a}(x,\nabla v_n)}{(1+|T_n(v_n)|)^{\varrho(x)}}\nabla\varphi = \int_\Omega H_n\varphi, \quad \forall\varphi\in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega),$$

where H_n is bounded in $L^1(\Omega)$. Furthermore,

$$\nabla T_k(v_n) \longrightarrow \nabla T_k(v) \text{ strongly in } (L^{p(.)}(\Omega))^N. \tag{2.2}$$

Remark 2.2 Notice that by Fatou's lemma, the condition (4) in Lemma 2.1 implies that

$$\|v_n\|_{W_0^{1,p(x)}(\Omega)} \le C,$$

after letting $k\to+\infty$, so, there exists a function v belongs to $W_0^{1,p(x)}(\Omega)$ such that, up to subsequences,

$$v_n \rightharpoonup v \quad \text{weakly in} \quad W_0^{1,p(x)}(\Omega), \tag{2.3}$$

$$v_n \longrightarrow v \quad \text{strongly in} \quad L^{p(x)}(\Omega), \tag{2.4}$$

$$v_n \longrightarrow v \quad \text{a.e in} \quad \Omega. \tag{2.5}$$

Proof of Lemma 2.1

Proof Let $(a_n(x))$,

$$a_n(x) = \frac{a(x)}{1+\frac{a(x)}{n}}, \tag{2.6}$$

be a sequence of bounded functions which converges strongly to $a(x)$ in $W^{1,p(x)}(\Omega)$. Note that we have, thanks to (1.1), there exist a constant C_λ such that

$$\frac{\lambda}{\lambda+1} = C_\lambda \le a_n(x) \le n. \tag{2.7}$$

Taking $\frac{\varphi}{a_n(x)}$ (belongs to $W_0^{1,p(x)}(\Omega)$) as test function in the weak formulation of $(*)$, we obtain

$$\int_\Omega \frac{a_n(x)}{a_n(x)}\frac{\widehat{a}(x,\nabla v_n)}{(1+|T_n(v_n)|)^{\varrho(x)}}\nabla\varphi - \int_\Omega \frac{a_n(x)}{a_n(x)^2}\frac{\widehat{a}(x,\nabla v_n)}{(1+|T_n(v_n)|)^{\varrho(x)}}\nabla a_n\varphi = \int_\Omega K_n\frac{\varphi}{a_n(x)},$$

which can be rewritten as

$$\begin{aligned}\int_\Omega \frac{\widehat{a}(x,\nabla v_n)}{(1+|T_n(v_n)|)^{\varrho(x)}}\nabla\varphi &= \int_\Omega \frac{\varphi}{a_n(x)}\frac{\widehat{a}(x,\nabla v_n)}{(1+|T_n(v_n)|)^{\varrho(x)}}\nabla a_n + \int_\Omega K_n\frac{\varphi}{a_n(x)} \\ &= \int_\Omega H_n(x)\varphi, \quad \forall\varphi\in W_0^{1,\infty}(\Omega),\end{aligned} \tag{2.8}$$

where

$$H_n(x) = \frac{1}{a_n(x)}\left(\frac{\widehat{a}(x,\nabla v_n)}{(1+|T_n(v_n)|)^{\varrho(x)}}\nabla a_n + K_n\right). \tag{2.9}$$

Thanks to (2.7), the sequence $\{\frac{1}{a_n(x)}\}_n$ is bounded by $\frac{1}{C_\lambda}$. Furthermore, since

$$\nabla a_n(x) = \frac{\nabla a(x)}{(1+\frac{a(x)}{2})^2},$$

we have that $|\nabla a_n(x)| \le |\nabla a(x)|$. Moreover, from the assumption $(\widehat{a_2})$ and the fact that $1+|T_n(v_n)| \ge 1$, we have

$$\begin{aligned}\int_\Omega |H_n(x)|dx &\le \frac{\alpha}{C_\lambda}\int_\Omega \Big(g+h|\nabla v_n|^{p(x)}\Big)^{1-\frac{1}{p(x)}}|\nabla a(x)|dx + \frac{1}{C_\lambda}\int_\Omega |K_n|dx \\ &\le C_1\int_\Omega \Big(g+|\nabla v_n|^{p(x)}\Big)^{1-\frac{1}{p(x)}}|\nabla a(x)|dx + C_2\int_\Omega |K_n|dx. \end{aligned} \tag{2.10}$$

where $C_1 = \frac{\alpha}{C_\lambda}(1+\|h\|_{L^\infty})^{1-\frac{1}{p^+}}$ and $C_2 = \frac{1}{C_\lambda}$. By application of Young's inequality, we have

$$\int_\Omega |H_n(x)|dx \le C_3\int_\Omega \Big(g+|\nabla v_n|^{p(x)}\Big)dx + C_4\int_\Omega |\nabla a(x)|^{p(x)}dx + C_2\int_\Omega |K_n|dx.$$

