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ORIGINAL RESEARCH

The Riemann problem with delta initial data with Dirac delta function in both components for a pressureless gas dynamic model

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Abstract In this paper, we study the Riemann problem with the initial data containing the Dirac delta function in both components for a nonstrictly hyperbolic system of two conservation laws from zero-pressure gas dynamics by introducing the new state variabe. In solutions, we find another kind of delta shocks on which both state variables simultaneously contain the Dirac delta functions. It is quite different from the previous ones on which only one state variable contains the Dirac delta function. This phenomenon is also captured numerically and experimentally by Krejić, Krunić and Nedeljkov [N. Krejić, T. Krunić, M. Nedeljkov, Numerical verification of delta shock waves for pressureless gas dynamics, J. Math. Anal. Appl. 345 (2008) 243-257]. Moreover, a new kind of nonclassical wave, namely a delta contact discontinuity, is discovered here, which is a Dirac delta function supported on a contact discontinuity.

Keywords Pressureless gas dynamic model · Riemann problem · Generalized Rankine-Hugoniot relation · Delta shock wave · Vacuum state · Generalized solution

Mathematics Subject Classification 35L65 · 35L67 · 76N01

1 Introduction

To get a deeper understanding of the celebrated pressureless gas dynamics system, Nedeljkov et al. [20] proposed the following pressureless gas dynamic model:

$$\begin{cases} \rho_t + v_x = 0, \\ v_t + (\frac{v^2}{\rho})_x = 0. \end{cases} \tag{1.1}$$

The system (1.1) can be obtained directly from the following celebrated zero-pressure gas dynamics system consisting of conservation laws of mass and momentum

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{1.2}$$

by introducing the new state variabe $v = \rho u$. Here ρ and u denote the gas density and velocity, respectively.

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The system (1.2) is also called the transport equations, or the zero-pressure gas dynamics. It can be used to describe the motion process of free particles sticking under collision in the low temperature [3] and the information of large-scale structures in the universe [28]. The solutions to the zero-pressure gas dynamics were investigated and analyzed extensively by many scholars [1,2,6,9,10,12–15,20,21,25,26,29,33–35,37]. Brenier and Grenier [3] and E, Rykov and Sinai [9] first obtained the global entropy solutions of the pressureless gas independently for the Cauchy problem, see also Wang, Huang and Ding [34] with a different class of initial data. The uniqueness was proved firstly by Wang and Ding [35] when the initial condition is bounded on the domain. Bouchut [2] obtained same result with a different technique. Huang and Wang [15] finally proved the uniqueness when the initial data is Radon measure, under an energy condition and Oleinik entropy conditions. For general pressureless type system, see Huang [12]. In particular, Sheng and Zhang [31] discovered that the delta shocks and vacuum states do occur in the Riemann solutions to the system (1.2) by the vanishing viscosity method.

It is well known that the delta shocks and vacuum states do occur in the Riemann solutions of (1.1) or (1.2), which can be obtained by the vanishing viscosity approach [31,40]. A distinctive feature for (1.1) is just that the delta shock wave with Dirac delta functions in both ρ and v appears in the Riemann solutions simultaneously. This fact was also captured numerically and experimentally by Nedeljkov et al. [20]. It is quite different from the previous ones on which only one state variable contains the Dirac delta function. For related researches of delta shock waves, we refer the readers to [5–8,10,11,14,16–22,24–27,29–33,36–63] and the references cited therein for more details. For the theory of the delta shock wave with Dirac delta function in multiple state variables, interested readers may refer to [6,25–27,39,40] for further details.

Because the delta shock wave has been captured numerically and experimentally by Krejić, Krunić and Nedeljkov [20] in the Riemann solutions of (1.1), in which the extreme concentrations appear in both state variables ρ and v simultaneously, it is natural to consider (1.1) with the initial data which contains Dirac delta functions. We can constructively find a new type of solution in case of additional Dirac delta measures sitting exactly at the points of discontinuity of the piecewies constant components of the initial data. This kind of Riemann problem is also called the Riemann problem with delta initial data. However, if the extreme concentration does not appear in Riemann solutions, then it is not possible to consider the Riemann problem with delta initial data. Thus, in this paper we mainly investigate the Riemann problem for (1.1) with initial data

$$\begin{cases} \rho(x,0) = \begin{cases} \rho_-, & x < 0 \\ \rho_+, & x > 0 \end{cases} + m_0\delta(x), \\ v(x,0) = \begin{cases} v_-, & x < 0 \\ v_+, & x > 0 \end{cases} + v_0\delta(x), \end{cases} \tag{1.3}$$

where $\delta(x)$ is the standard Dirac delta function, and $m_0 > 0$, $\rho_\pm > 0$, v_0 and $v_\pm$ are arbitrary constants. It is quite different from the previous works [10,21,33,37] on zero-pressure gas dynamics with the initial data containing the Dirac delta function in only one state variable ρ.

It is difficult to solve the Riemann problem (1.1) and (1.3), because it will give rise to a severe problem on how to overcome a problem with the vacuum state on the left- or right-hand side. To overcome this problem, we adopt the idea proposed by Liu and Smoller [23] when they considered the vacuum problem for isentropic gas dynamic equations, and then approximate them by a set of small non-admissible shocks like as the front tracking algorithm [4]. The same method has been used in [10,29]. Thus, when a vacuum state is on the right-hand side, we first consider the initial value problem (1.1) with the following initial data:

$$\begin{cases} \rho(x,0) = \begin{cases} \rho_-, & x < 0 \\ \varepsilon m_0,\ qx > 0 \end{cases} + m_0\delta(x), \\ v(x,0) = \begin{cases} v_-, & x < 0 \\ \varepsilon v_0, & x > 0 \end{cases} + v_0\delta(x), \end{cases} \tag{1.4}$$

where $\varepsilon > 0$ is sufficiently small, and $m_0 > 0$, $\rho_- > 0$, v_0 and v_- are arbitrary constants. We constructively obtain the solutions, global structure and large time-asymptotic behaviors of soltuions for problem (1.1)–(1.4). Letting $\varepsilon \to 0$, under the stability theory of weak solutions, we obtain solutions of the Riemann problem (1.1) and (1.3) when a vacuum state is on the right-hand side. This method of constructing global generalized solutions for the Riemann problem with delta initial data on the basis of the stability theory of generalized solutions has been firstly used in Wang and Zhang [36] to study the Riemann problem with delta initial data for the 1-D Chaplygin gas equations.



With the help of the generalized Rankine-Hugoniot conditions and the entropy condition, we constructively obtain the global generalized solutions for the problem (1.1) and (1.3) containing a new kind of nonclassical wave, namely, a delta contact discontinuity, which is a Dirac delta function supported on a contact discontinuity and was also discovered in the study of other systems, see [24,30], etc.

The paper is organized as follows. In Sect. 2, for readers' convenience we review the Riemann solutions of (1.1) with constant initial data and the solution concept in the sense of distributions to the Riemann problem for system (1.1) with delta measure initial data. In Sect. 3, we construct the generalized solutions of (1.1) with delta initial data (1.3) case by case.

2 Preliminaries

In this section, we briefly review the Riemann problem of (1.1) with initial data

$$(\rho, v)(x, 0) = (\rho_\pm, v_\pm), \qquad \pm x > 0, \tag{2.1}$$

where $\rho_\pm > 0$, which was studied in [40]. As in [40], for convenience, we set $r = \frac{v}{\rho}$.

It is easy to see that system (1.1) is non-strictly hyperbolic, because it has a double eigenvalue

$$\lambda = r$$

and only one right eigenvector

$$\overrightarrow{r} = (1, r)^T.$$

By a direct calculation,

$$\nabla\lambda \cdot \overrightarrow{r} \equiv 0.$$

Therefore, λ is linearly degenerate for $\rho > 0$.

Since system (1.1) and the Riemann data (2.1) are invariant under stretching of coordinates: $(x, t) \to (\alpha x, \alpha t)$ (α is constant), we seek the self-similar solution

$$(\rho, v)(x, t) = (\rho, v)(\xi), \quad \xi = \frac{x}{t}.$$

Then Riemann problem (1.1) and (2.1) is reduced to the following boundary value problem of ordinary differential equations:

$$\begin{cases} -\xi\rho_\xi + v_\xi = 0, \\ -\xi v_\xi + \left(\frac{v^2}{\rho}\right)_\xi = 0, \end{cases} \tag{2.2}$$

with $(\rho, v)(\pm\infty) = (\rho_\pm, v_\pm)$.

