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ORIGINAL RESEARCH

Logarithmically improved blow-up criteria for the 3D Navier-Stokes/Poisson-Nernst-Planck system

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Abstract In this paper, we investigate two sufficient conditions, in terms of the horizontal components $u^h = (u^1, u^2)$ of the velocity field, for the breakdown of local smooth solutions to the 3D incompressible Navier-Stokes/Poisson-Nernst-Planck system arising from electrohydrodynamics. More precisely, we prove that if

$$\int_0^T \frac{\|\nabla_h u^h(\cdot,t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1+\ln(e+\|\nabla_h u^h(\cdot,t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}})} dt < \infty \quad \text{for} \quad 0<\alpha<1$$

or

$$\int_0^T \frac{\|\nabla_h u^h(\cdot,t)\|_{\dot{B}^{0}_{\infty,\infty}}}{\sqrt{1+\ln(e+\|\nabla_h u^h(\cdot,t)\|_{\dot{B}^{0}_{\infty,\infty}})}} dt < \infty,$$

then the local solution can be smoothly extended past the time $t = T$.

Keywords Electrohydrodynamics · Navier-Stokes equations · Poisson-Nernst-Planck equations · Weak solutions · Regularity criteria

Mathematics Subject Classification 35B45 · 35B65 · 76W05

1 Introduction

Electrohydrodynamics (EHD) is an interdisciplinary field that studies the interaction between the electric fields and different fluid media, including dielectric liquids, gases, and crystals. The influence of moving media on electric fields, as well as the effect of the electric fields on moving media, can lead to many interesting phenomena of electric current (cf. [6,21]). In this paper, we are interested in the regularity of weak solutions to the 3D nonlinear dissipative system modeling electro-diffusion, where the fluid motion is given by the incompressible

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Navier-Stokes equations with an external forcing term, and the self-consistent charge transport is described by the Poisson-Nernst-Planck equations. The 3D Cauchy problem writes

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \mu\Delta u + \nabla\pi = \varepsilon\Delta\phi\nabla\phi, \\ \nabla \cdot u = 0, \\ \partial_t v + (u \cdot \nabla)v = \nabla \cdot (D_1\nabla v - \nu_1 v\nabla\phi), \\ \partial_t w + (u \cdot \nabla)w = \nabla \cdot (D_2\nabla w + \nu_2 w\nabla\phi), \\ \varepsilon\Delta\phi = v - w, \\ (u, v, w)|_{t=0} = (u_0, v_0, w_0), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^3$, $t > 0$, the unknown functions $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $\pi = \pi(x,t)$ stand for the velocity field and the pressure of the incompressible fluid, respectively, $v = v(x,t)$ and $w = w(x,t)$ stand for the densities of a negatively and positively charged species, respectively, and $\phi = \phi(x,t)$ is the electrostatic potential. The constants μ, ε, D_1, D_2, ν_1, ν_2 denote the kinematic viscosity, the dielectric constant, the diffusion and mobility coefficients of the charged particles, respectively, which shall be assumed to be all equal to one under the consideration of their concrete values playing no roles in our analysis.

The first two equations of system (1.1) are the momentum conservation and the mass conservation equations of the incompressible flow, which reduces to the following 3D incompressible Navier-Stokes equations if the flow is charge-free (i.e., $v = w = \phi = 0$):

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \mu\Delta u + \nabla\pi = 0, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1.2}$$

In [20], Leray constructed a global weak solution $u \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^3))$ for arbitrary initial data $u_0 \in L^2(\mathbb{R}^3)$ and questioned whether the weak solution is smooth for a given smooth and compactly supported initial data u_0. Until now, this problem still remains unsolved and is an outstanding open problem in mathematical fluid mechanics. In the celebrated work [26], Serrin introduced the class $L^q(0,T; L^p(\mathbb{R}^3))$ and showed that any weak solution u satisfies

$$\int_0^T \|u(\cdot,t)\|_{L^p}^q dt < \infty \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 1 \quad \text{and} \quad 3 < p \leq \infty, \tag{1.3}$$

then u is regular on $(0,T]$. The limit case $p = 3$ was addressed by Escauriaza et al. [8]. Later on, Beirão da Veiga [4] showed that if the gradient of the velocity satisfies

$$\int_0^T \|\nabla u(\cdot,t)\|_{L^p}^q dt < \infty \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 2 \quad \text{and} \quad \frac{3}{2} < p \leq \infty, \tag{1.4}$$

then u is still regular on $(0,T]$. In 1984, Beale, Kato and Majda in their celebrated work [3] showed that if the smooth solution u blows up at the time $t = T$, then

$$\int_0^T \|\omega(\cdot,t)\|_{L^\infty} \, dt = \infty, \tag{1.5}$$

where $\omega = \nabla \times u$ is the vorticity. Since then, many interesting sufficient conditions were proposed to guarantee the regularity of weak solutions, or to control the breakdown of the local smooth solutions, see [5,7,9,12,13,18, 19,23,27,30–33] and the references cited there.

The global existence of weak solutions to all kinds of initial/boundary-value problems of the Navier-Stokes/Poisson-Nernst-Planck system have already been established, we refer the readers to see the mixed Dirichlet boundary conditions [14,15], no-flux boundary conditions [24] and Neumann boundary conditions [25]. However, similar to the 3D incompressible Navier-Stokes equations (1.2), the global regularity of weak solution and the global existence of smooth solutions are still the challenging open problems. In [11], the authors realized that the velocity field plays the dominant role in the regularity issue of weak solutions of system (1.1), and they proved that if the velocity field u satisfies

$$\int_0^T \|u(\cdot,t)\|_{L^p}^q dt < \infty \quad \text{with} \quad \frac{2}{q}+\frac{3}{p}=1 \quad \text{and} \quad 3<p\leq\infty \tag{1.6}$$

