ORIGINAL RESEARCH

On certain k-AP van der Waerden game

Kai An Sim[1] · Wan Muhammad Afif Wan Ruzali[1] · Kok Bin Wong[2] · Chee Kit Ho[1]

Received: 22 August 2023 / Accepted: 30 September 2025 / Published online: 14 October 2025

Abstract
In 1980, J. Beck considered the following 2-player van der Waerden game on a sequence of integers $[1, N]$. Player 1 and Player 2 alternately pick a previously unpicked integers in $[1, N]$. Player 1 wins if he has selected all members of a k-arithmetic progression. Player 2 wins if Player 1 cannot obtain all members of a k-arithmetic progression. Let $W(k)$ be the least integer N so that the first player has a winning strategy and Beck has showed that $\lim_{k\to\infty} (W(k))^{1/k} = 2$. In this paper, we consider the situation where Player 2 will select t integers instead of one integer in his turn and Player 1 will remain to pick only one integer in his turn. Let $W_t(k)$ be the least integer N so that the first player has a winning strategy in this setting and we show that $\lim_{k\to\infty} (W_t(k))^{1/k} = t + 1$.

Keywords Van der Waerden number · k-term arithmetic progression · k-AP game

Mathematics Subject Classification 05A17 · 05D10

1 Introduction

For an arithmetic progression $A = \{a + ld : 0 \le l \le k - 1\}$, we say that A is a *k-term arithmetic progression (AP) with difference d*, or we write $\{a + ld\}_{0\le l\le k-1}$. For a positive integer t, we denote the set $\{1, 2, 3, \ldots, t\}$ by $[1, t]$. An *r-colouring* of a set S is a surjective function $\chi : S \to [1, r]$. A *monochromatic k-term arithmetic progression* refers to a k-term arithmetic progression such that all of its elements are of the same colour. One the most fundamental Ramsey-type theorem is the van der Waerden theorem. Van der Waerden's theorem [19] states that there exists a least positive integer $w = w(k; r)$ such that for any $n \ge w$, every r-colouring of $[1, n]$ admits a monochromatic k-term arithmetic progression. For more results on van der Waerden numbers, see [1, 5, 6, 13, 17, 18].

Theorem 1.1 *[19] (Van der Waerden's Theorem) Let $k, r \ge 2$ be integers. There exists a least positive integer $w = w(k; r)$ such that for any $n \ge w$, every r-colouring of $[1, n]$ admits a monochromatic k-term arithmetic progression.*

Note that exact values of $w(k; 2)$ are only known for small value of k where $w(3; 2) = 9, w(4; 2) = 35, w(5; 2) = 178, w(6; 2) = 1132$, see [13]. The exact value of $w(k; r)$ is not known in general. We refer the readers to [5, 6, 14, 16–18].

There are many variants of Ramsey games that have been investigated, see [3, 4, 8–12, 15]. Consider one of the games here, which is inspired by Ramsey Theory: Player 1 (red) and Player 2 (blue) alternate colouring edges of a complete graph K_m with their colours. Let H be a subgraph of K_m. The first player to avoid a monochromatic H

Communicated by Rahul Roy

https://doi.org/10.1007/s13226-025-00875-6

in his colour wins. A player is said to have a winning strategy if he will win the game by choosing certain integers at each of his turn regardless of what integer the other player chooses at the other player's turn. This motivates the investigations on similar van der Waerden games.

We describe a van der Waerden game here. A k-AP van der Waerden game (or k-AP game) is a game between two players, Player 1 and Player 2, on the set of integers $[1, n]$. Initially, all the integers $1, 2, \ldots, n$ are uncoloured. Player 1 will first select an integer and colour it with red. Player 2 will then select another integer and colour it with blue. The two players continue this alternating colouring process on uncoloured integers. Each player's aim is to form a monochromatic k-term arithmetic progression as soon as possible. The player who first obtains a monochromatic k-term arithmetic progression wins, thus ending the game. We assume all players play optimally where both players never purposefully avoid a monochromatic k-term arithmetic progression and at the same time, they would like to prevent the opponent from forming a monochromatic k-term arithmetic progression.

For the rest of the section, let T_i be the ith turn. Player 1 makes his move on the odd-numbered turns, whereas Player 2 makes his move on the even-numbered turns.

2 A variant of van der Waerden game

Beck [2] considered the following variant of the k-AP game: Initially, all the integers $1, 2, \ldots, n$ are uncoloured. Player 1 will first select an integer and colour it with red. Player 2 will then select another integer and colour it with blue. The two players continue this alternating colouring process on uncoloured integers. The aim of Player 1 is to form a red monochromatic k-term arithmetic progression and the aim of Player 2 is to prevent Player 1 from doing so. Player 1 wins if he obtains a red monochromatic k-term arithmetic progression and the game ends. Player 2 wins if Player 1 cannot obtain a red monochromatic k-term arithmetic progression after all the integers in $1, 2, \ldots, n$ are coloured.

Let $W(k)$ be the least positive integer n such that Player 1 has a winning strategy. Beck [2] showed that

$$\lim_{k\to\infty} (W(k))^{1/k} = 2. \tag{1}$$

In this paper, we consider the situation where Player 2 will select t integers instead of just one integer. Formally, at the beginning, all the integers $[n] = 1, 2, \ldots, n$ are uncoloured. Player 1 will first select an integer and colour it with red. Player 2 will then select t integers and colour them with blue. The two players continue this alternating colouring process on uncoloured integers. If there are less than t coloured integers at turn T_{2l}, then Player 2 will colour all the remaining integers with blue and the game ends. As before, the aim of Player 1 is to form a red monochromatic k-term arithmetic progression and the aim of Player 2 is to prevent Player 1 from doing so. Player 1 wins if he obtains a red monochromatic k-term arithmetic progression and the game ends. Player 2 wins if Player 1 cannot obtain a red monochromatic k-term arithmetic progression after all the integers in $1, 2, \ldots, n$ are coloured.

Note that we can write $n = (p-1)t + p + r$ where $0 \le r \le t$. Suppose $r = 0$. If Player 1 is not able to win, then the game ends at turn T_{2p-1} after Player 1 colours the last integer, say x_p with red. At this moment, the integers coloured with red are $\{x_1, x_2, \ldots, x_p\}$ and the integers coloured with blue are

$$\{y_{11}, y_{12}, \ldots, y_{1t}, y_{21}, y_{22}, \ldots, y_{2t}, \ldots, y_{p-1,1}, y_{p-1,2}, \ldots, y_{p-1,t}\}.$$

Suppose $1 \le r \le t$. If Player 1 is not able to win, then the game ends at turn T_{2p} after Player 2 colours the remaining integers, say $y_{p1}, y_{p2}, \ldots, y_{pr}$ with blue. At this moment, the integers coloured with red are $\{x_1, x_2, \ldots, x_p\}$ and the integers coloured with blue are

$$\{y_{11}, y_{12}, \ldots, y_{1t}, y_{21}, y_{22}, \ldots, y_{2t}, \ldots, y_{p-1,1}, y_{p-1,2}, \ldots, y_{p-1,t}, y_{p1}, y_{p2}, \ldots, y_{pr}\}.$$

 https://doi.org/10.1007/s13226-025-00875-6

Let $W_t(k)$ be the least positive integer n such that Player 1 has a winning strategy. Note that when $t = 1$, then $W_t(k) = W_1(k) = W(k)$. We will show that (see Corollary 2.9)

$$\lim_{k\to\infty} (W_t(k))^{1/k} = t + 1.$$

This is a generalization of equation (1).

2.1 Good system of sets

We begin by introducing some concepts and notations that will be used later. Let $[n] = \{1, 2, \ldots, n\}$ and $2^{[n]}$ be the collection of all subsets of $[n]$ including the empty set $\varnothing$. A hypergraph $\mathfrak{F}$ with vertex set $[n]$ is a collection of non-empty subsets of $[n]$, i.e., $\mathfrak{F} \subseteq 2^{[n]} \setminus \{\varnothing\}$. An element in a hypergraph is called an edge. For examples, if $n = 9$, then $\mathfrak{F}_1 = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4, 5, 9\}\}$ and $\mathfrak{F}_2 = \{\{1, 8, 9\}, \{1, 2, 4\}, \{2, 6, 9\}\}$ are hypergraphs and $\{1, 2, 3\}$ is an edge of $\mathfrak{F}_1$ but it is not an edge of $\mathfrak{F}_2$.

Let $\mathfrak{F}$ be a hypergraph with vertex set $[n]$. Let $t \geq 1$ be an integer. Let $X, Y \subseteq [n]$ with $|X| = p, |Y| = t(p-1)+r$ where $0 \leq r \leq t$ and $X \cap Y = \varnothing$, say

$$\begin{aligned} X &= \{x_i \ : \ 1 \leq i \leq p\}; \\ Y &= \left\{y_{ij} \ : \ 1 \leq i \leq p-1, 1 \leq j \leq t\right\} \cup \left\{y_{pj} \ : \ 1 \leq j \leq r\right\}, \end{aligned}$$

with the convention that if $r = 0$, then $\left\{y_{pj} \ : \ 1 \leq j \leq r\right\} = \varnothing$. For convenience, for each $i_0 \in [p-1]$, we set

$$\begin{aligned} X_{i_0} &= \{x_i \ : \ 1 \leq i \leq i_0\}; \\ Y_{i_0} &= \left\{y_{ij} \ : \ 1 \leq i \leq i_0, 1 \leq j \leq t\right\}, \end{aligned}$$

and we also set $X_0 = \varnothing$ and $Y_0 = \varnothing$, and $X_p = X$ and $Y_p = Y$. Note that

$$X_{i_0+1} = X_{i_0} \cup \{x_{i_0+1}\}, \qquad \text{for all } 0 \leq i_0 \leq p-1,$$

and

$$Y_{i_0+1} = \begin{cases} Y_{i_0} \cup \{y_{i_0+1,1}, y_{i_0+1,2}, \ldots, y_{i_0+1,t}\}, & \text{if } 1 \leq i_0 \leq p-2; \\ Y_{i_0} \cup \{y_{i_0+1,1}, y_{i_0+1,2}, \ldots, y_{i_0+1,r}\}, & \text{if } i_0 = p-1. \end{cases}$$

Therefore, $Y = Y_p = Y_{p-1} \cup \left\{y_{pj} \ : \ 1 \leq j \leq r\right\}$. We shall call

$$(\mathfrak{F}, \{X_i\}_{1\leq i\leq p}, \{Y_i\}_{1\leq i\leq p}),$$

a system of sets.

