Indian J Pure Appl Math (2026) 57:1303–1317
https://doi.org/10.1007/s13226-024-00631-2

ORIGINAL RESEARCH

Approximation of bivariate Lipschitz continuous functions by hidden variable fractal functions

Vijender Nallapu

Received: 23 August 2023 / Accepted: 18 June 2024 / Published online: 11 July 2024
© The Indian National Science Academy 2024

Abstract The crux of the present paper is approximation of bivariate Lipschitz continuous functions by hidden variable fractal functions. We propose the construction of hidden variable fractal perturbation associated with a given bivariate Lipschitz continuous function defined on a rectangle $\mathcal{D}$ in the Euclidean space. This procedure yields a fractal operator on the space of all $\mathbb{R}^2$-valued Lipschitz continuous functions defined on a rectangle $\mathcal{D}$. Some basic and important properties of this fractal operator will be discussed. Subsequently, we extend this fractal operator to the norm preserving bounded linear operator on the the space of all $\mathbb{R}^2$-valued continuous functions defined on a rectangle $\mathcal{D}$. We investigate the stability of hidden variable fractal functions with respect to a perturbation in the scaling factors. Finally, existence of optimal hidden variable fractal function which approximates the given bivariate Lipschitz continuous function is discussed.

Keywords Fractals · Iterated function systems · Bivariate Lipschitz continuous functions · Hidden variable fractal functions

1 Introduction

Iterated Function System (IFS) introduced by Hutchinson [1] is a central concept of fractal theory. Given a set of data$\{(x_i, z_i) \in \mathbb{R}^2 : i = 0, 1, \ldots, N\}$, where $x_0 < x_1 < \cdots, x_{N-1} < x_N$. Barnsley [2] introduced the concept of fractal interpolation using the iterated function system $\{[x_0, x_N] \times \mathbb{R}; w_i, i = 1, 2, \cdots, N\}$, where $w_i(x, z) = \Big(L_i(x), F_i(x, z)\Big)$, $(x, z) \in [x_0, x_N] \times \mathbb{R}$, L_i is homeomorphism from $[x_0, x_N]$ to $[x_{i-1}, x_i]$ satisfying

$$L_i(x_0) = x_{i-1}, L_i(x_N) = x_i, \tag{1}$$

F_i is a continuous function from $[x_0, x_N] \times \mathbb{R}$ to $\mathbb{R}$, which is contractive in the z-direction such that

$$F_i(x_0, z_0) = z_{i-1}, \quad F_i(x_N, z_N) = z_i. \tag{2}$$

According to [2], fractal interpolation function (FIF) corresponding to the above data and IFS $\{[x_0, x_N] \times \mathbb{R}; w_i, i = 1, 2, \cdots, N\}$, is a unique function $\sigma : [x_0, x_N] \to \mathbb{R}$ satisfying $\sigma(L_i(x)) = F_i(x, \sigma(x))$, and $\sigma(x_i) = z_i$ for all $i = 0, 1, 2, \cdots, N$. From [2], we can see that the graph $G(\sigma)$ of σ satisfies:

$$G(\sigma) = \underset{i \in \mathbb{N}_N}{\cup} w_i(G(\sigma)),$$

Communicated by NM Bujurke.

V. Nallapu (✉)
Department of Mathematics, Visvesvaraya National Institute of Technology Nagpur, South Ambazari Road, Nagpur, Maharashtra 440006, India
E-mail: vijendernallapu@gmail.com

and hence the FIF σ is self-referential. The existing FIFs are obtained by considering L_i and F_i as follows:

$$L_i(x) = a_i x + b_i, \;\; F_i(x, z) = \alpha_i z + q_i(x),$$

where $q_i : [x_0, x_N] \to \mathbb{R}$ is a suitable continuous function and $\alpha_i \in \mathbb{R}$ such that $|\alpha_i| < 1$. Let $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_N) \in (-1, 1)^N$. Let $h : [x_0, x_N] \to \mathbb{R}$ and $b : [x_0, x_N] \to \mathbb{R}$ be continuous functions such that $b \neq h$, $b(x_1) = h(x_1)$, and $b(x_N) = h(x_N)$. By taking $q_i(x) = h(L_i(x)) - \alpha_i b(x)$ in the IFS $\{[x_0, x_N] \times \mathbb{R}; w_i, i = 1, 2, \cdots, N\}$, the α-fractal function h^α that approximates the function h was constructed in [2,3].The process of constructing the α-fractal function h^α corresponding to the function h, yields a fractal operator $\mathcal{F}^\alpha_{\Delta,b}$ on the space of all continuous functions defined on $[x_0, x_N]$. Some basic properties such as linearity, boundedness, and invertibility of the fractal operators $\mathcal{F}^\alpha_{\Delta,b}$ are studied in [3–5]. This fractal operator plays a vital role in connecting the theory of fractal functions with other branches of mathematics such as approximation theory, complex analysis, and operator theory. Since fractal functions are constructed through a suitable IFS, these functions exhibit self-similarity and generally these functions are self-referential in the sense that the graph of the function is a union of transformed copies of itself. Owing to this reason, fractal functions may not be suitable for approximation of a function that shows no self-similarity or less self-similarity. Barnsley et al. [6] initiated the construction of non-self-referential univariate FIFs which are more diverse, appealing and irregular than self-referential FIFs for the given interpolation data. Subsequently, a huge amount of research towards the theory of non-self-referential univariate fractal approximation and interpolation has been carried out, for instance, see [7–9].

Recently, Ruan and Xu [10] developed a general framework for the construction of fractal interpolation surface (FIS) which is the graph of a continuous function defined on a rectangle in $\mathbb{R}^2$. Prior to Ruan and Xu, there are different approaches [11–14] for the construction of FISs and each of the above approaches has its own specific strengths and weaknesses. Based on the FIS construction by Ruan and Xu [10], bivariate α-fractal functions and their approximation properties, shape preserving aspects, and fractal dimension are reported in [15–17]. Rate of approximation and many other local approximation properties of positive operators are studied in [18–20]. In this paper, we study approximation of bivariate Lipschitz continuous functions by bivariate hidden variable fractal functions.

For a given $\mathcal{D} = [a, b] \times [c, d] \subset \mathbb{R}^2$, let us define

$$\begin{aligned}
\mathcal{C}(\mathcal{D}) &= \{h : \mathcal{D} \to \mathbb{R} : h \text{ is continuous on } \mathcal{D}\},\\
\mathcal{C}_2(\mathcal{D}) &= \{(\tau_1, \tau_2) : \tau_1, \tau_2 \in \mathcal{C}(\mathcal{D})\},\\
\mathrm{Lip}(\mathcal{D}) &= \{h : \mathcal{D} \to \mathbb{R} : h \text{ is Lipschitz continuous on } \mathcal{D}\},\\
\mathrm{Lip}_2(\mathcal{D}) &= \{(\tau_1, \tau_2) : \tau_1, \tau_2 \in \mathrm{Lip}(\mathcal{D})\}.
\end{aligned}$$

For a given Lipschitz continuous function $(f, g) : \mathcal{D} \to \mathbb{R}^2$, we construct a family of continuous functions $(f^{\alpha,\beta,\gamma}, g^\gamma)$, where α, β, and γ are the suitable square real matrices. It is worthwhile to mention that the above procedure is a natural extension of the notion of bivariate α-fractal functions. We refer $(f^{\alpha,\beta,\gamma}, g^\gamma)$ as the fractal function corresponding to the function (f, g). The vantage of the above construction is that the function $f^{\alpha,\beta,\gamma}$ provides a non-self-referential approximation to the function $f : \mathcal{D} \to \mathbb{R}$ in contrast to the self-referential fractal approximation obtained through bivariate α-fractal functions. We call the function $f^{\alpha,\beta,\gamma}$ as bivariate hidden variable function corresponding the function $f : \mathcal{D} \to \mathbb{R}$ since the graph of $f^{\alpha,\beta,\gamma}$ is the orthogonal projection of the graph of $(f^{\alpha,\beta,\gamma}, g^\gamma)$. A cheerful potent is that by choosing the matrices α, β, and γ of the hidden variable IFS, the graph of $f^{\alpha,\beta,\gamma}$ can be made self-referential as well, thus providing more flexibility and diversity in the process of approximation. The process of constructing the fractal function $(f^{\alpha,\beta,\gamma}, g^\gamma)$ corresponding to the given Lipschitz continuous function (f, g) elicits a fractal operator $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} : \mathrm{Lip}_2(\mathcal{D}) \to \mathcal{C}_2(\mathcal{D})$. We study the basic and important properties of $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$. We then obtain the fractal analogue of a vector-valued function $(f, g) \in \mathcal{C}_2(\mathcal{D})$ by extending the operator $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ to a bounded linear operator on $\mathcal{C}_2(\mathcal{D})$.

Nature of the bivariate hidden variable fractal function $f^{\alpha,\beta,\gamma}$ corresponding to $f \in \mathrm{Lip}(\mathcal{D})$ depends on the choice of α, β, and γ, and this naturally leads to the question that whether the function $f^{\alpha,\beta,\gamma}$ is stable with respect to a perturbation in α, β, and γ. We answer this question by obtaining an upper bound for the error caused by a perturbation of the parameters α, β, and γ occurring in the fractal function $(f^{\alpha,\beta,\gamma}, g^\gamma)$. The presence of the scaling matrices α, β, and γ in the bivariate hidden variable fractal function $f^{\alpha,\beta,\gamma}$ obviously offers different approximants for the continuous function $f \in \mathrm{Lip}(\mathcal{D})$, and at the same time it naturally raises the question of optimal bivariate hidden variable fractal function for the function $f \in \mathrm{Lip}(\mathcal{D})$. We prove that obtaining a

bivariate hidden variable fractal function $f^{\alpha,\beta,\gamma}$ close to the function $f \in \mathrm{Lip}(\mathcal{D})$ is a constrained non-linear convex optimization problem.