or

$$\int_0^T \|\nabla u(\cdot,t)\|_{L^p}^q dt < \infty \quad \text{with} \quad \frac{2}{q}+\frac{3}{p}=2 \quad \text{and} \quad \frac{3}{2}<p\leq\infty, \tag{1.7}$$

then the weak solution (u, v, w) of system (1.1) is regular on $(0, T]$. Later on, Fan, Nakamura and Zhou [10] established the following BKM type criterion, and they proved that if the smooth solution (u, v, w) blows up at the time $t = T$, then

$$\int_0^T \|\omega(\cdot,t)\|_{\dot{B}^0_{\infty,\infty}} dt = \infty, \tag{1.8}$$

where $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$ is the homogeneous Besov space. Zhao and Bai [29] subsequently improved the criterion (1.7) as

$$\int_0^T \|\nabla_h u^h(\cdot,t)\|_{\dot{B}^0_{\infty,\infty}} dt = \infty, \tag{1.9}$$

where $\nabla_h := (\partial_1, \partial_2)$, and $u^h := (u^1, u^2)$ is the horizontal components of the velocity field u. Recently, Zhao [28] further proved that if the vorticity ω satisfies

$$\int_0^T \frac{\|\omega(\cdot,t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}}{1+\ln(e+\|\omega(\cdot,t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}})} dt < \infty \quad \text{for} \quad 0<\alpha<2 \tag{1.10}$$

or in the marginal case $\alpha = 0$, if the vorticity field ω satisfies

$$\int_0^T \frac{\|\omega(\cdot,t)\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\ln(e+\|\omega(\cdot,t)\|_{\dot{B}^0_{\infty,\infty}})}} dt < \infty, \tag{1.11}$$

then the solution (u, v, w) is smooth up to time T.

In this paper, we are intend to establish an improvement of the above blow-up criteria (1.9)–(1.11), and the refined blow-up criteria in terms of the horizontal components of the velocity field can be stated as follows:

Theorem 1.1 *Let $u_0 \in H^3(\mathbb{R}^3)$ with $\nabla\cdot u_0 = 0$, $v_0, w_0 \in L^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)$ and $v_0, w_0 \geq 0$. Let $T_* > 0$ be the maximum existence time such that the system* (1.1) *admits an unique solution* (u, v, w) *satisfying*

$$\begin{cases} u \in C([0,T], H^3(\mathbb{R}^3)) \cap L^\infty(0,T; H^3(\mathbb{R}^3)) \cap L^2(0,T; H^4(\mathbb{R}^3)), \\ v, w \in C([0,T], H^2(\mathbb{R}^3)) \cap L^\infty(0,T; H^2(\mathbb{R}^3)) \cap L^2(0,T; H^3(\mathbb{R}^3)) \end{cases} \tag{1.12}$$

for any $0 < T < T_*$. If $T_* < \infty$, then

$$\int_0^{T_*} \frac{\|\nabla_h u^h(\cdot,t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1+\ln(e+\|\nabla_h u^h(\cdot,t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}})} dt = \infty \tag{1.13}$$

for all $0 < \alpha < 1$.

Remark 1.1. Under the assumptions of Theorem 1.1, if for some $0 < T < \infty$ and $0 < \alpha < 1$ such that

$$\int_0^T \frac{\|\nabla_h u^h(\cdot,t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1+\ln(e+\|\nabla_h u^h(\cdot,t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}})} dt < \infty,$$

then the solution (u, v, w) can be extended past the time $t = T$. Thus (1.13) can be regarded as an improvement of regularity criteria (1.10).

In the marginal case $\alpha = 0$, we obtain the following blow-up criterion.

Theorem 1.2 *Let* $u_0 \in H^3(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$, $v_0, w_0 \in L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ *and* $v_0, w_0 \geq 0$. *Let* $T_* > 0$ *be the maximum existence time such that the system* (1.1) *admits an unique solution* (u, v, w) *satisfying* (1.12) *for any* $0 < T < T_*$. *If* $T_* < \infty$, *then*

$$\int_0^{T_*} \frac{\|\nabla_h u^h(\cdot, t)\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \ln(e + \|\nabla_h u^h(\cdot, t)\|_{\dot{B}^0_{\infty,\infty}})}} dt = \infty. \tag{1.14}$$

Remark 1.2. Under the assumptions of Theorem 1.2, if for some $0 < T < \infty$ such that

$$\int_0^{T} \frac{\|\nabla_h u^h(\cdot, t)\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \ln(e + \|\nabla_h u^h(\cdot, t)\|_{\dot{B}^0_{\infty,\infty}})}} dt < \infty,$$

then the solution (u, v, w) can be extended past the time $t = T$. It is clear that (1.14) is a logarithmically improved regularity criterion of (1.9) and (1.11).

The remainder of this paper is organized as follows. In Sect. 2, we shall recall the Littlewood-Paley decomposition theory and the definition of homogeneous Besov spaces, then we present some analytical tools used in the proofs of our main results. In Sect. 3, we present the proof of Theorem 1.1, and the proof of Theorem 1.2 will be given in Sect. 4. Throughout the paper, we denote by C a generic positive constant which may vary from one line to another, and the special dependence will be pointed out explicitly in the text if necessary.