For any smooth solution, system (2.2) can be written as

$$\begin{pmatrix} -\xi & 1 \\ -r^2 & 2r - \xi \end{pmatrix} \begin{pmatrix} \rho_\xi \\ v_\xi \end{pmatrix} = 0. \tag{2.3}$$

It provides either the general solution (constant state)

$$(\rho, v)(\xi) = \text{constant} \quad (\rho > 0),$$

or the singular solution

$$\begin{cases} \xi = r, \\ \rho r_\xi = 0, \\ v r_\xi = 0, \end{cases} \tag{2.4}$$

which, by integrating, yields

$$\xi = r = r_-. \tag{2.5}$$

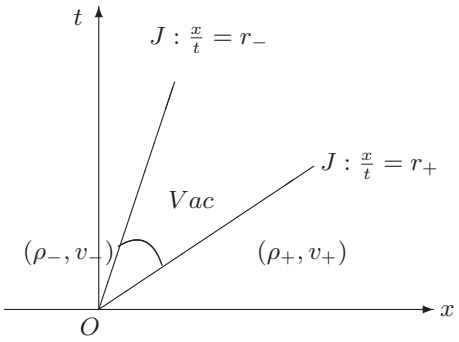


Fig. 1 Riemann solution of case $r_- < r_+$

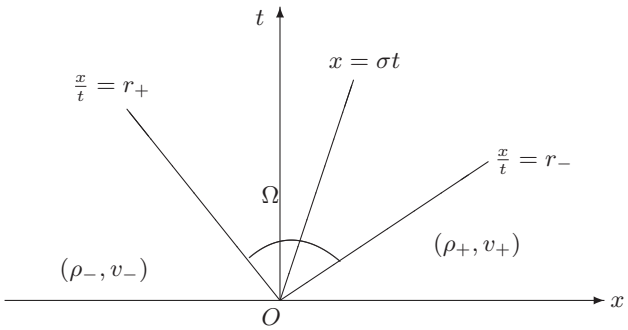


Fig. 2 Characteristic analysis of delta shock wave

The singular solution (2.4) contains a violated point $(\rho, v) = (0, 0)$ which satisfies the system (1.1). In terms of the vacuum state in fluid dynamics, it is called a vacuum state, denoted by Vac (see [31]).

For a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot condition holds:

$$\begin{cases} -\sigma[\rho] + [v] = 0, \\ -\sigma[v] + \left[\frac{v^2}{\rho}\right] = 0, \end{cases} \tag{2.6}$$

where $[q] = q_+ - q_-$ denotes the jump of q across the discontinuity. By solving (2.6), we obtain

$$J : \xi = \sigma = r_-(= \lambda_-) = r_+(= \lambda_+), \tag{2.7}$$

which is a contact discontinuity. It is a slip line and just the characteristic of solutions on both its sides in (t, x)-plane. Two states (ρ_-, v_-) and (ρ_+, v_+) can be connected by a contact discontinuity if and only if $r_- = r_+$, namely, $\frac{v_-}{\rho_-} = \frac{v_+}{\rho_+}$.

We now can construct the Riemann solutions of (1.1) and (2.1) by contact discontinuities, vacuum or δ-shock wave connecting two constant states $(\rho_\pm, v_\pm)$.

For the case $r_- < r_+$, there is no characteristic passing through the region $r_- t < x < r_+ t$, therefore the vacuum should appear in this region. In the (ρ, v)-phase plane, we can draw lines $\frac{v}{\rho} = \frac{v_-}{\rho_-}$ and $\frac{v}{\rho} = \frac{v_+}{\rho_+}$ from (ρ_-, v_-) and (ρ_+, v_+), respectively. The two lines intersect only at $(0, 0)$, an isolated point. Thus we obtain a Riemann solution which consists of two contact discontinuities and a vacuum besides two constant states (see Fig. 1). The solution can be expressed as

$$(\rho, v)(\xi) = \begin{cases} (\rho_-, v_-), & -\infty < \xi < r_-, \\ Vac, & r_- \le \xi \le r_+, \\ (\rho_+, v_+), & r_+ < \xi < +\infty. \end{cases} \tag{2.8}$$

For the case $r_- = r_+$, it is easy to see that the constant states $(\rho_\pm, v_\pm)$ can be connected by a contact discontinuity.

For the case $r_- > r_+$, the characteristics from initial data will overlap in a domain Ω as shown in Fig. 2. So, the singularity of solutions must develop in Ω. We assert that there exist no solutions in bounded variation space, that is, the solution cannot be constructed by classical waves. In fact, at this moment, the delta shock wave with Dirac delta functions in both ρ and v will appear in solutions. Recently, Nedeljkov et al. [20] presented the numerical results, obtained by specially designed procedure that is based on wave propagation method implemented in CLAWPACK (see [20]), to confirm this type of delta shock wave.

By the above analysis, for the case $r_- > r_+$, the delta shock wave with Dirac delta functions in both ρ and v will be considered. Thus we define the δ-measure solution, which was given by Nilsson [25,26] and Wang [32] where the framework was introduced by Danilov and Shelkovich [7,8] and improved by Kalisch and Mitrovic [16,17].

In order to define the measure solution, we define the following.

Definition 2.1 The two-dimensional weighted delta measure $w(s)\delta_\Gamma$ supported on a smooth curve $\Gamma = \{(x(s), t(s)) : a < s < b\}$ can be defined by

$$\langle w(s)\delta_\Gamma, \varphi(x(s), t(s))\rangle = \int_a^b w(s)\varphi(x(s), t(s))ds, \tag{2.9}$$

for all test functions $\varphi \in C_0^\infty(R \times [0, +\infty))$.

Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a graph in the closed upper half-plane $\{(x, t) : x \in R, t \in [0, +\infty)\} \in R^2$ containing smooth arcs γ_i, $i \in I$, and I is a finite set. Let I_0 be a subset of I such that an arc γ_i for $i \in I_0$ starts from the points of the $x-$axis; $\Gamma_0 = \{x_k^0 : i \in I_0\}$ is the set of initial points of arcs γ_i, $i \in I_0$. In what follows, we define the solutions in the sense of distributions to the Cauchy problem for system (1.1) with delta measure initial data.

Definition 2.2 Let $(\rho(x, t), v(x, t))$ be a pair of distributions and Γ be a graph, where $(\rho(x, t), v(x, t))$ is represented in form of the sum

$$\rho(x, t) = \widehat{\rho}(x, t) + w(x, t)\delta(\Gamma) \quad \text{and} \quad v(x, t) = \widehat{v}(x, t) + e(x, t)\delta(\Gamma),$$

$\widehat{\rho}, \widehat{v} \in L^\infty(R \times R_+)$ and $w(x, t)\delta(\Gamma) = \sum_{i\in I} w_i(x, t)\delta(\gamma_i)$, $e(x, t)\delta(\Gamma) = \sum_{i\in I} e_i(x, t)\delta(\gamma_i)$, $w_i(x, t), e_i(x, t) \in C(\Gamma)$ for $i \in I$. Consider δ-shock wave type initial data $(\rho^0(x), v^0(x))$, where

$$\rho^0(x) = \rho_0(x) + w^0\delta(\Gamma_0), \quad v^0(x) = v_0(x) + e^0\delta(\Gamma_0),$$

$\rho_0, v_0 \in L^\infty(R)$, $w^0\delta(\Gamma_0) = \sum_{k\in I_0} w_k^0\delta(x - x_k^0)$ and $e^0\delta(\Gamma_0) = \sum_{k\in I} e_k^0\delta(x - x_k^0)$, w_k^0 and e_k^0 are constants for $k \in I_0$. The above pair of distributions (ρ, v) is called a generalized δ-shock wave type solution of system (1.1) with the δ-shock wave type initial data $(\rho^0(x), v^0(x))$ if the integral identities

$$\begin{cases} \int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\widehat{\rho}\phi_t + \widehat{v}\phi_x\right)dxdt + \sum_{i\in I}\int_{\gamma_i} w_i(x, t)\frac{d\phi(x,t)}{dl}dl \\ \quad + \int_{-\infty}^{+\infty}\rho_0(x)\phi(x, 0)dx + \sum_{k\in I_0} w_k^0\phi(x_k^0, 0) = 0, \\ \int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\widehat{v}\phi_t + \frac{\widehat{v}^2}{\widehat{\rho}}\phi_x\right)dxdt + \sum_{i\in I}\int_{\gamma_i} e_i(x, t)\frac{d\phi(x,t)}{dl}dl \\ \quad + \int_{-\infty}^{+\infty} v_0(x)\phi(x, 0)dx + \sum_{k\in I_0} e_k^0\phi(x_k^0, 0) = 0, \end{cases} \tag{2.10}$$

hold for all test functions $\phi(x, t) \in \mathcal{D}(R \times [0, +\infty))$, where $\frac{\partial\phi(x,t)}{\partial l}$ is the tangential derivative on the graph γ_i, $\int_{\gamma_i} dl$ is a line integral over the arc γ_i.

Remark 1 Definition 2.2 is a natural generalization of the classical definition of a weak solution.

With Definitions 2.1–2.2, a δ-measure solution with the discontinuity $x = \sigma t$ of (1.1) and (2.1) in the case $r_- > r_+$ can be expressed as

$$\begin{cases} \rho(x,t) = \widehat{\rho}(x,t) + w(t)\delta_\Gamma, \\ v(x,t) = \widehat{v}(x,t) + r_\delta w(t)\delta_\Gamma, \end{cases}$$

where $\Gamma = \{(\sigma t, t) : 0 \le t < +\infty\}$,

$$\widehat{\rho}(x,t) = \rho_- + [\rho]H(x-\sigma t), \ \ \widehat{v}(x,t) = v_- + [v]H(x-\sigma t),$$
$$\sigma = r_\delta = \frac{[v] + \left(\frac{v_-}{\rho_-} - \frac{v_+}{\rho_+}\right)\sqrt{\rho_+\rho_-}}{[\rho]},$$
$$w(t) = \left(\frac{v_-}{\rho_-} - \frac{v_+}{\rho_+}\right)\sqrt{\rho_+\rho_-}\, t, \ \ \text{if } \rho_+ \neq \rho_-,$$

and

$$\sigma = r_\delta = \frac{v_+ + v_-}{2\rho_+}, \ \ w(t) = (v_- - v_+)t, \ \ \text{if } \rho_+ = \rho_-,$$

in which $H(x)$ is the Heaviside function that equals 0 for $x < 0$ and 1 for $x > 0$, δ_Γ is the standard Dirac delta function supported on the curve $\Gamma = \{(\sigma t, t) : 0 \le t < +\infty\}$.