2 Non-self-referential bivariate fractal function

In this section, we construct the non-self-referential bivariate fractal function corresponding to the given $f \in \mathrm{Lip}(\mathcal{D})$. Throughout the manuscript, we consider (i) the supremum norm, namely $\|f\|_\infty = \sup\{|f(x, y)| : (x, y) \in \mathcal{D}\}$ for $f \in \mathcal{C}(\mathcal{D})$ or $f \in \mathrm{Lip}(\mathcal{D})$, (ii) the supremum norm, namely $\|(f, g)\|_\infty = \|f\|_\infty + \|g\|_\infty$ for $(f, g) \in \mathcal{C}_2(\mathcal{D})$ or $(f, g) \in \mathrm{Lip}_2(\mathcal{D})$. For $r \in \mathbb{N}$, let $\Sigma_r = \{1, 2, \ldots, r\}$ and $\Sigma_{r,0} = \{0, 1, 2, \ldots, r\}$. Suppose that $f, g \in \mathcal{C}(\mathcal{D})$. Let $\Delta = \{x_0, x_1, \ldots, x_{N-1}, x_N, y_0, y_1, \ldots, y_{M-1}, y_N\}$ be a partition of $\mathcal{D}$, where $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$; $c = y_0 < y_1 < \cdots < y_{M-1} < y_M = d$. Let the interpolation data $\Delta_1 = \{(x_i, y_j, z_{i,j}, t_{i,j});\ (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}\}$ and $\Delta_2 = \{(x_i, y_j, t_{i,j});\ (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}\}$ be generated from $f, g \in \mathrm{Lip}(\mathcal{D})$ such that $f(x_i, y_j) = z_{i,j}$ and $g(x_i, y_j) = t_{i,j}$. Throughout the manuscript, we follow the following notation:

$\partial\Sigma_{N,0} = \{0, N\}$, $\mathrm{int}\Sigma_{N,0} = \{1, 2, \ldots, N-1\}$, $\partial\Sigma_{M,0} = \{0, M\}$, and $\mathrm{int}\Sigma_{M,0} = \{1, 2, \ldots, M-1\}$. Denote $I = [x_0, x_N]$, $J = [y_0, y_M]$, $I_i = [x_{i-1}, x_i]$, and $J_j = [y_{j-1}, y_j]$. For any $i \in \Sigma_N$, let $u_i : I \to I_i$ be a contractive homeomorphism satisfying

$$\begin{cases} u_i(x_0) = x_{i-1},\ u_i(x_N) = x_i, \text{ if } i \text{ is odd,} \\ u_i(x_0) = x_i,\ u_i(x_N) = x_{i-1}, \text{ if } i \text{ is even, and} \\ |u_i(x^*) - u_i(x^{**})| \le \zeta_i |x^* - x^{**}| \ \ \forall\, x^*, x^{**} \in I, \end{cases} \tag{3}$$

where $0 < \zeta_i < 1$ is a given constant. Similarly, for any $j \in \Sigma_M$, let $v_j : J \to J_j$ be a contractive homeomorphism satisfying

$$\begin{cases} v_j(y_0) = y_{j-1},\ v_j(y_M) = y_j, \text{ if } j \text{ is odd,} \\ v_j(y_0) = y_j,\ v_j(y_M) = y_{j-1}, \text{ if } j \text{ is even, and} \\ |v_j(y^*) - v_j(y^{**})| \le \varrho_j |y^* - y^{**}| \ \ \forall\, y^*, y^{**} \in J, \end{cases} \tag{4}$$

where $0 < \varrho_i < 1$ is a given constant. By definitions of u_i and v_j, it is easy to check that

$$u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i)\ \forall\, i \in \mathrm{int}\Sigma_{N,0},$$
$$v_j^{-1}(y_j) = v_{j+1}^{-1}(y_j)\ \forall\, j \in \mathrm{int}\Sigma_{M,0}.$$

Let $\tau : \mathbb{Z} \times \{0, N, M\} \to \mathbb{Z}$ be defined by

$$\tau(i, 0) = \begin{cases} i-1, \tau(i, N) = \tau(i, M) = i, \text{ if } i \text{ is odd,} \\ i, \tau(i, N) = \tau(i, M) = i-1, \text{ if } i \text{ is even.} \end{cases}$$

Let $\mathbb{R}^2$ be endowed with the Manhattan metric

$$d_M\big((x^*, y^*), (x^{**}, y^{**})\big) = |x^* - x^{**}| + |y^* - y^{**}|.$$

Let $K = I \times J \times \mathbb{R}^2$. For each $(i, j) \in \Sigma_N \times \Sigma_M$, let $F_{i,j} : K \to \mathbb{R}^2$ be a continuous function satisfying

$$F_{i,j}(x_k, y_l, z_{k,l}, t_{k,l}) = \big(z_{\tau(i,k),\tau(j,l)}, t_{\tau(i,k),\tau(j,l)}\big)\ \forall (k, l) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0},$$
$$d_M\big(F_{i,j}(x, y, z, t), F_{i,j}(x^*, y^*, z, t)\big) \le c_{i,j} d_M\big((x, y), (x^*, y^*)\big) \text{ for some } 0 < c_{i,j} < 1,$$
$$d_M\big(F_{i,j}(x, y, z^*, t^*), F_{i,j}(x, y, z^{**}, t^{**})\big) \le s_{i,j} d_M((z^*, t^*), (z^{**}, t^{**})),$$

where $0 < s_{i,j} < 1$ is a given constant. Now, for each $(i, j) \in \Sigma_N \times \Sigma_M$, we define $W_{i,j} : K \to I_i \times J_j \times \mathbb{R}^2$ by

$$W_{i,j}(x, y, z, t) = \big(u_i(x), v_j(y), F_{i,j}(x, y, z, t)\big).$$

Then $\{K, W_{i,j} : (i, j) \in \Sigma_N \times \Sigma_M\}$ is an IFS. The following theorem can be proved using the arguments which are similar to those used in Theorem 3.1 in [11], and hence it is omitted.



Theorem 1 *Let $\{K, W_{i,j} : (i, j) \in \Sigma_N \times \Sigma_M\}$ be the IFS defined as above. Assume that the map $F_{i,j}, (i, j) \in \Sigma_N \times \Sigma_M$ satisfies the following matching conditions:*

- *for all $i \in int\ \Sigma_{N,0},\ j \in \Sigma_M$ and $\tilde{x} = u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i)$,*

$$F_{i,j}(\tilde{x}, y, z, t) = F_{i+1,j}(\tilde{x}, y, z, t)\ \forall\ y \in J, z, t \in \mathbb{R},$$

- *for all $j \in int\ \Sigma_{M,0}$, $i \in \Sigma_N$ and $\tilde{y} = v_j^{-1}(y_j) = v_{j+1}^{-1}(y_j)$,*

$$F_{i,j}(x, \tilde{y}, z, t) = F_{i,j+1}(x, \tilde{y}, z, t)\ \forall\ x \in I, z, t \in \mathbb{R}.$$

Then, there exists a unique continuous function $\mathbf{f}^* : \mathcal{D} \to \mathbb{R}^2$ such that $\mathbf{f}^*(x_i, y_j) = (z_{i,j}, t_{i,j})$ for all $(i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}$ and $G = \underset{(i,j)\in\Sigma_N\times\Sigma_M}{\cup} W_{i,j}(G)$, where $G = \big\{(x, y, \mathbf{f}^*(x, y)) : (x, y) \in \mathcal{D}\big\}$ is the graph of $\mathbf{f}^*$.

Definition 1 The continuous function $\mathbf{f}^*$ defined in the previous theorem is called bivariate FIF corresponding to the data set Δ_1.

Now, we derive a functional equation for the bivariate FIF $\mathbf{f}^*$.
Let $\mathcal{G} = \big\{\mathbf{h} = (h_1, h_2) \in \mathcal{C}_2(\mathcal{D}) \mid \mathbf{h}(x_i, y_j) = (z_{i,j}, t_{i,j}), (i, j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}\big\}$. Then $\mathcal{G}$ is a complete metric space with respect to the metric ρ which is induced by supremum norm, i.e.,

$$\rho(\mathbf{g}, \mathbf{h}) = \|g_1 - h_1\|_\infty + \|g_2 - h_2\|_\infty,\ \forall\ \mathbf{g} = (g_1, g_2), \mathbf{h} = (h_1, h_2) \in \mathcal{G}.$$

Define the Read-Bajraktarević operator T on $(\mathcal{G}, \rho)$ as

$$T\mathbf{h}(x, y) = F_{i,j}\big(u_i^{-1}(x), v_j^{-1}(y), \mathbf{h}(u_i^{-1}(x), v_j^{-1}(y))\big),\ (x, y) \in \mathcal{D}_{i,j},\ (i, j) \in \Sigma_N \times \Sigma_M. \tag{5}$$

The bivariate FIF $\mathbf{f}^*$ in the previous theorem is the unique fixed point of the operator T and it satisfies

$$\mathbf{f}^*(x, y) = F_{i,j}\big(u_i^{-1}(x), v_j^{-1}(y), \mathbf{f}^*(u_i^{-1}(x), v_j^{-1}(y))\big),\ (x, y) \in \mathcal{D}_{i,j},\ (i, j) \in \Sigma_N \times \Sigma_M. \tag{6}$$

Let $\mathbf{L} = (L1, L2) : \mathcal{C}_2(\mathcal{D}) \to \mathbb{R}^2$ be a bounded linear map satisfying

$$L_1 f \neq f, L_2 g \neq g, \big(\mathbf{L}(f, g)\big)(x_i, y_j) = (f, g)(x_i, y_j)\ \forall\ (i, j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}. \tag{7}$$