Given a system of sets $(\mathfrak{F}, \{X_i\}_{1\leq i\leq p}, \{Y_i\}_{1\leq i\leq p})$, any integer $m \geq 1$ and each $i \in \{1, 2, \ldots, p\}$, we define

$$\Phi_{m,i} = \sum_{\substack{A\in\mathfrak{F}, \\ A\cap Y_{i-1}=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}}. \tag{2}$$

Here, we use the convention that $\Phi_{m,i} = 0$ if the set $\{A \in \mathfrak{F} \ : \ A \cap Y_{i-1} = \varnothing\} = \varnothing$.

Next, for any integer $m \geq 1$, any $u \in [n]$, all (i, j) with $1 \leq i \leq p-1$ and $1 \leq j \leq t$, and all (i, j) with $i = p$ and $1 \leq j \leq r$, we define

$$\phi_{m,i}^{(j)}(u) = \sum_{\substack{A \in \mathfrak{F}, u \in A, \\ A \cap (Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{i,j-1}\}) = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}},$$

and use the convention that $\phi_{m,i}^{(j)}(u) = 0$ if

$$\left\{A \in \mathfrak{F} \ : \ u \in A \ \text{ and } \ A \cap (Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{i,j-1}\}) = \varnothing\right\} = \varnothing.$$

Note that if $j = 1$, then $\{y_{i1}, y_{i2}, \ldots, y_{i,j-1}\} = \varnothing$ and

$$\phi_{m,i}^{(j)}(u) = \phi_{m,i}^{(1)}(u) = \sum_{\substack{A \in \mathfrak{F}, u \in A, \\ A \cap Y_{i-1} = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}}.$$

A *m-good* system of sets is a system of sets $(\mathfrak{F}, \{X_i\}_{1 \leq i \leq p}, \{Y_i\}_{1 \leq i \leq p})$ that satisfies the following conditions: for all $1 \leq i \leq p-1$ and $1 \leq j \leq t$,

$$\phi_{m,i}^{(j)}(y_{ij}) = \max_{u \in (X \cup Y_{p-1}) \setminus (X_i \cup Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{i,j-1}\})} \phi_{m,i}^{(j)}(u). \tag{3}$$

Lemma 2.1 *If $(\mathfrak{F}, X, Y)$ is a m-good system of sets, then $\Phi_{m,i+1} \leq \Phi_{m,i}$ for all $1 \leq i \leq p-1$. In particular,*

$$\Phi_{m,i} \leq \Phi_{m,1} \leq \frac{|\mathfrak{F}|}{(t+1)^{m-1}} \ \text{ for all } 1 \leq i \leq p.$$

Proof Since $m - |A \cap X_1| = m - 1$ if $x_1 \in A$, $m - |A \cap X_1| = m$ if $x_1 \notin A$, we have

$$\Phi_{m,1} = \sum_{\substack{A \in \mathfrak{F}, \\ A \cap Y_0 = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_1|}} = \sum_{A \in \mathfrak{F}} \frac{1}{(t+1)^{m-|A \cap X_1|}} \leq \sum_{A \in \mathfrak{F}} \frac{1}{(t+1)^{m-1}} = \frac{|\mathfrak{F}|}{(t+1)^{m-1}}.$$

Note that

$$\begin{aligned}
\phi_{m,i}^{(j+1)}(y_{i,j+1}) &= \sum_{\substack{A \in \mathfrak{F}, y_{i,j+1} \in A, \\ A \cap (Y_i \cup \{y_{i1}, y_{i2}, \ldots, y_{i,j-1}, y_{ij}\}) = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}} \\
&\leq \sum_{\substack{A \in \mathfrak{F}, y_{i,j+1} \in A, \\ A \cap (Y_i \cup \{y_{i1}, y_{i2}, \ldots, y_{i,j-1}\}) = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}} \\
&= \phi_{m,i}^{(j)}(y_{i,j+1}) \\
&\leq \phi_{m,i}^{(j)}(y_{ij}),
\end{aligned}$$

where the last inequality follows from (3). So, we have

$$\phi_{m,i}^{(t)}(y_{it}) \leq \phi_{m,i}^{(t-1)}(y_{i,t-1}) \leq \cdots \leq \phi_{m,i}^{(1)}(y_{i1}) \ \text{ for } 1 \leq i \leq p-1. \tag{4}$$

Now, we will show that

$$\sum_{\substack{A\in\mathfrak{F},x_{i+1}\in A,\\ A\cap Y_i=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}} \leq \phi_{m,i}^{(t)}(y_{it}) \text{ for } 1\leq i\leq p-1. \tag{5}$$

In fact,

$$\begin{aligned}
\sum_{\substack{A\in\mathfrak{F},x_{i+1}\in A,\\ A\cap Y_i=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}} &= \sum_{\substack{A\in\mathfrak{F},x_{i+1}\in A,\\ A\cap(Y_{i-1}\cup\{y_{i1},y_{i2},\ldots,y_{i,t-1},y_{it}\})=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}}\\
&\leq \sum_{\substack{A\in\mathfrak{F},x_{i+1}\in A,\\ A\cap(Y_{i-1}\cup\{y_{i1},y_{i2},\ldots,y_{i,t-1}\})=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}}\\
&= \phi_{m,i}^{(t)}(x_{i+1})\\
&\leq \phi_{m,i}^{(t)}(y_{it}),
\end{aligned}$$

where the last inequality follows from (3).

Let $1\leq i\leq p-1$ be fixed. We will show that $\Phi_{m,i+1}\leq\Phi_{m,i}$. Now,

$$\begin{aligned}
\Phi_{m,i} &= \sum_{\substack{A\in\mathfrak{F},y_{i1}\notin A,\\ A\cap Y_{i-1}=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}} + \sum_{\substack{A\in\mathfrak{F},y_{i1}\in A,\\ A\cap Y_{i-1}=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}}\\
&= \left(\sum_{\substack{A\in\mathfrak{F},\\ A\cap(Y_{i-1}\cup\{y_{i1}\})=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}}\right) + \phi_{m,i}^{(1)}(y_{i1})\\
&= \left(\sum_{\substack{A\in\mathfrak{F},y_{i2}\notin A,\\ A\cap(Y_{i-1}\cup\{y_{i1}\})=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}} + \sum_{\substack{A\in\mathfrak{F},y_{i2}\in A,\\ A\cap(Y_{i-1}\cup\{y_{i1}\})=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}}\right) + \phi_{m,i}^{(1)}(y_{i1})\\
&= \left(\left(\sum_{\substack{A\in\mathfrak{F},y_{i2}\notin A,\\ A\cap(Y_{i-1}\cup\{y_{i1}\})=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}}\right) + \phi_{m,i}^{(2)}(y_{i2})\right) + \phi_{m,i}^{(1)}(y_{i1})\\
&= \left(\sum_{\substack{A\in\mathfrak{F},\\ A\cap(Y_i\cup\{y_{i1},y_{i2}\})=\varnothing}} \frac{1}{(t+1)^{m-|A\cap X_i|}}\right) + \phi_{m,i}^{(2)}(y_{i2}) + \phi_{m,i}^{(1)}(y_{i1})\\
&\quad\vdots\\
&\quad\vdots
\end{aligned}$$

$$= \left(\sum_{\substack{A \in \mathfrak{F}, \\ A \cap (Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{it}\}) = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}} \right) + \sum_{1 \le j \le t} \phi_{m,i}^{(j)}(y_{ij})$$

$$= \left(\sum_{\substack{A \in \mathfrak{F}, \\ A \cap Y_i = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}} \right) + \sum_{1 \le j \le t} \phi_{m,i}^{(j)}(y_{ij}).$$

So,

$$\sum_{\substack{A \in \mathfrak{F}, \\ A \cap Y_i = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}} = \Phi_{m,i} - \sum_{1 \le j \le t} \phi_{m,i}^{(j)}(y_{ij}). \tag{6}$$

On the other hand,

$$\begin{aligned}
\Phi_{m,i+1} &= \sum_{\substack{A \in \mathfrak{F}, x_{i+1} \notin A, \\ A \cap Y_i = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_{i+1}|}} + \sum_{\substack{A \in \mathfrak{F}, x_{i+1} \in A, \\ A \cap Y_i = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_{i+1}|}} \\
&= \left(\sum_{\substack{A \in \mathfrak{F}, x_{i+1} \notin A, \\ A \cap Y_i = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}} \right) + \sum_{\substack{A \in \mathfrak{F}, x_{i+1} \in A, \\ A \cap Y_i = \varnothing}} \frac{t+1}{(t+1)^{m-|A \cap X_i|}} \\
&= \left(\sum_{\substack{A \in \mathfrak{F}, x_{i+1} \notin A, \\ A \cap Y_i = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}} + \sum_{\substack{A \in \mathfrak{F}, x_{i+1} \in A, \\ A \cap Y_i = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}} \right) + \sum_{\substack{A \in \mathfrak{F}, x_{i+1} \in A, \\ A \cap Y_i = \varnothing}} \frac{t}{(t+1)^{m-|A \cap X_i|}} \\
&= \left(\sum_{\substack{A \in \mathfrak{F}, \\ A \cap Y_i = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}} \right) + \sum_{\substack{A \in \mathfrak{F}, x_{i+1} \in A, \\ A \cap Y_i = \varnothing}} \frac{t}{(t+1)^{m-|A \cap X_i|}} \\
&= \left(\Phi_{m,i} - \sum_{1 \le j \le t} \phi_{m,i}^{(j)}(y_{ij}) \right) + t \left(\sum_{\substack{A \in \mathfrak{F}, x_{i+1} \in A, \\ A \cap Y_i = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_i|}} \right) && \text{(from (6))} \\
&\le \Phi_{m,i} - \sum_{1 \le j \le t} \phi_{m,i}^{(j)}(y_{ij}) + t\phi_{m,i}^{(t)}(y_{it}) && \text{(from (5))} \\
&= \Phi_{m,i} - \sum_{1 \le j \le t-1} \left(\phi_{m,i}^{(j)}(y_{ij}) - \phi_{m,i}^{(t)}(y_{it}) \right) \\
&\le \Phi_{m,i},
\end{aligned}$$

where the last inequality follows from (4). Hence, the lemma follows. ◻

Lemma 2.2 *Suppose $(\mathfrak{F}, \{X_i\}_{1 \le i \le p}, \{Y_i\}_{1 \le i \le p})$ is a m-good system of sets. If $|\mathfrak{F}| < (t+1)^{m-1}$ and*

$$\min_{A \in \mathfrak{F}} |A| \ge m,$$

then for each $1 \leq i \leq p$ *and all* $A \in \mathfrak{F}$*, either*

(a) $A \cap Y_{i-1} = \varnothing$ *and* $|A \cap X_i| \leq m-1$*, or*
(b) $A \cap Y_{i-1} \neq \varnothing$.