By assumption (2), the $L^1(\Omega)$-bound on $g(x)$, and $L^{p(\cdot)}(\Omega)$-bound on ∇v_n (since(4)) and the fact that the sequence $\{K_n\}_n$ is bounded in $L^1(\Omega)$ (since (1)), we conclude that H_n is bounded in $L^1(\Omega)$. Now, to show that

$$\nabla T_k(v_n) \longrightarrow \nabla T_k(v) \ \text{ strongly in } \ (L^{p(\cdot)}(\Omega))^N. \tag{2.11}$$

We fix $k > 0$. Plugging $T_{2k}(T_k(v_n) - T_k(v))$ as a test function in $(*)$, we get

$$\int_\Omega a_n(x)\frac{\widehat{a}(x,\nabla v_n)}{(1+|T_n(v_n)|)^{\varrho(x)}}\nabla(T_{2k}(T_k(v_n)-T_k(v)))dx = \int_\Omega K_n T_{2k}(T_k(v_n)-T_k(v))dx. \tag{2.12}$$

For the first integral in (2.12), we can write it as

$$\begin{aligned}&\int_\Omega a_n(x)\frac{\widehat{a}(x,\nabla v_n)}{(1+|T_n(v_n)|)^{\varrho(x)}}\nabla(T_{2k}(T_k(v_n)-T_k(v)))dx \\ &\ge \frac{C_\lambda}{(1+k)^{\varrho^+}}\int_{\{|u_n|\le k\}}\widehat{a}(x,\nabla T_k(u_n))\nabla(T_k(v_n)-T_k(v))dx \\ &\quad - \int_{\{|v_n|>k\}} a_n(x)\frac{\widehat{a}(x,\nabla v_n)}{(1+|T_n(v_n)|)^{\varrho(x)}}\nabla(T_k(v))dx \\ &\ge \frac{C_\lambda}{(1+k)^{\varrho^+}}\int_{\{|v_n|\le k\}}\widehat{a}(x,\nabla T_k(v_n))\nabla(T_k(v_n)-T_k(v))dx \\ &\quad - \int_{\{|v_n|>k\}} a_n(x)\widehat{a}(x,\nabla v_n)\nabla(T_k(v))dx.\end{aligned}$$

It follows from the above inequality that

$$\begin{aligned}&\frac{C_\lambda}{(1+k)^{\varrho^+}}\int_\Omega \widehat{a}(x,\nabla T_k(v_n))\nabla(T_k(v_n)-T_k(v))dx \\ &\le \int_{\{|v_n|>k\}} a_n(x)\widehat{a}(x,\nabla v_n)\nabla(T_k(v))dx + \int_\Omega K_n T_{2k}(T_k(v_n)-T_k(v))dx.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\frac{C_\lambda}{(1+k)^{\varrho^+}}\int_\Omega [\widehat{a}(x,\nabla T_k(v_n)) - \widehat{a}(x,\nabla T_k(v))]\nabla(T_k(v_n)-T_k(v))dx \\ &\le \int_{\{|v_n|>k\}} a_n(x)\widehat{a}(x,\nabla v_n)\nabla(T_k(v))dx - \frac{C_\lambda}{(1+k)^{\varrho^+}}\int_\Omega \widehat{a}(x,\nabla T_k(v))\nabla(T_k(v_n)-T_k(v))dx \\ &\quad + \int_{Q_T} K_n T_{2k}(T_k(v_n)-T_k(v))dx = I_1 + I_2 + I_3.\end{aligned}$$

Limit of I_1.By the assumption $\widehat{a_2}$, the $L^1(\Omega)$-bound on $g(x)$ and $L^\infty(\Omega)$-bound on $h(x)$, and $L^{p(x)}(\Omega)$-bound on ∇u_n for $p(x)$ (since (4)), we have

$$|\widehat{a}(x, \nabla v_n)| \leq \alpha\Big(g(x) + h(x)|\nabla v_n|^{p(x)}\Big)^{1-\frac{1}{p(x)}} \in L^{p'(x)}(\Omega). \tag{2.13}$$

The strong convergence of $a_n(x)$ to $a(x)$ in $L^{p(.)}(\Omega)$, and by the dominated convergence theorem

$$a_n(x)|\nabla(T_k(v))|\chi_{\{|v_n|>k\}} \longrightarrow a(x)|\nabla T_k(v)|\chi_{\{|v|>k\}} \quad \text{strongly in} \quad L^{p(.)}(\Omega),$$

which is zero, as $n \to +\infty$. Thus, we obtain

$$\lim_{n\longrightarrow+\infty} I_1 = 0. \tag{2.14}$$

Limit of I_2. Since (4), we obtain $\nabla T_k(v_n) \rightharpoonup \nabla T_k(v)$ weakly in $(L^{p(x)}(\Omega))^N$ and the boundedness of $\widehat{a}(x, \nabla T_k(v))$ in $L^{p'(x)}(\Omega)$, we obtain

$$\lim_{n\longrightarrow+\infty} I_2 = 0. \tag{2.15}$$

Limit of I_3. Notice that

$$\begin{aligned}
|I_3| &\leq \int_\Omega |K_n - K||T_{2k}(T_k(v_n) - T_k(v))|dx + \int_\Omega |K_n T_{2k}(T_k(v_n) - T_k(v))|dx \\
&\leq 2k\int_\Omega |K_n - K|dx + \int_\Omega |f T_{2k}(T_k(v_n) - T_k(v))|dx.
\end{aligned}$$