Consider the contractive homeomorphisms u_i and v_j satisfying (3)-(4) respectively as

$$u_i(x) = a_i x + b_i, v_j(y) = c_j y + d_j,$$

where

$$\begin{aligned}
a_i &= \frac{x_{i-\epsilon_i} - x_{i-1+\epsilon_i}}{x_N - x_0}, b_i = \frac{x_N x_{i-1+\epsilon_i} - x_0 x_{i-\epsilon_i}}{x_N - x_0}, c_j = \frac{y_{j-\epsilon_j^*} - y_{j-1+\epsilon_j^*}}{y_M - y_0}, d_j = \frac{y_M y_{j-1+\epsilon_j^*} - y_0 y_{j-\epsilon_j^*}}{y_M - y_0},\\
\epsilon_i &= \begin{cases} 0 & \text{if } i \text{ is odd,} \\ 1 & \text{if } i \text{ is even,} \end{cases}\\
\epsilon_j^* &= \begin{cases} 0 & \text{if } j \text{ is odd,} \\ 1 & \text{if } j \text{ is even.} \end{cases}
\end{aligned} \tag{8}$$

Define $F_{i,j} : K \to \mathbb{R}^2$ as

$$\begin{aligned}
F_{i,j}(x, y, z, t) = \Big(&\alpha_{i,j} z + \beta_{i,j} t + f(u_i(x), v_j(y)) - \alpha_{i,j}(L_1 f)(x, y) - \beta_{i,j}(L_2 g)(x, y),\\
&\gamma_{i,j} t + g(u_i(x), v_j(y)) - \gamma_{i,j}(L_2 g)(x, y)\Big),
\end{aligned} \tag{9}$$

where $\alpha_{i,j}, \beta_{i,j}$, and $\gamma_{i,j}$ are real numbers such that $|\alpha_{i,j}| < 1$ and $|\beta_{i,j}| + |\gamma_{i,j}| < 1$. The numbers $\alpha_{i,j}, \beta_{i,j}$, and $\gamma_{i,j}$ are called scaling factors. We define the IFS $\{K, W_{i,j} : (i, j) \in \Sigma_N \times \Sigma_M\}$, where $W_{i,j}(x, y, z, t) = \big(u_i(x), v_j(y), F_{i,j}(x, y, z, t)\big)$, and the above IFS is called a hidden variable IFS. One can verify that $F_{i,j}$ defined in (9) satisfy the matching conditions given in the previous theorem. As a result, we have the following theorem.

Theorem 2 *Let* $\{K, W_{i,j} : (i,j) \in \Sigma_N \times \Sigma_M\}$ *be the IFS. Then, there exists a unique continuous function* $\boldsymbol{\phi}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} = (f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2}) : \mathcal{D} \to \mathbb{R}^2$ *such that* $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(x_i, y_j) = f(x_i, y_j)$ *and* $g^{\gamma}_{\Delta,L_2}(x_i, y_j) = g(x_i, y_j)$ *for all* $(i,j) \in \Sigma_{N,0} \times \Sigma_{M,0}$ *and* $G = \underset{(i,j)\in\Sigma_N\times\Sigma_M}{\cup} W_{i,j}(G)$, *where* $G = \big\{(x, y, \boldsymbol{\phi}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}) : (x, y) \in \mathcal{D}\big\}$ *is the graph of* $\boldsymbol{\phi}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$.

Owing to (5), the aforementioned bivariate FIFs $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ and g^{γ}_{Δ,L_2} respectively satisfy the following functional equations: for all $(x, y) \in \mathcal{D}$, $(i,j) \in \Sigma_N \times \Sigma_M$,

$$\begin{aligned} f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(u_i(x), v_j(y)) &= \alpha_{i,j} f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(x, y) + \beta_{i,j} g^{\gamma}_{\Delta,L_2}(x, y) + f(u_i(x), v_j(y)) \\ &\quad -\alpha_{i,j}(L_1 f)(x, y) - \beta_{i,j}(L_2 g)(x, y), \end{aligned} \tag{10}$$

$$g^{\gamma}_{\Delta,L_2}(u_i(x), v_j(y)) = \gamma_{i,j} g^{\gamma}_{\Delta,L_2}(x, y) + g(u_i(x), v_j(y)) - \gamma_{i,j}(L_2 g)(x, y). \tag{11}$$

From (10), one can verify that the fractal function $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ interpolates f at the points (x_i, y_j), $(i,j) \in \Sigma_{N,0} \times \Sigma_{M,0}$. Further, the graph $Graph(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$ of $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ is projection of the graph $Graph(\boldsymbol{\phi}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$ of $\boldsymbol{\phi}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$, and $Graph(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$ is not always the union of transformed copies of itself. Hence, $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ is non-self-referential and we call the function $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ as non-self-referential bivariate fractal interpolation function corresponding to the function f.
From (11), we can verify that the fractal function g_γ interpolates g at the points (x_i, y_j), $(i,j) \in \Sigma_{N,0} \times \Sigma_{M,0}$. It is plain to see that the graph $Graph(g^{\gamma}_{\Delta,L_2})$ of g^{γ}_{Δ,L_2} satisfies

$$Graph(g^{\gamma}_{\Delta,L_2}) = \underset{(i,j)\in\Sigma_N\times\Sigma_M}{\cup} W^2_{i,j}\big(Graph(g^{\gamma}_{\Delta,L_2})\big),$$

where

$$W^2_{i,j}(x, y, t) = \big(u_i(x), v_j(y), F^2_{i,j}(x, y, t)\big),\ F^2_{i,j}(x, y, t) = \gamma_{i,j} t + g(u_i(x), v_j(y)) - \gamma_{i,j}(L_2 g)(x, y).$$

Hence, g^{γ}_{Δ,L_2} is self-referential and we call the function g^{γ}_{Δ,L_2} as self-referential fractal function corresponding to the function g. If the elements of the hidden variable IFS are chosen such that $z_{i,j} = t_{i,j}$ and $\alpha_{i,j} + \beta_{i,j} = \gamma_{i,j}$, for all $(i,j) \in \Sigma_N \times \Sigma_M$, and $f = g$, then we get that $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} = g^{\gamma}_{\Delta,L_2}$, and hence in this case one obtains a self-referential FIF. Similarly, if $\beta_{i,j} = 0$ for all $(i,j) \in \Sigma_N \times \Sigma_M$, then

$$Graph(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}) = \underset{(i,j)\in\Sigma_N\times\Sigma_M}{\cup} W^1_{i,j}\big(Graph(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})\big),$$

where

$$W^1_{i,j}(x, y, z) = \big(u_i(x), v_j(y), F^1_{i,j}(x, y, z)\big),\ F^1_{i,j}(x, y, z) = \alpha_{i,j} z + f(u_i(x), v_j(y)) - \alpha_{i,j}(L_1 f)(x, y).$$

It is worthwhile to mention that $f_{\alpha,\beta,\gamma}$ is self-referential in this case as well. Let us introduce the following notation: $\alpha = [\alpha_{i,j}]_{N\times M}$, $\beta = [\beta_{i,j}]_{N\times M}$, $\gamma = [\gamma_{i,j}]_{N\times M}$, $|\alpha|_\infty = \max\{|\alpha_{i,j}| : (i,j) \in \Sigma_N \times \Sigma_M\}$, $|\beta|_\infty = \max\{|\beta_{i,j}| : (i,j) \in \Sigma_N \times \Sigma_M\}$, and $|\gamma|_\infty = \max\{|\gamma_{i,j}| : (i,j) \in \Sigma_N \times \Sigma_M\}$.
From (11), we get that

$$g^{\gamma}_{\Delta,L_2}(u_i(x), v_j(x)) - g(u_i(x), v_j(x)) = \gamma_{i,j}\big(g^{\gamma}_{\Delta,L_2}(x, y) - g(x, y) + g(x, y) - (L_2 g)(x, y)\big)\ \forall\ (x, y) \in \mathcal{D}.$$

Now, it follows that

$$\begin{aligned} |g^{\gamma}_{\Delta,L_2}(u_i(x), v_j(x)) - g(u_i(x), v_j(x))| &\le |\gamma_{i,j}|\big(|g^{\gamma}_{\Delta,L_2})(x, y) - g(x, y)| + |g(x, y) - L_2 g(x, y)|\big) \\ &\le |\gamma|_\infty\big(\|g^{\gamma}_{\Delta,L_2} - g\|_\infty + \|g - L_2 g\|_\infty\big). \end{aligned} \tag{12}$$

The above inequality yields

$$\|g^{\gamma}_{\Delta,L_2} - g\|_\infty \le \frac{|\gamma|_\infty}{1 - |\gamma|_\infty}\|g - L_2 g\|_\infty. \tag{13}$$

From (10), we obtain that for all $(x, y) \in \mathcal{D}$,

$$f_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma}(u_i(x), v_j(x)) - f(u_i(x), v_j(x)) = \alpha_{i,j}(f_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma}(x, y) - (L_1 f)(x, y)) + \beta_{i,j}(g_{\Delta,L_2}^{\gamma}(x, y) - L_2 g(x, y)).$$

This implies that

$$\begin{aligned}|f_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma}(u_i(x), v_j(x)) - f(u_i(x), v_j(x))| &\leq |\alpha_{i,j}|\big(|f_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma}(x, y) - f(x, y)| + |f(x, y) - (L_1 f)(x, y)|\big)\\ &\quad + |\beta_{i,j}|\big(|g_{\Delta,L_2}^{\gamma}(x, y) - g(x, y)| + |g(x, y) - (L_2 g)(x, y)|\big),\\ &\leq |\alpha|_\infty\big(\|f_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma} - f\|_\infty + \|f - L_1 g\|_\infty\big)\\ &\quad + |\beta|_\infty\big(\|g_{\Delta,L_2}^{\gamma} - g\|_\infty + \|g - L_2 g\|_\infty\big).\end{aligned}$$

Using (13) in the above inequality, we get:

$$\|f_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma} - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|f - L_1 f\|_\infty + \frac{|\beta|_\infty}{(1 - |\alpha|_\infty)(1 - |\gamma|_\infty)}\|g - L_2 g\|_\infty. \tag{14}$$

3 Fractal linear operator on $\mathbf{Lip_2(\mathcal{D})}$

Definition 2 For a fixed partition Δ of $\mathcal{D}$, scaling matrices α, β, and γ, and a linear operator $\mathbf{L}$, we define the fractal operator

$$\mathcal{F}_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma} : \mathrm{Lip}_2(\mathcal{D}) \to \mathcal{C}_2(\mathcal{D}),\ \mathcal{F}_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma}(f, g) = (f_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma}, g_{\Delta,L_2}^{\gamma}).$$

To discuss some basic and important properties of the fractal operator $\mathcal{F}_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma}$, we need the following results and definitions [22].