2 Preliminaries

We recall some basic notions of the homogeneous Littlewood-Paley decomposition. Let (φ, χ) be a couple of smooth functions valued in [0, 1], such that φ is supported in the shell $\{\xi \in \mathbb{R}^3,\ \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, χ is supported in the ball $B(0, \frac{4}{3})$ and

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3 \backslash \{0\}$$

and

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3 \backslash \{0\}.$$

Write $h = \mathcal{F}^{-1}\varphi$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$. The homogeneous dyadic blocks Δ_j and the low-frequency cutoff operators S_j are defined for all $j \in \mathbb{Z}$ by

$$\Delta_j f := 2^{3j} \int_{\mathbb{R}^3} h(2^j y) f(x - y) dy, \quad S_j f := 2^{3j} \int_{\mathbb{R}^3} \widetilde{h}(2^j y) f(x - y) dy.$$

Let $\mathcal{S}'_h(\mathbb{R}^3)$ be the space of tempered distribution $f \in \mathcal{S}'(\mathbb{R}^3)$ such that

$$\lim_{j \to -\infty} S_j f(x) = 0.$$

Then one has the unit decomposition for any tempered distribution $f \in \mathcal{S}'_h(\mathbb{R}^3)$:

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f. \tag{2.1}$$

Using the above homogeneous Littlewood-Paley decomposition, the homogeneous Besov spaces can be defined as follows:

Definition 2.1 Let $s \in \mathbb{R}$, $1 \le p, r \le \infty$ and $f \in \mathcal{S}'(\mathbb{R}^3)$, we set

$$\|f\|_{\dot{B}^s_{p,r}} := \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\Delta_j f\|^r_{L^p}\right)^{\frac{1}{r}} & \text{for } 1 \le r < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{L^p} & \text{for } r = \infty. \end{cases}$$

Then the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined by

- For $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define

$$\dot{B}^s_{p,r}(\mathbb{R}^3) := \left\{ f \in \mathcal{S}'_h(\mathbb{R}^3) : \ \|f\|_{\dot{B}^s_{p,r}} < \infty \right\}.$$

- If $k \in \mathbb{N}$ and $\frac{3}{p} + k \le s < \frac{3}{p} + k + 1$ (or $s = \frac{3}{p} + k + 1$ if $r = 1$), then $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined as the subset of distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\partial^\beta f \in \dot{B}^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta| = k$.

The proofs of Theorems 1.1 and 1.2 require some crucial analytic tools, thus we briefly recall them here. We start with the well-known Gagliardo-Nirenberg type inequalities.

Lemma 2.2 *Let $k < m$ and $1 \le p, q < \infty$. Assume that $k \le j \le m$, $1 \le r \le \infty$ and $0 \le \theta \le 1$ satisfying*

$$j - \frac{3}{r} = \theta(k - \frac{3}{p}) + (1-\theta)(m - \frac{3}{q}).$$

Then for all $f \in W^{k,p}(\mathbb{R}^3) \cap W^{m,q}(\mathbb{R}^3)$, we have $f \in W^{j,r}(\mathbb{R}^3)$, and there exists a positive constant C such that

$$\|\nabla^j f\|_{L^r} \le C\|\nabla^k f\|^\theta_{L^p}\|\nabla^m f\|^{1-\theta}_{L^q}. \tag{2.2}$$

The next result is the generalization of the refined Sobolev interpolation result, where the proof can be found in [1] (see Theorem 2.42).

Lemma 2.3 *Let $\alpha > 0$ and $1 \le q < p < \infty$. Then there exists a positive constant C depending only on α, β, p and q such that*

$$\|f\|_{L^p} \le C\|f\|^{1-\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}}\|f\|^\theta_{\dot{B}^\beta_{q,q}} \quad \text{with} \quad \beta = \alpha(\frac{p}{q} - 1) \quad \text{and} \quad \theta = \frac{q}{p} \tag{2.3}$$

holds for all $f \in \dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^3) \cap \dot{B}^\beta_{q,q}(\mathbb{R}^3)$.

Especially, we shall use the following specific case of (2.3) by taking $p = 3$, $q = 2$ and using the fact $\dot{H}^\beta(\mathbb{R}^3) = \dot{B}^\beta_{2,2}(\mathbb{R}^3)$:

$$\|f\|_{L^3} \le C\|f\|^{\frac{1}{3}}_{\dot{B}^{-\alpha}_{\infty,\infty}}\|f\|^{\frac{2}{3}}_{\dot{H}^{\frac{\alpha}{2}}}. \tag{2.4}$$

The inequality (2.4) is originally derived from Meyer-Gerard-Oru [22], then was revisited by Guo-Gala [13] (see Lemma 2.3).

We also need to use the following well-known commutator estimates due to [16] (see Lemma X1 in Appendix) and Kenig-Ponce-Vega [17] (see Lemma 2.10).

Lemma 2.4 *Let $s > 1$ and $1 < p < \infty$. Then we have*

$$\|\nabla^s(fg) - f\nabla^s g\|_{L^p} \le C\big(\|\nabla f\|_{L^{p_1}}\|\nabla^{s-1}g\|_{L^{q_1}} + \|\nabla^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\big) \tag{2.5}$$

with $1 < q_1, p_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

To prove Theorem 1.1, the crucial step is to establish the higher-order derivative estimates of the solutions by using the following logarithmic Sobolev inequality, which the proof can be found in [2], and it is a slightly modification of the one in [19].

Lemma 2.5 *For all* $\nabla f \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$*,* $f \in H^s(\mathbb{R}^3)$ *with* $s > \frac{3}{2}$*, we have*

$$\|f\|_{L^\infty} \leq C\big(1 + \|\nabla f\|_{\dot{B}^{-1}_{\infty,\infty}} \sqrt{\log^+ \|f\|_{H^s}}\big), \tag{2.6}$$

where $H^s(\mathbb{R}^3)$ *denotes the standard Sobolev space and*

$$\log^+ x = \begin{cases} \log x, & x > e, \\ 1, & 0 \leq x \leq e. \end{cases}$$

Finally, let us prove the following embedding result.