Proof Suppose the lemma is false. Then there is an $1 \leq i_0 \leq p$ and an $A_0 \in \mathfrak{F}$ such that

$$A_0 \cap Y_{i_0-1} = \varnothing \ \text{ and } \ |A_0 \cap X_{i_0}| \geq m.$$

This implies that

$$\Phi_{m,i_0} = \sum_{\substack{A \in \mathfrak{F}, \\ A \cap Y_{i_0-1} = \varnothing}} \frac{1}{(t+1)^{m-|A \cap X_{i_0}|}} = \sum_{\substack{A \in \mathfrak{F}, \\ A \cap Y_{i_0-1} = \varnothing}} (t+1)^{|A \cap X_{i_0}|-m} \geq (t+1)^{|A_0 \cap X_{i_0}|-m} \geq 1.$$

On the other hand, by Lemma 2.1,

$$\Phi_{m,i_0} \leq \frac{|\mathfrak{F}|}{(t+1)^{m-1}} < \frac{(t+1)^{m-1}}{(t+1)^{m-1}} = 1,$$

a contradiction. Hence, the lemma holds. □

2.2 Lower bound of $W_t(k)$

In this subsection, we will find a lower bound of $W_t(k)$. Let us begin by stating the following lemma in [2, Lemma 4].

Lemma 2.3 *Let us be given* h $(h \geq 13^5)$ *arithmetic progressions* $P_1, P_2, \ldots, P_h$ *of length k, having the property that for every two progressions* P_i, P_j*, either* $P_i \cap P_j = \varnothing$ *or* P_i *and* P_j *have distinct differences. Then one can select a subsequence* $P_{i_1}, P_{i_2}, \ldots, P_{i_l}$*, where* $l = \lfloor h^{1/5} \rfloor$*, such that*

$$\left| \bigcup_{1 \leq j \leq l} P_{i_j} \right| \geq lk - k.$$

We note here that if $l \geq l_0$, $\left| \bigcup_{1 \leq j \leq l} P_{i_j} \right| \geq lk - k$ and $|P_{i_j}| = k$, then

$$(l - l_0)k + \left| \bigcup_{1 \leq j \leq l_0} P_{i_j} \right| = \sum_{l_0+1 \leq j \leq l} |P_{i_j}| + \left| \bigcup_{1 \leq j \leq l_0} P_{i_j} \right| \geq \left| \bigcup_{1 \leq j \leq l} P_{i_j} \right| \geq lk - k.$$

So, we have

$$\left| \bigcup_{1 \leq j \leq l_0} P_{i_j} \right| \geq l_0 k - k. \tag{7}$$

Given an arithmetic progressions P, we shall denote its difference by $d(P)$.

Let $k' = k - 2\lfloor k^{8/9}\rfloor$ and $\mathfrak{P}(N, k')$ be the set of all arithmetic progressions of the form $\{a, a+d, \ldots, a+(k'-1)d\}$ with $a + (k'-1)d \le N$ and $a = 2\lfloor k'^{8/9}\rfloor ld + r$ where $0 \le r \le d-1$ and $l \ge 1$. Since a can take values from 1 up to $N - (k'-1)$ and d can take values from 1 up to $\frac{N-1}{k'-1}$, we see that

$$\left|\mathfrak{P}(N, k')\right| \le \big(N - (k'-1)\big)\left(\frac{N-1}{k'-1}\right) < N^2. \tag{8}$$

Lemma 2.4 *Let* $P_1 \in \mathfrak{P}(N, k')$ *and*

$$\mathfrak{Q} = \left\{P \in \mathfrak{P}(N, k') \ : \ d(P) = d(P_1) \ \text{ and } \ P \cap P_1 \ne \varnothing\right\}.$$

Then $|\mathfrak{Q}| \le \frac{k'-1}{\lfloor k'^{8/9}\rfloor} + 1$.

Proof Let $P_1 = \{a_1, a_1 + d_1, \ldots, a_1 + (k'-1)d_1\}$ and $a_1 = 2\lfloor k'^{8/9}\rfloor l_1 d_1 + r_1$ where $0 \le r_1 \le d_1 - 1$ and $l_1 \ge 1$. If $a_1 + jd_1 = 2\lfloor k'^{8/9}\rfloor l_2 d_1 + r_1$ for some positive integer l_2, then $j = 2\lfloor k'^{8/9}\rfloor c$ for some positive integer c. This implies that

$$c \le \frac{k'-1}{2\lfloor k'^{8/9}\rfloor}.$$

So, the number of elements excluding a_1 in P_1 of the form $2\lfloor k'^{8/9}\rfloor ld_1 + r_1$ where $0 \le r_1 \le d_1 - 1$ and $l \ge 1$ is at most $\frac{k'-1}{2\lfloor k'^{8/9}\rfloor}$. Let's denote these elements by $b_1, b_2, \ldots, b_{c_0}$ where $c_0 = \left\lfloor \frac{k'-1}{2\lfloor k'^{8/9}\rfloor}\right\rfloor$.

Now, we count the number of $P \in \mathfrak{P}(N, k')$ with $d(P) = d(P_1) = d_1$ and $P \ne P_1$. Let the first term in P be a. If $a \in P_1$, then a must be equal to one of the b_i's. Therefore, there are at most c_0 of such P's. If $a \notin P_1$, we must have $a_1 \in P$. Using similar argument as in the previous paragraph, we see that the number of elements in P of the form $2\lfloor k'^{8/9}\rfloor ld_1 + r_1$ where $0 \le r_1 \le d_1 - 1$ and $l \ge 1$ is at most $\frac{k'-1}{2\lfloor k'^{8/9}\rfloor}$. Thus, the number of such P's is also at most $\frac{k'-1}{2\lfloor k'^{8/9}\rfloor}$. Hence, $|\mathfrak{Q}| \le 2c_0 + 1 \le \frac{k'-1}{2\lfloor k'^{8/9}\rfloor} + \frac{k'-1}{2\lfloor k'^{8/9}\rfloor} + 1 = \frac{k'-1}{\lfloor k'^{8/9}\rfloor} + 1$. □

A subset $\{A_1, A_2, \ldots, A_l\} \subseteq \mathfrak{P}(N, k')$ is said to be connected if there is $A_0 \in \mathfrak{P}(N, k')$ such that $A_i \cap A_0 \ne \varnothing$ for all i. Consider the hypergraph

$$\mathfrak{L} = \left\{ \bigcup_{1 \le i \le \lfloor k'^{1/9}\rfloor} A_i \ : \ \left| \bigcup_{1 \le i \le \lfloor k'^{1/9}\rfloor} A_i \right| \ge \lfloor k'^{1/9}\rfloor k' - k' \ \text{ and } \right.$$
$$\left. \left\{A_1, A_2, \ldots, A_{\lfloor k'^{1/9}\rfloor}\right\} \ \text{ is connected in } \mathfrak{P}(N, k') \right\}. \tag{9}$$

Lemma 2.5 *There exists a* k_1 *such that for all* $k' \ge k_1$*, if* $N \le (t+1)^{k' - 5k'^{8/9}}$*, then*

$$|\mathfrak{L}| < (t+1)^{(k'\lfloor k'^{1/9}\rfloor - 2k') - 1}.$$

Proof Let $A_0 \in \mathfrak{P}(N, k')$. Set

$$\mathfrak{H}(A_0) = \left\{A \in \mathfrak{P}(N, k') \ : \ A \cap A_0 \ne \varnothing\right\}.$$

We will show that $\mathfrak{H}(A_0) < k'^2 N$. Let

$$A_0 = \{a_0, a_0 + d_0, \ldots, a_0 + (k' - 1)d_0\}.$$

Now, we count the number of elements in $\mathfrak{H}(A_0)$. Let $A = \{a, a+d, \ldots, a+(k'-1)d\} \in \mathfrak{H}(A_0)$. If $a \in A \cap A_0$, then $a = a_0 + j_0 d_0$ for some $0 \leq j_0 \leq k' - 1$. So, there are at most k' choices for a. For each of these choices, $d \leq \frac{N-a}{k'-1} < N$. So, there are at most Nk' of such A's with $a \in A \cap A_0$. If $a \notin A \cap A_0$ and $a + d \in A \cap A_0$, then $a + d = a_0 + j_1 d_0$ for some $0 \leq j_1 \leq k' - 1$ and there are at most k' choices for $a + d$. For each of these choices, $d \leq \frac{N-a}{k'-1} < N$. So, there are at most Nk' of such A's with $a \notin A \cap A_0$ and $a + d \in A \cap A_0$. Similarly, for each $2 \leq j \leq k' - 1$, there are at most Nk' of such A's with $a + j_1 d \notin A \cap A_0$ for $0 \leq j_1 \leq j - 1$ and $a + jd \in A \cap A_0$. Thus, there are at most $k'^2 N$ of such A's. This means $|\mathfrak{H}(A_0)| \leq k'^2 N$. Therefore, the number of $\left\{A_1, A_2, \ldots, A_{\lfloor k'^{1/9} \rfloor}\right\} \subseteq \mathfrak{P}(N, k')$ with $A_i \cap A_0 \neq \varnothing$ is at most

$$\binom{k'^2 N}{\lfloor k'^{1/9} \rfloor} = \frac{k'^2 N (k'^2 N - 1) \ldots \left(k'^2 N - \lfloor k'^{1/9} \rfloor + 1\right)}{\left(\lfloor k'^{1/9} \rfloor\right)!} < k'^{2\lfloor k'^{1/9} \rfloor} N^{\lfloor k'^{1/9} \rfloor}.$$