Lemma 3.1 *[22] If U is a bounded linear operator from a Banach space into itself such that $\|U\| < 1$, then $(Id - U)^{-1}$ exists and it is bounded.*

Lemma 3.2 *[22] If U is a bounded linear operator and S is a compact operator on a normed linear space, then US and SU both are compact operators.*

Following [22], we use the product notation for the value of a linear functional on an element: $\langle x, \phi\rangle = \langle \phi, x\rangle = \phi(x)$ for $x \in X$ and $\phi \in X^*$ the dual of X. Let $S : X \to Y$ be a bounded linear operator. The adjoint or dual S^* of S is the unique map $S^* : Y^* \to X^*$ such that

$$\langle x, S^*\phi\rangle = \langle Sx, \phi\rangle = \phi(x)\ \forall\ x \in X, \phi \in Y^*.$$

Lemma 3.3 *[22] Let X and Y be normed linear spaces. Let $S : X \to Y$ be a bounded linear operator. If S invertible, then S^* is also invertible.*

Definition 3 [22] An operator U is Fredholm if: (i) Range(U) is closed, and (ii) $Ker(U^*)$ is finite-dimensional. Further, the index of a Fredholm operator is defined as: index (U) =dim$\big(Ker(U)\big) - Ker\big(U^*\big)$, where U^* is adjoint of U.

Now, we discuss our results in the following:

Theorem 3 *Let $\mathcal{C}_2(\mathcal{D})$ be endowed with the uniform norm $\|.\|_\infty$. Let Id be the identity operator on $\mathcal{C}_2(\mathcal{D})$. Then,*

1. *The fractal operator $\mathcal{F}_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma}$ is a bounded linear operator with respect to the uniform norm on $\mathcal{C}_2(\mathcal{D})$. Further, the the operator norm satisfies*

$$\|\mathcal{F}_{\Delta,\mathbf{L}}^{\alpha,\beta,\gamma}\| \leq 2 + \frac{(|\alpha|_\infty + |\beta|_\infty + |\gamma|_\infty)}{(1 - |\alpha|_\infty)(1 - |\gamma|_\infty)}\max\{\|I_d - L_1\|, \|I_d - L_2\|\}.$$

2. *If* $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < \|\mathbf{L}\|^{-1}$, *then the fractal operator* $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} : Lip_2(\mathcal{D}) \subset \mathcal{C}_2(\mathcal{D}) \to \mathcal{C}_2(\mathcal{D})$ *is not closed.*
3. *Let* $Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$ *be the range space of* $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$. *If* $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < \|\mathbf{L}\|^{-1}$, *then the fractal operator* $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ *is bounded below. Further, the inverse operator* $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1} : Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}) \subset \mathcal{C}_2(\mathcal{D}) \to Lip_2(\mathcal{D}) \subset \mathcal{C}_2(\mathcal{D})$ *is a bounded and closed operator.*
4. *Null space* $N((\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1})$ *of* $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}$ *is a closed subspace of* $Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$.
5. *For* $(f, g) \in Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$, *define* $\|(f, g)\|_{(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}} = \|(f, g)\|_\infty + \|(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}(f, g)\|_\infty$. *Then,* $(f, g) \mapsto \|(f, g)\|_{(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}}$ *is a complete norm on* $Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$.
6. *If* $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} \neq 0$, *then the fixed points of* $\mathbf{L}$ *are the fixed points of the fractal operator* $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$.

Proof 1. Let $(\rho_1, \rho_2), (\psi_1, \psi_2) \in \mathrm{Lip}_2(\mathcal{D})$, and $k_1, k_2 \in \mathbb{R}$. For $(x, y) \in \mathcal{D}_{i,j}$, we have

$$\begin{aligned}\Big(k_1\rho_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(x, y),\ k_1\rho_2{}^{\gamma}_{\Delta,L_2}(x, y)\Big) = \Big(&k_1\rho_1(x, y) + k_1\alpha_{i,j}\Big(\rho_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - (L_1\rho_1)\Big)(u_i^{-1}(x), v_j^{-1}(y))\\ &+ k_1\beta_{i,j}\Big(\rho_2{}^{\gamma}_{\Delta,L_2} - (L_2\rho_2)\Big)(u_i^{-1}(x), v_j^{-1}(y)),\\ &k_1\rho_2(x, y) + k_1\gamma_{i,j}\Big(\rho_2{}^{\gamma}_{\Delta,L_2} - (L_2\rho_2)\Big)(u_i^{-1}(x), v_j^{-1}(y))\Big),\end{aligned}$$

$$\begin{aligned}\Big(k_2\psi_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(x, y),\ k_2\psi_2{}^{\gamma}_{\Delta,L_2}(x, y)\Big) = \Big(&k_1\psi_1(x, y) + k_2\alpha_{i,j}\Big(\psi_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - (L_1\psi_1)\Big)(u_i^{-1}(x), v_j^{-1}(y))\\ &+ k_2\beta_{i,j}\Big(\psi_2{}^{\gamma}_{\Delta,L_2} - (L_2\psi_2)\Big)(u_i^{-1}(x), v_j^{-1}(y)),\\ &k_2\psi_2(x, y) + k_2\gamma_{i,j}\Big(\psi_2{}^{\gamma}_{\Delta,L_2} - (L_2\psi_2)\Big)(u_i^{-1}(x), v_j^{-1}(y))\Big)\end{aligned}$$

Addition of the above two equations and linearity of the operator $\mathbf{L}$ yield

$$\begin{aligned}&\Big(\Big(k_1\rho_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} + k_2\psi_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}\Big)(x, y),\ \Big(k_1\rho_2{}^{\gamma}_{\Delta,L_2} + k_2\psi_2{}^{\gamma}_{\Delta,L_2}\Big)(x, y)\Big) =\\ &\Big((k_1\rho_1 + k_2\psi_1)(x, y) + \alpha_{i,j}\Big(k_1\rho_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} + k_2\psi_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})(x, y) - L_1(k_1\rho_1 + k_2\psi_1)\Big)(u_i^{-1}(x), v_j^{-1}(y))\\ &+ \beta_{i,j}\Big(k_1\rho_2{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} + k_2\psi_2{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})(x, y) - L_2(k_1\rho_2 + k_2\psi_2)\Big)(u_i^{-1}(x), v_j^{-1}(y)),\\ &\gamma_{i,j}\Big(k_1\rho_2{}^{\gamma}_{\Delta,L_2} + k_2\psi_2{}^{\gamma}_{\Delta,L_2} - L_2(k_1\rho_2 + k_2\psi_2)\Big)(u_i^{-1}(x), v_j^{-1}(y))\Big).\end{aligned}$$

The previous equation implies that $\Big(k_1\rho_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} + k_2\psi_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}},\ k_1\rho_2{}^{\gamma}_{\Delta,L_2} + k_2\psi_2{}^{\gamma}_{\Delta,L_2}\Big)$ is fixed point of the Read-Bajraktarević operator T (cf. (5)). Next, from the construction of the hidden variable bivariate FIFs, we can notice that $\Big(\Big(k_1\rho_1 + k_2\psi_1\Big)^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}},\ \Big(k_1\rho_2 + k_2\psi_2\Big)^{\gamma}_{\Delta,L_2}\Big)$ is the fixed point of of the Read-Bajraktarević operator T (cf. (5)). Since the fixed point of the Read-Bajraktarević operator T is unique, it follows that:

$$\Big(k_1\rho_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} + k_2\psi_1{}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}},\ k_1\rho_2{}^{\gamma}_{\Delta,L_2} + k_2\psi_2{}^{\gamma}_{\Delta,L_2}\Big) = \Big(\Big(k_1\rho_1 + k_2\psi_1\Big)^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}},\ \Big(k_1\rho_2 + k_2\psi_2\Big)^{\gamma}_{\Delta,L_2}\Big),$$

which implies the linearity of the fractal operator $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$. Next, one can rewrite (13)-(14) as follows:

$$\|g^{\gamma}_{\Delta,L_2}\|_\infty \leq \|g\|_\infty + \frac{|\gamma|_\infty}{1 - |\gamma|_\infty}\|g - L_2 g\|_\infty. \tag{15}$$

$$\|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}\|_\infty \leq \|f\|_\infty + \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|f - L_1 f\|_\infty + \frac{|\beta|_\infty}{(1 - |\alpha|_\infty)(1 - |\gamma|_\infty)}\|g - L_2 g\|_\infty. \tag{16}$$

By definition of the fractal operator $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$, we get

$$\|\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(f,g)\|_\infty = \|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}\|_\infty + \|g^{\gamma}_{\Delta,L_2}\|_\infty.$$

Substituting (15)-(16) in the previous equation, we obtain:

$$\begin{aligned}\|\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(f,g)\|_\infty &\le \Big(1+\frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|I_d - L_1\|\Big)\|f\|_\infty \\ &\quad + \Big(1+\frac{|\beta|_\infty+|\gamma|_\infty}{(1-|\alpha|_\infty)(1-|\gamma|_\infty)}\|I_d - L_2\|_\infty\Big)\|g\|_\infty \\ &\le \Big(2+\frac{(|\alpha|_\infty+|\beta|_\infty+|\gamma|_\infty)}{(1-|\alpha|_\infty)(1-|\gamma|_\infty)}\max\{\|I_d-L_1,\|I_d-L_2\|\}\Big)\|(f,g)\|_\infty.\end{aligned}$$

Therefore, $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ is a bounded linear operator.