Since $K_n \longrightarrow K$ strongly in $L^1(\Omega)$, using (4) and the Lebesgue dominated convergence theorem, we have

$$\lim_{n\longrightarrow+\infty} I_3 = 0. \tag{2.16}$$

Now, passing to the limits in (2.13) as n tend to infinity, and by (2.14)-(2.16), we deduce that

$$\lim_{n\longrightarrow+\infty} I_n^k = 0,$$

where

$$I_n^k = \int_\Omega [\widehat{a}(x, \nabla T_k(v_n)) - \widehat{a}(x, \nabla T_k(v))]\nabla(T_k(v_n) - T_k(v))dx.$$

Using the following well-known inequalities that hold for any two real vectors ξ, η and a real $p \geq 2$:

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq \begin{cases} 2^{2-p}|\xi - \eta|^p, & \text{if } p \geq 2, \\ (p-1)\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}}, & \text{if } 1 < p < 2, \end{cases} \tag{2.17}$$

we find,

$$2^{2-p^+}\int_{\{x\in\Omega,\, p(x)\geq 2\}} |\nabla(T_k(v_n) - T_k(v))|^{p(x)}dx \leq I_n^k.$$

Now, on the set $E = \{x \in \Omega, 1 < p(x) < 2\}$, we have

$$\begin{aligned}
&\int_E |\nabla(T_k(v_n) - T_k(v))|^{p(x)}\, dx \\
&\leq \int_E \frac{|\nabla(T_k(v_n) - T_k(v))|^{p(x)}}{(|\nabla(T_k(v_n))| + |\nabla T_k(v)|)^{\frac{p(x)(2-p(x))}{2}}} \times (|\nabla(T_k(v_n))| + |\nabla T_k(v)|)^{\frac{p(x)(2-p(x))}{2}}\, dx \\
&\leq 2\Big\|\frac{|\nabla(T_k(v_n)) - \nabla T_k(v)|^{p(x)}}{(|\nabla(T_k(v_n))| + |\nabla T_k(v)|)^{\frac{p(x)(2-p(x))}{2}}}\Big\|_{L^{\frac{2}{p(\cdot)}}(E)} \times \Big\|(|\nabla(T_k(v_n))| + |\nabla T_k(v)|)^{\frac{p(x)(2-p(x))}{2}}\Big\|_{L^{\frac{2}{2-p(\cdot)}}(E)}
\end{aligned}$$

$$\leq 2\max\Big\{\Big(\int_\Omega \frac{|\nabla(T_k(v_n))-\nabla T_k(v)|^2}{\big(|\nabla(T_k(v_n))|+|\nabla T_k(v)|\big)^{2-p(x)}}\,dx\Big)^{\frac{p^-}{2}},\Big(\int_\Omega \frac{|\nabla(T_k(v_n))-\nabla T_k(v)|^2}{(|\nabla(T_k(v_n))|+|\nabla T_k(v)|)^{2-p(x)}}\,dxdt\Big)^{\frac{p^+}{2}}\Big\}$$
$$\times\max\Big\{\Big(\int_\Omega \big(|\nabla(T_k(v_n))|+|\nabla T_k(v)|\big)^{p(x)}\,dx\Big)^{\frac{2-p^+}{2}},\Big(\int_\Omega \big(|\nabla(T_k(v_n))|+|\nabla T_k(v)|\big)^{p(x)}\,dx\Big)^{\frac{2-p^-}{2}}\Big\}$$
$$\leq 2\max\Big\{(p^- -1)^{-\frac{p^+}{2}}(I_n^k)^{\frac{p^+}{2}},(p^- -1)^{-\frac{p^-}{2}}(I_n^k)^{\frac{p^-}{2}}\Big\}\times\max\Big\{\Big(\int_\Omega \big(|\nabla(T_k(v_n))|+|\nabla T_k(v)|\big)^{p(x)}\,dx\Big)^{\frac{2-p^+}{2}},$$
$$\times\Big(\int_\Omega \big(|\nabla(T_k(v_n))|+|\nabla T_k(v)|\big)^{p(x)}\,dx\Big)^{\frac{2-p^-}{2}}\Big\}. \tag{2.18}$$

Since $I_n^k \longrightarrow 0$ as $n \longrightarrow +\infty$ and $(T_k(v_n))_n$ is bounded in $W_0^{1,p(x)}(\Omega)$ (see lemma 2.1), we conclude (2.11). From this result, we deduce that (up to subsequences)

$$\nabla v_n \longrightarrow \nabla v \quad \text{almost everywhere in} \quad \Omega. \tag{2.19}$$

