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ORIGINAL RESEARCH

Multiple solutions for a critical Biharmonic elliptic equation of Kirchhoff type

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Abstract In this paper, we deal with the critical Biharmonic elliptic equation of Kirchhoff type. By transforming this equation into an equivalent system and utilizing the concentration compactness principle, we obtain the existence and multiplicity of nontrivial solutions.

Keywords Biharmonic elliptic equation · Kirchhoff type · Critical growth · Multiple solutions

Mathematics Subject Classification 35B33 · 35J35 · 35A15

1 Introduction and main results

We are interested in the existence of nontrivial solutions for the following Biharmonic elliptic equation

$$\left(a + b\int_{\mathbb{R}^N} |\Delta u|^2 dx\right) \Delta^2 u + u = |u|^{2^*-2}u + \mu f(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 5$, $a \geq 0$, $b > 0$, Δ^2 is the Biharmonic operator, $\mu > 0$ is a parameter. f is a continuous function and $2^* = \frac{2N}{N-4}$ is the critical Sobolev exponent.

Equation (1.1) is related to the problem of a model introduced by Woinowsky-Krieger in [27]. More precisely, the authors introduced the following equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI\partial^4 u}{\partial x^4} - \left(\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = f(x,u),$$

where E, I, H, ρ, A, L are constants. The model was proposed to refine the theory of the dynamic Euler-Bernoulli beam. It has also found applications in describing physical motivations, such as the large deflection of plates, and highlighting its importance in engineering, physics, and material mechanics. Recently, many researchers have focused on a stationary analogue of the fourth-order Kirchhoff type equation

$$\Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), \tag{1.2}$$

where $a, b > 0$. Under the conditions of $f(x,u)$, by the symmetric mountain pass theorem, the existence of infinitely many solutions for (1.2) was obtained in [1,21]. In [11,24,26], the authors proved the existence of

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positive solutions of (1.2). The authors [12,18,29] obtained the existence of sign-changing solutions for (1.2) by employing constraint variational method. Many papers have studied the existence of nontrivial solutions to (1.2), for example [6,9,10,13,15,17,19,20,23,25,30]. In addition, the multiplicity of solutions for (1.2) involving concave-convex nonlinearities was considered in [16,28]. In the case where $a = 1$ and $b = 0$, equation (1.1) was treated in [2,7,8] by the variational method. Aouaout [4] proved that equation (1.1) has two nontrivial solutions in $\mathbb{R}^4$.

Based on the above references, we consider the existence of nontrivial solutions for the critical Biharmonic elliptic equation of Kirchhoff type. Since equation (1.1) contains a critical growth term, which leads to the lack of compactness, by means of a concentration-compactness argument, we establish the existence and multiplicity of nontrivial solutions of equation (1.1).

To obtain the existence of solutions of (1.1), we transform equation (1.1) into an equivalent system. More precisely, we conclude the existence of nontrivial solutions by solving the following system in (u, λ)

$$\begin{cases} \Delta^2 u + u = |u|^{2^*-2}u + \mu f(x,u), & \text{in } \mathbb{R}^N, \\ a + b\lambda^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |\Delta u|^2 dx = \lambda^2, & \text{in } (0,+\infty). \end{cases} \tag{1.3}$$

We make the following assumptions on f:

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $f(x,t) = o(t)$ as $t \to 0$;

(f_2) there exists $2 < q < 2^*$ such that $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{q-2}t} = 0$;

(f_3) $tf(x,t) \geq 2F(x,t)$, where $F(x,t) = \int_0^t f(x,s)ds$ for all $t \in \mathbb{R}$;

(f_4) there exists $\kappa > 0$ such that $f(x,t) \geq \kappa t^{q-1}$ for all $t \geq 0$ and $2 < q < 2^*$.

Now we state our main results.

Theorem 1.1 *Assume that* $(f_1) - (f_4)$ *hold. Then*

(i) *if* $5 \leq N \leq 7$, *for all* $a, b > 0$, *equation* (1.1) *has a nontrivial solution;*

(ii) *if* $N = 8$, *there exists* $b_0 > 0$, *such that for all* $a > 0$, *equation* (1.1) *has a nontrivial solution for* $b < b_0$, *and equation* (1.1) *has no nontrivial solution for* $b \geq b_0$;

(iii) *if* $N > 8$, *there exists* $\alpha_0 > 0$, *such that equation* (1.1) *has two nontrivial solutions* u_1 *and* u_2 *for* $ab^{\frac{4}{N-8}} < \alpha_0$, *has one nontrivial solution for* $ab^{\frac{4}{N-8}} = \alpha_0$, *and has no nontrivial solution for* $ab^{\frac{4}{N-8}} > \alpha_0$.

For the degenerate case $a = 0$ of equation (1.1), we have the following Theorem.