2. From the functional analysis, it is known that: if X_0 is a subspace of the normed linear space X; Y is a Banach space, and $S : X_0 \subset X \to Y$ is a closed and linear bounded operator, then X_0 is a closed subspace of X. Since $\mathrm{Lip}_2(\mathcal{D})$ is not a closed subspace of the Banach space $\mathcal{C}_2(\mathcal{D})$ and $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ is a bounded linear operator, using the above result, it follows that $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ is not closed.
3. From (10)-(11), we get:

$$\|g^{\gamma}_{\Delta,L_2} - g\|_\infty \le |\gamma|_\infty \|g^{\gamma}_{\Delta,L_2} - (L_2 g)\|_\infty, \tag{17}$$

$$\|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - f\|_\infty \le |\alpha|_\infty \|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - (L_1 f)\|_\infty + |\beta|_\infty \|g^{\gamma}_{\Delta,L_2} - (L_2 g)\|_\infty. \tag{18}$$

Using (17)-(18), one obtains:

$$\begin{aligned}\|(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2}) - (f,g)\|_\infty &= \|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - f\|_\infty + \|g^{\gamma}_{\Delta,L_2} - g\|_\infty \\ &\le |\alpha|_\infty \|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - (L_1 f)\|_\infty + \big[|\beta|_\infty + |\gamma|_\infty\big]\|g^{\gamma}_{\Delta,L_2} - (L_2 g)\|_\infty.\end{aligned} \tag{19}$$

Using the previous inequality, we get

$$\begin{aligned}\|(f,g)\|_\infty - \|(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2}\|_\infty &\le \|(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2}) - (f,g)\|_\infty \\ &\le \max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\}\Big\{\|(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2})\|_\infty + \|(L_1 f, L_2 g)\|_\infty\Big\} \\ &\le \max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\}\Big\{\|(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2})\|_\infty + \|\mathbf{L}\|\|(f,g)\|_\infty\Big\}.\end{aligned}$$

Now, from the aforementioned inequality and definition of $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$, it follows that

$$\|\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(f,g)\|_\infty \ge \frac{1-\max\{|\alpha|_\infty, |\beta|_\infty+|\gamma|_\infty\}\|\mathbf{L}\|}{1+\max\{|\alpha|_\infty, |\beta|_\infty+|\gamma|_\infty\}}\|(f,g)\|_\infty,$$

which ensure that $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1} : Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}) \subset \mathcal{C}_2(\mathcal{D}) \to \mathcal{C}_2(\mathcal{D})$ is a bounded operator if $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < \frac{1}{\|\mathbf{L}\|}$. Now using the result mentioned in item 2, in order to prove that $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}$ is a closed operator, it is enough to prove that $Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$ is a closed subspace of $\mathcal{C}_2(\mathcal{D})$. For this, let $\{\boldsymbol{\sigma}_n\}$ be a sequence in $Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\boldsymbol{L}})$ such that $\boldsymbol{\sigma}_n \to \boldsymbol{\sigma} \in \mathcal{C}_2(\mathcal{D})$. Suppose that $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}(\boldsymbol{\sigma}_n) \to \boldsymbol{\omega} \in \mathcal{C}_2(\mathcal{D})$. Denoting $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}(\boldsymbol{\sigma}_n) = \boldsymbol{\omega}_n$, we obtain

$$\boldsymbol{\omega}_n \to \boldsymbol{\omega} \text{ and } \mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(\boldsymbol{\omega}_n) \to \boldsymbol{\sigma}.$$

Now by boundedness of $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$, we obtain $\boldsymbol{\omega}_n \to \boldsymbol{\omega}$ implies $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(\boldsymbol{\omega}_n) \to \mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(\boldsymbol{\omega})$. Finally, by uniqueness of limit, it follows that $\boldsymbol{\sigma} = \mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(\boldsymbol{\omega}) \in Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$. This completes the proof.

4. To prove that $N((\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1})$ is a closed subspace of $Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$, let $\{\mathbf{s}_n\}$ be a sequence in $N((\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1})$ such that $\mathbf{s}_n \to \mathbf{s} \in Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$. We show that $\mathbf{s} \in N((\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1})$. Since $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}(\mathbf{s}_n) = \mathbf{0}$ for all n, by definition of closedness of $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}$, it follows that $\mathbf{s} \in Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$ and $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}(\mathbf{s}) = \mathbf{0}$.
5. Notice that $\|(f,g)\|_{(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}} = \|(f,g)\|_\infty + \|(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}(f,g)\|_\infty$ is the graph norm on $Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$. To see that it is complete, let $\{\boldsymbol{\pi}_n\}$ be a Cauchy sequence in $Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$ with respect to $\|.\|_{(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}}$. Then it follows that $\{\boldsymbol{\pi}_n\}$ and $\{(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}(\boldsymbol{\pi}_n)\}$ are Cauchy sequences in $\mathcal{C}_2(\mathcal{D})$. Since $\mathcal{C}_2(\mathcal{D})$ is a Banach space, there exist $\boldsymbol{\pi}, \boldsymbol{\varphi} \in \mathcal{C}_2(\mathcal{D})$ such that

$$\|\boldsymbol{\pi}_n - \boldsymbol{\pi}\|_\infty \to 0 \ \text{ and } \ \|(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}(\boldsymbol{\pi}_n) - \boldsymbol{\varphi}\|_\infty \to 0 \ \text{ as } \ n \to \infty.$$

Now using the fact that $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}$ is closed operator, we have $\boldsymbol{\pi} \in Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$ and $(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}(\boldsymbol{\pi}) = \boldsymbol{\varphi}$. Thus,

$$\|\boldsymbol{\pi}_n - \boldsymbol{\pi}\|_\infty + \|(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}(\boldsymbol{\pi}_n - \boldsymbol{\pi})\|_\infty \to 0 \ \text{ as } \ n \to \infty,$$

showing that the sequence $\{\boldsymbol{\pi}_n\}$ converges in $Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$ with respect to $\|.\|_{(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})^{-1}}$.
6. Let (f,g) be a fixed point of $\mathbf{L}$. Using (19) and the definition of $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$, we get

$$\|(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(f,g) - (f,g)\|_\infty \le \max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\}\Big\{\|(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(f,g) - (f,g)\|_\infty\Big\}.$$

Since $0 < \max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < 1$, the above inequality implies that $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(f,g) = (f,g)$.

□

Theorem 4 *[22] Let $S_0 : X_0 \subset X \to Y$ be a bounded linear operator, where X_0 is dense in X, and Y is a Banach space. Then, there exists a unique bounded linear operator $S : X \to Y$ such that S is an extension of S_0. Further, $\|S_0\| = \|S\|$, and for $x \in X$, $Sx = \lim_{n\to\infty} S_0 x_n$, where $\{x_n\}$ is a sequence in X_0 such that $x_n \to x$.*

We first prove the denseness of $\mathrm{Lip}_2(\mathcal{D})$ in $\mathcal{C}_2(\mathcal{D})$ to apply the previous theorem in the present study. Let $(f,g) \in \mathcal{C}_2(\mathcal{D})$ and $\epsilon > 0$. Since $f, g \in \mathrm{Lip}(\mathcal{D})$ are continuous, by the Weierstrass theorem, there exist polynomials $p_i, i = 1, 2$, such that $\|f - p_1\|_\infty < \frac{\epsilon}{2}$ and $\|g - p_2\|_\infty < \frac{\epsilon}{2}$. Therefore, we get

$$\|(f,g) - (p_1,p_2)\|_\infty = \|f - p_1\|_\infty + \|g - p_2\|_\infty < \epsilon,$$

establishing the denseness of $\mathrm{Lip}_2(\mathcal{D})$ in $\mathcal{C}_2(\mathcal{D})$.
Let $(f,g) \in \mathcal{C}_2(\mathcal{D})$. Let $\{(f_n,g_n)\}$ be a sequence in $\mathrm{Lip}_2(\mathcal{D})$ such that $(f_n,g_n) \to (f,g)$ as $n \to \infty$. Let $\tilde{\mathbf{L}}$ be the norm preserving extension of $\mathbf{L}$ to $\mathcal{C}_2(\mathcal{D})$. We define $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} : \mathcal{C}_2(\mathcal{D}) \to \mathcal{C}_2(\mathcal{D})$ as

$$(f^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}, g^{\gamma}_{\Delta,\tilde{L}_2}) = \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(f,g) = \lim_{n\to\infty} \mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(f_n,g_n).$$

Theorem 5 *Let $\tilde{\mathbf{L}}$ be the norm preserving extension of $\mathbf{L}$ to $\mathcal{C}_2(\mathcal{D})$. For the operator $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}} : \mathcal{C}_2(\mathcal{D}) \to \mathcal{C}_2(\mathcal{D})$, the following holds:*

1. *For $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < (1 + \|Id - \mathbf{L}\|)^{-1}$, then the fractal operator $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is a topological automorphism on $\mathcal{C}_2(\mathcal{D})$.*
2. *If $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty|\} < \|\mathbf{L}\|^{-1}$, then the fractal operator $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is not a compact operator.*
3. *If $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < (1 + \|Id - \mathbf{L}\|)^{-1}$, then the fractal operator $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is Fredholm and its index is 0.*