Lemma 2.6 *For any* $0 \leq \alpha < 1$*, we have*

$$\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \leq C(\|u\|_{L^2} + \|\nabla^3 u\|_{L^2}). \tag{2.7}$$

Proof Let Δ_j be the Littlewood-Paley dyadic operator, which is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$. Then we deduce from the classical Bernstein inequality that

$$\begin{aligned}
\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}} &\leq \sup_{j\leq -1} 2^{-\alpha j}\|\Delta_j \nabla_h u^h\|_{L^\infty} + \sup_{j\geq 0} 2^{-\alpha j}\|\Delta_j \nabla_h u^h\|_{L^\infty} \\
&\leq C \sup_{j\leq -1} 2^{-\alpha j} 2^{\frac{5}{2}j}\|\Delta_j u^h\|_{L^2} + C\sup_{j\geq 0} 2^{(\frac{5}{2}-\alpha)j} 2^{-3j}\|\Delta_j \nabla^3 u^h\|_{L^2} \\
&\leq C \sup_{j\leq -1} 2^{(\frac{5}{2}-\alpha)j}\|u\|_{L^2} + C\sup_{j\geq 0} 2^{-(\frac{1}{2}+\alpha)j}\|\nabla^3 u\|_{L^2} \\
&\leq C(\|u\|_{L^2} + \|\nabla^3 u\|_{L^2}).
\end{aligned}$$

We complete the proof of Lemma 2.6. □

3 Proof of theorem 1.1

We prove Theorem 1.1 by contradiction. It suffices to show that if the condition (1.13) does not hold, then there exists a constant $K > 0$ such that for some $0 < \alpha < 1$,

$$\int_0^{T_*} \frac{\|\nabla_h u^h(\cdot,t)\|^{\frac{2}{1-\alpha}}_{\dot{B}^{-\alpha}_{\infty,\infty}}}{1 + \ln(e + \|\nabla_h u^h(\cdot,t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}})} dt < K, \tag{3.1}$$

then one can prove that

$$\sup_{0\leq t\leq T_*} \big(\|u(\cdot,t)\|_{H^3} + \|(v(\cdot,t), w(\cdot,t))\|_{H^2}\big) \leq C_0, \tag{3.2}$$

where C_0 is a constant depending only on $\|u_0\|_{H^3}$, $\|(v_0, w_0)\|_{L^1\cap H^2}$, T_* and K. This implies that the solution (u, v, w) can be extended beyond the time T_*, which contradicts to the maximality of T_*. We state the above fact as the following theorem.

Theorem 3.1 *Let* $u_0 \in H^3(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$, $v_0, w_0 \in L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ *with* $v_0, w_0 \geq 0$*. Assume that* (u, v, w) *is the corresponding local smooth solution of* (1.1) *on* $\mathbb{R}^3 \times [0, T)$ *and satisfies the condition* (3.1)*. Then*

$$\sup_{0\leq t<T} (\|u(\cdot,t)\|_{H^3} + \|(v(\cdot,t), w(\cdot,t))\|_{H^2}) \leq C, \tag{3.3}$$

where C *is a constant depending on* $\|u_0\|_{H^3}$, $\|(v_0, w_0)\|_{L^1\cap H^2}$ *and* T.

Proof We first notice that the fundamental energy inequalities have already been established, e.g., see [29], thus we have for all $0 \le t < T$,

$$\|v(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + 2\int_0^t \big(\|\nabla v(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2\big)d\tau \le \|v_0\|_{L^2}^2 + \|w_0\|_{L^2}^2, \tag{3.4}$$

$$\|u(t)\|_{L^2}^2 + \|\nabla\phi(t)\|_{L^2}^2 + 2\int_0^t \big(\|\nabla u(\tau)\|_{L^2}^2 + \|\Delta\phi(\tau)\|_{L^2}^2\big)d\tau \le C, \tag{3.5}$$

where C is a constant depending only on $\|u_0\|_{L^2}$ and $\|(v_0, w_0)\|_{L^1\cap L^2}$. Moreover, by the maximum principle, we know that if v_0 and w_0 are nonnegative, then v and w are also nonnegative.

Next we derive the desired L^2 bounds of the vorticity $\omega = \nabla \times u$, which leads to the desired bounds of ∇u by the Biot-Savart law. It clear that the vorticity ω satisfies the following equations:

$$\partial_t\omega + (u\cdot\nabla)\omega - \Delta\omega = (\omega\cdot\nabla)u + \nabla\times(\Delta\phi\nabla\phi). \tag{3.6}$$

Multiplying (3.6) by ω, and integrating over $\mathbb{R}^3$, we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 &= \int_{\mathbb{R}^3}(\omega\cdot\nabla)u\cdot\omega dx - \int_{\mathbb{R}^3}(\Delta\phi\nabla\phi)\cdot(\nabla\times\omega)dx\\ &:= I_1 + I_2.\end{aligned} \tag{3.7}$$

We can rewrite the term I_1 as

$$\begin{aligned}I_1 = \sum_{i,j=1}^3\int_{\mathbb{R}^3}\omega_i\partial_i u^j\omega_j dx &= \sum_{i,j=1}^2\int_{\mathbb{R}^3}\omega_i\partial_i u^j\omega_j dx + \sum_{i=1}^2\int_{\mathbb{R}^3}\omega_i\partial_i u^3\omega_3 dx\\ &\quad+\sum_{j=1}^2\int_{\mathbb{R}^3}\omega_3\partial_3 u^j\omega_j dx + \int_{\mathbb{R}^3}\omega_3\partial_3 u^3\omega_3 dx,\end{aligned}$$

by further using the facts

$$\partial_3 u^3 = -(\partial_1 u^1 + \partial_2 u^2), \quad \omega_3 = \partial_1 u^2 - \partial_2 u^1,$$

we can easily obtain

$$I_1 \le \int_{\mathbb{R}^3}|\nabla_h u^h||\omega|^2 dx. \tag{3.8}$$

Notice that, via the Biot-Savart law, we have the following two equalities:

$$\frac{\partial u}{\partial x_j} = R_j(R\times\omega) \ \text{ for } \ j = 1, 2, 3,$$

and

$$\nabla\times\omega = \nabla\times(\nabla\times u) = \nabla(\nabla\cdot u) - \Delta u = -\Delta u,$$

where $R = (R_1, R_2, R_3)$, $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ $(j = 1, 2, 3)$ are the Riesz transforms. Since the Riesz operators are bounded in $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$, we know that for any $1 < p < \infty$,

$$\|\nabla u\|_{L^p} \le C\|\omega\|_{L^p} \ \text{ and } \ \|\omega\|_{L^p} \le C\|\nabla u\|_{L^p}.$$