□

3 Existence of finite weak solution to problem (P)

Definition 3.1 Let $a(x) \in W^{1,p(x)}(\Omega)$. A function u is a weak solution of problem (P) if $u \in W_0^{1,p(x)}(\Omega)$ and

$$\int_\Omega a(x)\frac{\widehat{a}(x,\nabla u)}{(1+|u|)^{\varrho(x)}}\nabla\varphi dx = \int_\Omega f\varphi dx, \quad \forall\varphi \in W_0^{1,\infty}(\Omega). \tag{3.1}$$

Theorem 3.2 *Let $p(x) \geq 1$ and let $f \in L^{p_*(x)}(\Omega)$. Then, the problem (P) has at least one weak solution $u \in W_0^{1,p(x)}(\Omega)$ in the sense of Definition 3.1.*

3.1 Proof of Theorem 3.2

Let $(a_n(x))$ be as in (2.6) converges strongly to $a(x)$ in $W^{1,p(x)}(\Omega)$. We consider the approximate problem

$$\begin{cases} -\text{div}\Big(a_n(x)\frac{\widehat{a}(x,\nabla u_n)}{(1+|T_n(u_n)|)^{\varrho(x)}}\Big) = f, \quad \text{in } \Omega \\ \qquad\qquad u_n = 0, \text{ on } \partial\Omega. \end{cases} \tag{P_n}$$

Note that

$$\frac{1}{(1+|T_n(u_n)|)^{\varrho(x)}} \geq \frac{1}{(1+n)^{\varrho^+}}, \quad \varrho^+ = \max_{x\in\overline{\Omega}}\varrho(x), \tag{3.2}$$

so that the operator $B_n : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by

$$\int_\Omega \langle B_n v,\varphi\rangle dx = \int_\Omega a_n(x)\frac{\widehat{a}(x,\nabla u_n)}{(1+|T_n(u_n)|)^{\varrho(x)}}\nabla\varphi dx, \quad \forall\varphi \in W_0^{1,p(x)}(\Omega).$$

is coercive due to the fact that $a_n(x) \geq C_\lambda > 0$ and is pseudo-monotone on $W_0^{1,p(x)}(\Omega)$. Thus, the existence of the approximate solution is proved as in [8].

Now, Taking $T_k(u_n)$ as test function in (P_n) and using Hölder's inequality, we obtain

$$\int_\Omega a_n(x)\frac{\widehat{a}(x,\nabla u_n)}{(1+|T_n(u_n)|)^{\varrho(x)}}\nabla T_k(u_n)dx \leq \int_\Omega |f||T_k(u_n)|dx \leq 2k\|f\|_{L^{p_*(x)}(\Omega)}\|1\|_{L^{p'_*(x)}(\Omega)}. \tag{3.3}$$

According to assumption $(\widehat{a_1})$ and the condition (2.7), we get for $n > k > 0$

$$\frac{\beta C_\lambda}{(1+k)^{\varrho^+}}\int_\Omega |\nabla T_k(u_n)|^{p(x)}dx \leq 2k\|f\|_{L^{p_*(x)}(\Omega)}\|1\|_{L^{p^*(x)}(\Omega)}. \tag{3.4}$$

Therefore, there exists $C = C(f) > 0$ such that

$$\|T_k(u_n)\|_{W_0^{1,p(x)}(\Omega)} \leq C. \tag{3.5}$$

Up to subsequences, there exists a function $u \in W_0^{1,p(x)}(\Omega)$ such that

$$u_n \longrightarrow u \ \text{ a.e. in } \ \Omega. \tag{3.6}$$

Hence, assumption (4) of Lemma 2.1 holds for the sequence $\{T_k(u_n)\}_n$. Since f is bounded in $L^1(\Omega)$, then we have that assumption (1) of Lemma 2.1 holds. So, we can apply the Lemma 2.1 to have that,

$$\nabla u_n \longrightarrow \nabla u \ \text{ a.e. in } \ \Omega. \tag{3.7}$$

Moreover, by the estimate (2.13) and the strong convergence of $a_n(x)$ to $a(x)$ in $L^{p(.)}(\Omega)$, we can pass to the limit for $n \longrightarrow +\infty$ in the weak formulation for (P_n)

$$\int_\Omega a_n(x)\frac{\widehat{a}(x,\nabla u_n)}{(1+|T_n(u_n)|)^{\varrho(x)}}\nabla\varphi dx = \int_\Omega f\varphi dx, \qquad \forall\varphi \in W_0^{1,\infty}(\Omega),$$

we obtain that u is a weak solution for (P).

4 Existence of weak solutions to problems (P_1) and (P_2)

As mentioned in Section.1. In this section we will prove existence of solutions for the problem (P_1). We will use the following function defined for by $G_k(r) = r - T_k(r)$ for every $r \in \mathbb{R}$.

Definition 4.1 Let $a(x) \in W^{1,p(x)}(\Omega)$. A function u is a weak solution of problem (P_1) if $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, and

$$\int_\Omega a(x)\frac{\widehat{a}(x,\nabla u)}{(1+|u|)^{\varrho(x)}}\nabla\varphi dx + \int_\Omega e(x)|u|^{p(x)-2}u\varphi dx = \int_\Omega f\varphi dx, \tag{4.1}$$

for every $\varphi \in W_0^{1,\infty}(\Omega)$.