By Equation (8), there at most N^2 of such A_0's. Therefore,

$$|\mathfrak{L}| < \left(k'^{2\lfloor k'^{1/9} \rfloor} N^{\lfloor k'^{1/9} \rfloor}\right) N^2 = k'^{2k'^{1/9}} N^{2+\lfloor k'^{1/9} \rfloor} < (t+1)^{k'} N^{2+\lfloor k'^{1/9} \rfloor},$$

provided that $(1+t)^{k'^{8/9}} > k'^2$ which is true for sufficiently large k' depending on t. Hence,

$$\begin{aligned}
|\mathfrak{L}| &< (t+1)^{k'} N^{2+\lfloor k'^{1/9} \rfloor} \leq (t+1)^{k'} \left((t+1)^{k'-5k'^{8/9}}\right)^{2+\lfloor k'^{1/9} \rfloor} \\
&= (t+1)^{k'\lfloor k'^{1/9} \rfloor + 3k' - 10k'^{8/9} - 5k'^{8/9} \lfloor k'^{1/9} \rfloor} \\
&\leq (t+1)^{k'\lfloor k'^{1/9} \rfloor + 3k' - 10k'^{8/9} - 5k'^{8/9}\left(k'^{1/9} - 1\right)} \\
&= (t+1)^{k'\lfloor k'^{1/9} \rfloor - 2k' - 5k'^{8/9}} \\
&< (t+1)^{(k'\lfloor k'^{1/9} \rfloor - 2k') - 1}.
\end{aligned}$$

□

Now, note that every arithmetic progression of length k in $[N] = \{1, 2, \ldots, N\}$ contains an element from $\mathfrak{P}(N, k')$. In fact, $\{a_1, a_1 + d_1, \ldots, a_1 + (k-1)d_1\}$ contains the element

$$\left\{2z\lfloor k'^{8/9} \rfloor d_1 + r, (2z\lfloor k'^{8/9} \rfloor + 1)d_1 + r, \ldots, (2(z-1)\lfloor k'^{8/9} \rfloor + k' - 1)d_1 + r\right\} \in \mathfrak{P}(N, k'),$$

where $a_1 = qd_1 + r$ with $0 \leq r \leq d_1 - 1$ and $q = 2(z-1)\lfloor k'^{8/9} \rfloor + r'$ with $0 \leq r' \leq 2\lfloor k'^{8/9} \rfloor - 1$.

Now, recall that Player 1 chooses an element from $[N]$ and colours it with red at his turn and Player 2 chooses t elements from $[N]$ and colours them with blue at his turn. The aim of Player 1 is to obtain a monochromatic arithmetic progression of length k with red colour whereas the aim of Player 2 is to prevent Player 1 from doing so. Suppose that Player 1 wins. This means Player 1 would obtain a monochromatic arithmetic progression of length k with red colour. Since this arithmetic progression contains an element in $\mathfrak{P}(N, k')$, this means Player 1 would also be able to colour an element in $\mathfrak{P}(N, k')$ with red. Here, an element $A \in \mathfrak{P}(N, k')$ is said to be coloured with red if each integer in A is coloured with red.

We are going to show that if $N \leq (t+1)^{k-7k^{8/9}}$, then no matter what Player 1 does, he will not be able to colour any element in $\mathfrak{P}(N, k')$ with red. The following theorem then follows.

Theorem 2.6 *There is a positive integer $k_0 = k_0(t)$ such that for all $k \geq k_0$,*

$$W_t(k) > (t+1)^{k-7k^{8/9}}.$$

Note that

$$N \leq (t+1)^{k-7k^{8/9}} \leq (t+1)^{k'+2\lfloor k^{8/9}\rfloor - 7k^{8/9}} \leq (t+1)^{k'-5\lfloor k^{8/9}\rfloor} \leq (t+1)^{k'-5\lfloor k'^{8/9}\rfloor}. \tag{10}$$

Let's say after some turns, there are at most t uncoloured integers and it is Player 2's turn. Then Player 2 will colour the remaining integers with blue and the game ends. In this scenario, Player 1 loses the game. If Player 1 wins, then the game ends after Player 1 colours an integer with red and he obtains an element in $\mathfrak{P}(N, k')$ with red. So, we may assume that at each of the Player 2's turns, he will be able to colour t integers with blue.

We will devise a strategy for Player 2, so that no matter what Player 1 does, Player 1 will not be able to get an element in $\mathfrak{P}(N, k')$ with red.

At the beginning, just before Player 1 colours any integer with red, let $X_0 = Y_0 = \varnothing$ and $B_0 = \varnothing = V_{00} = W_{00}$. Suppose that after some turns (just before Player 1 chooses any uncoloured integer and colours it with red), we have obtained $\{X_i\}_{0\leq i\leq r_1}$, $\{Y_i\}_{0\leq i\leq r_1}$, $\{B_i\}_{0\leq i\leq r_2}$, $\{V_{ij}\}_{0\leq i\leq r_2, 0\leq j\leq s_i}$ and $\{W_{ij}\}_{0\leq i\leq r_2, 0\leq j\leq s_i}$ such that

(i) $X_i = X_{i-1} \cup \{x_i\}$ and $Y_i = Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{it}\}$ for $1 \leq i \leq r_1$;
(ii) $V_{i0} = \varnothing$ and $W_{i0} = \varnothing$ for $0 \leq i \leq r_2$;
(iii) $V_{ij} = V_{i,j-1} \cup \{v_{ij}\}$ and $W_{ij} = W_{i,j-1} \cup \{w_{ij1}, w_{ij2}, \ldots, w_{ijt}\}$ for $1 \leq i \leq r_2$ and $1 \leq j \leq s_i$;
(iv) for each $1 \leq i \leq r_2$, $|B_i| = \lfloor k'^{8/9}\rfloor$ and there is a $P_i \in \mathfrak{P}(N, k')$ with $B_i = P_i \setminus X_{r_1}$;
(v) for each $1 \leq i \leq r_2$ and $1 \leq j \leq s_i$, $V_{ij} \subseteq B_i \setminus (B_0 \cup B_1 \cup \cdots \cup B_{i-1})$;
(vi) if $[N] \setminus (X_{r_1} \cup Y_{r_1}) \subseteq \bigcup_{1\leq i\leq r_2} B_i$, then at most t different i's with $B_i \cap Y_{r_1} \neq \varnothing$;
(vii) if $[N] \setminus (X_{r_1} \cup Y_{r_1}) \nsubseteq \bigcup_{1\leq i\leq r_2} B_i$, then $B_i \cap Y_{r_1} = \varnothing$ for all $1 \leq i \leq r_2$.

Note that either (vi) or (vii) holds.

Now, Player 1 chooses an uncoloured integer, say z and colours it with red. We consider two cases.

Case 1. $z \notin \bigcup_{0\leq i\leq r_2} B_i$.

In this scenario, (vii) holds. We label z as x_{r_1+1} and set

$$X_{r_1+1} = X_{r_1} \cup \{x_{r_1+1}\}.$$

Player 2 chooses $y_{r_1+1,1}, y_{r_1+1,2}, \ldots, y_{r_1+1,t}$ and colour them with blue according to the two subcases:

Case 1.1. Suppose there are more than t uncoloured integers in $[N] \setminus \left(\bigcup_{1\leq i\leq r_2} B_i\right)$. Then we choose $y_{r_1+1,1}, y_{r_1+1,2}, \ldots, y_{r_1+1,t}$ from $[N] \setminus \left(\bigcup_{1\leq i\leq r_2} B_i\right)$.

Let $m_1 = k'\lfloor k'^{1/9}\rfloor - 2k'$. Suppose $y_{r_1+1,1}, y_{r_1+1,2}, \ldots, y_{r_1+1,j-1}$ have been chosen. Define

$$\alpha^{(j)}_{r_1+1}(u) = \sum_{\substack{A\in\mathfrak{L}, u\in A,\\ A\cap(Y_{r_1}\cup\{y_{r_1+1,1}, y_{r_1+1,2},\ldots,y_{r_1+1,j-1}\})=\varnothing}} \frac{1}{(t+1)^{m_1-|A\cap X_{r_1}|}},$$

for all $u \in [N] \setminus \left(\bigcup_{1\le i\le r_2} B_i\right)$. Player 2 chooses a $y_{r_1+1,j} \in [N] \setminus \left(\bigcup_{1\le i\le r_2} B_i\right)$ satisfying

$$\alpha_{r_1+1}^{(j)}(y_{r_1+1,j}) = \max_{\substack{u\in[N]\setminus\left(\bigcup_{1\le i\le r_2} B_i\right),\\ \text{u is uncoloured}}} \alpha_{r_1+1}^{(j)}(u). \tag{11}$$

Next, after Player 2 has chosen $y_{r_1+1,1}, y_{r_1+1,2}, \ldots, y_{r_1+1,t}$, set

$$Y_{r_1+1} = Y_{r_1} \cup \{y_{r_1+1,1}, y_{r_1+1,2}, \ldots, y_{r_1+1,t}\}.$$

Since $z = x_{r_1+1} \notin B_i$, we have $B_i = P_i \setminus X_{r_1} = P_i \setminus X_{r_1+1}$ for $1 \le i \le r_2$. Furthermore, $y_{r_1+1,j} \notin \bigcup_{1\le i\le r_2} B_i$ for all j implies that $B_i \cap Y_{r_1+1} = \varnothing$. Suppose these are the only

$$P_{r_2+1}, P_{r_2+2}, \ldots, P_{r_2+r_3} \in \mathfrak{P}(N, k') \setminus \{P_1, P_2, \ldots, P_{r_2}\},$$

with

$$|P_i \setminus X_{r_1+1}| = \lfloor k'^{8/9} \rfloor \text{ and } P_i \cap Y_{r_1+1} = \varnothing, \text{ for all } r_2+1 \le i \le r_2+r_3.$$

We set $B_i = P_i \setminus X_{r_1+1}$, $V_{i0} = \varnothing$, $W_{i0} = \varnothing$ and $s_i = 0$ for $r_2+1 \le i \le r_2+r_3$. So, we have obtained $\{X_i\}_{0\le i\le r_1+1}$, $\{Y_i\}_{0\le i\le r_1+1}$, $\{B_i\}_{0\le i\le r_2+r_3}$, $\{V_{ij}\}_{0\le i\le r_2+r_3, 0\le j\le s_i}$ and $\{W_{ij}\}_{0\le i\le r_2+r_3, 0\le j\le s_i}$ such that

(i') $X_i = X_{i-1} \cup \{x_i\}$ and $Y_i = Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{it}\}$ for $1 \le i \le r_1+1$;
(ii') $V_{i0} = \varnothing$ and $W_{i0} = \varnothing$ for $0 \le i \le r_2+r_3$;
(iii') $V_{ij} = V_{i,j-1} \cup \{v_{ij}\}$ and $W_{ij} = W_{i,j-1} \cup \{w_{ij1}, w_{ij2}, \ldots, w_{ijt}\}$ for $1 \le i \le r_2+r_3$ and $1 \le j \le s_i$;
(iv') for each $1 \le i \le r_2+r_3$, $|B_i| = \lfloor k'^{8/9} \rfloor$ and there is a $P_i \in \mathfrak{P}(N, k')$ with $B_i = P_i \setminus X_{r_1+1}$;
(v') for each $1 \le i \le r_2+r_3$ and $1 \le j \le s_i$, $V_{ij} \subseteq B_i \setminus (B_0 \cup B_1 \cup \cdots \cup B_{i-1})$;
(vi') if $[N] \setminus (X_{r_1+1} \cup Y_{r_1+1}) \subseteq \bigcup_{1\le i\le r_2+r_3} B_i$, then at most t different i's with $B_i \cap Y_{r_1+1} \ne \varnothing$;
(vii') if $[N] \setminus (X_{r_1+1} \cup Y_{r_1+1}) \nsubseteq \bigcup_{1\le i\le r_2+r_3} B_i$, then $B_i \cap Y_{r_1+1} = \varnothing$ for all $1 \le i \le r_2+r_3$.