Moreover, from (1.1), one can find that the function $r(x,t)$ satisfies the simplest quasilinear equation

$$r_t + \left(\frac{1}{2}r^2\right)_x = 0. \tag{2.11}$$

Then we consider Eq. (2.11) with initial data

$$r(x,0) = \begin{cases} r_-, & x < 0, \\ r_+, & x > 0. \end{cases} \tag{2.12}$$

For the case $r_- > r_+$, according to the left initial condition $t = 0 : r = r_-(x \le 0)$, by using the method of characteristics, on the domain

$$\Omega_- \overset{def}{=} \{(x,t) | 0 \le t < +\infty, x \le r_- t\},$$

one can construct the global solution of (2.11) as follows.

$$r(x,t) = r_-, \qquad \forall (x,t) \in \Omega_-.$$

Analogously, according to the right initial condition $t = 0 : r = r_+(x \ge 0)$, by using the method of characteristics, on the domain

$$\Omega_+ \overset{def}{=} \{(x,t) | 0 \le t < +\infty, x \ge r_+ t\},$$

one can construct the global solution of (2.11) as follows.

$$r(x,t) = r_+, \qquad \forall (x,t) \in \Omega_+.$$

Noting the fact that $r_- > r_+$, one has

$$\Omega_\# \overset{def}{=} \Omega_- \cap \Omega_+ = \{(x,t) | 0 \le t < +\infty, r_+ t \le x \le r_- t\},$$

see Fig. 3.

We wish to show that there exists a smooth curve $x = x_s(t)(x_s(0) = 0)$ in the (x,t) plane such that (1) the curve $x = x_s(t)$ divides $\Omega_\#$ into two parts; (2) on $x = x_s(t)$, the following Rankine-Hugoniot condition for Eq. (2.11):

$$\frac{dx_s(t)}{dt} = \frac{f(r^+(x_s(t),t)) - f(r^-(x_s(t),t))}{r^+(x_s(t),t) - r^-(x_s(t),t)}$$

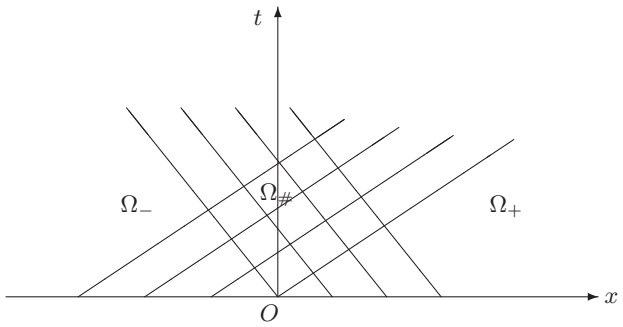


Fig. 3 Overlapping domain $\Omega_{\#}$ of characteristics

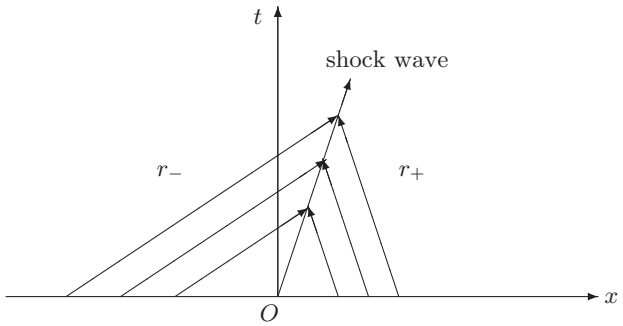


Fig. 4 The shock wave solution to (2.11) and (2.12) when $r_- > r_+$

holds, where $f(r) = \frac{1}{2}r^2$ and $r^{\pm}(x_s(t), t) \overset{def}{=} r(x_s(t) \pm 0, t)$. We see that, in our case, $r^{\pm}(x_s(t), t) = r_{\pm}$, Thus, the following Rankine-Hugoniot condition for Eq. (2.11) can be reduced to

$$\frac{dx_s(t)}{dt} = \frac{1}{2}(r_+ + r_-).$$

Noting the fact that $r_+ < \frac{r_+ + r_-}{2} < r_-$, we know that the Lax entropy condition: $r^+ < s < r^-$, i.e.,

$$f'(r^+) < s < f'(r^-)$$

holds, where $f(r) = \frac{1}{2}r^2$, $r^{\pm} = r^{\pm}(x_s(t), t) \overset{def}{=} r(x_s(t) \pm 0, t)$ and $s = \frac{dx_s(t)}{dt}$ is the speed of the discontinuity $x = x_s(t)$. So, the Riemann problem (2.11) and (2.12) admits a unique self-similar solution composed of two constant states r_-, r_+ separated by a shock wave $x = \frac{r_+ + r_-}{2}t$ in the class of elementary waves as follows:

$$r(x,t) = \begin{cases} r_-, & x < \frac{r_+ + r_-}{2}t, \\ r_+, & x > \frac{r_+ + r_-}{2}t, \end{cases} \tag{2.13}$$

see Fig 4. Thus, from (2.13), it can be observed that $r = \frac{v}{\rho}$ has a bounded variation. Thus we arrange r on the discontinuous line $x = \sigma t$ as

$$r|_{x=\sigma t} = r_\delta = \sigma.$$

Then we can multiply a delta function by a discontinuous function in a similar way to that in [39,40], etc. According to [40], the δ-measure solution (ρ, v) constructed above should satisfy

$$\begin{cases} \langle \rho, \varphi_t \rangle + \langle v, \varphi_x \rangle = 0, \\ \langle v, \varphi_t \rangle + \langle \frac{v^2}{\rho}, \varphi_x \rangle = 0, \end{cases}$$

for all test functions $\varphi(x,t) \in C_0^\infty(R \times [0,+\infty))$, where

$$\langle \rho, \varphi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \overline{\rho}(x,t)\varphi dxdt + \langle w\delta_S, \varphi \rangle,$$
$$\langle \frac{v^2}{\rho}, \varphi \rangle = \langle rv, \varphi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{\overline{v}^2}{\overline{\rho}}\right)(x,t)\varphi dxdt + \langle \sigma r_\delta w\delta_S, \varphi \rangle,$$

and v has the similar integral identities as above. Here, $S = \{(\sigma t, t) : 0 \le t < +\infty\}$,

$$\overline{\rho}(x,t) = \rho_- + [\rho]H(x-\sigma t), \quad \overline{v}(x,t) = v_- + [v]H(x-\sigma t),$$

$r|_S = \sigma = r_\delta$, σ is the tangential derivative of the curve S.

Theorem 2.3 *For the case $r_- > r_+$, the Riemann problem (1.1) and (2.1) has a δ-shock wave type solution in the form*

$$\begin{cases} \rho(x,t) = \begin{cases} \rho_-, & x < \sigma t \\ \rho_+, & x > \sigma t \end{cases} + w(t)\delta(x-\sigma t), \\ v(x,t) = \begin{cases} v_-, & x < \sigma t \\ v_+, & x > \sigma t \end{cases} + r_\delta w(t)\delta(x-\sigma t), \end{cases} \tag{2.14}$$

which satisfies the integral identities (2.10) *in the sense of Definition 2.2, where $\Gamma = \{(x,t) : x = x(t) = \sigma t,\ t \ge 0\}$, $\widehat{\rho}(x,t) = \rho_- + [\rho]H(x-\sigma t)$, $\widehat{v}(x,t) = v_- + [v]H(x-\sigma t)$, $r|_\Gamma = \sigma = r_\delta$,*

$$\int_\Gamma w(x,t)\frac{\partial \phi(x,t)}{\partial l}dl = \int_0^{+\infty} w(x(t),t)\frac{d\phi(x(t),t)}{dt}dt,$$

and $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x > 0. \end{cases}$$

In addition

$$\sigma = r_\delta = \frac{[v] + \left(\frac{v_-}{\rho_-} - \frac{v_+}{\rho_+}\right)\sqrt{\rho_+\rho_-}}{[\rho]}, \quad w(t) = \left(\frac{v_-}{\rho_-} - \frac{v_+}{\rho_+}\right)\sqrt{\rho_+\rho_-}\ t, \quad \text{as } \rho_+ \ne \rho_-,$$

and

$$\sigma = r_\delta = \frac{v_+ + v_-}{2\rho_+}, \quad w(t) = (v_- - v_+)t, \quad \text{as } \rho_+ = \rho_-.$$

Proof We need to check that the constructed δ-measure solution satisfies the Definition 2.2 in the sense of distributions, that is

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\widehat{\rho}\phi_t + \widehat{v}\phi_x\right)dxdt + \int_0^{+\infty} w(t)\frac{d\phi(\sigma t,t)}{dt}dt + \int_{-\infty}^{+\infty}\rho_0(x)\phi(x,0)dx = 0, \tag{2.15}$$
$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\widehat{v}\phi_t + \frac{\widehat{v}^2}{\widehat{\rho}}\phi_x\right)dxdt + \int_0^{+\infty} r_\delta w(t)\frac{d\phi(\sigma t,t)}{dt}dt + \int_{-\infty}^{+\infty} v_0(x)\phi(x,0)dx = 0, \tag{2.16}$$

for all test functions $\phi(x,t) \in \mathcal{D}(R \times [0,+\infty))$, where $\rho_0(x) = \rho_- + [\rho]H(x)$ and $v_0(x) = v_- + [v]H(x)$.