Thus applying (2.3) in Lemma 2.3 and the fact ($\Lambda = \sqrt{-\Delta}$)

$$\|\nabla_h u^h\|_{\dot{H}^{\frac{\alpha}{2}}} \approx \|\Lambda^{\frac{\alpha}{2}}\nabla_h u^h\|_{L^2} \le \|\Lambda^{\frac{\alpha}{2}}\nabla u\|_{L^2} \le C\|\Lambda^{\frac{\alpha}{2}}\omega\|_{L^2} \approx C\|\omega\|_{\dot{H}^{\frac{\alpha}{2}}},$$

the right-hand side of (3.8) can be bounded by

$$
\begin{aligned}
I_1 \le \int_{\mathbb{R}^3} |\nabla_h u^h||\omega|^2 dx &\le C\|\nabla_h u^h\|_{L^3}\|\omega\|_{L^3}^2 \\
&\le C\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{1}{3}} \|\nabla_h u^h\|_{\dot{H}^{\frac{\alpha}{2}}}^{\frac{2}{3}} \|\omega\|_{L^2}\|\nabla\omega\|_{L^2} \\
&\le C\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{1}{3}} \|\omega\|_{\dot{H}^{\frac{\alpha}{2}}}^{\frac{2}{3}} \|\omega\|_{L^2}\|\nabla\omega\|_{L^2} \\
&\le C\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{1}{3}} \|\omega\|_{L^2}^{\frac{5-\alpha}{3}} \|\nabla\omega\|_{L^2}^{\frac{3+\alpha}{3}} \\
&\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{3-\alpha}} \|\omega\|_{L^2}^{\frac{10-2\alpha}{3-\alpha}} \\
&\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C(\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}} + \|\omega\|_{L^2}^2)\|\omega\|_{L^2}^2 \\
&\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C(\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}} + \|\nabla u\|_{L^2}^2)\|\omega\|_{L^2}^2, \qquad (3.9)
\end{aligned}
$$

where we have used the following interpolation inequalities ($0 \le \alpha < 1$):

$$
\|f\|_{\dot{H}^{\frac{\alpha}{2}}} \le \|f\|_{L^2}^{1-\frac{\alpha}{2}}\|\nabla f\|_{L^2}^{\frac{\alpha}{2}} \quad \text{and} \quad \|f\|_{L^3} \le \|f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{1}{2}}.
$$

For the second term I_2 in (3.7), since ϕ obeys the Poisson equation $\Delta\phi = v - w$, we can use the energy inequalities (3.4)–(3.5) and the interpolation inequality $\|f\|_{L^4} \le \|f\|_{L^2}^{\frac{1}{4}}\|\nabla f\|_{L^2}^{\frac{3}{4}}$ to bound it as

$$
\begin{aligned}
-\int_{\mathbb{R}^3} (\Delta\phi\nabla\phi)\cdot(\nabla\times\omega)dx &\le C\|\nabla\phi\|_{L^4}\|\Delta\phi\|_{L^4}\|\nabla\omega\|_{L^2} \\
&\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C\|(v,w)\|_{L^4}^2\|\nabla\phi\|_{L^4}^2 \\
&\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C\|(v,w)\|_{L^2}^{\frac{1}{2}}\|(\nabla v,\nabla w)\|_{L^2}^{\frac{3}{2}}\|\nabla\phi\|_{L^2}^{\frac{1}{2}}\|\Delta\phi\|_{L^2}^{\frac{3}{2}} \\
&\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C\|(v,w)\|_{L^2}^2\|(\nabla v,\nabla w)\|_{L^2}^{\frac{3}{2}}\|\nabla\phi\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C\|(v,w)\|_{L^2}^2\|(\nabla v,\nabla w)\|_{L^2}^2 + C\|(v,w)\|_{L^2}^2\|\nabla\phi\|_{L^2}^2 \\
&\le \frac{1}{4}\|\nabla\omega\|_{L^2}^2 + C(\|(\nabla v,\nabla w)\|_{L^2}^2 + 1). \qquad (3.10)
\end{aligned}
$$

Plugging (3.9) and (3.10) into (3.7), then using the logarithmic Sobolev embedding inequality (2.7) in Lemma 2.6, we obtain

$$
\begin{aligned}
\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 &\le C\Big(1 + \|\nabla u\|_{L^2}^2 + \|(\nabla v,\nabla w)\|_{L^2}^2 + \|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}\Big)\Big(\|\omega\|_{L^2}^2 + 1\Big) \\
&\le C\Big(1 + \|\nabla u\|_{L^2}^2 + \|(\nabla v,\nabla w)\|_{L^2}^2 + \frac{\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1+\ln(e+\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}})}\Big) \\
&\quad \times \Big(1+\ln(e+\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}})\Big)\Big(\|\omega\|_{L^2}^2 + 1\Big) \\
&\le C\Big(1 + \|\nabla u\|_{L^2}^2 + \|(\nabla v,\nabla w)\|_{L^2}^2 + \frac{\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1+\ln(e+\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}})}\Big) \\
&\quad \times \ln\Big(e + \|\nabla^3 u\|_{L^2}^2 + \|(\nabla^2 v,\nabla^2 w)\|_{L^2}^2\Big)\Big(\|\omega\|_{L^2}^2 + 1\Big). \qquad (3.11)
\end{aligned}
$$

Based on the facts (3.4)–(3.5) and the growth condition (3.1), one deduces that for any small constant $\sigma > 0$, there exists $T_0 < T$ such that

$$\int_{T_0}^{T} \big(1 + \|\nabla u\|_{L^2}^2 + \|(\nabla v, \nabla w)\|_{L^2}^2\big) dt < \sigma$$

and

$$\int_{T_0}^{T} \frac{\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1 + \ln(e + \|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}})} dt < \sigma.$$