Proof 1. Since $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < \|\mathbf{L}\|^{-1}$, using the boundedness and linearity of $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$, one can verify that $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is linear and bounded. Let $(f, g) \in \mathcal{C}_2(\mathcal{D})$. Let $\{(f_n, g_n)\}$ be a sequence in $\mathrm{Lip}_2(\mathcal{D})$ such that $(f_n, g_n) \to (f, g)$ as $n \to \infty$. Then,

$$\begin{aligned}
\|\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(f, g) - (f, g)\|_\infty &= \lim_{n\to\infty} \|\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(f_n, g_n) - (f_n, g_n)\|_\infty, \\
&\leq \max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} \lim_{n\to\infty} \|\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(f_n, g_n) - \mathbf{L}(f_n, g_n)\|_\infty, \\
&= \max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} \lim_{n\to\infty} \|\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(f_n, g_n) - \tilde{\mathbf{L}}(f_n, g_n)\|_\infty, \\
&= \max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} \|\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(f, g) - \tilde{\mathbf{L}}(f, g)\|_\infty, \\
&\leq \max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\}. \\
&\Big(\|\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(f, g) - (f, g)\|_\infty + \|(f, g) - \tilde{\mathbf{L}}(f, g)\|_\infty\Big).
\end{aligned}$$

Therefore,

$$\|\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(f, g) - (f, g)\|_\infty \leq \frac{\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\}}{1 - \max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\}} \|(f, g) - \tilde{\mathbf{L}}(f, g)\|_\infty.$$

The previous inequality leads to

$$\|I_d - \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}\|_\infty \leq \frac{\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\}}{1 - \max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\}} \|I_d - \tilde{\mathbf{L}}\|,$$

where $I_d - \tilde{\mathbf{L}}$ is the norm preserving extension of $I_d - \mathbf{L}$ to $\mathcal{C}_2(\mathcal{D})$. Now, the hypothesis $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < (1 + \|Id - \mathbf{L}\|)^{-1}$ dictates via the previous inequality that $\|I_d - \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}\|_\infty < 1$. Now using the Lemma 3.1, it follows that the operator $I_d - (I_d - \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}) = \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ has bounded inverse.

2. Using the arguments similar to those used for showing $\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ is bounded below, one can show that $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is bounded below if $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty|\} < \|\mathbf{L}\|^{-1}$. Since $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is bounded below, it follows that $(\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}})^{-1} : \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(\mathcal{C}_2(\mathcal{D})) \to \mathcal{C}_2(\mathcal{D})$ is continuous. Without loss of generality assume that $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is compact. Using the Lemma 3.2, we can see that the identity operator $I_d = \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}})^{-1} : \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(\mathcal{C}_2(\mathcal{D})) \to \mathcal{C}_2(\mathcal{D})$ is compact. This implies that dimension of $\mathcal{C}_2(\mathcal{D})$ is finite and it is a contraction. Hence, the result follows.
3. Using the arguments similar to those used for showing $Rg(\mathcal{F}^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}})$ is closed (see item 3 of Theorem 3), one can verify that $Rg(\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}})$ is a closed subspace of $\mathcal{C}_2(\mathcal{D})$. In the previous item, it is notice that $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is invertible. Using the Lemma 3.3, it follows that the adjoint operator $\tilde{\mathcal{F}}^{*\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ of $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is invertible. According the definition 3 and the above discussion, it follows that $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is Fredholm, and

$$\mathrm{index}\Big(\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}\Big) = \dim\Big(\mathrm{Ker}(\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}})\Big) - \dim\Big(\mathrm{Ker}(\tilde{\mathcal{F}}^{*\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}})\Big).$$

Hence, $\mathrm{index}(\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}})$ is zero.

□

Definition 4 Let $p_1, p_2 \in \mathcal{C}(\mathcal{D})$ be bivariate polynomials. Then, $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(p_1, p_2) = (\tilde{p}^{\alpha,\beta,\gamma}_{1,\Delta,\tilde{L}}, \tilde{p}^{\gamma}_{2,\Delta,\tilde{L}_2})$. The hidden variable fractal function $\tilde{p}^{\alpha,\beta,\gamma}_{1,\Delta,\tilde{L}}$ and fractal function $\tilde{p}^{\gamma}_{2,\Delta,\tilde{L}_2}$ are respectively referred to as bivariate fractal polynomials of p_1 and p_2.

Let $\mathcal{P}_{m,n}(\mathcal{D})$ be the set of all bivariate polynomials of totall degree at most $m + n$ defined on $\mathcal{D}$. Let $\mathcal{P}_{2,m,n}(\mathcal{D}) = \{(p, q) : p, q \in \mathcal{P}_{m,n}(\mathcal{D})\}$ and $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(\mathcal{P}_{2,m,n}(\mathcal{D})) = \mathcal{P}^{\alpha,\beta,\gamma}_{2,m,n,\Delta,\tilde{L}}(\mathcal{D})$. For simplicity, we denote $\mathcal{P}^{\alpha,\beta,\gamma}_{2,m,n,\Delta,\tilde{L}}(\mathcal{D})$ by $\mathcal{P}^{\alpha,\beta,\gamma}_{2,m,n}(\mathcal{D})$.

Theorem 6 *If* $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < 1$, *then the space* $\mathcal{P}^{\alpha,\beta,\gamma}_{2,m,n}(\mathcal{D})$ *is finite dimensional. Further, if* $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < \|\mathbf{L}\|^{-1}$, *then dimension of* $\mathcal{P}^{\alpha,\beta,\gamma}_{2,m,n}(\mathcal{D})$ *is equal to the dimension of* $\mathcal{P}_{2,m,n}(\mathcal{D})$.

Proof Let the dimension of $\mathcal{P}_{2,m,n}(\mathcal{D})$ be λ. Let $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_\lambda\}$ be a basis for $\mathcal{P}_{2,m,n}(\mathcal{D})$. Since $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ is a linear operator, it follows that $\mathcal{P}^{\alpha,\beta,\gamma}_{2,m,n}(\mathcal{D})$ is spanned by $\{\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(\mathbf{p}_1), \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(\mathbf{p}_2), \cdots, \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(\mathbf{p}_\lambda)\}$, and it implies that dimension of $\mathcal{P}^{\alpha,\beta,\gamma}_{2,m,n}(\mathcal{D})$ is finite. Further, the condition $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < \|\mathbf{L}\|^{-1}$ makes the operator $\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}$ injective. It dictates that $\{\tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(\mathbf{p}_1), \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(\mathbf{p}_2), \cdots, \tilde{\mathcal{F}}^{\alpha,\beta,\gamma}_{\Delta,\tilde{\mathbf{L}}}(\mathbf{p}_\lambda)\}$ is a basis for $\mathcal{P}^{\alpha,\beta,\gamma}_{2,m,n}(\mathcal{D})$ if $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\} < \|\mathbf{L}\|^{-1}$. This completes the proof. □

4 Sensitivity and optimal choice

In this section, we discuss sensitivity of the hidden variable fractal function $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ with respect to a perturbation in the scaling factors and the existence of a optimal hidden variable fractal function $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$.

4.1 Sensitivity of hidden variable fractal function

For a given $(f, g) \in \mathrm{Lip}_2(\mathcal{D})$, the fractal function $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ depends on the scaling matrices α, β, and γ. This naturally raise the question that whether fractal function $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$ is stable with respect to a perturbation in the elements of the scaling matrices α, β, and γ. In the following, we address this question. Suppose that the scaling factors are perturbed as $\tilde{\alpha}_{i,j} = \alpha_{i,j} + \delta_{1,i,j}$, $\tilde{\beta}_{i,j} = \beta_{i,j} + \delta_{2,i,j}$, and $\tilde{\gamma}_{i,j} = \gamma_{i,j} + \delta_{3,i,j}$, such that

$$|\tilde{\alpha}_{i,j}| < 1 \text{ and } |\tilde{\beta}_{i,j}| + |\tilde{\gamma}_{i,j}| < 1 \text{ for all } (i, j) \in \Sigma_N \times \Sigma_M.$$

Hence, we get the perturbed hidden variable IFS as $\{K, \tilde{W}_{i,j} : (i, j) \in \Sigma_N \times \Sigma_M\}$, where $\tilde{W}_{i,j}(x, y, z, t) = \big(u_i(x), v_j(y), \tilde{F}_{i,j}(x, y, z, t)\big)$,

$$\begin{aligned}\tilde{F}_{i,j}(x, y, z, t) = \Big(&\tilde{\alpha}_{i,j}z + \tilde{\beta}_{i,j}t + f(u_i(x), v_j(y)) - \tilde{\alpha}_{i,j}(L_1 f)(x, y) - \tilde{\beta}_{i,j}(L_2 g)(x, y),\\ &\tilde{\gamma}_{i,j}t + g(u_i(x), v_j(y)) - \tilde{\gamma}_{i,j}(L_2 g)(x, y)\Big).\end{aligned}$$

We shall denote the fractal function corresponding to $(f, g) \in \mathrm{Lip}_2(\mathcal{D})$, generated through the above perturbed hidden variable IFS by $(f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}, g^{\tilde{\gamma}}_{\Delta,L_2})$. It is easy to verify that the fractal functions $(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2})$ and $(f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}, g^{\tilde{\gamma}}_{\Delta,L_2})$ interpolate (f, g) at the partition points in Δ. In the following theorem, we obtain an upper bound for perturbation error.

Theorem 7 *Let* $(f, g) \in Lip_2(\mathcal{D})$ *be the given function for which* $(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2})$ *and* $(f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}, g^{\tilde{\gamma}}_{\Delta,L_2})$ *are the fractal functions corresponding to two different sets of scaling factors as mentioned above. Then,*

$$\begin{aligned}\|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}\|_\infty \le{}& \frac{|\alpha - \tilde{\alpha}|_\infty \|f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}} - L_1 f\|_\infty + |\beta - \tilde{\beta}|_\infty \|g^{\tilde{\gamma}}_{\Delta,L_2} - L_2 g\|_\infty}{1 - |\tilde{\alpha}|_\infty}\\ &+ \frac{|\beta|_\infty |\gamma - \tilde{\gamma}|_\infty \|L_2 g\|_\infty}{(1 - |\tilde{\alpha}|_\infty)(1 - |\gamma - \tilde{\gamma}|_\infty)},\end{aligned} \tag{20}$$

where $|\alpha - \tilde{\alpha}|_\infty = \max\{|\alpha_{i,j} - \tilde{\alpha}_{i,j}| : (i, j) \in \Sigma_N \times \Sigma_M\}$, $|\beta - \tilde{\beta}|_\infty = \max\{|\beta_{i,j} - \tilde{\beta}_{i,j}| : (i, j) \in \Sigma_N \times \Sigma_M\}$, *and* $|\gamma - \tilde{\gamma}|_\infty = \max\{|\gamma_{i,j} - \tilde{\gamma}_{i,j}| : (i, j) \in \Sigma_N \times \Sigma_M\}$.