Denoting by A the left-hand side of (2.15), we have

$$A = \int_0^{+\infty}\int_{-\infty}^{\sigma t}\left(\rho_-\phi_t + v_-\phi_x\right)dxdt + \int_0^{+\infty}\int_{\sigma t}^{+\infty}\left(\rho_+\phi_t + v_+\phi_x\right)dxdt$$
$$+ \int_0^{+\infty} w(t)\frac{d\phi(\sigma t,t)}{dt}dt + \int_{-\infty}^{0}\rho_-\phi(x,0)dx + \int_0^{+\infty}\rho_+\phi(x,0)dx. \tag{2.17}$$

Without loss of generality, we assume $\sigma > 0$, then the first term on the right-hand side of (2.17) equals

$$\begin{aligned}
&\int_0^{+\infty}\int_{-\infty}^{0}\rho_-\phi_t dxdt + \int_0^{+\infty}\int_0^{\sigma t}\rho_-\phi_t dxdt + \int_0^{+\infty}\int_{-\infty}^{\sigma t} v_-\phi_x dxdt\\
&= -\int_{-\infty}^{0}\rho_-\phi(x,0)dx + \int_0^{+\infty}dx\int_{\frac{x}{\sigma}}^{+\infty}\rho_-\phi_t dt + \int_0^{+\infty} v_-\phi(\sigma t,t)dt\\
&= -\int_{-\infty}^{0}\rho_-\phi(x,0)dx - \int_0^{+\infty}\rho_-\phi(x,\frac{x}{\sigma})dx + \int_0^{+\infty} v_-\phi(\sigma t,t)dt\\
&= -\int_{-\infty}^{0}\rho_-\phi(x,0)dx + \int_0^{+\infty}(-\rho_-\sigma + v_-)\phi(\sigma t,t)dt. \qquad (2.18)
\end{aligned}$$

Similarly, the second term on the right-hand side of (2.17) equals

$$\begin{aligned}
&\int_0^{+\infty}dx\int_0^{\frac{x}{\sigma}}\rho_+\phi_t dt - \int_0^{+\infty} v_+\phi(\sigma t,t)dt\\
&= -\int_0^{+\infty}\rho_+\phi(x,0)dx + \int_0^{+\infty}(\rho_+\sigma - v_+)\phi(\sigma t,t)dt. \qquad (2.19)
\end{aligned}$$

The third term on the right-hand side of (2.17) equals

$$\begin{aligned}
&\big(\sigma[\rho]-[v]\big)t\phi(\sigma t,t)\Big|_0^{+\infty} - \int_0^{+\infty}\big(\sigma[\rho]-[v]\big)\phi(\sigma t,t)dt\\
&= -\int_0^{+\infty}\big(\sigma[\rho]-[v]\big)\phi(\sigma t,t)dt. \qquad (2.20)
\end{aligned}$$

From (2.17)–(2.20), we obtain

$$A = 0.$$

Similarly, we can obtain (2.16). So, we complete the proof. □

Now, let we consider a solution of (1.1) and (2.1) in the following form

$$\begin{cases}
\rho(x,t) = \left\{\begin{matrix}\rho_-, & x < x(t)\\ \rho_+, & x > x(t)\end{matrix}\right\} + w(t)\delta(x-x(t)),\\
v(x,t) = \left\{\begin{matrix}v_-, & x < x(t)\\ v_+, & x > x(t)\end{matrix}\right\} + r_\delta(t)w(t)\delta(x-x(t)),
\end{cases} \qquad (2.21)$$

where $x(t), w(t) \in C^1[0,+\infty)$, $r|_{x=x(t)} = r_\delta(t) = \sigma$, σ is the tangential derivative of the curve $x = x(t)$ and $\delta(\cdot)$ is the standard Dirac measure supported on the curve $x = x(t)$. Then, using Definition 2.2 and repeating the proof of Theorem 2.3 almost word-for-word, one can derive the following result.

Theorem 2.4 *For the case $r_- > r_+$, if the solution of the form (2.21) satisfies the following generalized Rankine-Hugoniot relation:*

$$\begin{cases}
\frac{dx(t)}{dt} = \sigma,\\
\frac{dw(t)}{dt} = \sigma[\rho] - [v],\\
\frac{d(r_\delta(t)w(t))}{dt} = \sigma[v] - \left[\frac{v^2}{\rho}\right],
\end{cases} \qquad (2.22)$$

and

$$r|_{x=x(t)} = r_\delta(t) = \sigma, \qquad (2.23)$$

where $[\rho] = \rho_+ - \rho_-$, *etc. Then it is a* δ*-shock wave type solution of (1.1) and (2.1) which satisfies the integral identities (2.10) in the sense of Definition 2.2, where* $\Gamma = \{(x,t) : x = x(t),\ t \geq 0\}$, $\widehat{\rho}(x,t) = \rho_- + [\rho]H(x - x(t))$, $\widehat{v}(x,t) = v_- + [v]H(x - x(t))$,

$$\int_\Gamma w(x,t)\frac{\partial \phi(x,t)}{\partial l}dl = \int_0^{+\infty} w(x(t),t)\frac{d\phi(x(t),t)}{dt}dt,$$

and $H(x)$ *is the Heaviside function*

$$H(x) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x > 0. \end{cases}$$

Remark 2 Eqs. (2.22)–(2.23) are called the generalized Rankine-Hugoniot relation. It reflects the exact relationship among the limit states on both sides of the discontinuity, location, propagation speed, weights and reassignment of $\frac{v}{\rho}$ on the discontinuity.

To guarantee uniqueness, we should propose the entropy condition

$$r_+ < \sigma < r_-, \tag{2.24}$$

which means that all characteristics on both sides of the discontinuity are in-coming.

Definition 2.5 A discontinuity satisfying (2.22)–(2.24) is called a delta shock wave of the system (1.1), denoted by δ.

Next, we employ the generalized Rankine-Hugoniot relation to solve Riemann problem (1.1) and (2.1) for the case $r_- > r_+$. At this moment, the Riemann problem is reduced to solve the initial value problem for (2.22) and (2.23) with initial data

$$t = 0 : x(0) = 0, \quad w(0) = 0. \tag{2.25}$$

In view of our knowledge about delta shock waves, we know that for the solution of the Riemann problem with constant initial data for all governing systems, if the delta shock wave appears in Riemann solution, then σ is a constant and $w(t)$ is a linear function of t. So we can find a delta shock wave solution to the Riemann problem (1.1) and (2.1) of the form

$$\begin{cases} \rho(x,t) = \begin{cases} \rho_-, & x < \sigma t \\ \rho_+, & x > \sigma t \end{cases} + w_0 t\delta(x - \sigma t), \\ v(x,t) = \begin{cases} v_-, & x < \sigma t \\ v_+, & x > \sigma t \end{cases} + r_\delta w_0 t\delta(x - \sigma t), \end{cases} \tag{2.26}$$

where σ, w_0 and r_δ are constants uniquely determined by the algebraic equations

$$\begin{cases} r_\delta w_0 = \sigma[v] - \left[\frac{v^2}{\rho}\right], \\ w_0 = \sigma[\rho] - [v], \\ \sigma = r_\delta. \end{cases} \tag{2.27}$$

Under the entropy condition (2.24), it is easy to see from (2.27) that when $\rho_+ = \rho_-$,

$$\sigma = r_\delta = \frac{v_+ + v_-}{2\rho_+}, \quad w(t) = w_0 t = (v_- - v_+)t; \tag{2.28}$$

when $\rho_+ \neq \rho_-$,

$$\sigma = r_\delta = \frac{[v] + \left(\frac{v_-}{\rho_-} - \frac{v_+}{\rho_+}\right)\sqrt{\rho_+\rho_-}}{[\rho]}, \quad w(t) = \left(\frac{v_-}{\rho_-} - \frac{v_+}{\rho_+}\right)\sqrt{\rho_+\rho_-}\ t. \tag{2.29}$$

We also can justify that the delta-shock satisfies the entropy condition

$$r_+ < \sigma < r_-. \tag{2.30}$$

Thus, we have obtained the global solutions of the Riemann problem (1.1) and (2.1).