Theorem 1.2 *Assume that* $(f_1) - (f_4)$ *hold. Then*

(i) *if* $5 \leq N \leq 7$ *or* $N > 8$, *for all* $b > 0$, *equation* (1.1) *has one nontrivial solution;*

(ii) *if* $N = 8$, *there exists* $b_0 > 0$, *such that equation* (1.1) *has one nontrivial solution* (u, λ) *for* $b = b_0$, *and equation* (1.1) *has no nontrivial solution for* $b \neq b_0$.

Example 1.3 We consider the function f defined by

$$f(x,t) := \begin{cases} ct + dqt^{q-1}, & \text{if } x \in \mathbb{R}^N, \ t \geq 0, \\ ct - dq(-t)^{q-1}, & \text{if } x \in \mathbb{R}^N, \ t < 0, \end{cases}$$

for each $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $2 < q < 2^*$ and c, d are two positive constants.

2 Proofs of Theorems 1.1-1.2

Let $D^{2,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the semi-norm

$$\|u\|_{D^{2,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx \right)^{\frac{1}{2}}.$$

We define the following Sobolev space

$$H^2(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) | \int_{\mathbb{R}^N} |\Delta u|^2 dx < \infty \text{ and } \int_{\mathbb{R}^N} |\nabla u|^2 dx < \infty \right\},$$

with the norm

$$\|u\|_{H^2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\Delta u|^2 + |\nabla u|^2 + |u|^2 dx \right)^{\frac{1}{2}}.$$

By [22], the norm $\|u\|_{H^2(\mathbb{R}^N)}$ is equivalent to the norm $\|u\|$ by

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\Delta u|^2 + |u|^2) dx \right)^{\frac{1}{2}}.$$

In this paper, we consider the equation (1.1) in a subspace $H_r^2(\mathbb{R}^N)$, that is

$$H_r^2(\mathbb{R}^N) = \left\{ u \in H^2(\mathbb{R}^N) : u(x) = u(|x|) \right\},$$

and use $\|\cdot\|$ as the norm of $H_r^2(\mathbb{R}^N)$. The space $L^p(\mathbb{R}^N)(1 \le p < \infty)$ denotes the Lebesgue space with the norm $\|u\|_p^p = \int_{\mathbb{R}^N} |u|^p dx$, $C, C_1, C_2, \ldots$ denote various positive constants, which may vary from line to line. Let S be the best constant for Sobolev embedding $D^{2,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, namely

$$S = \inf_{u \in D^{2,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}.$$

In order to obatin the results of system (1.3), we first prove the existence of solutions for the following equation

$$\Delta^2 u + u = |u|^{2^*-2} u + \mu f(x, u), \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

and we solve the algebra equation when u is known

$$a + b\lambda^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |\Delta u|^2 dx = \lambda^2, \quad \text{in } (0, +\infty). \tag{2.2}$$

The energy functional associated with (2.1) has the form

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u|^2 + |u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \int_{\mathbb{R}^N} F(x, u) dx.$$

We say that $u \in H_r^2(\mathbb{R}^N)$ is a weak solution of (2.1) if

$$\int_{\mathbb{R}^N} \Delta u \Delta \varphi dx + \int_{\mathbb{R}^N} u \varphi dx = \int_{\mathbb{R}^N} |u|^{2^*-2} u \varphi dx + \mu \int_{\mathbb{R}^N} f(x, u) \varphi dx,$$

for all $\varphi \in H_r^2(\mathbb{R}^N)$.

In the following, we prove that the energy functional I possesses the mountain-pass geometry.

Lemma 2.1 *The energy functional I satisfies the following properties:*

(i) *There exist constants $\alpha, \rho > 0$ such that $I(u) \ge \alpha > 0$ for $\|u\| = \rho$;*

(ii) *There exists $e \in H_r^2(\mathbb{R}^N)$ with $\|e\| > \rho$ such that $I(e) < 0$.*

Proof (i) For all $u \in H_r^2(\mathbb{R}^N)$, by (f_1), (f_2) and the Sobolev inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u|^2 + |u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \int_{\mathbb{R}^N} F(x, u) dx \\ &\ge \frac{1}{2} \|u\|^2 - C_1 \|u\|^{2^*} - \mu C_2 \|u\|^q, \end{aligned}$$

where the constants $C_1, C_2 > 0$ and $2 < q < 2^*$. Consequently, by choosing $\rho > 0$ small enough, fix any $\mu > 0$, we deduce that $I(u) \ge \alpha > 0$ for $\|u\| = \rho$.