Theorem 4.2 *Let $p(x) \geq 1$ and let $f \in L^1(\Omega)$, suppose that* (1.1), (1.2) *holds true. Then, the problem* (P_1) *has at least one weak solution $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ in the sense of Definition 4.1.*

4.1 Proof of Theorem 4.2

Let the sequence $a_n(x)$ be previously defined by the relation (2.6). Consider the following approximated problem

$$\begin{cases} -\mathrm{div}\left(a_n(x)\frac{\widehat{a}(x,\nabla u_n)}{(1+|T_n(u_n)|)^{\varrho(x)}}\right) + e(x)|u_n|^{p(x)-2}u_n = f_n & \text{in } \Omega, \\ u_n = 0 \text{ on } \partial\Omega. \end{cases} \tag{P_{1n}}$$

with (f_n), $f_n = T_n(f)$ be a sequence of bounded functions defined in Ω, such that

$$\begin{cases} \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \\ f_n \to f \quad \text{strongly in } \ L^1(\Omega)\ . \end{cases} \tag{4.2}$$

Note that by (1.2), we have

$$|f_n(x)| \leq |f(x)| \leq k_0.e(x) \tag{4.3}$$

Thus, from (3.2) and the boundary of the sequence $\{a_n\}_n$, the existence of the approximate solution of (P_{1n}) is proved as in [8](see also [10]). We choose $G_R(u_n)$ as test function in (P_{1n}) with $R = \max_{x\in\overline{\Omega}}\{k_0^{\frac{1}{p(x)-1}}\}$, we get

$$\int_\Omega a_n(x)\frac{\widehat{a}(x,\nabla u_n)}{(1+|T_n(u_n)|)^{\varrho(x)}}\nabla(G_R(u_n))dx + \int_\Omega e(x)|u_n|^{p(x)-2}u_nG_R(u_n)dx = \int_\Omega f_nG_R(u_n)dx. \tag{4.4}$$

By assumption $\widehat{a_1}$, we find that

$$\beta C_\lambda \int_\Omega \frac{1}{(1+|T_n(u_n)|)^{\varrho(x)}} |\nabla G_R(u_n)|^{p(x)} dx + \int_\Omega e(x)|u_n|^{p(x)-2} u_n G_R(u_n) dx \\ \leq \int_\Omega |f_n||G_R(u_n)| dx. \tag{4.5}$$

Using that $u_n G_R(u_n) \geq 0$ and (4.3), we obtain

$$\beta C_\lambda \int_\Omega \frac{1}{(1+|u_n|)^{\varrho^+}} |\nabla G_R(u_n)|^{p(x)} dx + \int_\Omega e(x)(|u_n|^{p(x)-1} - k_0)|G_R(u_n)| dx \leq 0.$$

We remark that the integrand in the second integral is zero if $|u_n(x)| \leq k_0^{\frac{1}{p(x)-1}} \leq R$, so we deduce that

$$\beta C_\lambda \int_\Omega \frac{1}{(1+|u_n|)^{\varrho^+}} |\nabla G_R(u_n)|^{p(x)} dx = 0,$$

which implies

$$|u_n| \leq R. \tag{4.6}$$

This implies that $(u_n)_n$ is bounded in $L^\infty(\Omega)$.

Now, to show that the sequence $\{T_k(u_n)\}_n$ is bounded in $W_0^{1,p(x)}(\Omega)$, inserting $T_k(u_n)$ in (P_{1n}), and from (4.6), we get

$$\beta C_\lambda \frac{1}{(1+R)^{\varrho^+}} \int_\Omega |\nabla T_k(u_n)|^{p(x)} dx + \int_\Omega e(x)|u_n|^{p(x)-1} u_n T_k(u_n) dx \leq \int_\Omega f_n T_k(u_n) dx.$$

Using that $u_n T_k(u_n) \geq 0$, so after dropping nonnegative terms, we can rewrite the above inequality as follows

$$\beta C_\lambda \frac{1}{(1+R)^{\varrho^+}} \int_\Omega |\nabla T_k(u_n)|^{p(x)} dx \leq k\|f\|_{L^1(\Omega)}.$$

Thus, the sequence $\{T_k(u_n)\}_n$ satisfies the assumption (4) of Lemma 2.1. Furthermore, thanks again to (4.6), we have

$$\int_\Omega e(x)|u_n|^{p(x)-2} u_n dx \leq \int_\Omega e(x)|u_n|^{p(x)-1} dx \leq (R^{p^+}+1) \int_\Omega e(x) dx.$$

Since $e(x) \in L^1(\Omega)$, the sequence $\{e(x)|u_n|^{p(x)-2}u_n\}_n$ is bounded in $L^1(\Omega)$ and thanks to the boundedness of the sequence $\{u_n\}_n$ in $L^\infty(\Omega)$, we have that

$$e(x)|u_n|^{p(x)-2}u_n \to e(x)|u|^{p(x)-2}u \quad \text{strongly in} \quad L^1(\Omega). \tag{4.7}$$