3 Riemann problem with delta initial data

In this section, we construct Riemann solutions of (1.1) and (1.3). For simplicity, we hereinafter denote a particular particle by its Lagrangian coordinate x_0. According to the relations among r_-, r_0 and r_+, where $r_0 = \frac{v_0}{m_0}$, we discuss the Riemann problem case by case.

Case 3.1. $r_- \le r_0 \le r_+$.
To construct the solution of (1.1) and (1.3), here we first consider the initial value problem (1.1) with the following initial data:

$$(\rho, v)(x, 0) = \begin{cases} (\rho_-, v_-), & x < -\varepsilon, \\ \left(\frac{m_0}{2\varepsilon}, \frac{v_0}{2\varepsilon}\right), & -\varepsilon < x < \varepsilon, \\ (\rho_+, v_+), & x > \varepsilon, \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ is sufficiently small. On the basis of the stability theory of weak solutions, if we obtain a solution of (1.1) and (3.1), then by letting $\varepsilon \to 0^+$, we can get a solution of (1.1) and (1.3).

Without loss of generality, we may assume that $r_- < r_0 < r_+$. In this case, when t is small, the solution of the initial value problem (1.1) and (3.1) can be expressed as follows (see Fig. 5):

$$(\rho_-, v_-) + J^- + \text{Vac} + J_1^- + \left(\frac{m_0}{2\varepsilon}, \frac{v_0}{2\varepsilon}\right) + J_1^+ + \text{Vac} + J^+ + (\rho_+, v_+),$$

where "+" means "followed by", $J^\pm$ and $J_1^\pm$ denote two contact discontinuities. The propagation speed of J_1^- is r_0, and that of J_1^+ is also r_0. This implies that the contact discontinuity J_1^- will never overtake the contact discontinuity J_1^+ in a finite time. So, we have completely constructed a solution of (1.1) and (3.1).

Next, letting $\varepsilon \to 0^+$, we have

$$\lim_{\varepsilon \to 0^+} \rho_0 = \lim_{\varepsilon \to 0^+} \frac{m_0}{2\varepsilon} = \infty, \tag{3.2}$$

$$\lim_{\varepsilon \to 0^+} \int_{r_0 t - \varepsilon}^{r_0 t + \varepsilon} \rho_0 dx = m_0. \tag{3.3}$$

The equalities (3.2) and (3.3) show that ρ_0 possesses a singularity as a weighted Dirac delta function with weight $w(t) = m_0$ when $\varepsilon \to 0^+$. Similarly, we have

$$\lim_{\varepsilon \to 0^+} \frac{v_0}{2\varepsilon} = \infty, \tag{3.4}$$

$$\lim_{\varepsilon \to 0^+} \int_{r_0 t - \varepsilon}^{r_0 t + \varepsilon} \frac{v_0}{2\varepsilon} dx = v_0. \tag{3.5}$$

The equalities (3.4) and (3.5) show that $\frac{v_0}{2\varepsilon}$ possesses a singularity as a weighted Dirac delta function when $\varepsilon \to 0^+$. Through the above analysis, it is clear that as $\varepsilon \to 0^+$, J_1^- and J_1^+ coincide to form a new type of nonlinear hyperbolic wave δJ, namely, a delta contact discontinuity on which both state variables ρ and v simultaneously contain the Dirac delta functions. This type of delta contact discontinuity is quite different from the previous ones on which only one state variable contains the Dirac delta function. Thus, letting $\varepsilon \to 0^+$, we obtain a solution of (1.1) and (1.3) as follows (see Fig. 6):

$$(\rho_-, v_-) + J^- + \text{Vac} + \delta J + \text{Vac} + J^+ + (\rho_+, v_+),$$

where the propagation speed of the δJ is r_0. Note that the obtained wave δJ is not overcompressive. It is usually called "delta contact discontinuity" (see [24,30]), which is a Dirac delta function supported on a contact discontinuity. So we now introduce the delta contact discontinuity, as in [24,30], as follows.

Definition 3.1 Consider a region Ω and a curve Γ in Ω of slope $\lambda = r$. A pair of distributions $(\rho, v) \in D'(\Omega) \times D'(\Omega)$ is called a delta contact discontinuity, if both ρ and v are sums of a locally integrable function on Ω and a delta function on Γ which solve (1.1) in the sense of distributions.

Remark 3 A distinctive feature for this delta contact discontinuity is that weighted Dirac delta functions develop in both state variables ρ and v simultaneously.

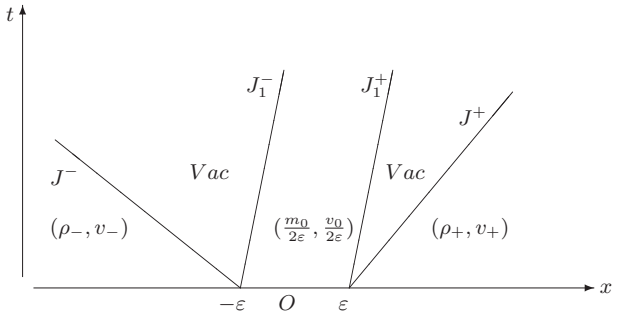


Fig. 5 The solution of the initial value problem (1.1) and (3.1) in Case $r_- < r_0 < r_+$

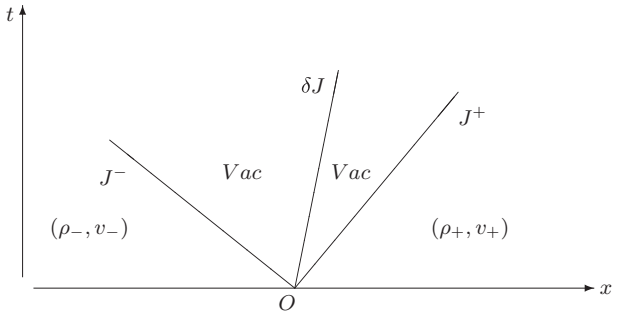


Fig. 6 The solution of Riemann problem (1.1) and (1.3) in Case $r_- < r_0 < r_+$

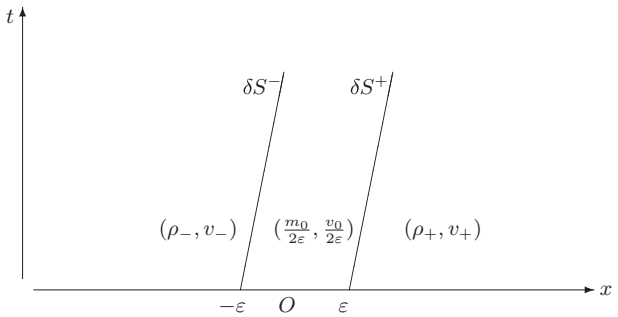


Fig. 7 The solution of the initial value problem (1.1) and (3.1) in Case $r_+ < r_0 < r_-$

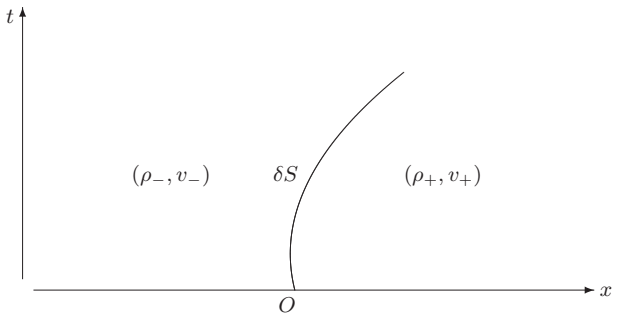


Fig. 8 The solution of Riemann problem (1.1) and (1.3) in Case $r_+ < r_0 < r_-$

Case 3.2. $r_+ < r_0 < r_-$.

This is a typical case. A delta shock wave emits from the origin (see [21,37,38]). Now, we seek the solution of (1.1) and (1.3) in the following form

$$\begin{cases} \rho(x,t) = \begin{cases} \rho_-, & x < x(t) \\ \rho_+, & x > x(t) \end{cases} \Bigg\} + w(t)\delta(x - x(t)), \\ v(x,t) = \begin{cases} v_-, & x < x(t) \\ v_+, & x > x(t) \end{cases} \Bigg\} + r_\delta(t)w(t)\delta(x - x(t)), \end{cases} \tag{3.6}$$

where $x(t)$, $w(t)$ and $r_\delta(t)$ satisfy the following generalized Rankine-Hugoniot relation:

$$\begin{cases} \frac{dx(t)}{dt} = \sigma, \\ \frac{dw(t)}{dt} = \sigma[\rho] - [v], \\ \frac{d(r_\delta(t)w(t))}{dt} = \sigma[v] - \left[\frac{v^2}{\rho}\right] \end{cases}$$

and

$$r|_{x=x(t)} = r_\delta(t) = \sigma,$$

i.e.,

$$\begin{cases} \frac{dx(t)}{dt} = r_\delta(t), \\ \frac{dw(t)}{dt} = r_\delta(t)[\rho] - [v], \\ \frac{d(r_\delta(t)w(t))}{dt} = r_\delta(t)[v] - \left[\frac{v^2}{\rho}\right], \end{cases} \tag{3.7}$$