We note here that if the game ends after Player 1 colours the integer z, then $Y_{r_2+1} = Y_{r_2}$. Also conditions (vi') and (vii') hold because by construction, $B_i \cap Y_{r_1+1} = \varnothing$ for all $1 \le i \le r_2+r_3$.

Case 1.2. Suppose there are less than t uncoloured integers in $[N] \setminus \left(\bigcup_{1\le i\le r_2} B_i\right)$, say t_1. Then we choose all these integers and label them as $y_{r_1+1,1}, y_{r_1+1,2}, \ldots, y_{r_1+1,t_1}$, then simply choose $t - t_1$ uncoloured integers from $\bigcup_{1\le i\le r_2} B_i$ and label them as $y_{r_1+1,t_1+1}, y_{r_1+1,t_1+2}, \ldots, y_{r_1+1,t}$.

Next, after Player 2 has coloured $y_{r_1+1,1}, y_{r_1+1,2}, \ldots, y_{r_1+1,t}$ with blue, set

$$Y_{r_1+1} = Y_{r_1} \cup \{y_{r_1+1,1}, y_{r_1+1,2}, \ldots, y_{r_1+1,t}\}.$$

Since $z = x_{r_1+1} \notin B_i$, we have $B_i = P_i \setminus X_{r_1} = P_i \setminus X_{r_1+1}$ for $1 \le i \le r_2$. Now, $0 \le t_1 < t$ implies that there are at most t different i's with $B_i \cap Y_{r_1+1} \ne \varnothing$ where $1 \le i \le r_2$. Suppose these are the only

$$P_{r_2+1}, P_{r_2+2}, \ldots, P_{r_2+r_3} \in \mathfrak{P}(N, k') \setminus \{P_1, P_2, \ldots, P_{r_2}\},$$

with

$$|P_i \setminus X_{r_1+1}| = \lfloor k'^{8/9} \rfloor \text{ and } P_i \cap Y_{r_1+1} = \varnothing, \text{ for all } r_2+1 \le i \le r_2+r_3.$$

We set $B_i = P_i \setminus X_{r_1+1}$, $V_{i0} = \varnothing$, $W_{i0} = \varnothing$ and $s_i = 0$ for $r_2 + 1 \leq i \leq r_2 + r_3$. So, we have obtained $\{X_i\}_{0\leq i\leq r_1+1}$, $\{Y_i\}_{0\leq i\leq r_1+1}$, $\{B_i\}_{0\leq i\leq r_2+r_3}$, $\{V_{ij}\}_{0\leq i\leq r_2+r_3, 0\leq j\leq s_i}$ and $\{W_{ij}\}_{0\leq i\leq r_2+r_3, 0\leq j\leq s_i}$ such that

(i') $X_i = X_{i-1} \cup \{x_i\}$ and $Y_i = Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{it}\}$ for $1 \leq i \leq r_1 + 1$;
(ii') $V_{i0} = \varnothing$ and $W_{i0} = \varnothing$ for $0 \leq i \leq r_2 + r_3$;
(iii') $V_{ij} = V_{i,j-1} \cup \{v_{ij}\}$ and $W_{ij} = W_{i,j-1} \cup \{w_{ij1}, w_{ij2}, \ldots, w_{ijt}\}$ for $1 \leq i \leq r_2 + r_3$ and $1 \leq j \leq s_i$;
(iv') for each $1 \leq i \leq r_2 + r_3$, $|B_i| = \lfloor k'^{8/9} \rfloor$ and there is a $P_i \in \mathfrak{P}(N, k')$ with $B_i = P_i \setminus X_{r_1+1}$;
(v') for each $1 \leq i \leq r_2 + r_3$ and $1 \leq j \leq s_i$, $V_{ij} \subseteq B_i \setminus (B_0 \cup B_1 \cup \cdots \cup B_{i-1})$;
(vi') if $[N] \setminus (X_{r_1+1} \cup Y_{r_1+1}) \subseteq \bigcup_{1\leq i\leq r_2+r_3} B_i$, then at most t different i's with $B_i \cap Y_{r_1+1} \neq \varnothing$;

Similarly, as in Case 1.1, if the game ends after Player 1 colours the integer z, then $Y_{r_2+1} = Y_{r_2}$. Only conditions (vi') holds because by construction, there are at most t different i's ($1 \leq i \leq r_2$) with $B_i \cap Y_{r_1+1} \neq \varnothing$, and $B_i \cap Y_{r_1+1} = \varnothing$ for $r_2 + 1 \leq i \leq r_2 + r_3$.

Case 2. $z \in \bigcup_{0\leq i\leq r_2} B_i$.

In this scenario, we set

$$B_i^* = B_i \setminus (B_0 \cup B_1 \cup \cdots \cup B_{i-1}), \quad \text{for all} 1 \leq i \leq r_2.$$

Clearly, $B_1^*, B_2^*, \ldots, B_{r_2}^*$ are disjoint subsets of $[N]$. Suppose $z \in B_{i_2}^*$ for some $1 \leq i_2 \leq r_2$. We set

$$\mathfrak{S}_{i_2} = \left\{ P \cap B_{i_2}^* \ : \ P \in \mathfrak{P}(N, k') \ \text{ and } \ |P \cap B_{i_2}^*| \geq k'^{1/9} \right\}.$$

Label z as $v_{i_2, s_{i_2}+1}$ and set

$$V_{i_2, s_{i_2}+1} = V_{i_2 s_{i_2}} \cup \{v_{i_2, s_{i_2}+1}\}.$$

Player 2 chooses $w_{i_2, s_{i_2+1}, 1}, w_{i_2, s_{i_2+1}, 2}, \ldots, w_{i_2, s_{i_2+1}, t}$ and colour them with blue according to the following rules:

Let $m_2 = \lfloor k'^{1/9} \rfloor$. Suppose $w_{i_2, s_{i_2+1}, 1}, w_{i_2, s_{i_2+1}, 2}, \ldots, w_{i_2, s_{i_2+1}, t_1-1}$ have been chosen. Define

$$\beta_{i_2, s_{i_2+1}}^{(t_1)}(u) = \sum_{\substack{A \in \mathfrak{S}_{i_2}, u \in A, \\ A \cap (W_{i_2 s_{i_2}} \cup \{w_{i_2, s_{i_2+1}, 1}, w_{i_2, s_{i_2+1}, 2}, \ldots, w_{i_2, s_{i_2+1}, t_1-1}\}) = \varnothing}} \frac{1}{(t+1)^{m_2 - |A \cap V_{i_2 s_{i_2}}|}},$$

for all $u \in [N]$. Player 2 chooses a w_{i_2, s_{i_2+1}, t_1} satisfying

$$\beta_{i_2, s_{i_2+1}}^{(t_1)}(w_{i_2, s_{i_2+1}, t_1}) = \max_{\substack{u \in [N], \\ \text{u is uncoloured}}} \beta_{i_2, s_{i_2+1}}^{(t_1)}(u). \tag{12}$$

Next, after Player 2 has chosen $w_{i_2, s_{i_2+1}, 1}, w_{i_2, s_{i_2+1}, 2}, \ldots, w_{i_2, s_{i_2+1}, t}$, set

$$W_{i_2, s_{i_2}+1} = W_{i_2, s_{i_2}} \cup \{w_{i_2, s_{i_2+1}, 1}, w_{i_2, s_{i_2+1}, 2}, \ldots, w_{i_2, s_{i_2+1}, t}\}.$$

Let $s'_{i_2} = s_{i_2} + 1$ and $s'_i = s_i$ for $i \neq i_2$. We have obtained $\{X_i\}_{0\leq i\leq r_1}$, $\{Y_i\}_{0\leq i\leq r_1}$, $\{B_i\}_{0\leq i\leq r_2}$, $\{V_{ij}\}_{0\leq i\leq r_2, 0\leq j\leq s'_i}$ and $\{W_{ij}\}_{0\leq i\leq r_2, 0\leq j\leq s'_i}$ such that

(i") $X_i = X_{i-1} \cup \{x_i\}$ and $Y_i = Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{it}\}$ for $1 \le i \le r_1$;
(ii") $V_{i0} = \varnothing$ and $W_{i0} = \varnothing$ for $0 \le i \le r_2$;
(iii") $V_{ij} = V_{i,j-1} \cup \{v_{ij}\}$ and $W_{ij} = W_{i,j-1} \cup \{w_{ij1}, w_{ij2}, \ldots, w_{ijt}\}$ for $1 \le i \le r_2$ and $1 \le j \le s_i'$;
(iv") for each $1 \le i \le r_2$, $|B_i| = \lfloor k'^{8/9} \rfloor$ and there is a $P_i \in \mathfrak{P}(N, k')$ with $B_i = P_i \cap X_{r_1}$;
(v") for each $1 \le i \le r_2$ and $1 \le j \le s_i$, $V_{ij} \subseteq B_i \setminus (B_0 \cup B_1 \cup \cdots \cup B_{i-1})$.
(vi") if $[N] \setminus (X_{r_1} \cup Y_{r_1}) \subseteq \bigcup_{1 \le i \le r_2} B_i$, then at most t different i's with $B_i \cap Y_{r_1} \neq \varnothing$;
(vii") if $[N] \setminus (X_{r_1} \cup Y_{r_1}) \nsubseteq \bigcup_{1 \le i \le r_2} B_i$, then $B_i \cap Y_{r_1} = \varnothing$ for all $1 \le i \le r_2$.