(ii) Fix $v_0 \in C_0^\infty(\mathbb{R}^N)\backslash\{0\}$ with $v_0 \geq 0$ in $\mathbb{R}^N$, by (f_4), for all $t > 0$, we have

$$I(tv_0) \leq \frac{t^2}{2}\|v_0\|^2 - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}|v_0|^{2^*}dx - \frac{\mu\kappa t^q}{q}\int_{\mathbb{R}^N}|v_0|^q dx \to -\infty,$$

as $t \to +\infty$, which implies that $I(tv_0) < 0$ for $t > 0$ large enough. Therefore, there exists $e = t_0 v_0$ for $t_0 > 0$ such that $\|e\| > \rho$ and $I(e) < 0$. The proof is complete. □

Lemma 2.2 *Assume that $(f_1) - (f_3)$ and $2 < q < 2^*$ hold. Then the energy functional I satisfies the $(PS)_c$ condition for all $c < c^* = \frac{2}{N}S^{\frac{N}{4}}$ and $\mu > 0$.*

Proof Let $\{u_n\} \subset H_r^2(\mathbb{R}^N)$ be a $(PS)_c$ sequence for I at the level c, that is

$$I(u_n) \to c, \text{ and } I'(u_n) \to 0 \text{ as } n \to \infty. \tag{2.3}$$

From (f_3) and (2.3) it follows that

$$\begin{aligned}|c| + 1 + o(1) &\geq I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N}|u_n|^{2^*}dx + \mu\int_{\mathbb{R}^N}\frac{1}{2}f(x,u_n)u_n - F(x,u_n)dx\\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\|u_n\|_{2^*}^{2^*},\end{aligned}$$

which implies that $\{u_n\}$ is bounded in $L^{2^*}(\mathbb{R}^N)$. From (f_1) and (f_2), for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|f(x,u_n)| \leq \varepsilon|u_n| + C_\varepsilon|u_n|^{q-1},$$

we have

$$\begin{aligned}o(1) &= \langle I'(u_n), u_n\rangle\\ &= \int_{\mathbb{R}^N}|\Delta u_n|^2 + |u_n|^2 dx - \int_{\mathbb{R}^N}|u_n|^{2^*}dx - \mu\int_{\mathbb{R}^N}f(x,u_n)u_n dx\\ &\geq \int_{\mathbb{R}^N}|\Delta u_n|^2 + |u_n|^2 dx - \int_{\mathbb{R}^N}|u_n|^{2^*}dx - \mu\int_{\mathbb{R}^N}(\varepsilon|u_n|^2 + C_\varepsilon|u_n|^q)dx\\ &\geq \|u_n\|^2 - C\|u_n\|_{2^*}^{2^*},\end{aligned}$$

where the constant $C > 0$. Thus, $\{u_n\}$ is bounded in $H_r^2(\mathbb{R}^N)$ for all $2 < q < 2^*$. Hence, we may assume up to a subsequence, still denoted by $\{u_n\}$, that there exists $u \in H_r^2(\mathbb{R}^N)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } H_r^2(\mathbb{R}^N),\\ u_n \to u, & \text{strongly in } L^p(\mathbb{R}^N)\ (2 \leq p < 2^*),\\ u_n(x) \to u(x), & \text{a.e. in } \mathbb{R}^N,\end{cases} \tag{2.4}$$

as $n \to \infty$. Next, we prove that $u_n \to u$ strongly in $H_r^2(\mathbb{R}^N)$. By using the concentration compactness principle (see [14]), there exist some at most countable index set J, δ_{x_j} is the Dirac mass at $x_j \subset \mathbb{R}^N$ and positive numbers $\{\nu_j\}$, $\{\omega_j\}$, $j \in J$, such that

$$\begin{aligned}&|u_n|^{2^*} \rightharpoonup \nu = |u|^{2^*} + \sum_{j\in J}\nu_j\delta_{x_j},\\ &|\Delta u_n|^2 \rightharpoonup \omega \geq |\Delta u|^2 + \sum_{j\in J}\omega_j\delta_{x_j}, \quad S\nu_j^{\frac{2}{2^*}} \leq \omega_j.\end{aligned} \tag{2.5}$$

For $\varepsilon > 0$, let $\phi_{\varepsilon,j}(x) \in C_0^\infty(\mathbb{R}^N)$ be a smooth cut-off function centered at x_j such that $0 \leq \phi_{\varepsilon,j} \leq 1$, $|\nabla\phi_{\varepsilon,j}| \leq \frac{2}{\varepsilon}$, $|\Delta\phi_{\varepsilon,j}| \leq \frac{2}{\varepsilon^2}$, and

$$\phi_{\varepsilon,j}(x) = \begin{cases}1, & \text{if } |x - x_j| < \varepsilon,\\ 0, & \text{if } |x - x_j| \geq 2\varepsilon.\end{cases} \tag{2.6}$$

Notice that

$$\int_{\mathbb{R}^N} \Delta u_n \Delta (u_n \phi_{\varepsilon,j}) dx = \int_{\mathbb{R}^N} |\Delta u_n|^2 \phi_{\varepsilon,j} dx + 2\int_{\mathbb{R}^N} \Delta u_n \nabla u_n \nabla \phi_{\varepsilon,j} dx + \int_{\mathbb{R}^N} u_n \Delta u_n \Delta \phi_{\varepsilon,j} dx.$$