Moreover, it is clear that assumption (1) in the Lemma 2.1 is satisfied because that the sequence $\{K_n\}_n$ where

$$K_n(x) = f_n(x) - e(x)|u_n|^{p(x)-2}u_n,$$

is bounded in $L^1(\Omega)$. We can apply the Lemma 2.1 to have that $\{\nabla u_n\}$ converges almost everywhere to ∇u in Ω. Furthermore it, thanks to the boundedness of the sequence $(u_n)_n$ in $W_0^{1,p}(\Omega)$, we have that

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in} \quad L^{p'}(\Omega). \tag{4.8}$$

Since the sequence $(a_n)_n$ strongly converges to $a(x)$ in $L^p(\Omega)$ (being strongly convergent in $W^{1,p}(\Omega)$), we can use (4.8) and by the dominated convergence Theorem, we get

$$a_n(x)|\nabla u_n|^{p-2}\nabla u_n \to a(x)|\nabla u|^{p-2}\nabla u \quad \text{in} \quad L^1(\Omega).$$

and using (2.5), we have that

$$\begin{cases} \frac{1}{1+|T_n(u_n)|}\nabla\varphi \to \frac{1}{1+|u_n|}\nabla\varphi, \text{a.e in } \Omega, \\ |\frac{1}{1+|T_n(u_n)|}\nabla\varphi| \leq |\nabla\varphi| \in L^\infty(\Omega), \end{cases} .$$

So,

$$\lim_{n\to+\infty}\int_\Omega a_n(x)\frac{|\nabla u_n|^{p-2}\nabla u_n}{1+|T_n(u_n)|}\nabla\varphi dx=\int_\Omega a(x)\frac{|\nabla u|^{p-2}\nabla u}{1+|u|}\nabla\varphi dx.$$

Finally. Using (4.2), (4.7), we can easily pass to the limit for $n \longrightarrow +\infty$ in the weak formulation for (P_1)

$$\int_\Omega a_n(x)\frac{\widehat{a}(x,\nabla u_n)}{(1+|T_n(u_n)|)^{\varrho(x)}}\nabla\varphi dx+\int_\Omega e(x)|u_n|^{p(x)-2}u_n\varphi dx=\int_\Omega f_n\varphi dx,\quad \forall\varphi\in W_0^{1,\infty}(\Omega),$$

to have that (4.1). Then, the Theorem 4.2 is proved.

Now, as stated in Sect. 1, we study the problem (P_2), and prove existence of solutions.

Theorem 4.3 *Let* $p(x)\geq 1$ *and let* $f\in L^1(\Omega)$*, suppose that* (1.1)-(1.4) *holds true. Then, the problem* (P_2) *has at least one weak solution* $u\in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$ *such that*

$$\int_\Omega a(x)\widehat{a}(x,\nabla u)\nabla\varphi dx+\int_\Omega e(x)|u|^{p(x)-2}u\varphi dx=\int_\Omega \pi(x)F(x,u,\nabla u)\varphi dx+\int_\Omega f\varphi dx, \tag{4.9}$$

for every $\varphi\in W_0^{1,\infty}(\Omega)$.

4.2 Proof of Theorem 4.3

Consider the following approximated problem

$$\begin{cases} -\mathrm{div}(a_n(x)\widehat{a}(x,\nabla u_n))+e(x)|u_n|^{p(x)-2}u_n \\ \qquad\qquad = \pi_n(x)F_n(x,u_n,\nabla u_n)+f_n & \text{in } \Omega,\\ u_n=0 & \text{on } \partial\Omega. \end{cases} \tag{P_{2n}}$$

with f_n be as in (4.2) and $\pi_n,\ F_n$ defined by

$$\pi_n(x)=\frac{\pi(x)}{1+\frac{|\pi(x)|}{n}},\quad F_n(x,u_n,\nabla u_n)=\frac{F(x,u,\nabla u)}{1+\frac{|F(x,u,\nabla u)|}{n}}. \tag{4.10}$$

Note that the definition of a_n in (2.6) and since $\psi(s)=s(1+\frac{s}{n})^{-1}$ is increasing, we deduce by (1.3) that

$$|\pi_n(x)|=\frac{|\pi(x)|}{1+\frac{|\pi(x)|}{n}}\leq\frac{a(x)}{1+\frac{a(x)}{n}}=a_n(x). \tag{4.11}$$

Thanks to the Schauder fixed point Theorem (see [9]), for every $n\in\mathbb{N}$, there exists a solution $u_n\in W_0^{1,p(x)}(\Omega)$ of (P_{2n}). As [2] we define the real functions $\varphi\in C^1(\mathbb{R})$ by $\varphi(s)=(e^{\tau|s|}-1)\mathrm{sgn}(s)$, we choose $v=\varphi(G_R(u_n))$ as a test function in (P_{2n}) with $R=\max_{x\in\overline{\Omega}}\{k_0^{\frac{1}{p(x)-1}}\}$, we obtain

$$\begin{aligned}\tau\int_\Omega a_n(x)\widehat{a}(x,\nabla u_n)\nabla u_n e^{\tau|G_R(u_n)|}dx+\int_\Omega e(x)|u_n|^{p(x)-2}u_n\varphi(G_R(u_n))dx\\ \leq\int_\Omega|\pi_n(x)||F_n(x,u_n,\nabla u_n)|(e^{\tau|G_R(u_n)|}-1)dx+\int_\Omega|f_n|(e^{\tau|G_R(u_n)|}-1)dx,\end{aligned}$$