where $[\rho] = \rho_+ - \rho_-$, etc. with initial data

$$(x, w, r_\delta)(0) = (0, m_0, r_0). \tag{3.8}$$

Similar to analysis in Case $r_- < r_0 < r_+$, we consider the initial value problem (1.1) with initial data (3.1). Because $r_0 < r_-$, $r_+ < r_0$, when t is small, the solution of the initial value problem (1.1) and (3.1) can be expressd briefly as follows (see Fig. 7):

$$(\rho_-, v_-) + \delta S^- + \left(\frac{m_0}{2\varepsilon}, \frac{v_0}{2\varepsilon}\right) + \delta S^+ + (\rho_+, v_+),$$

where "+" means "followed by", $\delta S^\pm$ denote two delta shock waves. The weight and propagation speed of δS^- are

$$\begin{aligned} w_1(t) &= (r_- - r_0)\sqrt{\frac{m_0}{2\varepsilon}\rho_-}, \\ \sigma_1 &= \frac{(\frac{v_0}{2\varepsilon} - v_-) + (r_- - r_0)\sqrt{\frac{m_0}{2\varepsilon}\rho_-}}{\frac{m_0}{2\varepsilon} - \rho_-}; \end{aligned}$$

and the weight and propagation speed of of δS^+ are

$$\begin{aligned} w_2(t) &= (r_0 - r_+)\sqrt{\frac{m_0}{2\varepsilon}\rho_+}, \\ \sigma_2 &= \frac{(v_+ - \frac{v_0}{2\varepsilon}) + (r_0 - r_+)\sqrt{\frac{m_0}{2\varepsilon}\rho_+}}{\rho_+ - \frac{m_0}{2\varepsilon}}. \end{aligned}$$

A simple calculation yields

$$\lim_{\varepsilon \to 0^+} \sigma_1 = \lim_{\varepsilon \to 0^+} \sigma_2 = r_0,$$

which implies that as $\varepsilon \to 0^+$, δS^- and δS^+ coincide to form a new type of nonlinear hyperbolic wave δS, namely, a delta shock wave on which both state variables ρ and v simultaneously contain the Dirac delta functions. This type of delta shock wave is quite different from the previous ones on which only one state variable contains the Dirac delta function. Thus, letting $\varepsilon \to 0^+$, we obtain a solution of (1.1) and (1.3) as follows (see Fig. 8):

$$(\rho_-, v_-) + \delta S + (\rho_+, v_+).$$

This solution is called a fusion solution which is actually produced by two delta shock waves of the form (2.26) overtaking each other at a finite time (see [37,38]), as shown in Fig. 8.

Integrating (3.7) from 0 to t with initial data (3.8), we have

$$\begin{cases} w - m_0 = [\rho]x - [v]t, \\ wr_\delta - m_0 r_0 = [v]x - \left[\frac{v^2}{\rho}\right]t, \end{cases} \tag{3.9}$$

which yields

$$m_0 r_\delta - m_0 r_0 + [\rho]x r_\delta - [v]t r_\delta - [v]x + \left[\frac{v^2}{\rho}\right]t = 0$$

or

$$\frac{d}{dt}\left\{\frac{1}{2}[\rho]x^2 - ([v]t - m_0)x + \frac{1}{2}\left[\frac{v^2}{\rho}\right]t^2 - m_0 r_0 t\right\} = 0. \tag{3.10}$$

Integrating (3.10) from 0 to t, we obtain

$$\frac{1}{2}[\rho]x^2 - ([v]t - m_0)x + \frac{1}{2}\left[\frac{v^2}{\rho}\right]t^2 - m_0 r_0 t = 0. \tag{3.11}$$

When $[\rho] \neq 0$, (3.11) is just a quadratic equation with respect to $x(t)$ and the discriminant is

$$\begin{aligned} \Delta &= ([v]t - m_0)^2 - 2[\rho]\left(\frac{1}{2}\left[\frac{v^2}{\rho}\right]t^2 - m_0 r_0 t\right) \\ &= ([r\rho]t - m_0)^2 - [\rho][\rho r^2]t^2 + 2[\rho]m_0 r_0 t \\ &= \rho_+\rho_-(r_- - r_+)^2 t^2 + 2m_0 t(\rho_+(r_0 - r_+) + \rho_-(r_- - r_0)) + m_0^2 > 0. \end{aligned}$$

By virtue of (3.11), one can find two solutions

$$x(t) = \frac{-m_0 + [v]t + w(t)}{[\rho]}, \tag{3.12}$$

and

$$x(t) = \frac{-m_0 + [v]t - w(t)}{[\rho]}, \tag{3.13}$$

where

$$w(t) = \sqrt{\rho_+\rho_-(r_- - r_+)^2 t^2 + 2m_0 t(\rho_+(r_0 - r_+) + \rho_-(r_- - r_0)) + m_0^2}.$$

From $(3.9)_2$, one has

$$\sigma(t) = r_\delta(t) = \frac{m_0 r_0 + [v]x(t) - \left[\frac{v^2}{\rho}\right]t}{w(t)}.$$

Next, with the help of the entropy condition $r_+ < \lim_{t\to+\infty} \sigma(t) < r_-$, we will choose the admissible solution to the equation (3.11) from (3.12) and (3.13).



For the solution (3.12), one has

$$\lim_{t\to+\infty}\sigma(t)=\frac{[v]+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]},\qquad [\rho]\neq 0.$$

By a brief calculation, when $[\rho]\neq 0$, one has

$$\begin{aligned}\lim_{t\to+\infty}\sigma(t)-r_+&=\frac{[v]-r_+[\rho]+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]}=\frac{\rho_-(r_+-r_-)+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]}\\&=\frac{\sqrt{\rho_-}(r_--r_+)}{\sqrt{\rho_+}+\sqrt{\rho_-}}>0,\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to+\infty}\sigma(t)-r_-&=\frac{[v]-r_-[\rho]+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]}=\frac{\rho_+(r_+-r_-)+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]}\\&=\frac{-\sqrt{\rho_+}(r_--r_+)}{\sqrt{\rho_+}+\sqrt{\rho_-}}<0,\end{aligned}$$

which imply that the entropy condition $r_+<\lim\limits_{t\to+\infty}\sigma(t)<r_-$ is valid. For the solution (3.13), when $[\rho]\neq 0$, one has

$$\begin{aligned}\lim_{t\to+\infty}\sigma(t)&=\frac{([v])^2-[v]\sqrt{\rho_+\rho_-}(r_--r_+)-\left[\frac{v^2}{\rho}\right][\rho]}{[\rho]\sqrt{\rho_+\rho_-}(r_--r_+)}\\&=\frac{([\rho r])^2-[v]\sqrt{\rho_+\rho_-}(r_--r_+)-[\rho r^2][\rho]}{[\rho]\sqrt{\rho_+\rho_-}(r_--r_+)}\\&=\frac{-[v]+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]},\end{aligned}$$

we can not rigorously confirm that the entropy condition $r_+<\lim\limits_{t\to+\infty}\sigma(t)<r_-$ is valid. So, when $[\rho]\neq 0$, one has constructively obtained a solution of the equation (3.11) as follows:

$$x(t)=\frac{-m_0+[v]t+w(t)}{[\rho]},\qquad [\rho]\neq 0.$$

When $[\rho]=0$, (3.11) is just a linear equation of $x(t)$, and then, one has

$$x(t)=\frac{m_0r_0t-\frac{1}{2}\left[\frac{v^2}{\rho}\right]t^2}{m_0-[v]t},\qquad [\rho]=0.$$

Then, one has

$$x(t)=\begin{cases}\frac{m_0r_0t-\frac{1}{2}\left[\frac{v^2}{\rho}\right]t^2}{m_0-[v]t}, & [\rho]=0,\\ \frac{-m_0+[v]t+w(t)}{[\rho]}, & [\rho]\neq 0,\end{cases}$$

where

$$w(t)=\sqrt{\rho_+\rho_-(r_--r_+)^2t^2+2m_0t(\rho_+(r_0-r_+)+\rho_-(r_--r_0))+m_0^2}.$$

From $(3.9)_2$, it is easy to see that

$$\lim_{t\to+\infty}\sigma(t)=\begin{cases}\frac{v_++v_-}{2\rho_+}, & [\rho]=0,\\ \frac{[v]+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]}, & [\rho]\neq 0.\end{cases}$$

So, we have $r_+<\lim\limits_{t\to+\infty}\sigma(t)<r_-$.

In fact, if $[\rho]=0$, then from $r_->r_+$, it follows that

$$r_+ < \lim_{t\to+\infty}\sigma(t) < r_-, \qquad \text{as } [\rho]=0.$$

If $[\rho]\neq 0$, then, we have

$$\begin{aligned}\lim_{t\to+\infty}\sigma(t)-r_+ &= \frac{[v]-r_+[\rho]+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]} = \frac{\rho_-(r_+-r_-)+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]}\\ &= \frac{\sqrt{\rho_-}(r_--r_+)}{\sqrt{\rho_+}+\sqrt{\rho_-}} > 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to+\infty}\sigma(t)-r_- &= \frac{[v]-r_-[\rho]+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]} = \frac{\rho_+(r_+-r_-)+\sqrt{\rho_+\rho_-}(r_--r_+)}{[\rho]}\\ &= \frac{-\sqrt{\rho_+}(r_--r_+)}{\sqrt{\rho_+}+\sqrt{\rho_-}} < 0.\end{aligned}$$

Remark 4 It is not proved in this work that the solutions derived constructively in Cases 3.1–3.4 from the differential equation plus Rankine-Hugoniot conditions are uniqueness. In fact, it is interesting to study the question of uniqueness of solutions. We leave this problem for our future work.