Let us denote for any $t \in [T_0, T)$,

$$Y(t) := \sup_{T_0 \le \tau \le t} \big(\|\nabla^3 u(\cdot,\tau)\|_{L^2}^2 + \|(\nabla^2 v(\cdot,\tau), \nabla^2 w(\cdot,\tau))\|_{L^2}^2\big).$$

It should be observed that the function $Y(t)$ is nondecreasing, and for any $T_0 \le t < T$, we integrate (3.11) from T_0 to t to obtain that

$$\begin{aligned}
\|\omega(t)\|_{L^2}^2 \le (\|\omega(\cdot, T_0)\|_{L^2}^2 + 1) \exp\Big(\int_{T_0}^{t} C\big(1 + \|\nabla u\|_{L^2}^2 + \|(\nabla v, \nabla w)\|_{L^2}^2 \\
+ \frac{\|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1 + \ln(e + \|\nabla_h u^h\|_{\dot{B}^{-\alpha}_{\infty,\infty}})}\big) d\tau \ln(e + Y(t))\Big) \\
\le (\|\omega(\cdot, T_0)\|_{L^2}^2 + 1) \exp\big(2C\sigma \ln(e + Y(t))\big) \\
\le (\|\omega(\cdot, T_0)\|_{L^2}^2 + 1)(e + Y(t))^{2C\sigma}.
\end{aligned} \tag{3.12}$$

Finally, we are going to derive the estimates for$(\nabla^3 u, \nabla^2 v, \nabla^2 w)$. Applying ∇^3 on the first equation of system (1.1), taking the L^2 inner product of the resultant equations with $\nabla^3 u$, we see that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla^3 u\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2 = -\int_{\mathbb{R}^3} \nabla^3((u\cdot\nabla)u)\cdot\nabla^3 u dx \\
+ \int_{\mathbb{R}^3} \nabla^3(\Delta\phi\nabla\phi)\cdot\nabla^3 u dx.
\end{aligned} \tag{3.13}$$

We estimate two terms on the right-hand side of (3.13) one by one. By using the commutator estimate (2.6) in Lemma 2.5 and the interpolation inequality in Lemma 2.2 as

$$\|\nabla^3 u\|_{L^4} \le C\|\nabla u\|_{L^2}^{\frac{1}{12}} \|\nabla^4 u\|_{L^2}^{\frac{11}{12}},$$

the first term can be bounded as

$$\begin{aligned}
-\int_{\mathbb{R}^3} \nabla^3((u\cdot\nabla)u)\cdot\nabla^3 u dx &= -\int_{\mathbb{R}^3} \big[\nabla^3((u\cdot\nabla)u) - (u\cdot\nabla)\nabla^3 u\big]\cdot\nabla^3 u dx \\
&\le C\|\nabla^3((u\cdot\nabla)u) - (u\cdot\nabla)\nabla^3 u\|_{L^{\frac{4}{3}}} \|\nabla^3 u\|_{L^4} \\
&\le C\|\nabla u\|_{L^2}\|\nabla^3 u\|_{L^4}^2 \\
&\le C\|\nabla u\|_{L^2}^{\frac{7}{6}} \|\nabla^4 u\|_{L^2}^{\frac{11}{6}} \\
&\le \frac{1}{4}\|\nabla^4 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{14},
\end{aligned}$$

where we have used the incompressible condition $\nabla\cdot u = 0$ to derive that

$$\int_{\mathbb{R}^3} (u\cdot\nabla)\nabla^3 u\cdot\nabla^3 u dx = 0.$$

For the second term, we employ the Leibniz's rule and the specific relation of ϕ and (v, w) to bound it as

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla^3(\Delta\phi\nabla\phi)\cdot\nabla^3 u dx = -\int_{\mathbb{R}^3} \nabla^2((v-w)\nabla\phi)\cdot\nabla^4 u dx\\
&\le \frac{1}{4}\|\nabla^4 u\|_{L^2}^2 + C\|\nabla^2((v-w)\nabla\phi)\|_{L^2}^2\\
&\le \frac{1}{4}\|\nabla^4 u\|_{L^2}^2 + C\big(\|\nabla^2(v-w)\nabla\phi\|_{L^2}^2 + 2\|(\nabla v-\nabla w)\nabla^2\phi\|_{L^2}^2 + \|(v-w)\nabla^3\phi\|_{L^2}^2\big)\\
&\le \frac{1}{4}\|\nabla^4 u\|_{L^2}^2 + C\big(\|(\nabla^2 v, \nabla^2 w)\|_{L^3}^2\|\nabla\phi\|_{L^6}^2 + \|(v, w)\|_{L^3}^2\|(\nabla v, \nabla w)\|_{L^6}^2\big)\\
&\le \frac{1}{4}\|\nabla^4 u\|_{L^2}^2 + C\big(\|(\nabla^2 v, \nabla^2 w)\|_{L^2}\|(\nabla^3 v, \nabla^3 w)\|_{L^2} + \|(\nabla v, \nabla w)\|_{L^2}\|(\nabla^2 v, \nabla^2 w)\|_{L^2}^2\big)\\
&\le \frac{1}{4}\|\nabla^4 u\|_{L^2}^2 + \frac{1}{6}\|(\nabla^3 v, \nabla^3 w)\|_{L^2}^2 + C\big(1+\|(\nabla v, \nabla w)\|_{L^2}^2\big)\|(\nabla^2 v, \nabla^2 w)\|_{L^2}^2.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\frac{d}{dt}\|\nabla^3 u\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2 \le \frac{1}{3}\|(\nabla^3 v, \nabla^3 w)\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{14}\\
+ C\big(1+\|(\nabla v, \nabla w)\|_{L^2}^2\big)\|(\nabla^2 v, \nabla^2 w)\|_{L^2}^2. \qquad (3.14)
\end{aligned}$$