By using the Hölder inequality, $|\nabla \phi_{\varepsilon,j}| \le \frac{2}{\varepsilon}$ and $|\Delta \phi_{\varepsilon,j}| \le \frac{2}{\varepsilon^2}$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} \Delta u_n \nabla u_n \nabla \phi_{\varepsilon,j} dx \right| \\
&\quad \le \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |\Delta u_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 |\nabla \phi_{\varepsilon,j}|^2 dx \right)^{\frac{1}{2}} \\
&\quad \le C \lim_{\varepsilon\to 0} \left(\int_{B(x_j,2\varepsilon)} |\nabla u|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} \left(\int_{B(x_j,2\varepsilon)} |\nabla \phi_{\varepsilon,j}|^N dx \right)^{\frac{1}{N}} \\
&\quad \le C \lim_{\varepsilon\to 0} \left(\int_{B(x_j,2\varepsilon)} |\nabla u|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} \left(\int_{B(x_j,2\varepsilon)} \left(\frac{2}{\varepsilon}\right)^N dx \right)^{\frac{1}{N}} \\
&\quad \le C_1 \left(\int_{B(x_j,2\varepsilon)} |\nabla u|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} \\
&\quad = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} u_n \Delta u_n \Delta \phi_{\varepsilon,j} dx \right| \\
&\quad \le \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |\Delta u_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} u_n^2 |\Delta \phi_{\varepsilon,j}|^2 dx \right)^{\frac{1}{2}} \\
&\quad \le C \lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}^N} u^2 |\Delta \phi_{\varepsilon,j}|^2 dx \right)^{\frac{1}{2}} \\
&\quad \le C \lim_{\varepsilon\to 0} \left(\int_{B(x_j,2\varepsilon)} |u|^{\frac{2N}{N-4}} dx \right)^{\frac{N-4}{2N}} \left(\int_{B(x_j,2\varepsilon)} \left(\frac{2}{\varepsilon^2}\right)^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \\
&\quad \le C_1 \lim_{\varepsilon\to 0} \left(\int_{B(x_j,2\varepsilon)} |u|^{\frac{2N}{N-4}} dx \right)^{\frac{N-4}{2N}} \\
&\quad = 0,
\end{aligned}$$

where $C_1, C_2 > 0$ are constants. Consequently, from (2.4) and (2.5) it follows that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} \Delta u_n \Delta (u_n \phi_{\varepsilon,j}) dx &= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |\Delta u_n|^2 \phi_{\varepsilon,j} dx \\
&\ge \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} |\Delta u|^2 \phi_{\varepsilon,j} dx + \omega_j = \omega_j.
\end{aligned}$$

From (2.4), (2.5), (f_1) and (f_2), we also derive that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}\phi_{\varepsilon,j}dx=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}|u|^{2^*}\phi_{\varepsilon,j}dx+\nu_j=\nu_j,$$
$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^2\phi_{\varepsilon,j}dx=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}|u|^2\phi_{\varepsilon,j}dx\le\lim_{\varepsilon\to 0}\int_{B(x_j,2\varepsilon)}|u|^2dx=0,$$
$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_{\mathbb{R}^N}f(x,u_n)u_n\phi_{\varepsilon,j}dx\right|\le C\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^2\phi_{\varepsilon,j}+|u_n|^q\phi_{\varepsilon,j}dx$$
$$\le C\lim_{\varepsilon\to 0}\int_{B(x_j,2\varepsilon)}\left(|u|^2+|u|^q\right)dx=0.$$

Noting that $u_n\phi_{\varepsilon,j}$ is bounded in $H_r^2(\mathbb{R}^N)$ uniformly for n, from the above information, we get

$$\begin{aligned}0&=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle I'(u_n),u_n\phi_{\varepsilon,j}\rangle\\&=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{\int_{\mathbb{R}^N}\Delta u_n\Delta(u_n\phi_{\varepsilon,j})dx+\int_{\mathbb{R}^N}|u_n|^2\phi_{\varepsilon,j}dx-\int_{\mathbb{R}^N}|u_n|^{2^*}\phi_{\varepsilon,j}dx\right.\\&\quad\left.-\mu\int_{\mathbb{R}^N}f(x,u_n)u_n\phi_{\varepsilon,j}dx\right\}\\&\ge\lim_{\varepsilon\to 0}\left(\int_{\mathbb{R}^N}|\Delta u|^2\phi_{\varepsilon,j}dx+\omega_j-\int_{\mathbb{R}^N}|u|^{2^*}\phi_{\varepsilon,j}dx-\nu_j\right)\\&=\omega_j-\nu_j.\end{aligned}$$