Case 3.3. $r_0<r_-<r_+$. (If $r_-<r_+<r_0$, then the structure of the solution is similar).

It is easy to see that the particles at $x_0<0$ collide with the particle at $x_0=0$ at the start. Let $x(t)$, $w(t)$ and $\sigma(t)$ be the trajectory, mass and velocity of the fused particles, respectively. Since $r_-<r_+$ and $r_0<\sigma(t)<r_-$, we find that $\sigma(t)$ is always less than r_+, showing that the particles $x_0\le 0$ never collide with those $x_0>0$. Thus, the solution of the initial value problem (1.1) and (1.3) can be expressed as

$$\begin{cases}\rho(x,t)=\begin{cases}\rho_-, & x<x(t)\\ \text{Vac}, & x(t)<x<r_+t\\ \rho_+, & x>r_+t\end{cases}+w(t)\delta(x-x(t)),\\ v(x,t)=\begin{cases}v_-, & x<x(t)\\ \text{Vac}, & x(t)<x<r_+t\\ v_+, & x>r_+t\end{cases}+r_\delta(t)w(t)\delta(x-x(t)),\end{cases}\tag{3.14}$$

We denote the delta shock wave with δS in this region, when one constructs solutions, how to overcome a problem with the vacuum state on the left- or right-hand side? What to do when $r=\frac{0}{0}$? Indeed, we believe that a resolution of this difficulty will be a major step in solving the Riemann problem (1.1) and (1.3). In order to overcome a problem with the vacuum state on the left- or right-hand side, we adopt the idea proposed by Liu and Smoller [23] when they considered the vacuum problem for isentropic gas dynamic equations, and then approximate them by a set of small non-admissible shocks like as the front tracking algorithm [4]. Thus, we consider the initial value problem (1.1) with initial data as follows:

$$\begin{cases}\rho(x,0)=\begin{cases}\rho_-, & x<0\\ \varepsilon m_0, & x>0\end{cases}+m_0\delta(x),\\ v(x,0)=\begin{cases}v_-, & x<0\\ \varepsilon v_0, & x>0\end{cases}+v_0\delta(x),\end{cases}\tag{3.15}$$

where $\varepsilon>0$ is arbitrarily small.

In view of $r_->r_0$, a new delta shock wave will generate and we denote it with $\delta S^\varepsilon: x=x^\varepsilon(t)$, which can be expressed as

$$\begin{cases}\rho(x,t)=\begin{cases}\rho_-, & x<x^\varepsilon(t)\\ \varepsilon m_0, & x>x^\varepsilon(t)\end{cases}+w^\varepsilon(t)\delta(x-x^\varepsilon(t)),\\ v(x,t)=\begin{cases}v_-, & x<x^\varepsilon(t)\\ \varepsilon v_0, & x>x^\varepsilon(t)\end{cases}+r_\delta^\varepsilon(t)w^\varepsilon(t)\delta(x-x^\varepsilon(t)),\end{cases}\tag{3.16}$$

where $x^\varepsilon(t)$, $w^\varepsilon(t)$ and $r_\delta^\varepsilon(t)$ satisfy the following generalized Rankine-Hugoniot relation:

$$\begin{cases} \frac{dx^\varepsilon(t)}{dt} = \sigma^\varepsilon, \\ \frac{dw^\varepsilon(t)}{dt} = \sigma^\varepsilon[\rho] - [v], \\ \frac{d(r_\delta^\varepsilon(t)w^\varepsilon(t))}{dt} = \sigma^\varepsilon[v] - \left[\frac{v^2}{\rho}\right], \end{cases}$$

and

$$r|_{x=x^\varepsilon(t)} = r_\delta^\varepsilon(t) = \sigma^\varepsilon,$$

i.e.,

$$\begin{cases} \frac{dx^\varepsilon(t)}{dt} = r_\delta^\varepsilon(t), \\ \frac{dw^\varepsilon(t)}{dt} = r_\delta^\varepsilon(t)[\rho] - [v], \\ \frac{d(r_\delta^\varepsilon(t)w^\varepsilon(t))}{dt} = r_\delta^\varepsilon(t)[v] - \left[\frac{v^2}{\rho}\right], \end{cases} \tag{3.17}$$

where $[\rho] = \varepsilon m_0 - \rho_-$, etc, with initial data

$$(x^\varepsilon, w^\varepsilon, r_\delta^\varepsilon)(0) = (0, m_0, r_0). \tag{3.18}$$

The trajectory, velocity and weight $(x^\varepsilon(t), r_\delta^\varepsilon(t), w^\varepsilon(t))$ of δS^ε can be uniquely obtained by solving the initial value problem (3.17) and (3.18). Then letting $(x(t), r_\delta(t), w(t)) = \lim_{\varepsilon\to 0}(x^\varepsilon(t), r_\delta^\varepsilon(t), w^\varepsilon(t))$, we immediately get the trajectory, velocity and weight $(x(t), r_\delta(t), w(t))$ of δS under the stability theory of weak solutions.

Integrating (3.17) from 0 to t with initial data (3.18), we have

$$\begin{cases} w^\varepsilon - m_0 = (\varepsilon m_0 - \rho_-)x^\varepsilon - (\varepsilon v_0 - v_-)t, \\ w^\varepsilon r_\delta^\varepsilon - m_0 r_0 = (\varepsilon v_0 - v_-)x^\varepsilon - \left(\frac{\varepsilon v_0^2}{m_0} - \frac{v_-^2}{\rho_-}\right)t. \end{cases} \tag{3.19}$$

Taking the limit $\varepsilon \to 0$ in (3.19), we have

$$\begin{cases} w - m_0 = -\rho_- x + v_- t, \\ w r_\delta - m_0 r_0 = -v_- x + \frac{v_-^2}{\rho_-}t. \end{cases} \tag{3.20}$$

It follows that

$$m_0 r_\delta - \rho_- x r_\delta + v_- t r_\delta + v_- x - m_0 r_0 - \frac{v_-^2}{\rho_-}t = 0$$

or

$$\frac{d}{dt}\left\{\frac{1}{2}\rho_- x^2 - v_- t x - m_0 x + m_0 r_0 t + \frac{v_-^2}{2\rho_-}t^2\right\} = 0. \tag{3.21}$$

Integrating (3.21) from 0 to t, we obtain

$$\frac{1}{2}\rho_- x^2 - v_- t x - m_0 x + m_0 r_0 t + \frac{v_-^2}{2\rho_-}t^2 = 0. \tag{3.22}$$

Solving (3.22) gives

$$x(t) = \frac{m_0 + v_- t - w(t)}{\rho_-}, \tag{3.23}$$

where

$$w(t) = \sqrt{m_0^2 + 2t m_0 \rho_-(r_- - r_0)}. \tag{3.24}$$

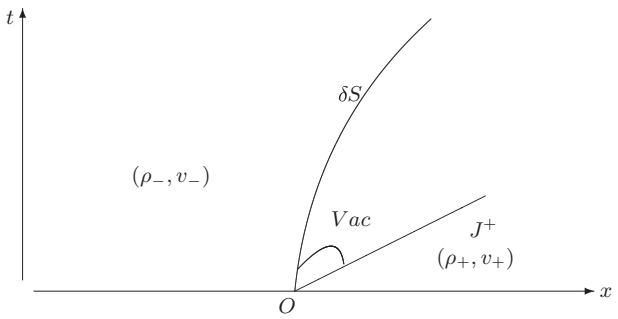


Fig. 9 The solution of Riemann problem (1.1) and (1.3) in Case $r_0 < r_- < r_+$

Substituting (3.23) into $(3.20)_2$, we have

$$\sigma(t) = r_\delta(t) = r_- - \frac{m_0(r_- - r_0)}{w(t)}. \tag{3.25}$$

From (3.25), it is clear that $\sigma(t) < r_- < r_+$. This implies that the delta shock δS will never overtake the contact discontinuity $J^+ : x = r_+t$ and finally has $x(t) = r_-t$ as its asymptote. So far, we have completely constructed a solution of (1.1) and (1.3) as follows (see Fig. 9):

$$(\rho_-, v_-) + \delta S + \text{Vac} + J^+ + (\rho_+, v_+).$$

Case 3.4. $r_0 < r_+ < r_-$. (If $r_+ < r_- < r_0$, then the structure of the solution is similar.)