We note here that if the game ends after Player 1 colours the integer z, then $W_{i_2, s_{i_2}+1} = W_{i_2, s_{i_2}}$.

At the end, Player 1 colours an integer and wins the game. At this moment we have obtained $\{X_i\}_{0 \le i \le R_1}$, $\{Y_i\}_{0 \le i \le R_1}$, $\{B_i\}_{0 \le i \le R_2}$, $\{V_{ij}\}_{0 \le i \le R_2, 0 \le j \le S_i}$ and $\{W_{ij}\}_{0 \le i \le R_2, 0 \le j \le S_i}$ such that

(I) $X_i = X_{i-1} \cup \{x_i\}$ and $Y_i = Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{it}\}$ for $1 \le i \le R_1$;
(II) $V_{i0} = \varnothing$ and $W_{i0} = \varnothing$ for $0 \le i \le R_2$;
(III) $V_{ij} = V_{i,j-1} \cup \{v_{ij}\}$ and $W_{ij} = W_{i,j-1} \cup \{w_{ij1}, w_{ij2}, \ldots, w_{ijt}\}$ for $1 \le i \le R_2$ and $1 \le j \le S_i$;
(IV) for each $1 \le i \le R_2$, $|B_i| = \lfloor k'^{8/9} \rfloor$ and there is a $P_i \in \mathfrak{P}(N, k')$ with $B_i = P_i \setminus X_{R_1}$;
(V) for each $1 \le i \le R_2$ and $1 \le j \le S_i$, $V_{ij} \subseteq B_i \setminus (B_0 \cup B_1 \cup \cdots \cup B_{i-1})$;
(VI) there are at most t different i's with $B_i \cap Y_{R_1} \neq \varnothing$ where $1 \le i \le R_2$.

Depending on Case 1 or Case 2, for (I), it is possible that $Y_{R_1} = Y_{R_1 - 1}$ and for (III), it is possible that $W_{i_2, S_{i_2}} = W_{i_2, S_{i_2} - 1}$. Note that (VI) is a consequence of the scenario in Case 1.2. It is also possible that $B_i \cap Y_{R_1} = \varnothing$ where $1 \le i \le R_2$.

Now, for each $A \in \mathfrak{L}$, $|A| \ge m_1$ (see (9)). So, $(\mathfrak{L}, \{X_i\}_{1 \le i \le R_1}, \{Y_i\}_{1 \le i \le R_1})$ is a m_1-good system of sets ((3) is satisfied with $\mathfrak{F} = \mathfrak{L}$). In fact, by equation (11), for $1 \le i \le R_1 - 1$ and $1 \le j \le t$, we have

$$\phi_{m,i}^{(j)}(y_{ij}) = \max_{u \in (X \cup Y_{R_1 - 1}) \setminus (X_i \cup Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{i,j-1}\})} \phi_{m,i}^{(j)}(u),$$

as the integers in $\big(X \cup Y_{R_1 - 1})\big) \setminus (X_i \cup Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{i,j-1}\})$ are uncoloured just before y_{ij} was chosen by Player 2. Furthermore, all the integers in $\big(X \cup Y_{R_1 - 1})\big) \setminus (X_i \cup Y_{i-1} \cup \{y_{i1}, y_{i2}, \ldots, y_{i,j-1}\})$ are in $[N] \setminus \big(\bigcup_{1 \le i \le R_2} B_i\big)$ by construction.

By Lemma 2.2, for all $A \in \mathfrak{L}$, either

(a) $A \cap Y_{R_1 - 1} = \varnothing$ and $|A \cap X_{R_1}| \le m_1 - 1 = k' \lfloor k'^{1/9} \rfloor - 2k' - 1$, or
(b) $A \cap Y_{R_1 - 1} \neq \varnothing$.

At the moment, let's assume that the following two facts are true.

Fact 1 **Fact 1.** *For each $P \in \mathfrak{P}(N, k')$, there are at most $k'^{7/9} - 1$ number of i's with $B_i \cap P \neq \varnothing$.*

Fact 2 *For each fixed i, there are at most $(t+1)^{\lfloor k'^{1/9} \rfloor - 1}$ edges $P \in \mathfrak{P}(N, k')$ with $|P \cap B_i| \ge 2$.*

Since Player 1 wins, there is a $Q \in \mathfrak{P}(N, k')$ with red. Therefore,

$$Q \subseteq X_{R_1} \cup \bigcup_{1 \le i \le R_2} V_{i S_i}.$$

By construction, if $Q = P_i$ for some $1 \le i \le R_2$, then $|Q \cap X_{R_1}| = k' - \lfloor k'^{8/9} \rfloor$. Since Q is red, $Q \cap Y_{R_1} = \varnothing$. Therefore, if $Q \neq P_i$ for all $1 \le i \le R_2$, then by construction, $|Q \setminus X_{R_1}| > \lfloor k'^{8/9} \rfloor$. This means $|Q \cap X_{R_1}| <$

$k' - \lfloor k'^{8/9} \rfloor$. In either case, $|Q \cap X_{R_1}| \le k' - \lfloor k'^{8/9} \rfloor$. This implies that

$$\left| Q \cap \bigcup_{1 \le i \le R_2} V_{i S_i} \right| \ge \lfloor k'^{8/9} \rfloor > k'^{8/9} - 1.$$

By (V), $V_{1S_1}, V_{2S_2}, \ldots, V_{R_2 S_{R_2}}$ are disjoint sets. Let i_0 be such that

$$|Q \cap V_{i_0 S_{i_0}}| = \max_{1 \le i \le R_2} |Q \cap V_{i S_i}|.$$

Since Q is red, $Q \cap Y_{R_1} = \varnothing$. Therefore, by Fact 1

$$\left| Q \cap \bigcup_{1 \le i \le R_2} V_{i S_i} \right| \le \left(k'^{7/9} - 1 \right) |Q \cap V_{i_0 S_{i_0}}|.$$

Thus,

$$|Q \cap V_{i_0 S_{i_0}}| > \frac{k'^{8/9} - 1}{k'^{7/9} - 1} > k'^{1/9}. \tag{13}$$

On the other hand, consider the set

$$\mathfrak{S}_{i_0} = \left\{ P \cap B^*_{i_0} \ : \ P \in \mathfrak{P}(N, k') \ \text{ and } \ |P \cap B^*_{i_0}| \ge k'^{1/9} \right\}.$$

Since $V_{i_0 S_{i_0}} \subseteq B^*_{i_0}$, it follows from (13) that $Q \cap B^*_{i_0} \in \mathfrak{S}_{i_0}$. Now, $k' = k - 2\lfloor k^{8/9} \rfloor \ge k - 2k^{8/9} = k\left(1 - \frac{2}{k^{1/9}}\right) \ge \frac{k}{2}$ for $k \ge 4^9$. This implies that $k' \ge 2^{17}$ and $k'^{1/9} > 2$. By Fact 2, $|\mathfrak{S}_{i_0}| \le (t+1)^{\lfloor k'^{1/9} \rfloor - 1}$. For each $A \in \mathfrak{S}_{i_0}$, $|A| \ge k'^{1/9} \ge \lfloor k'^{1/9} \rfloor$. By (12), it is not hard to see that $(\mathfrak{S}_{i_0}, \{V_{i_0 j}\}_{1 \le j \le S_{i_0}}, \{W_{i_0 j}\}_{1 \le j \le S_{i_0}})$ is a m_2-good system of sets ((3) is satisfied). By Lemma 2.2, for all $A \in \mathfrak{S}_{i_0}$, either

(c) $A \cap W_{i_0, S_{i_0} - 1} = \varnothing$ and $|A \cap V_{i_0 S_{i_0}}| \le m_2 - 1 = \lfloor k'^{1/9} \rfloor - 1 \le k'^{1/9} - 1$, or
(d) $A \cap W_{i_0, S_{i_0} - 1} \ne \varnothing$.

Since Q is red, $(Q \cap B^*_{i_0}) \cap W_{i_0, S_{i_0} - 1} = \varnothing$. This implies that

$$|Q \cap V_{i_0 S_{i_0}}| = |(Q \cap B^*_{i_0}) \cap V_{i_0 S_{i_0}}| \le k'^{1/9} - 1,$$

a contradiction (see (13)).

Now, it is left to show that Facts 1 and 2 are true.

Proof of Fact 1 Assume that Fact 1 is false. So, there is a $P_0 \in \mathfrak{P}(N, k')$ and $B_{a_1}, B_{a_2}, \ldots, B_{a_\nu}$ such that $B_{a_i} \cap P_0 \ne \varnothing$ for all $1 \le i \le \nu$ and $\nu > k'^{7/9} - 1$. By (VI), we may assume that $B_{a_i} \cap P_0 \ne \varnothing$ and $B_{a_i} \cap Y_{R_1} = \varnothing$ for all $1 \le i \le \nu'$ and $\nu' > k'^{7/9} - 1 - t$. By (IV), there are $P_{a_1}, P_{a_2}, \ldots, P_{a_{\nu'}}$ such that $B_{a_i} = P_{a_i} \setminus X_{R_1}$. So, $P_{a_i} \cap P_0 \ne \varnothing$. Let $b_1 = 1$ and set

$$\mathfrak{Q}_1 = \left\{ P_{a_i} \ : \ 1 \le i \le \nu', d(P_{a_i}) = d(P_{a_{b_1}}) \ \text{ and } \ P_{a_i} \cap P_{a_{b_1}} \ne \varnothing \right\}.$$

 https://doi.org/10.1007/s13226-025-00875-6

Let $P_{a_{b_2}} \notin \mathfrak{Q}_1$ and set

$$\mathfrak{Q}_2 = \left\{ P_{a_i} \;:\; 1 \le i \le \nu',\, P_{a_i} \notin \mathfrak{Q}_1,\, d(P_{a_i}) = d(P_{a_{b_2}}) \text{ and } P_{a_i} \cap P_{a_{b_2}} \neq \varnothing \right\}.$$