Consequently, we deduce that $\nu_j\ge\omega_j$. From (2.5) it follows that

$$\omega_j\ge S\nu_j^{\frac{2}{2^*}}\ge S\omega_j^{\frac{2}{2^*}},$$

which implies that

$$\nu_j\ge\omega_j\ge S^{\frac{N}{4}}.\tag{2.7}$$

To proceed further we show that (2.7) is impossible. Combining with (2.3) and (2.7), we have

$$\begin{aligned}c&=\lim_{n\to\infty}I(u_n)\\&\ge\lim_{n\to\infty}\left\{\frac{1}{2}\int_{\mathbb{R}^N}(|\Delta u_n|^2+|u_n|^2)dx\right.\\&\quad\left.-\frac{1}{2^*}\int_{\mathbb{R}^N}|u_n|^{2^*}dx-\frac{\mu}{2}\int_{\mathbb{R}^N}f(x,u_n)u_ndx\right\}\\&\ge\frac{1}{2}\omega_j-\frac{1}{2^*}\nu_j\ge\frac{2}{N}S^{\frac{N}{4}},\end{aligned}$$

which implies that $c\ge c^*$. It contradicts our assumption, so it indicates that $\nu_j=\omega_j=0$ for all $j\in J$, we obtain that $u_n\to u$ in $L^{2^*}(\mathbb{R}^N)$ as $n\to\infty$. According to (f_1), (f_2), the boundness of $\{u_n\}$ and the Hölder inequality, we have

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_{\mathbb{R}^N}f(x,u_n)(u_n-u)dx\right|\\&\le C\lim_{n\to\infty}\int_{\mathbb{R}^N}(|u_n|+|u_n|^{q-1})(u_n-u)dx\\&\le\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|u_n|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|u_n-u|^2dx\right)^{\frac{1}{2}}\\&\quad+\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|u_n|^qdx\right)^{\frac{q-1}{q}}\left(\int_{\mathbb{R}^N}|u_n-u|^qdx\right)^{\frac{1}{q}}\\&=0.\end{aligned}\tag{2.8}$$

Therefore, by view of (2.4) and (2.8), we have

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \left\langle I'(u_n), u_n - u \right\rangle \\
&= \lim_{n\to\infty} \left\{ \int_{\mathbb{R}^N} \Delta u_n \Delta(u_n - u)dx + \int_{\mathbb{R}^N} u_n(u_n - u)dx \right. \\
&\quad \left. - \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n(u_n - u)dx - \mu \int_{\mathbb{R}^N} f(x, u_n)(u_n - u)dx \right\} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \Delta u_n \Delta(u_n - u)dx,
\end{aligned}
$$

which implies that $u_n \to u$ strongly in $H_r^2(\mathbb{R}^N)$. The proof is complete. □

Choose the extremal function

$$
U_\varepsilon(x) = \frac{[N(N-4)(N^2-4)]^{\frac{N-4}{8}} \varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-4}{2}}}, \quad x \in \mathbb{R}^N, \ \ \varepsilon > 0.
$$

It is a solution of the following problem

$$
\Delta^2 U_\varepsilon = U_\varepsilon^{2^*-1}, \quad \text{in } \mathbb{R}^N,
$$

satisfies

$$
\int_{\mathbb{R}^N} |\Delta U_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*} dx = S^{\frac{N}{4}}.
$$

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that $\varphi(x) = 1$ on $B(0, r)$, $\varphi(x) = 0$ on $\mathbb{R}^N \setminus B(0, 2r)$ and $0 \le \varphi(x) \le 1$ on $\mathbb{R}^N$. Set $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$, from [2] and [5], we have

$$
\begin{cases}
\int_{\mathbb{R}^N} |\Delta u_\varepsilon|^2 dx = S^{\frac{N}{4}} + O(\varepsilon^{N-4}), \\
\int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx = S^{\frac{N}{4}} + O(\varepsilon^N),
\end{cases} \tag{2.9}
$$

and

$$
\int_{\mathbb{R}^N} |u_\varepsilon|^q dx = \begin{cases}
O\left(\varepsilon^{\frac{N-4}{2}q}\right), & \text{if } q < \dfrac{N}{N-4}, \\
O\left(\varepsilon^{N-\frac{N-4}{2}q} |ln\varepsilon|\right), & \text{if } q = \dfrac{N}{N-4}, \\
O\left(\varepsilon^{N-\frac{N-4}{2}q}\right), & \text{if } 2^* > q > \dfrac{N}{N-4}.
\end{cases} \tag{2.10}
$$

Moreover, we obtain that

$$
\int_{\mathbb{R}^N} |u_\varepsilon|^2 dx = \begin{cases}
O(\varepsilon^\gamma), & \text{if } N \ge 5, \ N \ne 8, \\
O(\varepsilon^\gamma |ln\varepsilon|), & \text{if } N = 8,
\end{cases} \tag{2.11}
$$

where $\gamma = \min\{4, N-4\}$. Then we have the following Lemma.