Similar to the analysis in Case 3.3, we know that, in this case, when t is small enough, the solution is the same as that in Case 3.3. More precisely, when $0 \le t < t^*$, the solution is the same as that in Case 3.3, which can be expressed as

$$\begin{cases} \rho(x,t) = \begin{cases} \rho_-, & x < x(t) \\ \text{Vac}, & x(t) < x < r_+t \\ \rho_+, & x > r_+t \end{cases} \end{cases} + w(t)\delta(x - x(t)),$$
$$\begin{cases} v(x,t) = \begin{cases} v_-, & x < x(t) \\ \text{Vac}, & x(t) < x < r_+t \\ v_+, & x > r_+t \end{cases} \end{cases} + r_\delta(t)w(t)\delta(x - x(t)), \tag{3.26}$$

for $0 \le t < t^*$, where t^* is the time that the delta shock begins to meet the particles from the right $x_0 > 0$, and t^* is determined by

$$\begin{cases} x(t^*) = r_+t^*, \\ x(t^*) = \frac{m_0 + v_-t^* - \sqrt{m_0^2 + 2t^*m_0\rho_-(r_- - r_0)}}{\rho_-}, \end{cases}$$

which means that

$$t^* = \frac{2m_0(r_+ - r_0)}{\rho_-(r_- - r_+)^2}. \tag{3.27}$$

At the time $t = t^*$, the particles $x_0 \le 0$ start to collide with those $x_0 > 0$. At that moment, we have

$$t = t^* : x_1(t^*) = x(t^*) := x^*,$$
$$w_1(t^*) = w(t^*) = \sqrt{m_0^2 + 2t^*m_0\rho_-(r_- - r_0)} = \frac{m_0(r_- + r_+ - 2r_0)}{r_- - r_+} := w^*, \tag{3.28}$$
$$\sigma_1(t^*) = r_{\delta 1}(t^*) = \sigma(t^*) = \frac{v_0 - m_0r_-}{w(t^*)} + r_- = \frac{2r_-r_+ - r_0r_- - r_0r_+}{r_- + r_+ - 2r_0} := \sigma^*, \tag{3.29}$$

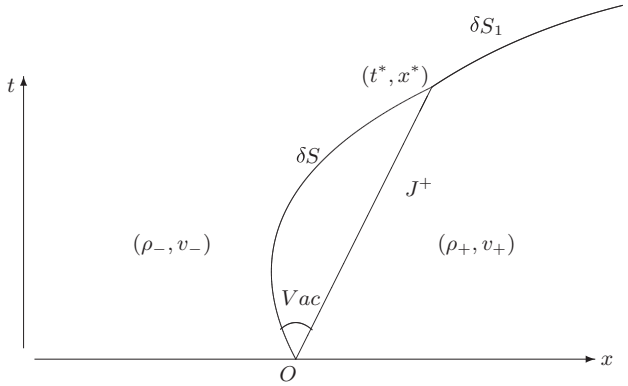


Fig. 10 The solution of Riemann problem (1.1) and (1.3) in Case $r_0 < r_+ < r_-$

and after that time, the solution of (1.1) and (1.3) can be expressed as

$$\begin{cases} \rho(x,t) = \begin{cases} \rho_-, & x < x_1(t) \\ \rho_+, & x > x_1(t) \end{cases} \Big\} + w_1(t)\delta(x - x_1(t)), \\ v(x,t) = \begin{cases} v_-, & x < x_1(t) \\ v_+, & x > x_1(t) \end{cases} \Big\} + r_{\delta 1}(t) w_1(t)\delta(x - x_1(t)) \end{cases} \tag{3.30}$$

for $t^* \le t < +\infty$, where $x_1(t)$, $w_1(t)$ and $r_{\delta 1}(t)$ satisfy the following generalized Rankine-Hugoniot relation:

$$\begin{cases} \frac{dx_1(t)}{dt} = \sigma_1, \\ \frac{dw_1(t)}{dt} = \sigma_1[\rho] - [v], \\ \frac{d(r_{\delta 1}(t)w_1(t))}{dt} = \sigma_1[v] - \left[\frac{v^2}{\rho}\right], \end{cases} \tag{3.31}$$

and

$$r|_{x=x_1(t)} = r_{\delta 1}(t) = \sigma_1, \tag{3.32}$$

i.e.,

$$\begin{cases} \frac{dx_1(t)}{dt} = r_{\delta 1}(t), \\ \frac{dw_1(t)}{dt} = r_{\delta 1}(t)[\rho] - [v], \\ \frac{d(r_{\delta 1}(t)w_1(t))}{dt} = r_{\delta 1}(t)[v] - \left[\frac{v^2}{\rho}\right], \end{cases} \tag{3.33}$$

where $[\rho] = \rho_+ - \rho_-$, etc. with initial data

$$(x_1, w_1, r_{\delta 1})(t^*) = (x^*, w^*, \sigma^*). \tag{3.34}$$

Similar to the Case 3.2, solving the initial value problem (3.33) and (3.34), we obtain

$$x_1(t) = \begin{cases} \frac{w^*\sigma^*(t-t^*) - \frac{1}{2}\left[\frac{v^2}{\rho}\right](t-t^*)^2}{w^* - [v](t-t^*)} + x^*, & [\rho] = 0, \\ \frac{[v](t-t^*) - w^* + w_1(t)}{[\rho]} + x^*, & [\rho] \neq 0, \end{cases} \tag{3.35}$$

$$\begin{aligned} w_1(t) &= \sqrt{(w^*)^2 + 2w^*(-[v] + [\rho]\sigma^*)(t-t^*) + \rho_+\rho_-(r_+ - r_-)^2(t-t^*)^2} \\ &= \sqrt{(w^*)^2 + 2m_0(t-t^*)(\rho_+(r_+ - r_0) + \rho_-(r_- - r_0)) + \rho_+\rho_-(r_+ - r_-)^2(t-t^*)^2}\,, \end{aligned} \tag{3.36}$$

and

$$\sigma_1(t) = r_{\delta 1}(t) = \frac{w^*\sigma^* - [v]x^* + [v]x_1(t) - [\frac{v^2}{\rho}](t-t^*)}{w_1(t)}. \tag{3.37}$$

When $[\rho] = 0$, it follows from $(3.35)_1$, (3.36) and (3.37) that

$$\begin{aligned}\lim_{t\to+\infty}\sigma_1(t) &= \lim_{t\to+\infty}\frac{[v]\left(\frac{w^*\sigma^*(t-t^*)-\frac{1}{2}\left[\frac{v^2}{\rho}\right](t-t^*)^2}{w^*-[v](t-t^*)} + x^*\right) - \left[\frac{v^2}{\rho}\right](t-t^*)}{\sqrt{(w^*)^2 + 2w^*(-[v]+[\rho]\sigma^*)(t-t^*) + \rho_+\rho_-(r_+-r_-)^2(t-t^*)^2}} \\ &= \frac{[v]\frac{-\frac{1}{2}\left[\frac{v^2}{\rho}\right]}{-[v]} - \left[\frac{v^2}{\rho}\right]}{\sqrt{\rho_+\rho_-}(r_- - r_+)} = \frac{-\frac{1}{2}[r^2\rho]}{\sqrt{\rho_+\rho_-}(r_- - r_+)} = \frac{1}{2}(r_+ + r_-) = \frac{v_+ + v_-}{2\rho_+}, \quad [\rho] = 0. \end{aligned} \tag{3.38}$$

Similarly, when $[\rho] \neq 0$, it follows from $(3.35)_2$, (3.36) and (3.37) that

$$\begin{aligned}\lim_{t\to+\infty}\sigma_1(t) &= \lim_{t\to+\infty}\frac{[v]\left(\frac{[v](t-t^*)-w^*+w_1(t)}{[\rho]} + x^*\right) - \left[\frac{v^2}{\rho}\right](t-t^*)}{\sqrt{(w^*)^2 + 2w^*(-[v]+[\rho]\sigma^*)(t-t^*) + \rho_+\rho_-(r_+-r_-)^2(t-t^*)^2}} \\ &= \frac{[v]\frac{[v]}{[\rho]} + \frac{[v]}{[\rho]}\sqrt{\rho_+\rho_-}(r_- - r_+) - \left[\frac{v^2}{\rho}\right]}{\sqrt{\rho_+\rho_-}(r_- - r_+)} \\ &= \frac{([r\rho])^2 + [r\rho]\sqrt{\rho_+\rho_-}(r_- - r_+) - [r^2\rho][\rho]}{[\rho]\sqrt{\rho_+\rho_-}(r_- - r_+)} \\ &= \frac{\sqrt{\rho_+\rho_-}(r_- - r_+) + r_+\rho_+ - r_-\rho_-}{[\rho]} = \frac{[v] + \sqrt{\rho_+\rho_-}(r_- - r_+)}{[\rho]}, \quad [\rho] \neq 0. \end{aligned} \tag{3.39}$$

From (3.38) and (3.39), we obtain

$$\lim_{t\to+\infty}\sigma_1(t) = \begin{cases} \frac{v_+ + v_-}{2\rho_+}, & [\rho] = 0, \\ \frac{[v] + \sqrt{\rho_+\rho_-}(r_- - r_+)}{[\rho]}, & [\rho] \neq 0. \end{cases} \tag{3.40}$$

So we have

$$r_+ < \lim_{t\to+\infty}\sigma_1(t) < r_-.$$

So far, we have completely constructed a solution of (1.1) and (1.3), see Fig. 10. Thus, we have finished the discussion for all the cases and constructively obtained the global existence of generalized solutions of (1.1) and (1.3) that explicitly exhibit four kinds of different structures involving delta shock waves.

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Data Availability No data was used for the research described in the article.

Declarations

Conflict of Interest The authors have no conflicts to disclose.



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