Let $P_{a_{b_3}} \notin \mathfrak{Q}_1 \cup \mathfrak{Q}_2$ and set

$$\mathfrak{Q}_3 = \left\{ P_{a_i} \;:\; 1 \le i \le \nu',\, P_{a_i} \notin \bigcup_{1\le j\le 2} \mathfrak{Q}_j,\, d(P_{a_i}) = d(P_{a_{b_3}}) \text{ and } P_{a_i} \cap P_{a_{b_3}} \neq \varnothing \right\}.$$

We may continue this process and obtain $\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots, \mathfrak{Q}_h$ at the end because $\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots, \mathfrak{Q}_h$ are disjoint and are subsets of $\{P_{a_1}, P_{a_2}, \ldots, P_{a_{\nu'}}\}$. By Lemma 2.4, $|\mathfrak{Q}_i| \le \frac{k'-1}{\lfloor k'^{8/9} \rfloor} + 1$ for all i. This means

$$\nu' \le \sum_{1\le i\le h} |\mathfrak{Q}_i| \le h\left(\frac{k'-1}{\lfloor k'^{8/9}\rfloor} + 1\right),$$

which implies that

$$h \ge \frac{\nu'}{\frac{k'-1}{\lfloor k'^{8/9}\rfloor}+1} > \frac{\lfloor k'^{8/9}\rfloor \left(k'^{7/9} - 1 - t\right)}{k' - 1 + \lfloor k'^{8/9}\rfloor} > \frac{\left(k'^{8/9} - 1\right)\left(k'^{7/9} - 1 - t\right)}{k' - 1 + k'^{8/9}} > k'^{5/9},$$

for sufficiently large k'. Now, each $P_{a_{b_i}} \in \mathfrak{Q}_i$. So, we have $P_{a_{b_1}}, P_{a_{b_2}}, \ldots, P_{a_{b_h}}$ such that either $P_{a_{b_i}} \cap P_{a_{b_j}} = \varnothing$ or $d(P_{a_{b_i}}) \neq d(P_{a_{b_j}})$ for all $i \neq j$.

By Lemma 2.3, we can select $l = \lfloor h^{1/5} \rfloor > \lfloor k'^{1/9} \rfloor$ elements from $\{P_{a_{b_1}}, P_{a_{b_2}}, \ldots, P_{a_{b_l}}\}$, say

$$P_{a_{c_1}}, P_{a_{c_2}}, \ldots, P_{a_{c_l}},$$

such that

$$\left| \bigcup_{1\le j\le l} P_{a_{c_j}} \right| \ge lk' - k'.$$

It follows from equation (7) that

$$\left| \bigcup_{1\le j\le \lfloor k'^{1/9}\rfloor} P_{a_{c_j}} \right| \ge \lfloor k'^{1/9} \rfloor k' - k'.$$

Now, $\left\{P_{a_{c_1}}, P_{a_{c_2}}, \ldots, P_{a_{c_{\lfloor k'^{1/9}\rfloor}}}\right\}$ is connected by P_0. So, $A_0 = \bigcup_{1\le j\le \lfloor k'^{1/9}\rfloor} P_{a_{c_j}} \in \mathfrak{L}$. Since $B_{a_i} \cap Y_{R_1} = \varnothing$ for all $1 \le i \le \nu'$, we must have $A_0 \cap Y_{R_1 - 1} = \varnothing$. Note that

$$|A_0 \cap X_{R_1}| = |A_0| - |A_0 \setminus X_{R_1}| \ge |A_0| - \sum_{1\le j\le \lfloor k'^{1/9}\rfloor} |P_{a_{c_j}} \setminus X_{R_1}|$$

$$
\begin{aligned}
&\geq \left(\lfloor k'^{1/9} \rfloor k' - k'\right) - \sum_{1\leq j\leq \lfloor k'^{1/9}\rfloor} \lfloor k'^{8/9} \rfloor \\
&= \left(\lfloor k'^{1/9} \rfloor k' - k'\right) - \lfloor k'^{1/9} \rfloor \lfloor k'^{8/9} \rfloor \\
&\geq \lfloor k'^{1/9} \rfloor k' - 2k',
\end{aligned}
$$

a contradiction (see (a)). □

Proof of Fact 2 Given any arithmetic progression $\{a, a+d, \ldots, a+(k'-1)d\}$, if $a+j_1 d$ and $a+j_2 d$ are known for different j_1 and j_2, then a and d can be determined. So, given any two fixed integers, there are at most $\binom{k'}{2}$ elements in $\mathfrak{P}(N, k')$ containing the two integers. This implies that for a fixed B_i, there are at most $\binom{k'}{2}\binom{|B_i|}{2}$ elements $P \in \mathfrak{P}(N, k')$ with $|P \cap B_i| \geq 2$. By (IV),

$$
\binom{k'}{2}\binom{|B_i|}{2} = \binom{k'}{2}\binom{\lfloor k'^{8/9} \rfloor}{2} < \binom{k'}{2}^2 < \frac{k'^4}{4} \leq 2^{k'^{1/9}-2} < 2^{\lfloor k'^{1/9} \rfloor -1} \leq (t+1)^{\lfloor k'^{1/9} \rfloor -1}.
$$

Here, $2^{k'^{1/9}-2} \geq \frac{k'^4}{4}$ provided that k' is sufficiently large. □

Hence, Theorem 2.6 holds.

2.3 Upper bound of $W_t(k)$

In this subsection, we will find the upper bound of $W_t(k)$. We begin with the following lemma.

Lemma 2.7 *Let $\mathfrak{B}_0$ be the set of all k-term arithmetic progressions in $[N]$. If $N \geq \frac{k\sqrt{2}}{\sqrt{2}-1}$, then*

$$
|\mathfrak{B}_0| > \frac{N^2}{4(k-1)}.
$$

Proof A typical element in $\mathfrak{B}_0$ is of the form

$$
\{a, a+d, \ldots, a+(k-1)d\}.
$$

For each $1 \leq a \leq N-(k-1)$, we have $d \leq \frac{N-a}{k-1}$. Therefore,

$$
\begin{aligned}
|\mathfrak{B}_0| &= \sum_{1\leq i\leq N-(k-1)} \left\lfloor \frac{N-i}{k-1} \right\rfloor > \sum_{1\leq i\leq N-(k-1)} \left(\frac{N-i}{k-1} - 1\right) \\
&= \frac{N(N-(k-1))}{(k-1)} - \frac{(N-(k-1))(N-(k-1)+1)}{2(k-1)} - (N-(k-1)) \\
&= \frac{(N-k+1)(N-k)}{2(k-1)} > \frac{(N-k)^2}{2(k-1)} \geq \frac{N^2}{4(k-1)},
\end{aligned}
$$

provided that $N \geq \frac{k\sqrt{2}}{\sqrt{2}-1}$. □

Theorem 2.8 *There is positive integer $k_0 = k_0(t)$ such that for all $k \geq k_0$,*

$$
(t+1)^{k-7k^{8/9}} < W_t(k) \leq 4k^3(t+1)^{k-2}.
$$

 https://doi.org/10.1007/s13226-025-00875-6

Proof Let $\mathfrak{B}_0$ denotes the set of all k-term arithmetic progressions in $[N]$ where $N \geq 4k^3(t+1)^{k-2}$.

We will devise a strategy for Player 1, so that no matter what Player 2 does, Player 1 will always be able to get an element in $\mathfrak{B}_0$ with red, or in other words, Player 1 will always win. The theorem then follows from this.

Now, just before Player 1 chooses the first integer and colours it with red, let $X_0 = Y_0 = \varnothing$. Suppose that after some turns (just before Player 1 chooses any uncoloured integer and colours it with red), we have obtained $\{X_i\}_{0\leq i\leq r_1}$, $\{Y_i\}_{0\leq i\leq r_1}$ such that

$$X_i = X_{i-1} \cup \{x_i\} \text{ and } Y_i = Y_{i-1} \cup \{y_{i1}, y_{i2}, \dots, y_{it}\}, \quad \text{for } 1 \leq i \leq r_1.$$

Now, Player 1 chooses an uncoloured integer x_{r_1+1} and colours it with red according to the following rules: Define

$$\theta_{r_1+1}(u) = \sum_{\substack{A\in\mathfrak{B}_0, u\in A,\\ A\cap Y_{r_1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_{r_1}|}},$$

and Player 1 chooses an integer x_{r_1+1} that satisfies

$$\theta_{r_1+1}(x_{r_1+1}) = \max_{\substack{u\in[N],\\ \text{u is uncoloured}}} \theta_{r_1+1}(u). \tag{14}$$

Set $X_{r_1+1} = X_{r_1} \cup \{x_{r_1+1}\}$.

If there are more than t uncoloured integers left, Player 2 will choose some t uncoloured integers and colour them with blue, say $y_{r_1+1,1}, y_{r_1+1,2}, \dots, y_{r_1+1,t}$. We set

$$Y_{r_1+1} = Y_{r_1} \cup \{y_{r_1+1,1}, y_{r_1+1,2}, \dots, y_{r_1+1,t}\}.$$

If there are more at most $s \leq t$ uncoloured integers left, say $y_{r_1+1,1}, y_{r_1+1,2}, \dots, y_{r_1+1,s}$, Player 2 will colour all of them with blue. We set

$$Y_{r_1+1} = Y_{r_1} \cup \{y_{r_1+1,1}, y_{r_1+1,2}, \dots, y_{r_1+1,s}\}.$$

Note that if $s = 0$, then $Y_{r_1+1} = Y_{r_1}$.

If Player 1 wins the game using this strategy, we are done. Assume that Player 1 loses the game, that is he is not able to obtain an element $A \in \mathfrak{B}_0$ with red.

Now, write $N = (p-1)t + p + r$ where p, r are positive integers and $0 \leq r \leq t$. The game ends after Player 2 colours the remaining r uncoloured integers. So, we have obtained $\{X_i\}_{0\leq i\leq p}$, $\{Y_i\}_{0\leq i\leq p}$ such that

$$\begin{aligned} X_i &= X_{i-1} \cup \{x_i\} \text{ for } 1 \leq i \leq p; \\ Y_i &= Y_{i-1} \cup \{y_{i1}, y_{i2}, \dots, y_{it}\}, \quad \text{for } 1 \leq i \leq p-1; \end{aligned}$$

and $Y_p = Y_{p-1} \cup \{y_{i1}, y_{i2}, \dots, y_{ir}\}$. Note that if $r = 0$, then $Y_p = Y_{p-1}$.