Lemma 2.3 *Assume that* (f_4) *and* $2 < q < 2^*$ *hold. Then*

$$
\sup_{t\ge 0} I(tu_\varepsilon) < \frac{2}{N} S^{\frac{N}{4}},
$$

for all $\mu > 0$.

Proof According to Lemma 2.1, we have $I(tu_\varepsilon) \to -\infty$ as $t \to +\infty$ and $I(tu_\varepsilon) < 0$ as $t \to 0$. Thus, there exists $t_\varepsilon > 0$ such that $I(t_\varepsilon u_\varepsilon) = \sup_{t \ge 0} I(tu_\varepsilon) \ge \alpha > 0$. Moreover, there exist positive constants $t_1, t_2 > 0$ such that $0 < t_1 < t_\varepsilon < t_2 < +\infty$. Now, we define

$$I(t_\varepsilon u_\varepsilon) = J(t_\varepsilon u_\varepsilon) - \mu \int_{\mathbb{R}^N} F(t_\varepsilon u_\varepsilon) dx,$$

where

$$J(t_\varepsilon u_\varepsilon) = \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} |\Delta u_\varepsilon|^2 + |u_\varepsilon|^2 dx - \frac{t_\varepsilon^{2^*}}{2^*} \int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx.$$

Let

$$h(t_\varepsilon) := J(t_\varepsilon u_\varepsilon) = \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} |\Delta u_\varepsilon|^2 + |u_\varepsilon|^2 dx - \frac{t_\varepsilon^{2^*}}{2^*} \int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx.$$

It is easy to see that $\lim_{t_\varepsilon \to 0} h(t_\varepsilon) = 0$ and $\lim_{t_\varepsilon \to \infty} h(t_\varepsilon) = -\infty$. Consequently, there exists $t_\varepsilon^* > 0$ such that $h(t_\varepsilon^*) = \max_{t_\varepsilon \ge 0} h(t_\varepsilon)$, that is

$$h'(t_\varepsilon)|_{t_\varepsilon^*} = t_\varepsilon^* \|u_\varepsilon\|^2 - (t_\varepsilon^*)^{2^*-1} \int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx = 0,$$

from which we have

$$t^* = \left(\frac{\|u_\varepsilon\|^2}{\int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx} \right)^{\frac{1}{2^*-2}}.$$

Therefore, from (f_4), (2.9), (2.10) and (2.11) it follows that

$$\begin{aligned}
\sup_{t \ge 0} I(tu_\varepsilon) = I(t_\varepsilon u_\varepsilon) &\le h(t_\varepsilon^*) - \frac{\mu \kappa t_\varepsilon^q}{q} \int_{\mathbb{R}^N} |u_\varepsilon|^q dx \\
&= \frac{(t_\varepsilon^*)^2}{2} \|u_\varepsilon\|^2 - \frac{(t_\varepsilon^*)^{2^*}}{2^*} \int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx - \frac{\mu \kappa t_\varepsilon^q}{q} \int_{\mathbb{R}^N} |u_\varepsilon|^q dx \\
&\le \frac{2}{N} \left(\frac{\|u_\varepsilon\|^2}{\int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx} \right)^{\frac{2}{2^*-2}} \|u_\varepsilon\|^2 - \mu C_1 \kappa \int_{\mathbb{R}^N} |u_\varepsilon|^q dx \\
&\le \frac{2}{N} S^{\frac{N}{4}} - \mu C_1 \kappa \int_{\mathbb{R}^N} |u_\varepsilon|^q dx + C_2 \begin{cases} \varepsilon^\gamma \\ \varepsilon^\gamma |ln \varepsilon| \end{cases} \\
&< \frac{2}{N} S^{\frac{N}{4}},
\end{aligned}$$

where $C_1, C_2 > 0$ are constants and $\gamma = \min\{4, N-4\}$. In fact, by (2.10), we have the following cases:
Case (i) if $5 \le N \le 7$, $2 < q < 2^*$ and $\gamma = \min\{4, N-4\} = N-4$, let $\mu = \varepsilon^{-4}$, we have

$$C_2 \varepsilon^{N-4} - \mu C_1 \kappa \int_{\mathbb{R}^N} |u_\varepsilon|^q dx < 0.$$

Case (ii) if $N = 8$, $\frac{N}{N-4} = 2 < q < 2^*$ and $\gamma = \min\{4, N-4\} = 4$, let $\mu = |ln\varepsilon|$, we get

$$C_2 \varepsilon^4 |ln \varepsilon| - \mu C_1 \kappa \varepsilon^{N - \frac{N-4}{2} q} = |ln\varepsilon| \left(C_2 \varepsilon^4 - C_1 \kappa \varepsilon^{N - \frac{N-4}{2} q} \right) < 0.$$