Observing that

$$0\leq u_n\varphi(G_R(u_n))=|u_n|(e^{\tau|G_R(u_n)|}-1),\quad e^{\tau|G_R(u_n)|}-1\leq e^{\tau|G_R(u_n)|}.$$

According assumption $(\widehat{a_1})$ and by (4.11) , we get

$$\begin{aligned}\beta\tau\int_\Omega a_n(x)|\nabla G_R(u_n)|^{p(x)}e^{\tau|G_R(u_n)|}dx+\int_\Omega e(x)|u_n|^{p(x)-1}(e^{\tau|G_R(u_n)|}-1)dx\\ \leq\int_\Omega a_n(x)|F_n(x,u_n,\nabla u_n)|e^{\tau|G_R(u_n)|}dx+\int_\Omega|f_n|(e^{\tau|G_R(u_n)|}-1)dx,\end{aligned}$$

So, by (1.4), we have

$$\frac{\tau\lambda}{\lambda+1}\int_\Omega(\beta-1)|\nabla G_R(u_n)|^{p(x)}e^{\tau|G_R(u_n)|}dx+\int_\Omega e(x)|u_n|^{p(x)-1}(e^{\tau|G_R(u_n)|}-1)dx$$
$$\leq\int_\Omega|f_n|(e^{\tau|G_R(u_n)|}-1)dx.$$

Moreover, from (4.3) we get

$$\frac{\tau\lambda(\beta-1)}{\lambda+1}\int_\Omega|\nabla G_R(u_n)|^{p(x)}e^{\tau|G_R(u_n)|}dx+\int_\Omega e(x)(|u_n|^{p(x)-1}-k_0)(e^{\tau|G_R(u_n)|}-1)dx\leq 0. \quad (4.12)$$

If $|u_n(x)|\leq k_0^{\frac{1}{p(x)-1}}\leq R$, it is straightforward to observe that

$$\int_\Omega e(x)(|u_n|^{p(x)-1}-k_0)(e^{\tau|G_R(u_n)|}-1)dx=0.$$

This proves that

$$\frac{\tau\lambda(\beta-1)}{\lambda+1}\int_\Omega|\nabla G_R(u_n)|^{p(x)}e^{\tau|G_R(u_n)|}dx=0.$$

Here, $\beta>1$ ensures that

$$|u_n|\leq R. \quad (4.13)$$

Now, taking $\varphi=(e^{\tau|T_k(u_n)|}-1)\mathrm{sgn}(T_k(u_n))$ as a test function in (P_{2n}), using $|T_k(u_n)|\leq k$, we obtain

$$\frac{\tau\lambda}{\lambda+1}\int_\Omega\beta|\nabla T_k(u_n)|^{p(x)}e^{\tau|T_k(u_n)|}dx+\int_\Omega e(x)|u_n|^{p(x)-2}u_n(e^{\tau|u_n|}-1)\mathrm{sgn}(T_k(u_n))dx$$
$$\leq\int_\Omega|f_n|(e^{\tau k}-1)dx. \quad (4.14)$$

The fact that $u_n(e^{\tau|T_k(u_n)|}-1)\mathrm{sgn}(T_k(u_n))\geq 0$ and dropping the non-negative term in (4.14), we derive

$$\beta\frac{\tau\lambda}{\lambda+1}\int_\Omega|\nabla T_k(u_n)|^{p(x)}e^{\tau|T_k(u_n)|}dx\leq(e^{\tau k}-1)\|f\|_{L^1(\Omega)}.$$

In particular, we deduce that

$$\|T_k(u_n)\|_{W_0^{1,p(x)}(\Omega)}\leq C. \quad (4.15)$$

Furthermore, after choosing again $\varphi=(e^{\tau|u_n|}-1)\mathrm{sgn}(u_n)$ as a test function in (P_{2n}) and using $|u_n|\leq R$, we find (after simplifications)

$$\tau\beta\int_\Omega a_n(x)|\nabla u_n|^{p(x)}e^{\tau|u_n|}dx\leq\int_\Omega|\pi_n(x)||F_n(x,u_n,\nabla u_n)|e^{\tau|u_n|}dx+(e^{\tau R}-1)\|f\|_{L^1(\Omega)}.$$

Since $|\pi_n(x)|\leq a_n(x)$ and (1.4), we have

$$(\beta-1)\int_\Omega|\pi_n(x)||F_n(x,u_n,\nabla u_n)|e^{\tau|u_n|}dx\leq(e^{\tau R}-1)\|f\|_{L^1(\Omega)}.$$