For the charged density v, we can exactly proceed as (3.14) to obtain that

$$\frac{1}{2}\frac{d}{dt}\|\nabla^2 v\|_{L^2}^2 + \|\nabla^3 v\|_{L^2}^2 = -\int_{\mathbb{R}^3}\nabla^2((u\cdot\nabla)v)\nabla^2 v dx - \int_{\mathbb{R}^3}\nabla^2\nabla\cdot(v\nabla\phi)\nabla^2 v dx. \qquad (3.15)$$

Based on the energy inequalities (3.4) and (3.5), we obtain from the Sobolev embedding inequality that

$$\|\nabla u\|_{L^\infty}^2 \le C(\|u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2) \le C(1+\|\nabla^3 u\|_{L^2}^2),$$

thus we can estimate two terms on the right-hand side of (3.15) one by one as follows:

$$\begin{aligned}
-&\int_{\mathbb{R}^3}\nabla^2((u\cdot\nabla)v)\nabla^2 v dx = \int_{\mathbb{R}^3}\nabla((u\cdot\nabla)v)\cdot\nabla^3 v dx\\
&\le \frac{1}{12}\|\nabla^3 v\|_{L^2}^2 + C\big(\|(\nabla u\cdot\nabla)v\|_{L^2}^2 + \|(u\cdot\nabla)\nabla v\|_{L^2}^2\big)\\
&\le \frac{1}{12}\|\nabla^3 v\|_{L^2}^2 + C\big(\|\nabla u\|_{L^\infty}^2\|\nabla v\|_{L^2}^2 + \|u\|_{L^4}^2\|\nabla^2 v\|_{L^4}^2\big)\\
&\le \frac{1}{12}\|\nabla^3 v\|_{L^2}^2 + C\big(\|\nabla v\|_{L^2}^2(1+\|\nabla^3 u\|_{L^2}^2) + \|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|v\|_{L^2}^{\frac{1}{6}}\|\nabla^3 v\|_{L^2}^{\frac{11}{6}}\big)\\
&\le \frac{1}{6}\|\nabla^3 v\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2\big(1+\|\nabla^3 u\|_{L^2}^2\big) + C\|\nabla u\|_{L^2}^{18},
\end{aligned}$$

$$\begin{aligned}
-&\int_{\mathbb{R}^3}\nabla^2\nabla\cdot(v\nabla\phi)\nabla^2 v dx = \int_{\mathbb{R}^3}\nabla^2(v\nabla\phi)\cdot\nabla^3 v dx\\
&\le \frac{1}{12}\|\nabla^3 v\|_{L^2}^2 + C\big(\|\nabla^2 v\nabla\phi\|_{L^2}^2 + \|\nabla v\nabla^2\phi\|_{L^2}^2 + \|v\nabla^3\phi\|_{L^2}^2\big)\\
&\le \frac{1}{12}\|\nabla^3 v\|_{L^2}^2 + C\big(\|\nabla^2 v\|_{L^3}^2\|\nabla\phi\|_{L^6}^2 + \|\nabla v\|_{L^6}^2\|\nabla^2\phi\|_{L^3}^2 + \|v\|_{L^3}^2\|\nabla(v-w)\|_{L^6}^2\big)\\
&\le \frac{1}{12}\|\nabla^3 v\|_{L^2}^2 + C\big(\|\nabla^2 v\|_{L^2}\|\nabla^3 v\|_{L^2} + \|v\|_{L^2}\|\nabla v\|_{L^2}\|(\nabla^2 v, \nabla^2 w)\|_{L^2}^2\big)\\
&\le \frac{1}{6}\|\nabla^3 v\|_{L^2}^2 + C\big(1+\|(\nabla v, \nabla w)\|_{L^2}^2\big)\big(1+\|(\nabla^2 v, \nabla^2 w)\|_{L^2}^2\big),
\end{aligned}$$

where we have used the interpolation inequality in Lemma 2.2 as

$$\|\nabla^2 v\|_{L^4} \le C\|v\|_{L^2}^{\frac{1}{12}}\|\nabla^3 v\|_{L^2}^{\frac{11}{12}}.$$

Therefore we obtain the estimate for v as

$$\frac{d}{dt}\|\nabla^2 v\|_{L^2}^2 + \frac{4}{3}\|\nabla^3 v\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^{18} + C\big(1 + \|(\nabla v, \nabla w)\|_{L^2}^2\big)\big(1 + \|\nabla^3 u\|_{L^2}^2 + \|(\nabla^2 v, \nabla^2 w)\|_{L^2}^2\big). \tag{3.16}$$

The estimate for w can be derived exactly as v, thus we obtain

$$\frac{d}{dt}\|\nabla^2 w\|_{L^2}^2 + \frac{4}{3}\|\nabla^3 w\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^{18} + C\big(1 + \|(\nabla v, \nabla w)\|_{L^2}^2\big)\big(1 + \|\nabla^3 u\|_{L^2}^2 + \|(\nabla^2 v, \nabla^2 w)\|_{L^2}^2\big). \tag{3.17}$$

Summarizing the above estimates (3.14), (3.16) and (3.17) together, after absorbing all the dissipative terms and using the fact that $\|\nabla u\|_{L^2} \le C\|\omega\|_{L^2}$, we can derive from (3.12) that

$$\begin{aligned}\frac{d(e + Y(t))}{dt} &\le C\big(\|\nabla u\|_{L^2}^{14} + \|\nabla u\|_{L^2}^{18}\big) + C\big(1 + \|(\nabla v, \nabla w)\|_{L^2}^2\big)\big(e + Y(t)\big)\\ &\le C\big(e + Y(t)\big)^{18C\sigma} + C\big(1 + \|(\nabla v, \nabla w)\|_{L^2}^2\big)\big(e + Y(t)\big).\end{aligned} \tag{3.18}$$

Choosing σ small enough such that $18C\sigma \le 1$, the above inequality (3.18) yields that for any $T_0 \le t < T$,

$$\frac{d}{dt}(e + Y(t)) \le C\big(1 + \|(\nabla v, \nabla w)\|_{L^2}^2\big)(e + Y(t)).$$