Case (iii) if $N > 8$, $\frac{N}{N-4} < 2 < q < 2^*$ and $\gamma = \min\{4, N-4\} = 4$, we have

$$C_2 \varepsilon^4 - \mu C_1 \kappa \varepsilon^{N - \frac{N-4}{2} q} < 0.$$

The proof is complete. □

Proposition 2.4 *Assume that $5 \leq N \leq 7$ or $N \geq 8$ and $2 < q < 2^*$. Then equation* (2.1) *has a nontrivial solution for all $\mu > 0$.*

Proof Applying the mountain pass lemma in [3], there exists a sequence $\{u_n\} \subset H_r^2(\mathbb{R}^N)$ such that

$$I(u_n) \to c > 0, \quad \text{and} \quad I'(u_n) \to 0 \text{ as } n \to \infty,$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

and

$$\Gamma = \left\{ \gamma \in C([0,1], H_r^2(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = e \right\}.$$

From Lemma 2.2, we know that $\{u_n\} \subset H_r^2(\mathbb{R}^N)$ has a convergent subsequence, still denoted by $\{u_n\}$, such that $u_n \to u_*$ in $H_r^2(\mathbb{R}^N)$ as $n \to \infty$. Thus, we have

$$I(u_*) = \lim_{n \to \infty} I(u_n) = c > 0,$$

which implies that $u_* \not\equiv 0$. Therefore, we infer that u_* is a nontrivial solution of (2.1). The proof is complete. □

Lemma 2.5 *Equation* (1.1) *has at least one nontrivial solution $v \in H_r^2(\mathbb{R}^N)$ if and only if system* (1.3) *has at least one nontrivial solution $(u, \lambda) \in H_r^2(\mathbb{R}^N) \times (0, +\infty)$.*

Proof Suppose that $v \in H_r^2(\mathbb{R}^N)$ is a nontrivial solution of equation (1.1), then

$$\left(a + b \int_{\mathbb{R}^N} |\Delta v|^2 dx \right) \Delta^2 v + v = |v|^{2^*-2} v + \mu f(x, v), \quad x \in \mathbb{R}^N.$$

Letting $u(x) = v(\lambda^{\frac{1}{2}} x) = v(y)$ and $\lambda^2 = a + b \int_{\mathbb{R}^N} |\Delta v|^2 dx$, we get

$$\lambda^2 = a + b\lambda^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |\Delta u|^2 dx,$$

and

$$\begin{aligned} \Delta^2 u(x) + u(x) &= \lambda^2 \Delta^2 v(y) + v(y) \\ &= \left(a + b \int_{\mathbb{R}^N} |\Delta v|^2 dx \right) \Delta^2 v(y) + v(y) \\ &= |v(y)|^{2^*-2} v(y) + \mu f(x, v) \\ &= |u(x)|^{2^*-2} u(x) + \mu f(x, u). \end{aligned}$$

Consequently, we obtain that (u, λ) is a nontrivial solution of system (1.3).

Now, let (u, λ) be a nontrivial solution of system (1.3), then

$$\Delta^2 u + u = |u|^{2^*-2} u + \mu f(x, u), \quad x \in \mathbb{R}^N,$$

and

$$\lambda^2 = a + b\lambda^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |\Delta u|^2 dx, \quad \lambda \in \mathbb{R}^+.$$

Define $v(x) = u(\lambda^{-\frac{1}{2}} x) = u(y)$, we have

$$\begin{aligned} &\left(a + b \int_{\mathbb{R}^N} |\Delta v|^2 dx \right) \Delta^2 v(x) + v(x) \\ &= \lambda^{-2} \left(a + b\lambda^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |\Delta u|^2 dx \right) \Delta^2 u(y) + u(y) \\ &= |u(y)|^{2^*-2} u(y) + \mu f(x, u) \\ &= |v(x)|^{2^*-2} v(x) + \mu f(x, v), \end{aligned}$$

which implies that v is a nontrivial solution of equation (1.1). The proof is complete. □

Proposition 2.6 (i) *If* $5 \leq N \leq 7$, *then system* (1.3) *has a nontrivial solution* (u, λ) *for all* $\mu, a, b > 0$*;*

(ii) *If* $N = 8$, *then there exists* $b_0 > 0$, *such that for all* $\mu, a > 0$, *system* (1.3) *has a nontrivial solution* (u, λ) *for* $b < b_0$, *and system* (1.3) *has no nontrivial solution for* $b \geq b_0$*;*

(iii) *If* $N > 8$, *then there exists* $\alpha_0 > 0$, *such that for all* $\mu > 0$, *system* (1.3) *has two nontrivial solutions* (u, λ_1) *and* (u, λ_2) *for* $ab^{\frac{4}{N-8}} < \alpha_0$, *has one nontrivial solution for* $ab^{\frac{4}{N-8}} = \alpha_0$, *and has no nontrivial solution for* $ab^{\frac{4}{N-8}} > \alpha_0$.