In particular, we have that the sequence $\{\pi_n(x)F_n(x,u_n,\nabla u_n)\}_n$ is bounded in $L^1(\Omega)$, so that if one defines

$$K_n(x)=f_n(x)+\pi_n(x)F_n(x,u_n,\nabla u_n)-e(x)|u_n|^{p(x)-2}u_n,$$

the sequence $\{K_n\}_n$ is bounded in $L^1(\Omega)$, so that assumption (1) of Lemma 2.1 holds. Since the sequence $\{T_k(u_n)\}_n$ is bounded in $W_0^{1,p(x)}(\Omega)$ by (4.15), we can apply again Lemma 2.1 to have that

$$\nabla u_n\longrightarrow\nabla u\ \text{ a.e. in }\ \Omega. \quad (4.16)$$

Now, to pass the limit in the weak formulation for (P_2), we take

$$\varphi = e^{u_n - u} \frac{T_k(a(x))}{a_n(x)} v, \quad \text{with} \quad 0 \le v \in W_0^{1,N}(\Omega) \cap L^\infty(\Omega),$$

as test function in (P_{2n}), we obtien

$$\int_\Omega \widehat{a}(x, \nabla u_n) \nabla (u_n - u) e^{u_n - u} T_k(a) v dx + \int_\Omega \widehat{a}(x, \nabla u_n) e^{u_n - u} \left[\frac{a_n \nabla T_k(a) - T_k(a) \nabla a_n}{a_n} v + T_k(a) \nabla v \right] dx$$
$$+ \int_\Omega e(x) |u_n|^{p(x)-2} u_n \varphi dx = \int_\Omega \pi_n(x) F_n(x, u_n, \nabla u_n) \varphi dx + \int_\Omega f_n \varphi dx,$$

so,

$$\int_\Omega \widehat{a}(x, \nabla u_n) \nabla (u_n - u) e^{u_n - u} T_k(a) v dx + \int_\Omega \widehat{a}(x, \nabla u_n) e^{u_n - u} \left[\nabla T_k(a) v + T_k(a) \nabla v \right] dx$$
$$+ \int_\Omega e(x) |u_n|^{p(x)-2} u_n \varphi dx$$
$$= \int_\Omega \widehat{a}(x, \nabla u_n) e^{u_n - u} \frac{T_k(a) \nabla a_n}{a_n} v + \int_\Omega \pi_n(x) F_n(x, u_n, \nabla u_n) \varphi dx + \int_\Omega f_n \varphi dx.$$

Since $\widehat{a}$ is a Carathéodory function and by (2.5),(4.13),(4.16), the strong convergence of a_n to a in $W^{1,p(x)}$ and the boundedness of $\frac{1}{a_n}$, we can pass to the limit in (P_{2n}) by Fatou's Lemma, to have that

$$\int_\Omega \widehat{a}(x, \nabla u) \left[\nabla T_k(a) v + T_k(a) \nabla v \right] dx + \int_\Omega e(x) |u|^{p(x)-2} u \varphi dx$$
$$\le \int_\Omega \widehat{a}(x, \nabla u) \frac{T_k(a) \nabla a}{a} v + \int_\Omega \pi(x) F(x, u, \nabla u) v dx + \int_\Omega f \frac{T_k(a)}{a} v dx.$$

Letting k tend to $+\infty$, we deduce that

$$\int_\Omega \widehat{a}(x, \nabla u) \left[\nabla a v + a \nabla v \right] dx + \int_\Omega e(x) |u|^{p(x)-2} u v dx$$
$$\le \int_\Omega \widehat{a}(x, \nabla u) \nabla s v + \int_\Omega \pi(x) F(x, u, \nabla u) v dx + \int_\Omega f v dx,$$

which implies that

$$\int_\Omega a(x) \widehat{a}(x, \nabla u) \nabla v dx + \int_\Omega e(x) |u|^{p(x)-2} u v dx \le \int_\Omega \pi(x) F(x, u, \nabla u) v dx + \int_\Omega f v dx. \tag{4.17}$$

To prove the opposite inequality, we choose

$$\varphi = e^{u - u_n} \frac{T_k(a(x))}{a_n(x)} v, \quad \text{with} \quad 0 \le v \in W_0^{1,N}(\Omega) \cap L^\infty(\Omega).$$

In the same way as above and letting first n tend to infinity and then k tend to infinity, we arrive at

$$\int_\Omega a(x) \widehat{a}(x, \nabla u) \nabla v dx + \int_\Omega e(x) |u|^{p(x)-2} u v dx$$
$$\ge \int_\Omega \pi(x) F(x, u, \nabla u) v dx + \int_\Omega f v dx. \tag{4.18}$$

Combining (4.17) and (4.18) , we deduce that

$$\int_\Omega a(x) \widehat{a}(x, \nabla u) \nabla v dx + \int_\Omega e(x) |u|^{p(x)-2} u v dx$$
$$= \int_\Omega \pi(x) F(x, u, \nabla u) v dx + \int_\Omega f v dx, \quad \forall v \in W_0^{1,N}(\Omega) \cap L^\infty(\Omega),$$

with $v \ge 0$. Thus, we have that (4.9) holds for every nonnegative test function. The case of a general test function v is then obtained by choosing v^+ and v^-, and then adding up the two equalities.

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