For each $0 \leq i \leq p$, we define

$$\Theta_i = \sum_{\substack{A\in\mathfrak{B}_0,\\ A\cap Y_i=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}}.$$

By Lemma 2.7,

$$\Theta_0 = \sum_{\substack{A\in\mathfrak{B}_0,\\ A\cap Y_0=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_0|}} = \sum_{A\in\mathfrak{B}_0} \frac{1}{(t+1)^{|A|}} = \frac{|\mathfrak{B}_0|}{(t+1)^k} \geq \frac{N^2}{4(k-1)(t+1)^k}. \tag{15}$$

Now, no elements in $\mathfrak{B}_0$ are red. So, $A \cap Y_p \neq \varnothing$ for all $A \in \mathfrak{B}_0$. This implies that

$$\Theta_p = \sum_{\substack{A\in\mathfrak{B}_0,\\ A\cap Y_p=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_p|}} = 0. \tag{16}$$

For $0 \leq i \leq p-1$, we will show that

$$\Theta_{i+1} \geq \Theta_i - \frac{tk^2}{2(t+1)^2}. \tag{17}$$

Note that

$$\begin{aligned}
\Theta_{i+1} &= \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_{i+1}|}} + \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\notin A,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_{i+1}|}} \\
&= \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_{i+1}=\varnothing}} \frac{(t+1)}{(t+1)^{|A\setminus X_i|}} + \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\notin A,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} \\
&= \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_{i+1}=\varnothing}} \frac{t}{(t+1)^{|A\setminus X_i|}} + \sum_{\substack{A\in\mathfrak{B}_0,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}}.
\end{aligned} \tag{18}$$

Now, the number of $A \in \mathfrak{B}_0$ with $x_{i+1} \in A$ and $c \in Y_{i+1} \setminus Y_i$ is at most $\binom{k}{2}$. So, if $Y_{i+1} \setminus Y_i \neq \varnothing$, we have

$$\begin{aligned}
\theta_{i+1}(x_{i+1}) &= \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_i=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} \\
&= \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} + \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_i=\varnothing, A\cap(Y_{i+1}\setminus Y_i)\neq\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} \\
&\leq \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} + \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_i=\varnothing, A\cap(Y_{i+1}\setminus Y_i)\neq\varnothing}} \frac{1}{(t+1)^2} \\
&\leq \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} + \frac{\binom{k}{2}}{(t+1)^2} \\
&< \sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} + \frac{k^2}{2(t+1)^2},
\end{aligned}$$

 https://doi.org/10.1007/s13226-025-00875-6

which implies that

$$\sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\in A,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} > \theta_{i+1}(x_{i+1}) - \frac{k^2}{2(t+1)^2}. \tag{19}$$

Next, for $0 \le i \le p-2$, we have

$$\begin{aligned}
\sum_{\substack{A\in\mathfrak{B}_0,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} &= \sum_{\substack{A\in\mathfrak{B}_0,\\ A\cap Y_i=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} - \sum_{\substack{A\in\mathfrak{B}_0,\\ A\cap Y_i=\varnothing, A\cap(Y_{i+1}\setminus Y_i)\neq\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}}\\
&= \Theta_i - \sum_{\substack{A\in\mathfrak{B}_0,\\ A\cap Y_i=\varnothing, A\cap(Y_{i+1}\setminus Y_i)\neq\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}}\\
&\ge \Theta_i - \sum_{1\le j\le t} \sum_{\substack{A\in\mathfrak{B}_0, y_{i+1,j}\in A\\ A\cap Y_i=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}}\\
&= \Theta_i - \sum_{1\le j\le t} \theta_{i+1}(y_{i+1,j}).
\end{aligned} \tag{20}$$

Similarly, for $i = p-1$, we have

$$\sum_{\substack{A\in\mathfrak{B}_0, x_{i+1}\notin A,\\ A\cap Y_{i+1}=\varnothing}} \frac{1}{(t+1)^{|A\setminus X_i|}} \ge \Theta_i - \sum_{1\le j\le r} \theta_{i+1}(y_{i+1,r}). \tag{21}$$

By (18), (19), and (20), for $0 \le i \le p-2$, we have

$$\Theta_{i+1} \ge t\theta_{i+1}(x_{i+1}) - \frac{tk^2}{2(t+1)^2} + \Theta_i - \sum_{1\le j\le t} \theta_{i+1}(y_{i+1,j}).$$

By equation (14), $\theta_{i+1}(x_{i+1}) \ge \theta_{i+1}(y_{i+1,j})$ for all j. Therefore

$$t\theta_{i+1}(x_{i+1}) \ge \sum_{1\le j\le t} \theta_{i+1}(y_{i+1,j})$$

and inequality (17) holds. Similarly, by (18), (19), and (21), inequality (17) also holds for $i = p-1$. Therefore,

$$\Theta_p \ge \Theta_0 - \frac{ptk^2}{2(t+1)^2}.$$

Recall that $N \ge 4k^3(t+1)^{k-2}$ and $N = (p-1)t + p + r \ge (t+1)p - t$. So, $p \le \frac{N+t}{t+1} < \frac{2N}{t+1}$ for sufficiently large k. Together with (15), we have

$$\Theta_p > \frac{N^2}{4(k-1)(t+1)^k} - \frac{Ntk^2}{(t+1)^3} > 0,$$

contradicting equation (16). Hence, Player 1 will win the game.

This completes the proof of the theorem. □

The following corollary is a direct consequence of Theorem 2.8.

Corollary 2.9 *For a fixed $t \geq 1$,*

$$\lim_{k\to\infty} (W_t(k))^{1/k} = t + 1.$$

3 Conclusions

In this paper, we show that $\lim_{k\to\infty}(W_t(k))^{1/k} = t + 1$ and Player 1 always has a winning strategy in the game. Here, we only consider 2-players games in this paper. So, it will be interesting to consider r-players games where each player is given one colour with certain number of choices at each turn. So, this will be a r-colouring of $[1, n]$ and more parameters of the game can be explored.

Acknowledgements This project is supported by Fundamental Research Grant Scheme (FRGS)- FRGS/1/2020/STG06/SYUC/03/1 by Malaysia Ministry of Higher Education.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflicts of Interest The authors have no relevant financial or non-financial interests to disclose.

References

1. M. Beeler and P. O'Neil, Some new van der Waerden numbers, *Discrete Math.* **28** (1979), 135–146.
2. J. Beck, Van der Waerden and Ramsey type games. *Combinatoria***1** (1981), 103–116.
3. J. Beck, Achievement games and the probabilistic method. *Combin. Paul Erdős is Eighty Bolyai Soci. Math. Stud.***1** (1993), 51–78.
4. J. Beck, Ramsey games, *Discrete Mathematics*,**249**(1-3) (2002), 3–30.
5. Z. Berikkyzy, A. Schulte, and M. Young, Anti-van der Waerden numbers of 3-term arithmetic progressions. *The Electronic Journal of Combinatorics* **24** (2017) #P2.39.
6. T. Blankenship, J. Cummings and V. Taranchuk, A new lower bound for van der Waerden numbers, *European Journal of Combinatorics* **69** (2018), 163–168.
7. T. C. Brown, B. M. Landman and A. Robertson, Bounds on some van der Waerden numbers, *J. Combin. Theory Ser. A.* **115**(7) (2008), 1304–1309.
8. J. Butterfield, T. Grauman, W.B. Kinnersley, K.G. Milans, C. Stocker, D.B. West, On-line Ramsey theory for bounded degree graphs. *Electron. J. Combin.* **18**(11) (2011), 136.
9. D. Conlon, On-line Ramsey numbers. *SIAM J. Discrete Math.* **23**(4) (2009), 1953–1954
10. J. Cyman, T. Dzido, J. Lapinskas, A. Lo, On-line Ramsey numbers of paths and cycles. *Electron. J. Combin.* **22** (2015), 1–15.
11. S. David, I. Hartarsky, M. Tiba, Strong Ramsey games in unbounded time, *European Journal of Combinatorics*, **86** (2020), 103096.
12. D. Hefetz, C. Kusch, L. Narins, A. Pokrovskiy, C. Requilé, A. Sarid, Strong Ramsey games: Drawing on an infinite board, *Journal of Combinatorial Theory, Series A*, **150** (2017), 248–266.
13. B. Landman & A. Robertson, *Ramsey Theory on the integers*, 2nd ed., (2015), American Mathematical Society, Province, Rhode Island.
14. X. Li, H. Broersma, and L. Wang, Integer colorings with no rainbow 3-term arithmetic progression. *The Electronic Journal of Combinatorics* **29** (2022) #P2.28.

https://doi.org/10.1007/s13226-025-00875-6

15. F. N. N. B. Mohd Latip, T. S. Tan, A Note on On-Line Ramsey Numbers of Stars and Paths. *Bull. Malays. Math. Sci. Soc.* **44** (2021), 3511—3521.
16. J. Pach and I. Tomon, Colorings with only rainbow arithmetic progressions. *Acta Mathematica Hungarica* **161** (2020) 507–515.
17. K.A. Sim, T.S. Tan, and K.B. Wong, A note on the mixed Van Der Waerden number, *Bulletin of the Korean Mathematical Society*, **58** (2021) 1341–1354.
18. K.A. Sim and K.B. Wong, Minimum Number of Colours to Avoid k-Term Monochromatic Arithmetic Progressions, *Mathematics*, **10** (2022) 247.
19. B. L. van der Waerden, Beweis einer baudetschen Vermutung, *Nieuw Arch. Wisk.* **15** (1927), 212–216.

Authors and Affiliations

Kai An Sim[1] · Wan Muhammad Afif Wan Ruzali[1] · Kok Bin Wong[2] · Chee Kit Ho[1]

✉ Kai An Sim
kaians@sunway.edu.my

Wan Muhammad Afif Wan Ruzali
wmawanruzali@proton.me

Kok Bin Wong
kbwong@um.edu.my

Chee Kit Ho
ckho@sunway.edu.my

1 School of Mathematical Sciences, Sunway University, 47500 Bandar Sunway, Selangor, Malaysia

2 Institute of Mathematical Sciences, Faculty of Science, Universiti Malaya, 50603 Kuala Lumpur, Malaysia

Indian J Pure Appl Math (2026) 57:1282–1302