Proof We define a function

$$g_u(\lambda) = \lambda^2 - a - b\lambda^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |\Delta u|^2 dx, \quad \forall\, \lambda \in \mathbb{R}^+.$$

(i) If $5 \leq N \leq 7$, by the definition of $g_u(\lambda)$, for all $a, b > 0$, we obtain that $g_u(\lambda) \to +\infty$ as $\lambda \to +\infty$. Since $g_u(\lambda) < 0$ for $\lambda \in (0, a^{\frac{1}{2}}]$, so there exists $\lambda > a^{\frac{1}{2}}$ such that $g_u(\lambda) = 0$. Thus, (u, λ) is a nontrivial solution of system (1.3).

(ii) If $N = 8$, we have $g_u(\lambda) = (1 - b\int_{\mathbb{R}^N} |\Delta u|^2 dx)\lambda^2 - a$. Let $b_0 = \frac{1}{\int_{\mathbb{R}^N} |\Delta u|^2 dx}$, if $b < b_0$, then $\lambda = \left(\frac{a}{1 - b\int_{\mathbb{R}^N} |\Delta u|^2 dx}\right)^{\frac{1}{2}}$ is a solution of $g_u(\lambda) = 0$ and (u, λ) is a nontrivial solution of system (1.3). If $b \geq b_0$, then system (1.3) has no nontrivial solution.

(iii) If $N > 8$, for all $a, b > 0$, it is easy to see that $g_u(\lambda) \to +\infty$ as $\lambda \to -\infty$ and $g_u(\lambda) < 0$ for $\lambda \in (0, a^{\frac{1}{2}}]$, for every $u \neq 0$, we get

$$\frac{dg_u(\lambda)}{d\lambda} = \lambda^{-1}\left(2\lambda^2 - b\frac{N-4}{2}\lambda^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |\Delta u|^2 dx\right),$$

which implies that $g_u(\lambda)$ has a unique maximum point

$$\lambda_u = \left(\frac{4}{b(N-4)\int_{\mathbb{R}^N} |\Delta u|^2 dx}\right)^{\frac{2}{N-8}} > 0,$$

and

$$\max_{\lambda \in \mathbb{R}^+} g_u(\lambda) = g_u(\lambda_u) = \frac{N-8}{N-4}\left(\frac{4}{b(N-4)\int_{\mathbb{R}^N} |\Delta u|^2 dx}\right)^{\frac{4}{N-8}} - a.$$

Let

$$\alpha_0 = \frac{N-8}{N-4}\left(\frac{4}{(N-4)\int_{\mathbb{R}^N} |\Delta u|^2 dx}\right)^{\frac{4}{N-8}}.$$

If $ab^{\frac{4}{N-8}} < \alpha_0$, there exist $\lambda_1 \in (a, \lambda_u)$, $\lambda_2 \in (\lambda_u, +\infty)$, such that (u, λ_1), (u, λ_2) solve system (1.3). If $ab^{\frac{4}{N-8}} = \alpha_0$, (u, λ_u) solve system (1.3). If $ab^{\frac{4}{N-8}} > \alpha_0$, system (1.3) has no solution. The proof is complete. □

For the degenerate case, i.e. $a = 0$, we have the following Proposition.

Proposition 2.7 (i) *If* $5 \leq N \leq 7$ *or* $N > 8$, *then system* (1.3) *has a nontrivial solution* (u, λ) *for all* $\mu, b > 0$*;*

(ii) *If* $N = 8$, *then there exists* $b_0 > 0$, *such that for all* $\mu > 0$, *system* (1.3) *has a nontrivial solution* (u, λ) *for* $b = b_0$, *and system* (1.3) *has no nontrivial solution for* $b \neq b_0$.

Proof We define a function

$$g_u(\lambda) = \lambda^2 - b\lambda^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |\Delta u|^2 dx, \quad \forall\, \lambda \in \mathbb{R}^+.$$

(i) If $5 \leq N \leq 7$ or $N > 8$, by $g_u(\lambda)$, there exist $\lambda = (b\int_{\mathbb{R}^N} |\Delta u|^2 dx)^{\frac{2}{8-N}}$ such that $g_u(\lambda) = 0$ for all $b > 0$. Hence, (u, λ) is a nontrivial solution of system (1.3).

(ii) If $N = 8$, we deduce that $g_u(\lambda) = (1 - b\int_{\mathbb{R}^N} |\Delta u|^2 dx)\lambda^2$. Let $b_0 = \frac{1}{\int_{\mathbb{R}^N} |\Delta u|^2 dx}$, if $b = b_0$, for all $\lambda > 0$ such that $g_u(\lambda) = 0$, then (u, λ) is a nontrivial solution of system (1.3). If $b \neq b_0$, then system (1.3) has no nontrivial solution.

Proofs of Theorems 1.1-1.2 Combining the facts with Proposition 2.4, Proposition 2.6 and Proposition 2.7 the proofs of Theorems 1.1-1.2 are complete. □

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Declarations

Conflict of interest The authors declare no conflict of interest in this paper.

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