Proof In view of the functional equation (10) of $f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}$, one can write the functional equation of $f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}$, as follows:

$$\begin{aligned} f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}(x,y) &= \tilde{\alpha}_{i,j} f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}(u_i^{-1}(x), v_j^{-1}(y)) + \tilde{\beta}_{i,j} g^{\tilde{\gamma}}_{\Delta,L_2}(u_i^{-1}(x), v_j^{-1}(y)) + f(x,y) \\ &\quad - \tilde{\alpha}_{i,j}(L_1 f)(u_i^{-1}(x), v_j^{-1}(y)) - \tilde{\beta}_{i,j}(L_2 g)(u_i^{-1}(x), v_j^{-1}(y)), (x,y) \in \mathcal{D}_{i,j},\ (i,j) \in \Sigma_N \times \Sigma_M. \end{aligned} \tag{21}$$

Using (10) and (21), one can write:

$$\begin{aligned} f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}(x,y) - f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}(x,y) &= (\alpha_{i,j} - \tilde{\alpha}_{i,j})[f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}} - L_1 f](u_i^{-1}(x), v_j^{-1}(y)) \\ &\quad + (\beta_{i,j} - \tilde{\beta}_{i,j})[g^{\tilde{\gamma}}_{\Delta,L_2} - L_2 g](u_i^{-1}(x), v_j^{-1}(y)) \\ &\quad + \tilde{\alpha}_{i,j}[f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}](u_i^{-1}(x), v_j^{-1}(y)) \\ &\quad + \beta_{i,j}[g^{\gamma}_{\Delta,L_2} - g^{\tilde{\gamma}}_{\Delta,L_2}](u_i^{-1}(x), v_j^{-1}(y)), (x,y) \in \mathcal{D}_{i,j},\ (i,j) \in \Sigma_N \times \Sigma_M. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}\|_\infty &\le |\alpha - \tilde{\alpha}|_\infty \|f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}} - L_1 f\|_\infty + |\beta - \tilde{\beta}|_\infty \|g^{\tilde{\gamma}}_{\Delta,L_2} - L_2 g\|_\infty \\ &\quad + |\tilde{\alpha}|_\infty \|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}\|_\infty + |\beta|_\infty \|g^{\gamma}_{\Delta,L_2} - g^{\tilde{\gamma}}_{\Delta,L_2}\|_\infty. \end{aligned}$$

This implies that

$$\|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}\|_\infty \le \frac{|\alpha - \tilde{\alpha}|_\infty \|f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}} - L_1 f\|_\infty + |\beta - \tilde{\beta}|_\infty \|g^{\tilde{\gamma}}_{\Delta,L_2} - L_2 g\|_\infty + |\beta|_\infty \|g^{\gamma}_{\Delta,L_2} - g^{\tilde{\gamma}}_{\Delta,L_2}\|_\infty}{1 - |\tilde{\alpha}|_\infty}. \tag{22}$$

Using (11), we can write the functional equation of $g^{\tilde{\gamma}}_{\Delta,L_2}$ as follows: for $(x,y) \in \mathcal{D}_{i,j},\ (i,j) \in \Sigma_N \times \Sigma_M$,

$$g^{\tilde{\gamma}}_{\Delta,L_2}(x,y) = \gamma_{i,j} g^{\tilde{\gamma}}_{\Delta,L_2}(u_i^{-1}(x), v_j^{-1}(y)) + g(x,y) - \tilde{\gamma}_{i,j}(L_2 g)(u_i^{-1}(x), v_j^{-1}(y)). \tag{23}$$

Exploiting (11) and (23), we get

$$\|g^{\tilde{\gamma}}_{\Delta,L_2} - g^{\gamma}_{\Delta,L_2}\|_\infty \le \frac{|\gamma - \tilde{\gamma}|_\infty}{1 - |\gamma - \tilde{\gamma}|_\infty} \|L_2 g\|_\infty. \tag{24}$$

Using (24) in (22), we get the desired result and it completes the proof. □

Since $\|f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}} - L_1 f\|_\infty < \infty$, $\|g^{\tilde{\gamma}}_{\Delta,L_2} - L_2 g\|_\infty < \infty$, and $\|L_2 g\|_\infty < \infty$, from (20), it follows that $\|f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}} - f^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}_{\Delta,\mathbf{L}}\|_\infty \to 0$ as $|\alpha - \tilde{\alpha}| \to 0$, $|\beta - \tilde{\beta}| \to 0$, and $|\gamma - \tilde{\gamma}| \to 0$. Hence, we conclude that any small perturbation in the scaling factors leads to a small error in the approximation.

4.2 Existence of optimal hidden variable fractal function

Let $(f, g) \in \mathrm{Lip}_2(\mathcal{D})$. In the following, we show that there exist the scaling matrices α^*, β^*, γ^* such that

$$\|(f,g) - (f^{\alpha^*,\beta^*,\gamma^*}_{\Delta,\mathbf{L}}, g^{\gamma^*}_{\Delta,L_2})\|_\infty \le \|(f,g) - (f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2})\|_\infty \ \forall\, \alpha, \beta, \gamma.$$

That is, we show that there exists an optimal fractal function $(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2})$ which approximates $(f, g) \in \mathrm{Lip}_2(\mathcal{D})$. We know that the fractal function $(f^{\alpha,\beta,\gamma}_{\Delta,\mathbf{L}}, g^{\gamma}_{\Delta,L_2})$ is the fixed point of the Read-Bajraktarević operator

T defined in (5), and $\max\{|\alpha|_\infty, |\beta|_\infty + |\gamma|_\infty\}$ is contractivity factor of T. To emphasize the dependence of T on α, β, and γ, we write $T^{\alpha,\beta,\gamma}$.

The Collage Theorem [22] states that if $(f, g) \in \mathrm{Lip}_2(\mathcal{D})$ is given, and α^*, β^*, and γ^* such that $\|T^{\alpha^*,\beta^*,\gamma^*}(f, g) - (f, g)\|_\infty \leq \delta$, then

$$\|(f, g) - (f_{\Delta,\mathbf{L}}^{\alpha^*,\beta^*,\gamma^*}, g_{\Delta,L_2}^{\gamma^*})\|_\infty = \|(f, g) - T^{\alpha^*,\beta^*,\gamma^*}(f_{\Delta,\mathbf{L}}^{\alpha^*,\beta^*,\gamma^*}, g_{\Delta,L_2}^{\gamma^*})\|_\infty \leq \frac{\delta}{1 - \max\{|\alpha^*|_\infty, |\beta^*|_\infty + |\gamma^*|_\infty\}}.$$

Based on the above result, we need to search for $(\alpha^*, \beta^*, \gamma^*) \in \Gamma = \Big\{(\alpha, \beta, \gamma) : \alpha = [\alpha_{i,j}]_{N\times M},\ \beta = [\beta_{i,j}]_{N\times M},\ \gamma = [\gamma_{i,j}]_{N\times M},\ |\alpha|_\infty \leq \mu < 1,\ |\beta|_\infty + |\gamma|_\infty \leq \mu < 1\Big\}$ for which $\|(f, g) - T^{\alpha,\beta,\gamma}(f, g)\|_\infty$ is minimum.

Proposition 1 *Let $(f, g) \in Lip_2(\mathcal{D})$. Then the map $\Phi : \Gamma \to \mathcal{C}_2(\mathcal{D})$ defined by $\Phi(\alpha, \beta, \gamma) = T^{\alpha,\beta,\gamma}(f, g)$ is Lipschitz continuous. Consequently, the map $F : \Gamma \to \mathbb{R}^+ \cup \{0\}$ defined by $F(\alpha, \beta, \gamma) = \|T^{\alpha,\beta,\gamma}(f, g) - (f, g)\|_\infty$ is continuous.*

Proof Let $(\alpha, \beta, \gamma), (\alpha^*, \beta^*, \gamma^*) \in \Gamma$. From (5) and (9), we have

$$\begin{aligned}
T^{\alpha,\beta,\gamma}(f, g)(x, y) = \Big(&\alpha_{i,j} f(u_i^{-1}(x), v_j^{-1}(y)) + \beta_{i,j} g(u_i^{-1}(x), v_j^{-1}(y)) + f(x, y)\\
&-\alpha_{i,j}(L_1 f)(u_i^{-1}(x), v_j^{-1}(y)) - \beta_{i,j}(L_2 g)(u_i^{-1}(x), v_j^{-1}(y)),\\
&\gamma_{i,j} g(u_i^{-1}(x), v_j^{-1}(y)) + g(x, y) - \gamma_{i,j}(L_2 g)(x, y)\Big), (x, y) \in \mathcal{D}_{i,j},\ (i, j) \in \Sigma_N \times \Sigma_M. \qquad (25)\\
T^{\alpha^*,\beta^*,\gamma^*}(f, g)(x, y) = \Big(&\alpha_{i,j}^* f(u_i^{-1}(x), v_j^{-1}(y)) + \beta_{i,j}^* g(u_i^{-1}(x), v_j^{-1}(y)) + f(x, y)\\
&-\alpha_{i,j}^*(L_1 f)(u_i^{-1}(x), v_j^{-1}(y)) - \beta_{i,j}^*(L_2 g)(u_i^{-1}(x), v_j^{-1}(y)),\\
&\gamma_{i,j}^* g(u_i^{-1}(x), v_j^{-1}(y)) + g(x, y) - \gamma_{i,j}^*(L_2 g)(x, y)\Big), (x, y) \in \mathcal{D}_{i,j},\ (i, j) \in \Sigma_N \times \Sigma_M. \qquad (26)
\end{aligned}$$