The Gronwall's inequality directly leads to

$$Y(t) \le (e + Y(T_0)) \exp\Big(C\int_{T_0}^t \big(1 + \|(\nabla v, \nabla w)\|_{L^2}^2\big)d\tau\Big) \le C,$$

where $Y(T_0) = \|\nabla^3 u(\cdot, T_0)\|_{L^2}^2 + \|(\nabla^2 v(\cdot, T_0), \nabla^2 w(\cdot, T_0))\|_{L^2}^2$. This together with the basic energy inequalities (3.4) and (3.5) finally show that

$$\sup_{0\le t<T}\big(\|u(\cdot, t)\|_{H^3} + \|(v(\cdot, t), w(\cdot, t))\|_{H^2}\big) \le C.$$

We complete the proof of Theorem 3.1. □

4 Proof of theorem 1.2

Similarly, we prove Theorem 1.2 by contradiction. If the condition (1.14) does not hold, then there exists a constant $K > 0$ such that

$$\int_0^{T_*} \frac{\|\nabla_h u^h(\cdot, t)\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \ln(e + \|\nabla_h u^h(\cdot, t)\|_{\dot{B}^0_{\infty,\infty}})}} dt < K, \tag{4.1}$$

then we can still prove that

$$\sup_{0\le t\le T_*}\big(\|u(\cdot, t)\|_{H^3} + \|(v(\cdot, t), w(\cdot, t))\|_{H^2}\big) \le C_0. \tag{4.2}$$

This implies that the solution (u, v, w) can be extended beyond the time T_*, which contradicts to the maximality of T_*. We state the above fact as the following theorem.

Theorem 4.1 *Under the assumptions of Theorem 3.1 if we replace the condition* (3.1) *by* (4.1)*, we still obtain the desired bound* (3.3).

Proof Based on the condition (4.1), we apply the logarithmical Sobolev inequality (2.6) in Lemma 2.5 to derive the bound of $\|\omega\|_{L^2}$ that

$$\begin{aligned} I_1 &\le C\|\nabla_h u^h\|_{L^\infty}\|\omega\|_{L^2}^2 \\ &\le C\big(1+\|\nabla\nabla_h u^h\|_{\dot{B}^{-1}_{\infty,\infty}}\sqrt{\ln(e+\|\nabla^3 u\|_{L^2})}\big)\|\omega\|_{L^2}^2 \\ &\le C\big(1+\|\nabla_h u^h\|_{\dot{B}^{0}_{\infty,\infty}}\sqrt{\ln(e+\|\nabla^3 u\|_{L^2})}\big)\|\omega\|_{L^2}^2. \end{aligned} \tag{4.3}$$

Taking (4.3) and (3.10) into (3.7), and employing (2.7) with $\alpha = 0$ in Lemma 2.6, we find that

$$\begin{aligned} \frac{d}{dt}\|\omega\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2 &\le C\Big(1+\|(\nabla v,\nabla w)\|_{L^2}^2+\|\nabla\nabla_h u^h\|_{\dot{B}^{-1}_{\infty,\infty}}\sqrt{\ln(e+\|\nabla^3 u\|_{L^2})}\Big)\Big(\|\omega\|_{L^2}^2+1\Big) \\ &\le C\Big(1+\|(\nabla v,\nabla w)\|_{L^2}^2+\frac{\|\nabla_h u^h\|_{\dot{B}^{0}_{\infty,\infty}}}{\sqrt{1+\ln(e+\|\nabla_h u^h\|_{\dot{B}^{0}_{\infty,\infty}})}}\sqrt{1+\ln(e+\|\nabla_h u^h\|_{\dot{B}^{0}_{\infty,\infty}})}\Big) \\ &\quad\times\Big(\sqrt{\ln(e+\|\nabla^3 u\|_{L^2})}\Big)\Big(\|\omega\|_{L^2}^2+1\Big) \\ &\le C\Big(1+\|(\nabla v,\nabla w)\|_{L^2}^2+\frac{\|\nabla_h u^h\|_{\dot{B}^{0}_{\infty,\infty}}}{\sqrt{1+\ln(e+\|\nabla_h u^h\|_{\dot{B}^{0}_{\infty,\infty}})}}\Big) \\ &\quad\times\ln\Big(e+\|\nabla^3 u\|_{L^2}^2+\|(\nabla^2 v,\nabla^2 w)\|_{L^2}^2\Big)\Big(\|\omega\|_{L^2}^2+1\Big). \end{aligned} \tag{4.4}$$

Under the growth condition (4.1), one deduces that for any small constant $\sigma > 0$, there exists $T_0 < T$ such that

$$\int_{T_0}^{T}\frac{\|\nabla_h u^h\|_{\dot{B}^{0}_{\infty,\infty}}}{\sqrt{1+\ln(e+\|\nabla_h u^h\|_{\dot{B}^{0}_{\infty,\infty}})}}dt < \sigma.$$

Therefore, integrating (4.4) from T_0 to t for any $T_0 \le t < T$, one obtains that

$$\begin{aligned} \|\omega(t)\|_{L^2}^2 &\le C\exp\Big(\int_{T_0}^{t}C\big(1+\|(\nabla v,\nabla w)\|_{L^2}^2+\frac{\|\nabla_h u^h\|_{\dot{B}^{0}_{\infty,\infty}}}{\sqrt{1+\ln(e+\|\nabla_h u^h\|_{\dot{B}^{0}_{\infty,\infty}})}}\big)d\tau\,\ln(e+Y(t))\Big) \\ &\le C\exp\big(2C\sigma\ln(e+Y(t))\big) \\ &\le C(e+Y(t))^{2C\sigma}. \end{aligned} \tag{4.5}$$

As soon as (4.5) is established, one can exactly follow the same procedures as that in the proof of Theorem 3.1 to get the desired bound (3.3), here we safely omit the proof. □

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