Consequently,

$$|T^{\alpha,\beta,\gamma}(f, g)(x, y) - T^{\alpha^*,\beta^*,\gamma^*}(f, g)(x, y)| \leq \max\{|\alpha_i - \alpha_i^*|, |\beta_i - \beta_i^*| + |\gamma_i - \gamma_i^*|\}\|(f, g) - \mathbf{L}(f, g)\|_\infty.$$

Hence, we have

$$\|T^{\alpha,\beta,\gamma}(f, g) - T^{\alpha^*,\beta^*,\gamma^*}(f, g)\|_\infty \leq \max\{|\alpha - \alpha^*|_\infty, |\beta - \beta^*|_\infty + |\gamma - \gamma^*|_\infty\}\|(f, g) - \mathbf{L}(f, g)\|_\infty,$$

where $|\alpha - \alpha^*|_\infty = \max\{|\alpha_{i,j} - \alpha_{i,j}^*| : (i, j) \in \Sigma_N \times \Sigma_M\}$, $|\beta - \beta^*|_\infty = \max\{|\beta_{i,j} - \beta_{i,j}^*| : (i, j) \in \Sigma_N \times \Sigma_M\}$, and $|\gamma - \gamma^*|_\infty = \max\{|\gamma_{i,j} - \gamma_{i,j}^*| : (i, j) \in \Sigma_N \times \Sigma_M\}$. That is, $\|\Phi(\alpha, \beta, \gamma) - \Phi(\alpha^*, \beta^*, \gamma^*)\|_\infty \leq \max\{|\alpha - \alpha^*|_\infty, |\beta - \beta^*|_\infty + |\gamma - \gamma^*|_\infty\}\|(f, g) - \mathbf{L}(f, g)\|_\infty$. Therefore, Φ is a Lipschitz continuous map with Lipschitz constant $\|(f, g) - \mathbf{L}(f, g)\|_\infty$. Thus, if $|F(\alpha, \beta, \gamma) - F(\alpha^*, \beta^*, \gamma^*)| < \epsilon$, then we can find $\max\{|\alpha - \alpha^*|_\infty, |\beta - \beta^*|_\infty + |\gamma - \gamma^*|_\infty\} < \frac{\epsilon}{\|(f,g)-\mathbf{L}(f,g)\|_\infty}$ to prove the continuity of F. □

Corollary 1 *There exists $(\alpha^\dagger, \beta^\dagger, \gamma^\dagger) \in \Gamma$ at which $F(\alpha, \beta, \gamma) = \|T^{\alpha,\beta,\gamma}(f, g) - (f, g)\|_\infty$ is minimum.*

Proof Since the function $F : \Gamma \to \mathbb{R}^+ \cup \{0\}$ is continuous and set Γ is compact, the existence of an optimal scaling vector $(\alpha^\dagger, \beta^\dagger, \gamma^\dagger) \in \Gamma$ such that

$$F(\alpha^\dagger, \beta^\dagger, \gamma^\dagger) = \min_{(\alpha,\beta,\gamma)\in\Gamma} F(\alpha, \beta, \gamma) = \min_{(\alpha,\beta,\gamma)\in\Gamma} \|T^{\alpha,\beta,\gamma}(f, g) - (f, g)\|_\infty$$

follows from the result: A continuous real function on a compact set attains its maximum and minimum. □

Now the following result provides a tool to find $(\alpha^\dagger, \beta^\dagger, \gamma^\dagger) \in \Gamma$.

Proposition 2 *The function* $F : \Gamma \to \mathbb{R}^+ \cup \{0\}$ *defined by* $F(\alpha, \beta, \gamma) = \|T^{\alpha,\beta,\gamma}(f, g) - (f, g)\|_\infty$ *is convex.*

Proof Let $(\alpha, \beta, \gamma), (\alpha^*, \beta^*, \gamma^*) \in \Gamma$ and $\lambda \in [0, 1]$. It follows that

$$\begin{aligned} F\Big((1-\lambda)(\alpha, \beta, \gamma) + \lambda(\alpha^*, \beta^*, \gamma^*)\Big) &= \|T^{(1-\lambda)\alpha+\lambda\alpha^*,(1-\lambda)\beta+\lambda\beta^*,(1-\lambda)\gamma+\lambda\gamma^*}(f, g) - (f, g)\|_\infty \\ &\leq (1-\lambda)\|T^{\alpha,\beta,\gamma}(f, g) - (f, g)\|_\infty + \lambda\|T^{\alpha^*,\beta^*,\gamma^*}(f, g) - (f, g)\|_\infty \\ &= (1-\lambda)F(\alpha) + \lambda F(\beta). \end{aligned} \tag{27}$$

□

It is straight forward to verify that Γ is a convex set. Using the previous proposition, the choice of $\alpha^\dagger, \beta^\dagger, \gamma^\dagger \in \Gamma$ such that $F(\alpha^\dagger, \beta^\dagger, \gamma^\dagger) = \min_{(\alpha,\beta,\gamma)\in\Gamma} F(\alpha, \beta, \gamma)$ is a constrained convex optimization problem.

Remark 1 The choice of $\alpha^\dagger, \beta^\dagger, \gamma^\dagger \in \Gamma$ such that $F(\alpha^\dagger, \beta^\dagger, \gamma^\dagger) = \min_{(\alpha,\beta,\gamma)\in\Gamma} F(\alpha, \beta, \gamma)$ gives the optimal hidden variable fractal function $f_{\Delta,\mathbf{L}}^{\alpha^\dagger,\beta^\dagger,\gamma^\dagger}$ corresponding to the given $f \in \mathrm{Lip}(\mathcal{D})$.

Author Contributions All authors contributed to the study conception and design.

Funding Author is thankful for the sponsored research project: MTR/2021/000905-MATRICS from the Science and Engineering Research Board (SERB), Government of India.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Competing Interests The author has no relevant financial or non-financial interests to disclose.

Data repositories The author ensures that datasets on which the conclusions of the paper rely are presented in the main manuscript.

References

1. J. E. Hutchinson, Fractals and self similarity, Indiana Univ. Math. J. 30(5), 713–747, 1981.
2. M. F. Barnsley, Fractal functions and interpolation, Constr. Approx. 2, 303–329, 1986.
3. M. A. Navascués, Non-smoothpolynomials, Int .J. Math. Anal. 1, 159–174, 2007.
4. M. A. Navascués, Fractal approximation, Complex Anal. Oper. Theory 4, 953–974, 2010.
5. M. A. Navascués and A. K. B. Chand, Fundamental sets of fractal functions, Acta Appl. Math.100, 247–261, 2008.
6. M. F. Barnsley, J. Elton, D. Hardin, and P. Massopust, Hidden variable fractal interpolation functions, SIAM J. Math. Anal. 20(5), 1218–1242, 1989.
7. A. K. B. Chand and G. P. Kapoor, Cubic spline coalescence fractal interpolation through moments, Fractals 15 (1), 41–53, 2007.
8. A. K. B. Chand, S. K. Katiyar, and P. Viswanathan, Approximation using hidden variable fractal interpolation function, J. Fractal Geom. 2, 81–114, 2015.
9. N. Vijender, Approximation by hidden variable fractal functions: A sequential approach, Results in Mathematics 74 (192), 2019.
10. H. J. Ruan, Q. Xu, Fractal interpolation surfaces on rectangular grids, Bull. Aust. Math. Soc. 91, 435–446, 2015.
11. J. S. Geronimo and D. P. Hardin, Fractal interpolation functions from $\mathbb{R}^n \to \mathbb{R}^m$ and their projections, Z. Anal. Anwend. 12, 535–548, 1993.
12. P. Bouboulis, L. Dalla, Closed Fractal Interpolation Surfaces, J. Math. Anal. Appl. 327, 116-126, 2007.
13. A. K. B. Chand, G. P. Kapoor, Hidden variable bivatiate fractal interpolation surfcaes, Fractals 11(3), 277-288, 2003.
14. Z. Feng, Y. Feng, and Z. Yuan, Fractal interpolation surfaces with function vertical scaling factors. Appl. Math. Lett. 25(11), 1896-1900, 2012.
15. S. Verma and P. Viswanathan, A fractal operator associated with bivariate fractal interpolation functions on rectangular grids, Results in Mathematics 75 (28), 2020.
16. N. Vijender and A. K. B. Chand, Shape preserving aspects of bivariate α-fractal function, Fractals 49 (7), 13 pages, 2021.
17. Sangita Jha, A. K. B. Chand, M. A. Navascués, and Abhilash Sahu, Approximation Properties of Bivariate α-fractal functions and dimension results, Applicable Analysis 100 (16), 2021.
18. R. Aslan and A. İzgi, Approximation by one and two variables of the Bernstein-Schurer-type operators and associated GBS operators on symmetrical mobile interval, Journal of Function Spaces 2021, Article ID 9979286, 12 pages, 2021.
19. R. Aslan and M. Mursaleen, Approximation by bivariate Chlodowsky type Szász-Durrmeyer operators and associated GBS operators on weighted spaces, Journal of Inequalities and Applications 2022:26, 2022.

20. R. Aslan, Rate of approximation of blending type modified univariate and bivariate λ-SchurerKantorovich operators, Kuwait Journal of Science 51, 2024.
21. B. Bollobás, Linear Analysis, an Introductory Course, Cambridge University Press, 2nd ed., 1999.
22. DL Torre and M. Rocca, Approximating Continuous Functions by Iterated Function Systems and Optimization Problems, Int. Math. J. 2(8), 801-811, 2002.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.